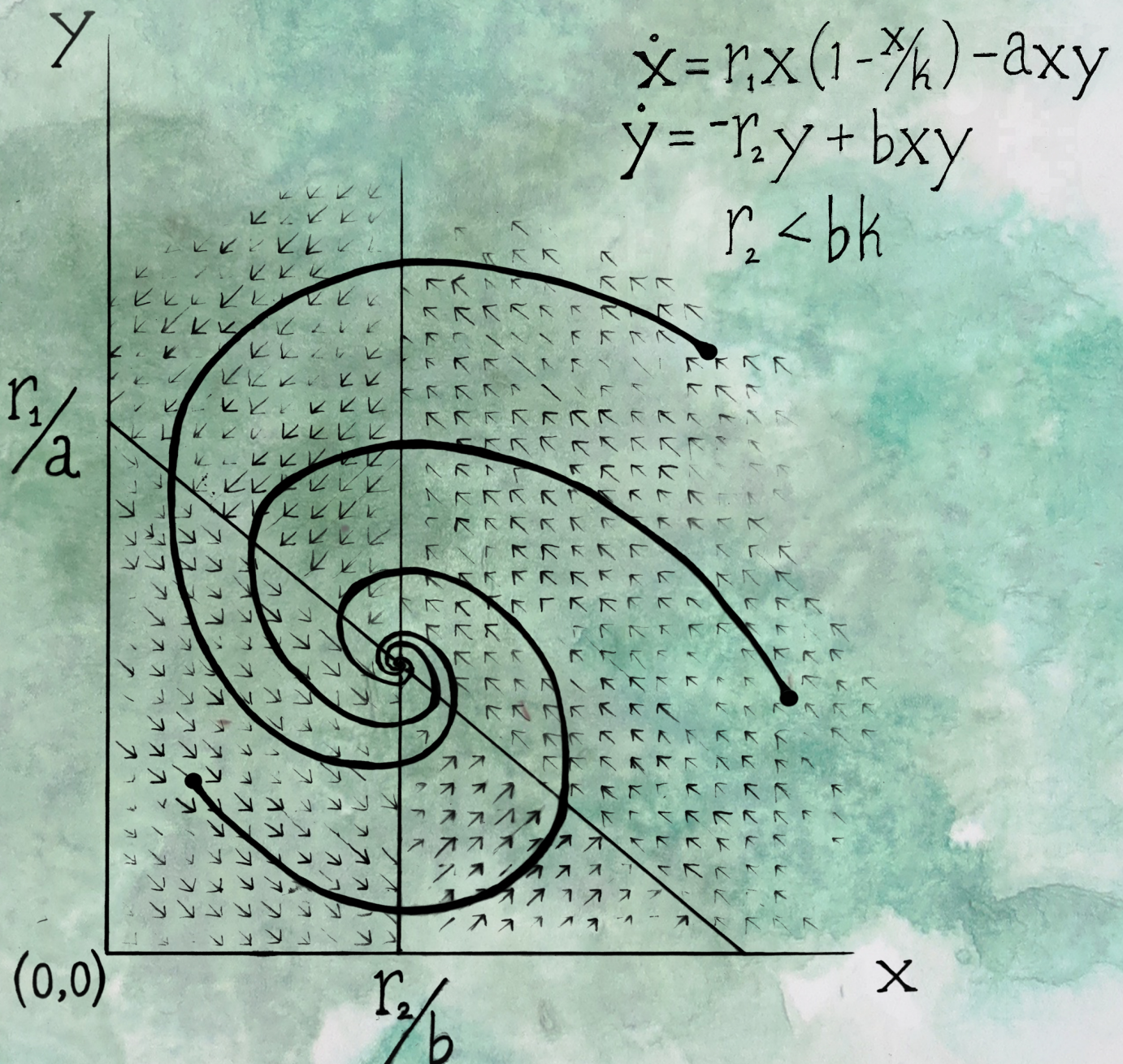


Differential Equations

A Toolbox for Modeling the World

Kurt Bryan



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1. Why Study Differential Equations?

To begin, we offer mathematical models of three quite different physical situations. Remarkably, all can be described by similar mathematics. These three examples and many others appear throughout the text and will help illustrate and motivate the mathematics to come.

1.1 The 2008 Olympic 100-Meter Dash

The material in this section is based on the SIMIODE project “Dash It All!” [33].

1.1.1 Usain Bolt’s Olympic Victory

Table 1.1 contains data from the 100-meter dash final at the Olympic Games in Beijing in 2008 [11]. The times belong to the gold medal winner Usain Bolt and represent a world record of 9.69 seconds, which he lowered in 2009 to 9.58 seconds. The data are in the form of (time, distance) pairs, where distance is measured in meters, horizontally along the track from the starting line, and time is measured in seconds elapsed from the firing of the starting gun. The initial (0.165, 0) data point indicates that Bolt had a reaction time of 0.165 seconds after the gun was fired before he started running and crossed the starting line. Between the 50 and 80 meter mark Bolt averaged 12.2 meters per second, an astonishing 27.3 miles per hour. After the 80 meter mark he actually eased up and looked back at the other runners; see [3].

Time (seconds)	0.165	1.85	2.87	3.78	4.65	5.50
Position (meters)	0	10	20	30	40	50
Time (seconds)	6.32	7.14	7.96	8.79	9.69	
Position (meters)	60	70	80	90	100	

Table 1.1: Race splits (seconds) every 10 meters for Usain Bolt’s 2008 Olympic gold medal final [11].

Our goals are to use this data to:

- (1) Develop a mathematical model of sprinting, a quantitative description that explains the data in Table 1.1. This description should be based on accepted physics, mathematics, and reasonable assumptions.
- (2) Test or validate this model by using it to make predictions about how fast Bolt or comparable sprinters could run other distances, and determine conditions under which the model is accurate.

Although we have data only for Bolt, we want our model to be more generally applicable. For the moment let us focus on sprinting, and assume that the runner applies maximum effort throughout the race.

1.1.2 Modeling a Sprint

We now consider a classic mathematical model for sprinting, the Hill-Keller model [66, 72]. The model is grounded in Newton's second law of motion, $F = ma$, where m denotes the mass of an object, a the acceleration of the object, and F the net force acting on the object. This is a fundamental law of physics, at least non-relativistic physics. (Bolt is fast, but not that fast.) In general, force and acceleration are three-dimensional vectors, but in our model they may be treated as scalars for reasons described below.

In order to explain the data in Table 1.1, we need to predict Bolt's position on the track as a function of time. We will thus introduce a variable t to denote time, in seconds, from the start of the race and x to denote position on the straight track, in meters, with $x = 0$ as the starting line and positive values of x in the direction of the finish line.

Mathematical models involve making assumptions and simplifications. In our case, matters are simplified by focusing only on the sprinter's horizontal motion along the track. Any other motion or forces, for example, vertical or side-to-side, will be ignored. As such, the sprinter's position, velocity, and acceleration are parallel to the track and may be considered scalar or one-dimensional quantities. We limit our attention to this component of motion. Our model will initially focus on the sprinter's velocity as a function of time, which can be described by some function $v(t)$, where v will be measured in meters per second. If $v(t)$ can be determined then we can then integrate $v(t)$ to predict the runner's position at any time, and in Bolt's case, compare this to the data.

Remark 1.1.1 In this text we will use various notations for the derivative of a function f . In most cases f will be a function of a single independent variable t , and t will denote time. We will write df/dt , f' , or \dot{f} ; this last notation is common in physics and is used only when the independent variable is time. In each case we may or may not explicitly write the independent variable, that is, we may write $f'(t)$ or just f' . Second derivatives are denoted by d^2f/dt^2 , f'' , or \ddot{f} .

We now apply Newton's second law of motion $F = ma$ to a sprinting race. Here m will be the sprinter's mass, which is not known and, happily, won't be needed. The variable a denotes the sprinter's acceleration, which will be a function of time, and F is the net force on the sprinter.

Reading Exercise 1.1.1 Express the sprinter's acceleration $a(t)$ in terms of velocity $v(t)$.

Reading Exercise 1.1.2 List in plain English all of the horizontal forces you can think of that might be relevant to the sprinter's progress down the track. Which forces aid progress down the track? Which impede progress?

The heart of the Hill-Keller model is an examination of the net horizontal force F on the runner, which is split into the sum of a propulsive force F_p that aids the runner and a resistive force F_r that impedes the runner's motion.

To quantify the propulsive force F_p , assume that F_p depends on the runner's level of exertion and is at a constant and maximum value throughout the race. That is, in a short race like a sprint,

the runner exerts a maximum propulsive force for the duration of the race and this force does not depend on the runner's current velocity. Moreover, in the Hill-Keller model this maximum propulsive force is treated as being proportional to the runner's mass m , under the assumption that if sprinter A is twice as massive as sprinter B then sprinter A should be capable of exerting twice the propulsive force of sprinter B. Under this assumption

$$F_p = mP, \tag{1.1}$$

where P is some constant that depends on the runner's maximum propulsive effort. Reading Exercise 1.1.3 gives some ideas on how we might interpret P physically.

Reading Exercise 1.1.3

- Given that F_p is a force and m is a mass, argue that P should have units of acceleration. Hint: consider Newton's second law. (See Section 1.5 for a more extensive discussion of the value of understanding the units or physical dimensions involved when modeling.)
- Suppose the runner is standing still, with no forces acting on the runner, who then suddenly applies maximum propulsive effort according to (1.1). Given that also $F = ma$, what physical interpretation can you give to P at this instant?

The quantity P has been measured for world-class sprinters like Bolt and is approximately 11.0 meters per second squared (see [99]). This is the value we'll use for P , at least for now. In doing so we are using meters, kilograms, and seconds for our analysis (SI units), so t will be measured in seconds and $v(t)$ in meters per second.

The resistive force F_r should, of course, oppose the runner's motion. In general we expect that the faster the runner moves, the greater the resistive force. Let's start with the simplest model that captures this idea: F_r should be proportional to the runner's velocity v , and in the opposite direction to v . Where do these resistive forces come from? In the Hill-Keller model these resistive forces are considered to be predominantly internal to the runner, a sort of friction of joints and muscles that opposes rapid motion, rather than external factors like air resistance. As such, like the propulsive force, the resistive force is modeled as proportional to the mass of the runner, under the reasoning that a runner who is twice as large has twice the internal resistance to motion. In summary, the resistive force F_r is proportional to both the runner's velocity v and the runner's mass m . That is, F_r is jointly proportional to v and m , and opposed to v . A simple model that captures this is

$$F_r = -kmv(t) \tag{1.2}$$

where k is a positive constant. The explicit minus sign on the right in (1.2) with the specification that $k > 0$ assures that F_r is opposed to v .

Modeling Tip 1.1.1 In mathematical modeling one encounters many physical constants, e.g., m and k as in (1.2). When a constant must be of one sign, positive or negative, it is common to take the constant as positive and explicitly add a minus sign if necessary. This helps to alert the reader that the constant in question is of one particular sign.

The value of k in (1.2) is not known, but can be estimated from data. For now think of k as a known but unspecified positive constant.

1.1.3 The Hill-Keller Differential Equation

We now have all the pieces necessary to construct a model that will (eventually) lead to an explanation of the data in Table 1.1, and provide broader insight into how a sprinter progresses down the track.

Reading Exercise 1.1.4 The total force F on the runner is $F = F_p + F_r$. Combine (1.1), (1.2), and $F = ma$ with the result of Reading Exercise 1.1.1 to show that the function $v(t)$ must satisfy

$$v'(t) = P - kv(t). \quad (1.3)$$

Note that m drops out.

The relation between $v'(t)$ and $v(t)$ in (1.3) must hold for all times during the race at which the runner is exerting maximum propulsive effort. The function $v(t)$ is the unknown we seek. Equation (1.3) is an **ordinary differential equation (ODE)**, that is, an equation involving an unknown function of one independent variable and that function's derivatives. The goal is to determine the unknown function $v(t)$ by using (1.3).

Reading Exercise 1.1.5 Based on your intuition, sketch what the graph of $v(t)$ would look like over the course of a race that lasts about 10 seconds.

Reading Exercise 1.1.6 Unfortunately, there are infinitely many solutions to (1.3). Verify that each of the following choices for $v(t)$ satisfies equation (1.3) (that is, $v'(t)$ is identically equal to $P - kv(t)$ as a function of t).

- (a) $v(t) = P/k$
- (b) $v(t) = P/k - Pe^{-kt}/k$
- (c) $v(t) = P/k - Ce^{-kt}$, where C is any constant.

In Reading Exercise 1.1.6 you may notice that parts (a) and (b) are special cases ($C = 0$ and $C = P/k$, respectively) of part (c). Indeed, the differential equation (1.3) has infinitely many solutions, all of the form in part (c) for some choice of the constant C . In this case $v(t) = P/k - Ce^{-kt}$ is called a **general solution** to the ODE (1.3). This means that all solutions to (1.3) can be expressed in the form $v(t) = P/k - Ce^{-kt}$ for some constant C . Given that there are infinitely many solutions, which one is relevant to the present case? It seems we need a bit more information, something that you may have noticed in Reading Exercise 1.1.5.

Reading Exercise 1.1.7 What piece of information is missing? Hint: look at the first few entries in Table 1.1. What was Bolt's velocity at the start of the race?

You should conclude from Reading Exercise 1.1.7 that $v(t)$ satisfies $v(0.165) = 0$. This is the **initial condition** for $v(t)$. After $t = 0.165$ the function $v(t)$ satisfies (1.3). Equation (1.3), together with the initial condition $v(0.165) = 0$ constitutes an **initial value problem**. It turns out that there is a unique (one, and only one) solution $v(t)$ to this initial value problem, and this is what we want to find. A proof of this fact is presented in more advanced differential equations texts, but further remarks will be made on this matter in Section 2.4. Once $v(t)$ is known we can integrate with respect to t to find Bolt's position as a function of time, and then adjust k to match this model to the data in Table 1.1. But it will first be helpful to develop some techniques for solving differential equations, and for determining the optimal value for k . Although we will initially use $P = 11$ meters per second squared, we may wish to adjust this value as well, to better fit the data in Table 1.1.

Reading Exercise 1.1.8 You might be tempted to state $v(0) = 0$ for the initial condition, which is certainly true since Bolt should not have been in motion when the gun was fired. But does (1.3) hold for all $t > 0$? Hint: What is P in (1.1) for $0 < t < 0.165$?

At this point we will leave the Hill-Keller model and return to it after mastering some techniques for analyzing and solving differential equations. The Hill-Keller model rests upon basic physics, specifically Newton's second law of motion. The next section illustrates another common modeling technique.

Reading Exercise 1.1.9 Review the derivation of (1.3) and list every assumption we made in constructing this model.

1.2 Intracochlear Drug Delivery

The material in this section is based on the SIMIODE project “Intracochlear Drug Delivery” [116].

1.2.1 The Challenge of Hearing Loss

Over 5% of the world’s population—or 466 million people—have disabling hearing loss [130]. The World Health Organization estimates that by 2050 over 900 million people, or about one in every ten people, will suffer from such hearing loss. Treating this hearing loss will be a significant challenge.

One aspect of this challenge is that the inner ear is surrounded by dense temporal bone and protected by the blood-cochlea barrier, as illustrated in Figure 1.1. The cochlea, with the shape of a snail, is the part of the inner ear involved in hearing. It is lined by sensory hair cells and is filled with fluid (about 0.2 milliliters, or 200 microliters). The cochlea is a particularly difficult target for drug therapy aimed at treating hearing loss. Oral medications and injections are typically blocked by the blood-cochlea barrier, and thus ineffective in reaching or delivering precise doses of drugs to the cochlea.

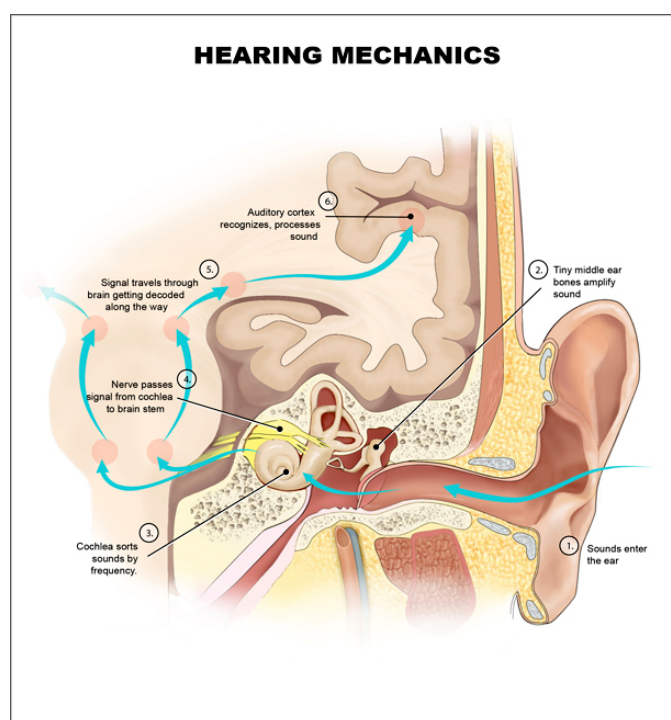


Figure 1.1: Anatomy of the human ear (from Wikimedia Commons, [41]).

As an alternative to systemic administration, localized drug delivery methods have emerged. One approach is the use of reciprocating perfusion systems based on microfluidic technologies [109]. This approach releases drugs directly to the inner ear, in order to support regeneration of the sensory hair cells and auditory nerves inside the cochlea, and enables precise targeting of drug concentrations within the therapeutic window for extended delivery. These implantable microfluidic devices are connected with a small tube to the cochlea. A battery-powered micropump pulses precise quantities of a drug from a small reservoir into the cochlea in a push-pull mode, i.e., infusing and withdrawing cochlear fluid in a cyclic manner nearly simultaneously so that the fluid volume inside the cochlea stays constant.

In order to avoid damage to hearing structures, limits on the maximum rate at which fluid can be pumped into the cochlea place stringent requirements on the system. It is challenging to design

reliable systems that are capable of maintaining control over drug concentrations for long-term drug release. To address the difficulties in drug delivery and achieve safety and efficiency, we need an effective quantitative model of the situation. In particular, we need to know how the concentration of the drug being administered varies over time inside the cochlea, and how the concentration depends on the physical parameters involved.

1.2.2 A Compartmental Model for the Cochlea

As a first approximation, consider Figure 1.2. This is an example of a **compartmental model**, a model which consists of one or more compartments with conduits into and out of each. The model in Figure 1.2 is a single-compartment model. Our task is to account for the rate at which some substance moves into and out of the compartment, and how the amount of the substance in the compartment changes with time. In this case the compartment represents the cochlea.

The input to the cochlea is through a single conduit in a push-pull or input-output operation. That is, a tiny amount of drug-containing fluid is introduced through the pipe into the cochlea; the drug then diffuses throughout the cochlea, and then a short time later the same amount of fluid is withdrawn.

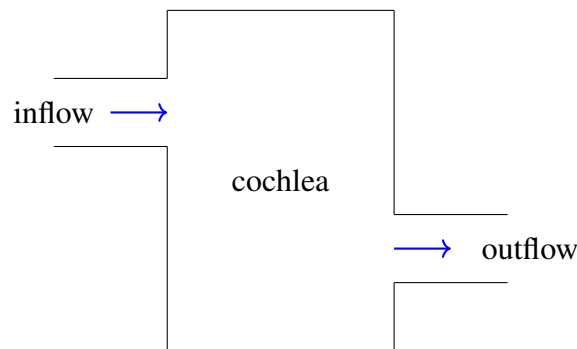


Figure 1.2: A compartmental model for cochlear drug delivery.

However, we will model the situation as if the drug-containing fluid is introduced through one input (the inflow pipe in Figure 1.2) and withdrawn through another (the outflow pipe) continually, at the same volumetric rate. The justification for this is as follows: we assume that when a tiny amount of drug-containing fluid is introduced into the cochlea, the drug diffuses rapidly and uniformly throughout the volume of the cochlea, so the concentration of the drug in the cochlea is always spatially constant. As a result, during the withdrawal phase of the push-pull cycle, the fluid removed has a constant concentration of the drug. The amount of fluid introduced and withdrawn during each cycle is very small, so the total volume of fluid in the cochlea remains essentially constant. On a sufficiently large time scale (say, several push-pull time cycles) the process may be viewed as the introduction of drug-containing fluid through one pipe; the drug is then considered to instantaneously diffuse to constant concentration, with fluid being withdrawn from another pipe at the same volumetric rate as the inflow. We also assume that the drug is not metabolized or otherwise destroyed (or created) in the cochlea.

Thus the only way into the cochlea for the drug is through the inflow pipe, the only way out is through the outflow pipe, and the drug is neither created nor destroyed in the cochlea. If during a time interval Δt an amount A_1 micrograms (μg) enters through the inflow pipe and an amount A_2 μg exits through the outflow pipe, then the amount of drug in the cochlea has changed (increased or decreased) by an amount $(A_1 - A_2)$ μg . Divide by Δt to obtain $(A_1 - A_2)/\Delta t$ as the average rate at which the amount of drug in the cochlea is changing during this time interval. The quantity $A_1/\Delta t$ is the average rate at which the drug enters through the inflow, and $-A_2/\Delta t$ is average rate the drug

leaves through the outflow. The equation $(A_1 - A_2)/\Delta t = A_1/\Delta t - A_2/\Delta t$ states that

The average rate of change of the amount of drug contained in the cochlea equals the average inflow rate minus the average outflow rate.

When $\Delta t \rightarrow 0$ the average rates of change become instantaneous rates of change, and we have

$$\begin{aligned} \text{The instantaneous rate of change of drug in the cochlea} &= \text{instantaneous rate drug enters} \\ &- \text{instantaneous rate drug exits.} \end{aligned} \quad (1.4)$$

Equation (1.4) is the basis of our mathematical model.

Reading Exercise 1.2.1 Let $u(t)$ denote the amount (in μg) of drug in the cochlea at time t , with t measured in minutes. What familiar mathematical quantity denotes the instantaneous rate of change of the amount of drug with respect to time? What units does this instantaneous rate of change have here?

Reading Exercise 1.2.1 quantifies the left side of (1.4). To quantify the right side of (1.4) we also need to know the instantaneous rate at which the drug is entering the cochlea through the inflow pipe, and the instantaneous rate at which the drug is leaving. Suppose that fluid is entering the cochlea through the inflow pipe in Figure 1.2 at a volumetric rate of r microliters per minute ($\mu\text{L}/\text{min}$). This fluid contains the drug at a constant concentration of c_1 micrograms per microliter ($\mu\text{g}/\mu\text{L}$).

Reading Exercise 1.2.2 At what rate is the drug entering the cochlea through the inflow pipe? Your answer should be in units of micrograms per minute ($\mu\text{g}/\text{min}$, physical dimensions of mass per unit time). Hint: the answer depends on r and c_1 .

Modeling Tip 1.2.1 Always keep track of the units or physical dimensions of the quantities of interest. They should always make sense, and in particular one should only ever have to add, subtract, or equate quantities with like dimension, e.g., add a mass and a mass. If you ever find yourself adding a mass and a meter, you messed up. In many equations there will also be dimensionless quantities, constants like π , e , or other real or complex numbers. The same logic applies to these quantities. You can only add or subtract a dimensionless quantity to another dimensionless quantity. This topic will be explored in more detail in Section 1.5.

Computing the rate at which the drug is leaving the cochlea is similar to computing the inflow rate. Suppose that at time t there are $u(t)$ μg of the drug in the cochlea. Suppose the volume of the cochlea is V μL . We have assumed that the drug is uniformly distributed throughout the cochlea, and hence will have a concentration of $\frac{u(t)}{V} \frac{\mu\text{g}}{\mu\text{L}}$ at any time. That is, each μL of fluid in the cochlea contains $\frac{u(t)}{V} \frac{\mu\text{g}}{\mu\text{L}} \times 1 (\mu\text{L}) = \frac{u(t)}{V} \mu\text{g}$ of the drug. Each minute r μL of this fluid exits the cochlea.

Reading Exercise 1.2.3 At what rate is the drug exiting the cochlea through the outflow pipe? Your answer should be in units of micrograms per minute ($\mu\text{g}/\text{min}$, which has physical dimensions of mass per unit time). Hint: the answer depends on $u(t)$, V , and r .

1.2.3 The Differential Equation

Let's now put it all together. Based on (1.4) and Reading Exercises 1.2.1-1.2.3 we have

$$\underbrace{u'(t)}_{\text{rate of change of } u(t)} = \underbrace{rc_1}_{\text{rate in}} - \underbrace{\frac{r}{V}u(t)}_{\text{rate out}}. \quad (1.5)$$

Each term in (1.5) has units of μg per minute.

It's worth noting that this type of reasoning, "rate of change equals rate in minus rate out," appears frequently in modeling. It rests on a **conservation law**, a principle that requires that a substance is neither created nor destroyed in a given situation. In this case the drug is neither created nor destroyed in the cochlea. In situations where the drug or other substance is created or destroyed (which we'll encounter) (1.5) must be modified to account for this.

Reading Exercise 1.2.4 A patient is implanted with a reciprocating perfusion device to treat hearing loss. Suppose that the drug reservoir is primed with a drug solution at a concentration of $1.2 \mu\text{g}/\mu\text{L}$ (micrograms per microliter). The drug solution is infused to the patient's cochlea at a steady rate of $1 \mu\text{L}$ of the drug solution every 30 minutes. Simultaneously the well-mixed fluid in the cochlea is withdrawn at the same rate. The fluid volume inside the cochlea stays constant at $200 \mu\text{L}$. What does (1.5) become in this case? If time t is measured in minutes with $t = 0$ corresponding to the moment drug delivery begins, what is the appropriate initial condition?

Reading Exercise 1.2.5 Verify that the function

$$u(t) = c_1 V(1 - e^{-rt/V}) \quad (1.6)$$

satisfies (1.5) with initial condition $u(0) = 0$. That is, if $u(t)$ is as defined in (1.6), then $u'(t)$ equals $rc_1 - ru(t)/V$. With the parameters r, V , and c_1 of Reading Exercise 1.2.4, use (1.6) to determine how much of the drug (μg) will be in the cochlea after one week, and after two weeks. What is the concentration (in μg per μL) of the drug in the cochlea at each of these times? Plot the amount and concentration of the drug in the cochlea over time. What do you observe? Does this seem reasonable?

Remark 1.2.1 At this point you should take note of an important fact, a theme that will recur over and over throughout this book and is well-illustrated by the previous two sections. The model of Usain Bolt's Olympic victory as described by the ODE (1.3) and the intracochlear drug delivery ODE (1.5) are more than just similar: for all practical purposes *they are exactly the same differential equation*. Each is of the general form

$$x'(t) = a + bx(t)$$

for constants a, b , and an unknown function $x(t)$. In the Hill-Keller ODE we have $a = P$ and $b = -k$, with function $x(t) = v(t)$. In the intracochlear model we have $a = rc_1$ and $b = -r/V$, with $x(t) = u(t)$. As a result, the analysis we perform on any one equation will be valid for the other, and this allows us to make some general conclusions that apply to these very different physical systems. These models illustrate the power of ODEs (and more generally, mathematics) to highlight and capture the commonality of situations that, on the surface, seem very different.

1.3 Population Growth and Fishery Management

The material in this section is based on the SIMIODE project "Fishery Harvesting" [43].

1.3.1 The Need to Manage Fish Harvesting

Fish are a valuable source of protein, and many people depend to a large extent on fish and other seafood for nourishment. However, overfishing has driven many species of fish to near extinction [64, 65]. This applies in particular to the Mediterranean Sea, the Baltic Sea, and the North Atlantic Ocean. The collapse of the Newfoundland or Baltic sea cod fisheries should be taken as a pointed warning that the fishing industry needs more careful controls [76]. With appropriate stock assessment data, mathematical models can be used to derive possible management strategies, which may aid the supervision and enduring success of this industry.

U.S. stocks of Atlantic cod came close to commercial collapse in the mid-1990s. This precipitous decline is illustrated in Figure 1.3. The 2012 assessments of Gulf of Maine and Georges Bank cod indicated that both stocks are seriously overfished and are not recovering as quickly as expected. Based on these assessments, quotas for fishing for both stocks were significantly reduced in 2013 to help ensure that overfishing does not occur and that these stocks rebuild. The Gulf of Maine cod quota was cut by 80%, and the Georges Bank cod quota was cut by 61%. National Oceanic and Atmospheric Administration (NOAA) Fisheries and the New England Fishery Management Council continue to work on management measures that will further protect cod stocks and provide opportunities for fishermen to target other healthy fish stocks instead of cod [2]. A quantitative model of how harvesting affects the fish population is an essential part of any program to manage this industry sustainably.

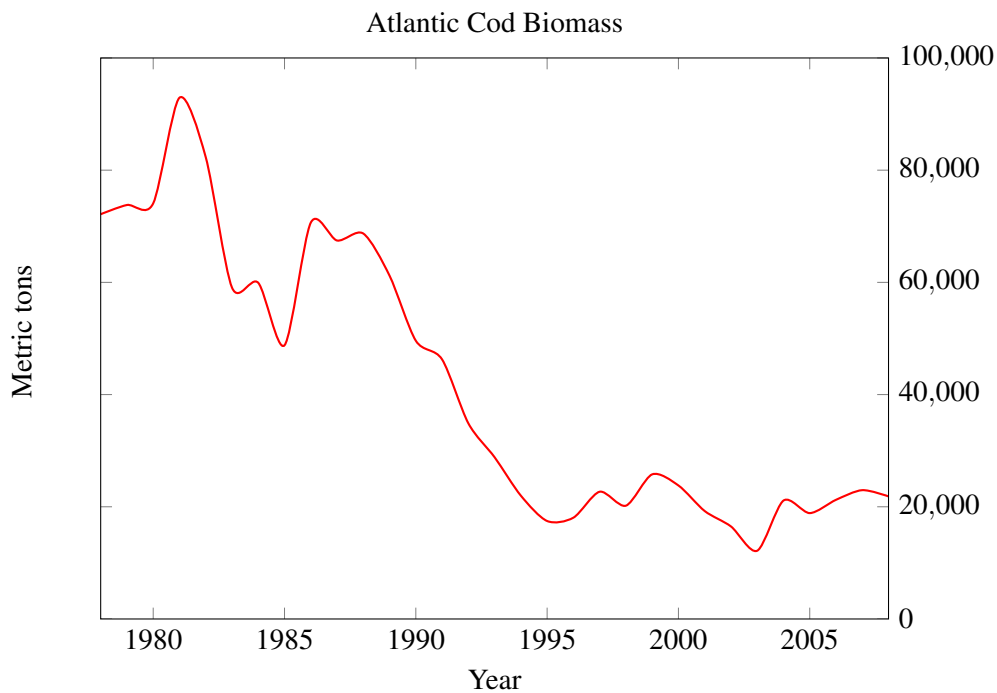


Figure 1.3: Atlantic Cod Biomass, 1978-2008.

1.3.2 Modeling Fish Population

Let us consider the cod population to be confined to a closed, finite region of the ocean. We will use $u(t)$ to denote the cod population in this region at time t ; units for u and t will be specified below. One of the simplest models for the population of an organism in a given environment, whether these organisms are bacteria, fish, or humans, is equation

$$u'(t) = ru(t). \quad (1.7)$$

This model is based on the assumption that at any time the population produces new individuals at a rate proportional to the number of individuals present at that time. Here r is a positive constant of proportionality, and is called the **intrinsic growth rate**. Equation (1.7) is a differential equation with solution

$$u(t) = u_0 e^{rt}, \quad (1.8)$$

where u_0 is the population at time $t = 0$. The drawback to the model (1.7) is that the solution (1.8) exhibits **exponential growth**, without limit, so the population tends to infinity. We need a better model.

Reading Exercise 1.3.1 Verify that $u(t)$ as defined by (1.8) satisfies (1.7) with $u(0) = u_0$. How long does it take the population to double from its initial population u_0 ? That is, at which time t is $u(t) = 2u_0$ satisfied? Hint: the answer depends on r . How long does it take for the population to quadruple? How long does it take to increase eight-fold over the initial population?

The difficulty with (1.7) is that it models the growth rate r as constant, regardless of how large the population becomes. In reality as the population increases, limits on space, food, and other resources (to say nothing of disease and predation) should slow the population growth. One common approach to capture this idea is to alter the growth rate r so that it decreases as the population increases. We thus assume that r tapers to zero at some maximum sustainable value for the population. This maximum sustainable population is commonly called the **carrying capacity** of the environment. Let us use K to denote this population value. Of course K is positive.

To incorporate these ideas into the model, write (1.7) in the form $u'(t)/u(t) = r$; this emphasizes that in our first model new individuals are produced at a constant rate of r individuals per unit time per individual in the population. But in the new model this rate should drop to zero when $u = K$, the maximum sustainable population. A simple modification to (1.7) that accomplishes this is

$$\frac{u'(t)}{u(t)} = r(1 - u(t)/K). \quad (1.9)$$

The right side in (1.9) is a modification to account for the limited resources available to the population that limits growth.

Reading Exercise 1.3.2

- If $u(t) \approx 0$ (but $u(t)$ is still positive) at some time t , what does the right side of (1.9) equal? What is the growth rate $u'(t)/u(t)$ of the population?
- If $u(t) = K$ at some time t (the population is at the carrying capacity), what does the right side of (1.9) equal? What is the growth rate $u'(t)/u(t)$ of the population?
- If $u(t) > K$ at some time t (the population is above the carrying capacity), show that $u'(t)/u(t) < 0$. What is the population $u(t)$ doing at this time?

With this modification the intrinsic growth rate r in (1.9) might be interpreted as a maximum growth rate, the growth rate that the organism is capable of when $u(t) \approx 0$ and environmental limitations have not come into play. Equation (1.9) is conventionally written in the form

$$u'(t) = ru(t)(1 - u(t)/K), \quad (1.10)$$

obtained by multiplying both sides of (1.9) by $u(t)$; (1.10) is called the **logistic equation**.

Reading Exercise 1.3.3 What units on r are necessary for (1.10) (or (1.7)) to be dimensionally consistent, if u measures the population in units of organisms and t is measured in days? What units are necessary for K ?

Reading Exercise 1.3.4 As you will compute in the next chapter, the solution $u(t)$ to (1.10) with initial condition $u(0) = u_0$ is

$$u(t) = \frac{K}{1 + e^{-rt}(K/u_0 - 1)}. \quad (1.11)$$

Take $K = 10$, $r = 1$, and $u_0 = 2$ in (1.11). Plot the solution $u(t)$ for $0 \leq t \leq 10$. Is it consistent with the modeling assumptions that were made? Try increasing or decreasing the value of r ; how does this affect the behavior of the solution? What happens if you take $u_0 > 10$?

1.3.3 Modeling Harvesting

Let's now consider the case in which the population quantified by $u(t)$ is harvested at some rate, that is, a certain portion of the population is taken out of the environment per unit time. To be specific, let's focus on the Atlantic cod population. We will assume that the cod are harvested by humans at a rate that is proportional to the number of cod present. The reasoning is that if fisherman put a certain amount of effort into catching fish for a certain period of time (e.g., the number of boats in the water), then the number of fish caught should be proportional to the number of fish present.

Let us call this constant of proportionality h , and so assume that the rate at which fish are harvested (fish per unit time) is $hu(t)$. Since the rate at which the fish are reproducing is quantified by the right side of (1.10) (fish per unit time) and humans are harvesting fish at rate $hu(t)$, the rate at which $u(t)$ is changing is the difference between these quantities, $ru(t)(1 - u(t)/K) - hu(t)$. This yields an ODE

$$u'(t) = ru(t)(1 - u(t)/K) - hu(t), \quad (1.12)$$

which is called the **logistic equation with harvesting**. See [38] for more discussion of this model.

Reading Exercise 1.3.5 Before considering the solution to the differential equation (1.12), what do you expect of the behavior of the fish population when $h > 0$? Will harvesting increase or decrease the long-term population? What might happen if the harvesting constant h is very large?

Reading Exercise 1.3.6 The solution to (1.12) (which we will deduce in the next chapter) is

$$u(t) = \frac{(1 - \frac{h}{r})K}{1 + e^{-(r-h)t}(\frac{K}{u_0}(1 - \frac{h}{r}) - 1)}. \quad (1.13)$$

When $h = 0$ this is the same as (1.11).

- Take $K = 10, r = 1, u_0 = 2$, and $h = 0.1$ in (1.13). Plot the solution for $0 \leq t \leq 10$, and compare to the solution (1.11) with these same choices for K, r , and u_0 . Are the results consistent with the modeling assumptions that were made?
- Repeat part (a) but increase h to 0.5. What happens?
- What is $\lim_{t \rightarrow \infty} u(t)$ in (1.13)? How large can h be before the population cannot survive?

1.3.4 Parameter Estimation and Harvesting

The estimated Atlantic cod biomass (in metric tons) and harvest rate h in Georges Bank from 1978 to 2008 are given in Table 1.2; see [131]. Note the fish population is estimated not in individuals, but in total mass. Let's assume that these quantities are proportional to each other, so that biomass can be used as a proxy for population and the logistic model with harvesting derived above should still hold, if we instead think of $u(t)$ in terms of mass, rather than individuals.

Reading Exercise 1.3.7 Although the harvest rate h in Table 1.2 varies, let us model this as a constant for the moment. The average value for h_t in Table 1.2 is $h \approx 0.200$ over the time period listed. If we treat 1978 as time $t = 0$ with t measured in years, then the initial condition is $u(0) = 72,148$, with $u(t)$ in units of metric tons. Plot the data in Table 1.2. With $h = 0.2$ and $u_0 = 72148$, can you find values for r and K that provide a reasonable fit to the data when you plot $u(t)$ from (1.13)? Hint: try something around $K = 10^5$, and r just a bit larger than 0.2. You may find that a different value for u_0 (or even h) gives better results.

The process of adjusting unspecified parameters in a model to fit data is known as **parameter estimation**. In Section 3.4 we'll look at more methodical and effective ways to estimate these parameters.

Year	u_t	h_t	Year	u_t	h_t	Year	u_t	h_t
1978	72,148	0.18847	1988	68,702	0.23154	1998	20,196	0.18953
1979	73,793	0.14974	1989	61,191	0.20860	1999	25,776	0.17011
1980	74,082	0.21921	1990	49,599	0.33565	2000	23,796	0.15660
1981	92,912	0.17678	1991	46,266	0.29534	2001	19,240	0.28179
1982	82,323	0.28203	1992	34,877	0.33185	2002	16,495	0.25287
1983	59,073	0.34528	1993	28,827	0.35039	2003	12,167	0.25542
1984	59,920	0.20655	1994	21,980	0.28270	2004	21,104	0.08103
1985	48,789	0.33819	1995	17,463	0.19928	2005	18,871	0.08740
1986	70,638	0.14724	1996	18,057	0.18781	2006	21,241	0.08195
1987	67,462	0.19757	1997	22,681	0.19357	2007	22,962	0.10518
						2008	21,848	unknown

Table 1.2: Annual (1978-2008) values of Atlantic cod biomass in metric tons, u_t , and harvest rate, $h(t)$, in Georges Bank, from [131].

Reading Exercise 1.3.8 What is the long-term behavior of the cod population with the parameters you found in Reading Exercise 1.3.7? What does harvesting do to the cod population? According to your model, can the cod population survive under these conditions? Using the data in Table 1.2, if we increase the constant harvest rate, h , to be 0.4, how will the population of Atlantic Cod change over time?

Acknowledgement

We used a simplified version of the models from W. Ding, G.E. Herrera, H.R. Joshi, S. Lenhart, and M.G. Neubert [44, 70].

1.4 Where Do We Go from Here?

1.4.1 A Toolbox for Describing the World

We've modeled a world-class sprinter, a microfluidic pump, and the population of a species occupying a swath of ocean a thousand miles wide. These phenomena evolve on vastly different scales in time and space, yet all can be described by an equation of the form

$$u'(t) = f(t, u(t)). \quad (1.14)$$

This is quite remarkable and provides testimony to the importance and ubiquity of differential equations as a tool for describing the world.

In each case the function $u(t)$ in (1.14) is considered as an unknown to be found, while the function $f(t, u)$ defines the precise ODE that arises from the physical model. In the Hill-Keller model $f(t, u) = P - ku$ (though there we used v instead of u), while in the intracochlear drug delivery model we had $f(t, u) = rc_1 - ru/V$, and in the fish harvesting model we had $f(t, u) = ru(1 - u/K) - hu$. In each case we used an additional piece of information: an initial condition of the form $u(t_0) = u_0$.

For the ODEs encountered so far, we presented an explicit closed-form or analytical solution with which to experiment. In the remainder of this text we'll look at how one can methodically find such solutions to ODEs like (1.14) and many others. In cases where an analytical solution cannot be found, we will explore other techniques for gleaning information about solutions. The models developed in this chapter, their extensions, and additional models we'll develop later will serve as templates to illuminate the techniques presented in the coming chapters.

1.4.2 Some Terminology

In discussing how to solve or otherwise analyze differential equations, the approach we take will depend greatly on the structure of the differential equation, so it's helpful to make a few definitions.

Scalar ODEs, Systems, and PDEs

The focus of this text is **ordinary differential equations**. In the scalar case this means there is an unknown function $u(t)$ of a single independent variable t . Equations (1.3), (1.5), and (1.12) are examples. But one may also consider systems of ordinary differential equations, for example,

$$\begin{aligned}u'(t) &= u(t) - u(t)v(t), \\v'(t) &= -2v(t) + 3u(t)v(t)\end{aligned}$$

in which two (or more) unknown functions of a single independent variable t appear. The focus of the first five chapters will be scalar ODEs. In Chapters 6 and 7 we will develop techniques for analyzing system of ODEs.

Ordinary differential equations stand in contrast to **partial differential equations** (or **PDE's**) in which the unknown function depends on two or more independent variables. An example is the **wave equation**

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0,$$

where $u(x, t)$ depends on two independent variables, x and t . One can even have systems of partial differential equations.

Order and Linearity

Equation (1.14) is an example of a **first-order** differential equation, that is, an equation involving an unknown function $u(t)$, in which the highest derivative of u that appears is the first derivative. We'll refer to (1.14), in which $u'(t)$ is given explicitly in terms of t and $u(t)$, as the **standard form** for a first-order ODE. More generally, the **order** of an ODE is the highest derivative of the unknown function that appears. Second-order differential equations (involving $u''(t)$) are also common, and form much of the mathematical basis for significant engineering applications. Occasionally higher-order equations make an appearance.

The distinction between linear and nonlinear ODEs is the last we need to make for the moment. An n th-order ODE for $u(t)$ is **linear** if it can be written as

$$a_n(t)u^{(n)}(t) + a_{n-1}(t)u^{(n-1)}(t) + \cdots + a_1(t)u'(t) + a_0(t)u(t) = b(t) \quad (1.15)$$

for some given functions $a_k(t)$, $b(t)$, $0 \leq k \leq n$, where $u^{(k)}$ denotes the k th derivative of $u(t)$. Equations that are not linear are **nonlinear**. Linear equations have a lot of structure and are often comparatively easy to analyze or solve. Nonlinear equations are everything else and their analysis can be a bit like the Wild West, although there are techniques of general utility.

For convenience, let's amalgamate the essential terminology into the following definition.

Definition 1.4.1 — Basic ODE Terminology. Let $u(t)$ be a function of a single independent variable t . A **scalar ordinary differential equation** for u is an equation of the form

$$F(t, u(t), u'(t), \dots, u^{(n)}(t)) = 0,$$

where F is a function of $n + 1$ variables that relates the independent variable t , the function $u(t)$, and the derivatives $u'(t)$, \dots , $u^{(n)}(t)$. The integer n indicates the highest derivative of u that appears in the equation and is called the **order** of the differential equation. If the ODE can

be written in the form of (1.15) then the differential equation is **linear**.

There are other important adjectives to describe ODEs, such as constant-coefficient, autonomous, separable, homogeneous, and more. Their definitions will be presented later, in the context where they naturally arise. Each type of equation demands different techniques for analysis.

1.4.3 You Already Know How to Solve Some Differential Equations

You are already familiar with the solution procedure for certain types of differential equations, specifically, those that can be solved with one or more straightforward applications of antidifferentiation.

First-Order Equations

Consider a first-order differential equation of the form

$$u'(t) = g(t), \quad (1.16)$$

where $g(t)$ is a specified function and $u(t)$ is the unknown to be found. Comparison of (1.16) to (1.14) shows that the right side in (1.16) does not involve the unknown function u . In this case we can antidifferentiate both sides of (1.16) with respect to t and find

$$u(t) = \int g(t) dt + C, \quad (1.17)$$

where C is an arbitrary constant of integration. Equation (1.17) is a **general solution** to the ODE (1.16): all functions that satisfy (1.16) can be expressed in the form of (1.17) for some choice of C . If an initial condition $u(t_0) = u_0$ is given then the constant C can be adjusted so that $u(t)$ in (1.17) satisfies this initial condition.

■ **Example 1.1** Let us find a general solution to the ordinary differential equation

$$u'(t) = 3t^2$$

and use this to find a solution with the initial condition $u(1) = 3$.

Antidifferentiating both sides of the ODE yields general solution $u(t) = t^3 + C$. The initial condition $u(1) = 3$ requires $u = 3$ when $t = 1$, so substitute this into this general solution to find $3 = 1 + C$ and solve for $C = 2$. The solution with the required initial condition is $u(t) = t^3 + 2$. ■

Reading Exercise 1.4.1 Verify that $u(t) = t^3 + 2$ does, in fact, satisfy $u'(t) = 3t^2$ with $u(1) = 3$. If the initial condition is changed to $u(t_0) = u_0$ for unspecified values of t_0 and u_0 , what would the solution $u(t)$ be (it depends on t_0 and u_0)? Can $u'(t) = 3t^2$ be solved with any initial data t_0 and u_0 ?

Reading Exercise 1.4.2 Find a general solution to $u'(t) = e^{2t}$. Use this general solution to find a particular solution with $u(0) = 8$.

Second- and Higher-Order Equations

At this time certain second-order differential equations are also within reach, specifically, second-order equations of the form

$$u''(t) = g(t). \quad (1.18)$$

To solve this ODE, first integrate both sides of (1.18) with respect to t to find

$$u'(t) = \int g(t) dt + C_1, \quad (1.19)$$

where C_1 is an arbitrary constant of integration. Let $G(t) = \int g(t) dt$ be any antiderivative for $g(t)$, so (1.19) becomes $u'(t) = G(t) + C_1$. Integrating both sides of (1.19) with respect to t then yields

$$u(t) = \int G(t) dt + C_1 t + C_2, \quad (1.20)$$

where C_2 is a second arbitrary constant of integration. Equation (1.20) is a general solution to (1.18). Finding the constants C_1 and C_2 requires two additional pieces of information about the solution. They typically (but not always) take the form of initial conditions such as $u(t_0) = u_0$ and $u'(t_0) = u'_0$ for some initial time t_0 and constants u_0 and u'_0 .

■ **Example 1.2** Let us find a general solution to the ordinary differential equation

$$u''(t) = e^t$$

and then find a particular solution with the initial conditions $u(1) = 2, u'(1) = 3$.

Antidifferentiating both sides of the ODE with respect to t yields

$$u'(t) = e^t + C_1.$$

Antidifferentiate again to find

$$u(t) = e^t + C_1 t + C_2.$$

This is a general solution to the ODE. The condition that $u'(1) = 3$ requires that $e + C_1 = 3$, so $C_1 = 3 - e$. The condition $u(1) = 2$ then requires $e + (3 - e) + C_2 = 2$, so $C_2 = -1$. The solution with the required initial condition is

$$u(t) = e^t + (3 - e)t - 1.$$

■

Reading Exercise 1.4.3 Verify that $u(t) = e^t + (3 - e)t - 1$ does in fact satisfy $u''(t) = e^t$ with $u(1) = 2$ and $u'(1) = 3$. If the initial conditions are $u(t_0) = u_0$ and $u'(t_0) = u'_0$ for given constants t_0, u_0 , and u'_0 , what would the solution be? (It depends on t_0, u_0 and u'_0 .) Can $u''(t) = e^t$ be solved with any initial conditions of this form?

Reading Exercise 1.4.4 Find a general solution to $u''(t) = \sin(t)$. Use this general solution to find a particular solution with initial data $u(0) = 2$ and $u'(0) = 4$.

It should be clear that this process can be extended to solve any differential equation of the form $u^{(n)}(t) = g(t)$, where $u^{(n)}(t)$ denotes the n th derivative of u . Integrate n times, and in the process pick up n constants of integration that require n additional pieces of information to find. This information is often in the form of initial data that specifies values for $u(t_0), u'(t_0), \dots, u^{(n-1)}(t_0)$.

Remark 1.4.1 You may have noticed that we usually refer to finding “a” general solution to an ODE rather than “the” general solution. The reason is that a general solution to an ODE may assume various superficially different forms, especially later in the text. For instance, in Example 1.2 a general solution $u(t) = e^t + C_1 t + C_2$ was found. But $u(t) = e^t + 17C_1 t - 3C_2$ could also be considered a general solution, since all solutions to $u''(t) = e^t$ are still of this form, and C_1 and C_2 can still be adjusted to obtain any initial conditions.

1.4.4 Exercises

Exercise 1.4.1 For each ODE and initial condition below, find a general solution to the ODE, and then find the specific solution with the given initial condition by using the technique of Example 1.1 or Example 1.2 as appropriate.

- (a) $u'(t) = t, u(0) = 3$
- (b) $u'(t) = \cos(t), u(0) = 0$
- (c) $u'(t) = e^t, u(0) = 4$
- (d) $u'(t) = 1/t, u(2) = 0$
- (e) $u'(t) = \cos(t), u(0) = 1$
- (f) $u'(t) = t \cos(t), u(0) = 1$
- (g) $u'(t) = 1/(t^2 + 1), u(1) = 2$
- (h) $v'(t) = g, v(0) = v_0$, where g and v_0 are some constants
- (i) $h'(t) = t^n, h(0) = 0$, where n is a positive integer
- (j) $u''(t) = t, u(0) = 1, u'(0) = 3$
- (k) $u''(t) = \sin(t), u(0) = 1, u'(0) = 0$
- (l) $y''(t) = -g, y(0) = 10, y'(0) = 0$, where g is a constant
- (m) $x''(t) = 5 - e^{-2t}, x(0) = 0, x'(0) = 0$

Exercise 1.4.2 The model for drug delivery to the cochlea in Section 1.2 is a special case of a more general compartmental model in which one has a tank of volume V (in our model, the tank was the cochlea) with an input pipe and an output pipe. These types of problems are often called **salt tank models**, since the input and output pipes are assumed to carry salt, dissolved in water. In our model the drug played the role of the salt. The general situation is still accurately depicted by Figure 1.2.

Consider a tank of volume $V = 100$ liters, into which a pipe delivers a salt solution at a rate of 5 liters per minute; this input salt solution has a concentration of 50 grams of salt per liter. The solution in the tank is well-stirred and always of uniform concentration. An output pipe carries away this well-stirred solution, also at 5 liters per minute. Let $u(t)$ denote the amount (grams) of salt in the tank at time t . If the tank starts with no salt at time $t = 0$, use the reasoning that led to (1.5) to formulate an appropriate ODE and initial condition for u . Use (1.6) to write out the solution. Plot the solution $u(t)$ for $0 \leq t \leq 200$ minutes. What limit does the amount of salt in the tank approach? What limit does the concentration approach? Does this make sense, in light of the incoming fluid concentration?

1.5 The Blessing of Dimensionality

1.5.1 Definition of Dimension

The subject of differential equations involves a lot of fundamental physical quantities such as distances, velocities, electric charge, mass, etc. Most of these quantities have units or physical dimensions. For example, mass, length, and time are fundamental dimensions. Other dimensions we'll encounter later are electric charge and temperature. These are the basic building blocks for the dimension of all other quantities in this text. In this section we'll look at how the consideration of the basic dimensions of physical quantities can aid mathematical modeling and setting up ODEs, and provide a sanity check for our work. This is the subject of **dimensional analysis**.

Mass, Length, and Time

To illustrate the notion of dimension, a variable r in a given problem may have the dimension of length, in which case we will write

$$[r] = L.$$

The notation $[r]$ indicates the physical dimension of the quantity r and L is the notation for the physical dimension of length. Note that L here is not the actual length of whatever r quantifies; L just stands for the dimension length. We will use T to denote the dimension of time and M to denote the dimension of mass. The dimension of many other common quantities can be derived from these. Further examples:

- If A denotes an area then $[A] = L^2$. If V denotes a volume then $[V] = L^3$.
- If v is a velocity then $[v] = LT^{-1}$, length (or distance) per time.
- If a is an acceleration then $[a] = LT^{-2}$.
- If ρ is a density (mass per volume) then $[\rho] = ML^{-3}$.

The dimension of a physical quantity will generally be expressed as $M^a L^b T^c$ where a, b , and c may be positive, negative, or zero. In many but not all cases a, b , and c will be integers.

Dimension Versus Units

The dimension of a physical variable is not quite the same as the units used to measure the variable. Thus length is a fundamental physical dimension, but it can be measured using many systems of units, e.g., meters, feet, or inches. If I ever get sloppy and refer to the dimension of a velocity as “meters per second” feel free to write me a stern email. However, specifying the units of a given quantity does allow us to determine its dimension.

Reading Exercise 1.5.1 Air is being pumped into a balloon at a fixed rate q liters per second. What is $[q]$? What is the dimension of the rate at which the balloon surface area is changing in time? What is the dimension of the rate at which the radius of the balloon is increasing?

1.5.2 The Algebra of Dimension

Add, Subtract, Multiply, Divide

In addition to mass, length and time, we'll later encounter charge (denoted by Q) and temperature (denoted by Θ .) It is a fundamental property of our mathematical framework for describing the world that both sides of any equation or inequality involving physical variables must have the same physical dimensions. It is nonsense to ask if the length of a string equals the mass of an apple or whether a certain time interval is longer than a stick. Similarly, it only makes sense to add or subtract physical quantities that have the same dimensions. You can add years to your life span, but you cannot add years to the mass of an apple. We can, however, take products and quotients of dimensionally dissimilar quantities, for example, dividing a length by a time interval to obtain a velocity. If a variable x has dimension $[x] = M^{a_1} L^{a_2} T^{a_3}$ and variable y has dimension $[y] = M^{b_1} L^{b_2} T^{b_3}$ then

$$[xy] = [x][y] = M^{a_1+b_1} L^{a_2+b_2} T^{a_3+b_3} \quad \text{and} \quad [x/y] = [x]/[y] = M^{a_1-b_1} L^{a_2-b_2} T^{a_3-b_3}. \quad (1.21)$$

Reading Exercise 1.5.2 If v has dimension $[v] = LT^{-1}$ (a velocity, perhaps) and $[\Delta t] = T$, what is the dimension of $v\Delta t$? What is a physical interpretation of this situation?

Dimensionless Constants

In many formulas certain **dimensionless** mathematical constants appear. For example, the formula for the area of a circle is $A = \pi r^2$. Here $[r] = L$, $[A] = L^2$, and π is a dimensionless constant. We

write $[\pi] = M^0L^0T^0$ to denote this, or just $[\pi] = 1$. (Careful: put the square brackets around π or else you're claiming $\pi = 1$.) In accordance with (1.21) it follows that

$$[A] = [\pi r^2] = [\pi][r^2] = M^0L^0T^0 M^0L^2T^0 = M^0L^2T^0 = L^2.$$

It's also worth noting that the angular measure radian is dimensionless. The definition of the radian involves the ratio of two lengths (the radius of a circle and the arc length of that portion of the circle subtended by the angle); the ratio of these two lengths is dimensionless.

Deducing Dimension from Common Formulas

It's frequently possible to determine the dimension of certain quantities by looking at familiar formulas that involve them. For example, what is the dimension of force? If you remember $F = ma$ and know that $[m] = M$ and $[a] = LT^{-2}$ then

$$[F] = [m][a] = MLT^{-2}.$$

What is the dimension of kinetic energy? You may recall the formula $E = \frac{1}{2}mv^2$ for the kinetic energy E of a mass m moving at speed v . Since $[m] = M$, $[v] = LT^{-1}$, and $1/2$ is dimensionless, we find

$$[E] = [1/2][m][v]^2 = ML^2T^{-2}.$$

Reading Exercise 1.5.3 Newton's universal law of gravitation specifies that the force F between two point masses m_1 and m_2 separated by a distance r is

$$F = \frac{Gm_1m_2}{r^2}.$$

Use this to determine $[G]$.

1.5.3 Derivatives, Integrals, Elementary Functions

Differentiation With Respect to Time

Suppose that $y = f(t)$ is a function with input argument t , a time, and f outputs a physical variable y with dimension $[y] = M^aL^bT^c$ for some constants a, b, c . What is the dimension of the derivative $f'(t)$? This is a common situation. The derivative is defined as

$$f'(t) = \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

The numerator $f(t + \Delta t) - f(t)$ has dimension $[f(t + \Delta t) - f(t)] = M^aL^bT^c$ and Δt has dimension $[\Delta t] = T$. As a result, the difference quotient $(f(t + \Delta t) - f(t))/\Delta t$ has dimension $M^aL^bT^{c-1}$ and $f'(t)$, as the limit of such quantities, also has this dimension. That is,

$$[f'(t)] = M^aL^bT^{c-1}.$$

For example, if $f(t)$ is a function that outputs a position (displacement from the origin) as a function of time t then $[f] = L$, while $[t] = T$. Then $[f'(t)] = LT^{-1}$, which has the dimension of velocity.

Reading Exercise 1.5.4 An object has kinetic energy $E(t)$ that varies with time. What is the dimension of $E'(t)$? What is the dimension of $E''(t)$?

Integration With Respect to Time

If $f(t)$ is a function that accepts time t as an input argument and outputs a variable y with dimension $[y] = M^a L^b T^c$ then

$$\left[\int_p^q f(t) dt \right] = M^a L^b T^{c+1}.$$

This isn't surprising, given that computing the integral typically involves computing an antiderivative, but this can also be deduced using the definition of the integral. Recall from basic calculus that the integral is defined as the limit of a Riemann sum,

$$\int_p^q f(t) dt = \lim \sum_k f(t_k) \Delta t_k.$$

We needn't go into the details of the nature of this limit here. It suffices to note that each term $f(t_k) \Delta t_k$ has dimension $[f(t_k)] [\Delta t_k] = M^a L^b T^{c+1}$ and hence so does the sum, and also the limit.

Reading Exercise 1.5.5 Water flows into a tank at a variable rate of $r(t)$ liters per second. What is the dimension of $r(t)$? What is the dimension of

$$\int_a^b r(t) dt?$$

What is the physical interpretation of this integral?

Elementary Functions

Consider expressions like $\sin(z)$, $\cos(z)$, e^z , each an elementary transcendental function of a variable z . These types of expressions require that the input argument z be dimensionless. The reason is that these expressions have Taylor series, typically of the form

$$a_0 + a_1 z + a_2 z^2 + \dots$$

(the a_k are dimensionless), and so each of $1, z, z^2, \dots$ must have the same dimension if they are to be added. This only occurs if z is dimensionless, and in this case the sum consists of dimensionless quantities, and hence is itself dimensionless. As an example, consider the expression $\cos(\omega t)$. Here ωt should be dimensionless; if t is time ($[t] = T$) then it must be the case that $[\omega] = T^{-1}$, reciprocal time. This is the case— ω is always some kind of radial frequency, with the dimension of reciprocal time.

1.5.4 Unit-Free Equations and Bending the Rules

Unit-Free Equations

In general when we write down fundamental laws of physics or differential equations the equations should be independent of any particular system of units. As an example, consider the formula $d = \frac{1}{2} g t^2$ for the distance an object falls in t time units under the influence of gravitational acceleration g , with no other forces acting on the object. This equation holds in any system of units. In the SI system however, the equation can be written approximately as $d = 4.9 t^2$, while in English units it becomes $d = 16 t^2$. These expressions hard-code the units into the equation. Equations that do not depend on the system of units are said to be **unit-free** and are usually a more desirable way to express the situation.

As another example, consider the Hill-Keller model (1.3), $v' = P - kv$. This model remains unchanged when the system of units changes. If we use $P = 11$ meters per second squared, however, the ODE $v' = 11 - kv$ is specific to SI units and won't be $v' = P - kv$ in, say, English units.

Bending the Rules

We won't always strictly adhere to these rules, so long as no confusion results. For example, we may wish to express the position of a particle moving along the x axis as a function of time t as $x = \cos(\omega t)$. Here ω has dimension T^{-1} and $[t] = T$, so the argument of the cosine function is dimensionless, in accordance with the discussion above. But then $\cos(\omega t)$ is a dimensionless quantity, while x should have dimension L . Writing $x = \cos(\omega t)$ requires choosing a unit for length; it would be more precise to say $x = A \cos(\omega t)$ where $[A] = L$ and $A = 1$ in whatever units we choose to measure length. In principle this is what we'll do, we just won't remark on it, except in cases where it might cause confusion.

1.5.5 Using Dimension to Find Plausible Models

The fact that physical quantities come with a dimension can be an incredibly powerful tool for figuring out things that we have no right to know. As an example, consider a black hole, a roughly spherical region in space-time left behind by the collapse of a massive star. How does the radius r of the black hole depend on its mass m ? Since $[r] = L$ and $[m] = M$, there must be other variables involved. Black holes are black because light cannot escape them, so maybe the speed of light c also plays a role; $[c] = LT^{-1}$. But light can't escape because of the intense gravitational field, so presumably the gravitational constant G is important. In Reading Exercise 1.5.3 you showed that $[G] = M^{-1}L^3T^{-2}$.

Let's put these observations together. Suppose a formula of the form

$$r = KG^\alpha c^\beta m^\gamma \tag{1.22}$$

holds for some constants α, β , and γ (that need not be integers) and dimensionless constant K . What choices for α, β, γ lead to a dimensionally consistent formula? To find out, note that

$$[KG^\alpha c^\beta m^\gamma] = [K][G]^\alpha [c]^\beta [m]^\gamma = (M^{-\alpha}L^{3\alpha}T^{-2\alpha})(L^\beta T^{-\beta})M^\gamma = M^{-\alpha+\gamma}L^{3\alpha+\beta}T^{-2\alpha-\beta},$$

after combining exponents. If the right side is to have the same dimension as r , namely $[r] = M^0L^1T^0$, then $-\alpha + \gamma = 0$ (to match the M exponents), $3\alpha + \beta = 1$ (to match the L exponents), and $-2\alpha - \beta = 0$ (to match the T exponents). The solution to these three equations in three unknowns is $\alpha = 1, \beta = -2, \gamma = 1$. From (1.22), a formula of the form

$$r = K \frac{Gm}{c^2}$$

is dimensionally consistent, and in fact the only dimensionally consistent formula involving G, m, c and r in the form (1.22). The dimensionless constant K can be anything.

With $K = 2$ the formula is correct. The radius r is called the **Schwarzschild radius** of the black hole. It is often the case that this kind of dimensional analysis leads to a formula of the correct form with one or more dimensionless constants that are simple, e.g., 2 or π or such. It seems rather amazing that we just derived an important result from physics that presumably requires an understanding of general relativity to truly understand, but we used nothing more than the dimensions of the variables involved.

1.5.6 Other Dimensions

Later in the text we will encounter other physical dimensions, specifically temperature, which has dimension denoted by Θ , and electric charge, which has dimension denoted by Q . These dimensions are independent from mass, length, and time. In certain specific instances it can be helpful to temporarily assign other dimensions. For example, in a problem involving money we could use V to denote the dimension value, that is, the worth of some quantity; it might be tempting

to use the symbol \$ but that is a unit, dollars. See the project “Money Matters” in Section 1.6 and [111]. In a population model we might use N to denote the dimension of population for some species. In a problem involving two or more species we might introduce a unique dimension for each. See Exercise 1.5.6.

1.5.7 Exercises

Exercise 1.5.1

- (a) What is the dimension of momentum?
- (b) What is the dimension of angular velocity?
- (c) What is the dimension of work (force times distance)?
- (d) What is the dimension of pressure?

Exercise 1.5.2 The energy of a photon with wavelength λ is $E = hc/\lambda$, where c is the speed of light and h is Planck’s constant. Find the dimension of Planck’s constant.

Exercise 1.5.3 What must be the dimension of the constant k in the Hill-Keller ODE (1.3)?

Exercise 1.5.4 Suppose $f(x)$ is a function that outputs a variable with dimension $[f] = M^a L^b T^c$ and that the input argument has dimension $[x] = M^\alpha L^\beta T^\gamma$. What is the dimension of $f'(x)$?

Exercise 1.5.5 Verify that the ODE (1.5) is dimensionally consistent. Then verify that the solution (1.6) is also dimensionally correct. In particular, is the argument of the exponential function dimensionless?

Exercise 1.5.6 Recall the fish harvesting model of Section 1.3, and in particular the ODE (1.10). The variable t in that equation is time, but u has no obvious dimension. Let us take $[u] = N$, where N denotes the dimension of “population.” (Although we could consider u as dimensionless since it simply counts how many fish are present, in other contexts we’ll encounter later it can be beneficial to think of $u(t)$ as having a specific dimension.) If $[u] = N$, then in the model leading to the ODE (1.10), what is the dimension of K ? What must be the dimension of r for the ODE to be dimensionally consistent?

Exercise 1.5.7 The orbital period P of an object in a circular orbit of radius r around a comparatively massive body like the earth, with mass m , is given by $P = 2\pi\sqrt{\frac{r^3}{Gm}}$. Verify that this formula is dimensionally correct.

Exercise 1.5.8 Find a plausible formula $v = G^a m^b r^c$ for the escape velocity v of a planet with mass m and radius r . Here G is the gravitational constant. Look up the correct formula and compare.

Exercise 1.5.9 Find a plausible formula for the period P of a pendulum as a function of its length ℓ , the mass m of the bob, and the earth's gravitational acceleration g , in the form $P = \ell^a m^b g^c$.

Exercise 1.5.10 Find a plausible formula for the speed of sound v in a gas as a function of its pressure P and density ρ , in the form $v = P^a \rho^b$. Note that pressure has the dimension of force per area.

Exercise 1.5.11 Find a plausible formula for the frequency f of a string's vibration in terms of its linear density λ , the tension τ in the string (which has the same units as force), and the length ℓ of the string, in the form $f = \lambda^a \tau^b \ell^c$.

Exercise 1.5.12 Dimensionless constants like 2 or π do not change value in physical formulas when the system of units is changed, and might therefore be considered very fundamental.

Four of the most important constants in physics are the speed of light c , Planck's constant \hbar , the charge e on an electron, and the Coulomb constant k_e in Coulomb's law. These constants have dimensions and approximate values (SI units) of

- $[c] = LT^{-1}$, value 299792458 meters per second
- $[\hbar] = ML^2T^{-1}$, value $1.054571817 \times 10^{-34}$ joule-seconds
- $[k_e] = ML^3T^{-2}Q^{-2}$, value 8.9875517923×10^9 kg-meters cubed per second squared per coulomb squared
- $[e] = Q$, value $e = 1.602176634 \times 10^{-19}$ coulomb

Show that the quantity $\alpha = \frac{e^2 k_e}{\hbar c}$ is dimensionless. This number is often called the *fine structure constant*. Compute its value using the data above (the value should be near $1/137$.) What does this constant signify about our universe? Is it related to π or other fundamental mathematical constants? No one knows.

1.6 Modeling Projects

1.6.1 Project: Hang Time

This project is based on the SIMIODE Modeling Scenario “Hang Time” [123].

The phrase “hang time” is common in sports. An announcer in a football game may refer to the hang time for a punter's kick, or a basketball announcer may refer to the amount of time a player appears to hang in the air during a jump. In this modeling project we'll take a closer look at this phenomenon. Why do objects sometimes appear to hang in midair, even to the point that they seem to defy the law of gravity?

Consider an object of mass m in an idealized one-dimensional situation in which the object goes straight up and comes straight down. In particular, let's focus on a basketball player about to take a jump shot. The best professional basketball players have vertical jumps of 40 inches, possibly even higher (that's how high their hips or head go, above the standing position). Let's go with 1 meter, a bit over 39 inches, as the height a good professional player might jump.

We use t for time and $g = 9.81$ meters per second squared for gravitational acceleration. Let $y(t)$ denote the height of the player's hips during the jump, with $y = 0$ corresponding to the height of the player's hips when standing (so all vertical displacements are relative to this) with $y > 0$ corresponding to upward displacement. If $t = 0$ is the time the jump starts (ignore $t < 0$ when the player may crouch before jumping) then the appropriate initial condition is $y(0) = 0$.

Modeling Exercise 6.1.1 Newton's Second Law of Motion is $F = ma$, where a is the acceleration

of an object of mass m and F is the sum of all forces acting on the object. In this case a is the vertical acceleration of the player. Express a in terms of $y(t)$.

Modeling Exercise 6.1.2 If the only force acting on the player is gravity, what is F in $F = ma$? Hint: be careful with the sign—make sure F points downward.

Modeling Exercise 6.1.3 Put Modeling Exercises 6.1.1 and 6.1.2 together to find a second-order differential equation for $y(t)$. One initial condition is $y(0) = 0$. Take the other as $y'(0) = v_0$, where v_0 is some (as yet unknown) initial velocity the player gets from crouching and jumping.

Modeling Exercise 6.1.4 Find a general solution to the differential equation in Modeling Exercise 6.1.3 by integrating twice, as was done in Section 1.4. Then find the particular solution that satisfies the initial conditions. Check your work by making sure your solution satisfies the ODE and initial conditions. Hint: the solution should be quadratic in t and should involve v_0 .

Modeling Exercise 6.1.5 Suppose that $y(t)$ attains a maximum value of $y(t_1) = 1$ meter for some unknown time t_1 (when the player attains peak altitude). Show that this yields the equation

$$v_0 t_1 - \frac{1}{2} g t_1^2 = 1 \quad (1.23)$$

in SI (metric) units, so the 1 on the right side in (1.23) signifies 1 meter. Verify that all terms in (1.23) have the same dimension, namely length.

Modeling Exercise 6.1.6 What is $y'(t_1)$ equal to, if t_1 is the time at which the player attains maximum altitude? Use this to find a second equation relating the unknowns t_1 and v_0 . Then use this equation along (1.23) from Modeling Exercise 6.1.5 to find v_0 and t_1 . Work in SI units with $g = 9.81$ meters per second squared.

Modeling Exercise 6.1.7 How long does the entire jump last? What percentage of the total jump time is spent in the top 25 percent of the jump? How might this explain why the player seems to hang near the top of the jump?

1.6.2 Project: Money Matters

Almost everyone, at some point, joins the workforce, works for a period of time, then retires. It is of course essential to plan for retirement, and to save for that day. How much should you be saving throughout your career, and how should you invest it?

As an illustration, let's say you start work at age 22 and work until age 67. Let us use time $t = 0$ to indicate age 22, with t measured in years, so $t = 45$ years is retirement time. Like most people, as you progress in your career you earn promotions, responsibility, and more and more money. Suppose your pre-tax income at time t is given by

$$p(t) = 50000e^{t/45} \quad (1.24)$$

dollars per year (so you make \$50,000 per year starting at age 22, typical for college graduates in 2021 according to [54]). Note that the argument $t/45$ of the exponential function in (1.24) is dimensionless, since t has the dimension time, as does 45 (years). We can write $[p] = V$ with V as the dimension of value. For simplicity let's ignore inflation in this first analysis. Suppose you diligently save 10 percent of your income each year throughout your career, which is harder than it sounds. That is, you put away money at a rate of $0.1p(t)$ dollars per year. As a supremely conservative investor, you invest your money in a bank account that pays no interest. Let $S(t)$ denote the amount of money you've saved at time t .

Modeling Exercise 6.2.1 Explain why $S(t)$ obeys the ODE

$$S'(t) = 0.1p(t), \quad (1.25)$$

with $p(t)$ given by (1.24). Then find a general solution to this ODE. What is the dimension of S ? Of p ? What is the dimension of the constant 0.1?

Modeling Exercise 6.2.2 At age 22 you inherit \$100,000 tax-free from your grandparents, to start you on the path to retirement. It seems you're set for life. What initial condition for $S(0)$ is appropriate?

Modeling Exercise 6.2.3 Find the particular solution to (1.25) that satisfies the initial condition from Modeling Exercise 6.2.2.

Modeling Exercise 6.2.4 How much money will you have saved at retirement (what is $S(45)$)? If social security is defunct at this time and you live to age 90, how much money do you have on which to live each year? Will that support the lifestyle to which you will have become accustomed?

1.6.3 Project: Ant Tunneling

This project is based on the SIMIODE Modeling Scenario "Ant Tunnel Building" [121]; see also [119].

If a well-meaning relative ever purchased an ant colony for you when you were in grade school, or even if you just watched an ant hill on a summer day, you've noticed that ants are extremely industrious, and tireless tunnel builders. How long does it take an ant to build a single tunnel? This seems like an interesting modeling question. To address the issue we need to narrow the scope of the problem, simplify, and identify some terms and parameters.

To begin, let's define a few crucial variables. Consider a single ant digging a tunnel into a hillside as illustrated in Figure 1.4. (The ant drawing was provided by Isaac H. All.) Let x denote the current length, in feet, of the tunnel that the ant is digging. Let $T(x)$ be the time, in hours, it has taken the ant to build the tunnel of length x .

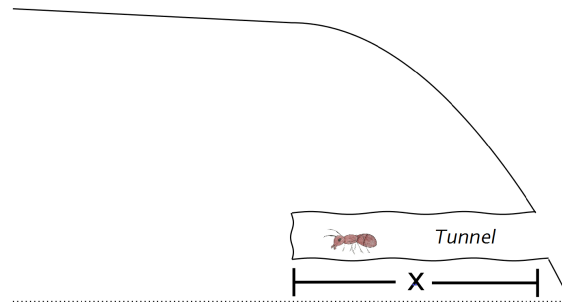


Figure 1.4: Illustration of an ant building a tunnel. Here x is the current length of the tunnel and $T(x)$ is the time it has taken the ant to build the tunnel of length x . Ant drawing provided by Isaac H. All.

Modeling Exercise 6.3.1 Write down several candidate functions for $T(x)$ and give one or two statements in each one's defense, and one or two statements against each.

Modeling Exercise 6.3.1 should convince you that jumping right to a defensible formula for $T(x)$ can be hard. So, instead of going after $T(x)$ directly, let us examine Figure 1.5, a depiction of the essential aspects of the situation. We'll make some assumptions that reflect the relevant geometry and physics, and might also make the model simpler to formulate. Specifically, when an ant digs a tunnel, the ant must extend the tunnel incrementally, by removing soil between coordinates x and $x+h$, carrying this soil back to the tunnel entrance, and then returning to remove

more soil. Let us consider how long it takes the ant to complete the extension of the tunnel from length x to length $x + h$.

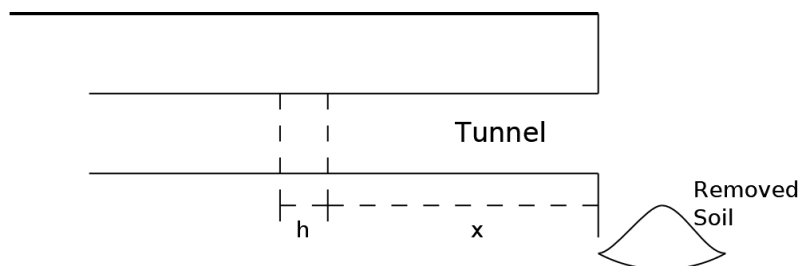


Figure 1.5: A useful diagram for modeling the time it takes to extend a small section of the ant tunnel from distance x to $x + h$.

Modeling Exercise 6.3.2 On the right side of (1.26), we seek an expression for how long it would require the ant to take a short length h of soil and then carry it a distance x to the mouth of the tunnel.

$$T(x+h) - T(x) = \text{_____} \quad (1.26)$$

Notice that $T(x+h) - T(x) \neq T(h)$, for $T(h)$ represents the time it takes to extend the tunnel a distance h from the mouth of the tunnel (at $x = 0$), while $T(x+h) - T(x)$ includes this time plus the time it takes to carry the soil to the mouth of the tunnel.

Below are a few possibilities for the right side of (1.26). Defend or reject each and offer your reasons. Modify one or two to improve them. When trying to reject a model consider some extreme cases and see if the model makes sense, e.g., $h = 0$ or $x = 0$, or either h or x very large.

- (a) $T(x+h) - T(x) = x + h$
- (b) $T(x+h) - T(x) = x - h$
- (c) $T(x+h) - T(x) = x^h$
- (d) $T(x+h) - T(x) = x \cdot h$
- (e) $T(x+h) - T(x) = h^x$
- (f) $T(x+h) - T(x) = c$

Modeling Exercise 6.3.3 Convert your model difference equation (1.26) to a related differential equation with appropriate initial conditions. It may be helpful to consider the familiar expression $(T(x+h) - T(x))/h$ and what happens as h approaches 0.

Modeling Exercise 6.3.4 Solve the differential equation you create in Modeling Exercise 6.3.3 for $T(x)$. Hint: what initial condition $T(0)$ will you use?

Modeling Exercise 6.3.5 Use your solution from Modeling Exercise 6.3.4 to determine how much longer it takes to build a tunnel that is twice as long as an original tunnel of length L . What would some of the original function models you set forth in (a)-(f) have told you here?

Modeling Exercise 6.3.6 Suppose two ants dig from opposite sides of the sand hill, directly toward each other along the same straight line. How would this alter the total time for digging the tunnel?

Modeling Exercise 6.3.7 Of course, the same principles can be applied to model tunnel building for engineers. If we were considering Modeling Exercise 6.3.6 as related to engineering the construction of a long tunnel of length L , outline some of the issues we should be aware of when having two crews (one from each end of the tunnel) working on it.

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