Notes on the Frobenius Method

T S L RADHIKA

Department of Mathematics,

BITS Pilani, Hyderabad Campus, Hyderabad, India.

Email: radhikatsl@hyderabad.bits-pilani.ac.in

Abstract: This article explains and demonstrates applications of the Frobenius theory as discussed in the book by Simmons on "Differential Equations with Applications and Historical Notes." We specifically discuss the existence of a second Frobenius solution to a linear second-order linear differential equation when the two roots of its indicial equation differ by a positive integer and explain it with some examples.

Keywords: Second order differential equation, Approximate solutions, Frobenius Method, Indicial equation.

1. Introduction

A second-order differential equation with variable coefficients, though linear, does not possess an analytical solution in most cases. This situation is because all the analytical methods demand a specific form for the equation (for example, Euler's equidimensional equation), which most of the equations fail to fit in. Thus, we resort to approximate solutions to these equations. The series solution method is the most frequently used to generate approximate solutions to second-order linear equations [1]. In this series solution method, we assume a relation where the dependent variable is a power series (about a point) in the independent variable as the solution to the differential equation. Now this point could be an ordinary point or a singular point for the differential equation. The reader may refer to [1-3] for more details on this classification.

If the series solution is to be carried out at a singular point, in particular, a regular singular point (refer [1-3] for details), of the given differential equation, we use the Frobenius method. The Frobenius method assumes the solution in the form $y(x)=\sum_{n=0}^{\infty} a_n\left(x-x_0\right)^{n+m}, a_0 \neq 0$ where x_0 the regular singular point of the differential equation is unknown coefficients to be computed. The other parameter m is the root of the indicial equation, the equation obtained by equating to zero, the coefficient of least power of x after substituting $y(x)$ (mentioned above) in the given differential equation. As the indicial equation for a second-order differential equation is a second-degree polynomial equation in m, it has two roots. This review would confine our discussion to the case where the indicial equation has real roots only. Further, in this case, we come across three subcases: case (I): the two roots are equal, case (II): the roots are distinct and do not differ by an integer, and case (III): the roots are distinct and differ by an integer.

Referring to the theorem-A in Section -30 on regular singular points as mentioned in Simmons [2], we understand that in case(I), we have only one Frobenius solution, while in case(II), we have two linearly independent Frobenius solutions, one at each value of m. In case(III), theorem-A states that one

Frobenius solution always exists at the greater of the two roots m of the indicial equation. However, the second Frobenius solution exists for some differential equations, whereas it is not for others. This review notes explain in detail the theory given in Simmons[2] on the existence of a second Frobenius solution. We first detail the Frobenius method and then solve a few examples to illustrate the method.

Frobenius method:

Consider the second-order linear equation given by $y'' + P(x)y' + Q(x)y = 0$ (1)

Without loss of generality, we assume that $x = 0$ is a regular singular point of (1).

For if, $x = x_0$ is a regular singular point (RSP) of (1), then, change of variable $X = x - x_0$ transforms (1) into an equation for which $X = 0$ is an RSP.

Let us assume the solution of (1) in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+m}, a_0 \neq 0 \quad (2)$$

Since $x = 0$ is an RSP of (1), by the definition of RSP, we have that $xP(x)$ and $x^2Q(x)$ are both analytic at $x = 0$ [1]. Thus, we have

$$xP(x) = \sum_{n=0}^{\infty} p_n x^n \quad (3)$$

and $$x^2Q(x) = \sum_{n=0}^{\infty} q_n x^n \quad (4)$$

that are both valid on some interval $|x| < R$.

Substituting (2), (3), and (4) in (1), we have

$$\begin{aligned} &\sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{n+m-2} + \\ &\left(\sum_{n=0}^{\infty} p_n x^{n-1}\right) \sum_{n=0}^{\infty} (m+n) a_n x^{n+m-1} + \\ &+\left(\sum_{n=0}^{\infty} q_n x^{n-2}\right) \sum_{n=0}^{\infty} a_n x^{n+m} = 0 \end{aligned} \quad (5)$$

Comparing the coefficient of least power of x on both sides (i.e., the coefficient of x^{m-2}), we get an equation termed as the indicial equation as

$$\left(m(m-1) + mp_0 + q_0\right) a_0 = 0 \quad (6)$$

Since $a_0 \neq 0$, we have

$$m(m-1) + mp_0 + q_0 = 0 \,. \quad (7)$$

Equation (7) results in two roots. Let the greater of the two roots be termed m_1, and the smaller as m_2. Since we are working on case (III), these two roots differ by an integer, and thus, we have $m_1 - m_2 = n,$ a positive integer.

From Theorem-A [2] stated in the Appendix, we understand that the Frobenius solution exists for $m = m_1$. To find out whether the second Frobenius solution exists or not for $m = m_2$, we proceed further as shown below:

Let us first write the coefficient of the x^{m-1} from equation (5) which is

$$\left(m(m+1) + (m+1)p_0 + q_0\right) a_1 = -\left(mp_1 + q_1\right) a_0 \quad (8)$$

The next higher coefficient gives the relation

$$\begin{aligned} &\left((m+2)(m+1) + (m+2)p_0 + q_0\right) a_2 = \\ &-\left(mp_2 + q_2\right) a_0 - \left((m+1)p_1 + q_1\right) a_1 \end{aligned} \quad (9)$$

Thus, the coefficient of x^{m+n-2} gives the recursion formula

$$\begin{aligned} &\left((m+n)(m+n-1) + (m+n)p_0 + q_0\right) a_n = \\ &-\sum_{k=0}^{n-1} \left((m+k)p_{n-k} + q_{n-k}\right) a_k \end{aligned} \quad (10)$$

Since $m_1 - m_2 = n$, a positive integer, using equation (10), we derive n- equations connecting the coefficients $a_0, a_1, \ldots a_{n-1}$ by giving values to n as $1, 2, \ldots n-1$ with m replaced by m_2. These equations help us to determine the values of the coefficients $a_1, a_2, \dots a_{n-1}$ in terms of a_0. Now, for the value of n, if equation (10) results in a possibility that its left side becomes zero while the right side assumes a non-zero value, then we conclude that there is no Frobenius solution for $m = m_2$. Otherwise, there is a second Frobenius solution that can be found by assigning $a_n = 0$ and proceeding further to compute $a_{n+1}, a_{n+2}, \dots$

Consider the following examples.

Example-1: Consider the Bessel's differential equation of order ½ given as

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0 \qquad (11)$$

Solution: It can be easily verified that $x = 0$ is an RSP. Further, we see that

$xP(x) = 1$, and hence

$$p_0 = 1, p_i = 0, \textit{ for } i \geq 1 . \qquad (12)$$

Also $x^2 Q(x) = x^2 - \frac{1}{4}$, and hence

$$q_0 = -\frac{1}{4}, q_1 = 0, q_2 = 1, q_i = 0 \textit{ for } i \geq 3 . \qquad (13)$$

Therefore, the indicial equation (7) becomes

$$m^2 - \frac{1}{4} = 0 . \qquad (14)$$

Solving this equation gives, $m_1 = \frac{1}{2}, m_2 = -\frac{1}{2}$ with $m_1 - m_2 = 1$. (15)

It is left to the reader to derive the solution to equation (11) for $m_1 = \frac{1}{2}$ which is

$$y_1(x) = x^{1/2}\left(1 - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right) \qquad (16)$$

To find whether the second Frobenius solution exists or not, we now, consider equation (10) with $m = -\frac{1}{2}, n = 1$.

We see that the LHS of equation (10) is zero and the RHS of equation (10) is $\left(-\frac{1}{2} p_1 + q_1\right) a_0 = 0$

Thus, from our above discussions, we conclude that a second Frobenius solution can be obtained by taking $a_1 = 0$ and computing $a_2, a_3, \dots$.

Thus, from straightforward calculations, we get $y_2(x) = x^{-1/2}\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots\right)$ (17)

(Pl. note that these solutions are constant multiples of the Bessel functions $J_{1/2}(x)$ and $J_{-1/2}(x)$, which are the solutions of the Bessel's equation considered in (11))

Example-2: Consider the Bessel's differential equation of order 2 as

$$x^2 y'' + xy' + \left(x^2 - 4\right) y = 0 . \qquad (18)$$

Solution: Here, again, $x = 0$ is an RSP. And

$xP(x) = 1$, and hence

$$p_0 = 1, p_i = 0, \textit{ for } i \geq 1 . \qquad (19)$$

Also $x^2 Q(x) = x^2 - 4$, and hence

$$q_0 = -4, q_1 = 0, q_2 = 1, q_i = 0 \textit{ for } i \geq 3 . \qquad (20)$$

Thus, the indicial equation (7) takes the form,

$$m^2 - 4 = 0 . \qquad (21)$$

Its roots are $m_1 = 2, m_2 = -2$ with $m_1 - m_2 = 4$. (22)

A straightforward calculation gives the solution for $m_1 = 2$ as

$$y_1(x) = x^2\left(1 - \frac{x^2}{2.3!} + \frac{x^4}{2!.4!.2^3} - \dots\right) \qquad (23)$$

To find whether the second Frobenius solution exists or not, we consider the equation (10) for $m=-2$ and when n takes the values $1,\ldots 4$.

For $n=1$, LHS of equation (10) gives $-3a_1$ and RHS of equation (10) is zero. Thus, we have

$$a_1 = 0. \tag{24}$$

For $n=2$, we get $a_2 = \frac{a_0}{4}$. (25)

Now, $n=3$ gives, $a_3 = 0$. (26)

Furthermore, for $n=4$, the LHS of equation (10) is zero, while the RHS is $a_2 = \frac{a_0}{4}$, which is a non-zero quantity.

Thus, we conclude that the second Frobenius solution does not exist for this equation. (You may recall that one solution of this equation is $J_2(x)$, the Bessel's function of the first kind and order 2, while its second solution is $Y_2(x)$ which is the Bessel's function of the second kind and order 2.)

For practice problems, the reader may refer to the textbooks [2,3].

References:

[1] Radhika T.S.L., Iyengar T. K. V. and RajaRani T., *Approximate Analytical Methods for Solving Ordinary Differential Equations*, Boca Raton: CRC Press, Taylor and Francis Group, 2015.

[2] George F. Simmons, John S. Robertson, *Differential Equations with Applications and Historical Notes*, 2nd Edition, Tata Mc. Graw Hill, 1991.

[3] Grewal B.S., *Higher Engineering Mathematics*, 44th Edition, Khanna Publishers, 2017.

Appendix

Theorem-A [1]:

Assume that $x=0$ is an RSP of the differential equation (1) and the power series expansions (3) and (4) of $xP(x)$ and $x^2Q(x)$ are valid on the interval $|x|<R$ with $R>0$. Let the indicial equation (6) have real roots m_1 and m_2 with $m_2 \le m_1$. Then equation (1) has at least one solution $y_1 = x^{m_1} \sum\limits_{n=0}^{\infty} a_n x^n, a_0 \ne 0$ on the interval $0<x<R$, where the a_n are determined in terms of a_0 by the recursion formula (10) with m replaced by m_1, and the series $\sum\limits_{n=0} a_n x^n$ converges for $|x|<R$: Furthermore, if $m_1 - m_2$ is not zero or a positive integer, the equation (1) has a second independent solution $y_2 = x^{m_2} \sum\limits_{n=0}^{\infty} a_n x^n, a_0 \ne 0$ on the same interval, wherein this case the a_n are determined in terms of a_0 by the formula (10) with m replaced by m_2, and again the series $\sum\limits_{n=0} a_n x^n$ converges for $|x|<R$.