STUDENT VERSION
The Lorenz System Simulation

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In the Memory of Edward Norton Lorenz

Edward Norton Lorenz (May 23, 1917 – April 16, 2008) was an American mathematician and meteorologist [R1] who established the theoretical basis of weather and climate predictability, as well as the basis for computer-aided atmospheric physics and meteorology [R2, R3]. He is best known as the founder of modern chaos theory [R4, R5], a branch of mathematics focusing on the behavior of dynamical systems that are highly sensitive to initial conditions.

His discovery of deterministic chaos "profoundly influenced a wide range of basic sciences and brought about one of the most dramatic changes in mankind's view of nature since Sir Isaac Newton," according to the committee that awarded him the 1991 Kyoto Prize for basic sciences in the field of earth and planetary sciences [R1].

STATEMENT

Part I: The Lorenz Attractor Model

This case study is designed to introduce you to numerical modeling of the attractors in chaotic dynamical systems observed in weather forecast, turbulence, and socio-economic system development [1-4]. You will study bifurcations of a simplified system of nonlinear ordinary differential equations modeling atmospheric dynamics (the Lorenz attractor) that was first studied by E. N. Lorenz in [5]. It was derived from a simplified model of convection in the Earth's atmosphere originally described as a system of twelve equations that E. N. Lorenz [5] and B. Saltzman [R6] had developed to model complex atmospheric processes. It also arises naturally in models of lasers and dynamos [2]. The solutions of this system of twelve equations exhibited what

Lorenz described as the "butterfly effect" [1, 5], later interpreted as the "sensitive dependence on initial conditions" [6-9] (see Figure 1, below). While the simpler system of three equations was not intended to model any natural phenomenon, it was later found to accurately describe a simplified model of atmospheric convection. In this model, a torus-shaped tube forming a closed loop is filled with a fluid and heated from below, causing the heated fluid to rotate to the top of the tube. The rate of heating and the rate of cooling can be adjusted so that the fluid will occasionally change direction, never exhibiting any predictable pattern. This motion is accurately modeled by the Lorenz equations (see (1), (2), and (3) below) that can be solved numerically (see the MATLAB code in Appendix A).

E. N. Lorenz [5] started with an overview of the system of the equations [R6] governing finite-amplitude convection in a 3D incompressible liquid. The liquid is considered to be of height H, with a rigid lower boundary and a free or rigid upper boundary, between which a temperature contrast $\Delta T = T(0) - T(H)$ is maintained externally. To simplify the problem, the assumption is made that the convective motions are developed only in the form of two-dimensional "rolls" in the X-Z plane. This approximation allows to define a stream function ψ and the departure of temperature θ from that occurring in the state of no convection, formulate the vorticity equation [R6], and, after introducing non-dimensional variables, transform the original equations into a system of two non-dimensional partial differential equations (that include functions ψ and θ and their derivatives) [5, R6] with two general similarity parameters, the Prandtl number $\sigma = v/\kappa$ and the Rayleigh number $R_a = g\varepsilon H^2 \Delta T/(\kappa v)$, where v is kinematic viscosity, κ – coefficient of thermal diffusivity, g – acceleration of gravity, and ε – coefficient of thermal diffusivity.

L. Rayleigh [R7] found that fields of bi-harmonic motion ("rolls") (with d as a parameter) in the X-Z plane would develop if the Rayleigh number quantity R_a exceeded a critical value $R_c = \pi^4 d^{-2}(1+d^2)^3$, and the minimum value of $R_{c,min}$, namely $27\pi^4/4 \approx 657.51$, occurs when $d^2 = \frac{1}{2}$.

ASSIGNMENT-1: Recommended Preliminary Readings

You are encouraged to read a few articles before starting the analysis of the Lorenz System of Differential Equations:

[5] Lorenz, Edward N. Deterministic Nonperiodic Flow. *Journal of Atmospheric Sciences*. 1963, 20(2): 130–141. DOI: 10.1175/1520-0469(1963)020<0130:DNF>2.0.CO;2. Freely available from:
https://journals.ametsoc.org/view/journals/atsc/20/2/1520-0469_1963_020_0130_dnf_2_0_co_2.xml.

- o Formulation of finite systems of deterministic ordinary nonlinear differential equations that represent a forced dissipative hydrodynamic flow.
- o Study general properties of solutions of these equations that can be identified with trajectories in phase space.
- o Understanding why the simplified version of the equation system that represents the cellular convection has been selected for the numerical analysis.

[R8] Chaos Theory. [Online] https://en.wikipedia.org/wiki/Chaos_theory.

- o *Chaos theory* is a branch of mathematics focusing on the study of *chaos* — dynamical systems whose apparently random states of disorder and irregularities are actually governed by underlying patterns and deterministic laws that are highly sensitive to initial conditions.
- o The *butterfly effect*, an underlying principle of chaos, describes how a small change in one state of a deterministic nonlinear system (due to errors in measurements or due to rounding errors in numerical computation) can result in large differences in a later state (meaning that there is sensitive dependence on initial conditions).
- o The deterministic nature of these systems (with the behavior that follows a unique evolution and is fully determined by their initial conditions) does not make them predictable. This behavior is known as *deterministic chaos*, or simply *chaos*.

The Lorenz System of Differential Equations

The Lorenz system is most commonly expressed as the following three coupled nonlinear ordinary differential equations [5]:

$$\mathrm{d}x / \mathrm{d}t = a\,(y - x) \tag{1}$$
$$\mathrm{d}y / \mathrm{d}t = x\,(b - z) - y \tag{2}$$
$$\mathrm{d}z / \mathrm{d}t = xy - c\,z \tag{3}$$

These equations relate the properties of a two-dimensional fluid layer uniformly warmed from below and cooled from above. In particular, the equations describe the rate of change of three quantities with respect to time: x is proportional to the rate of convection, y to the horizontal temperature variation, and z to the vertical temperature variation.

In (1)-(3) the non-dimensional variables x, y, and z are functions of time alone and they are the same as Saltzman's variables A, D, and G [R6]; "t" is the dimensionless time variable using the normalization parameter $\pi^2 H^{-2}(1+d^2)\kappa$; "$a$" is the Prandtl number σ; "b" is the normalized Rayleigh number R_a/R_c, and "c" is the system parameter $4(1+d^2)^{-1}$ that represents physical dimensions of the layer itself.

As mentioned in [5], in these convection equations (1), (2), and (3), the function x is proportional to the intensity of the convective motion, while y is proportional to the temperature difference between the ascending and descending currents (similar signs of x and y denoting that warm fluid is rising and cold fluid is descending). The variable z is proportional to the distortion of the vertical temperature profile from linearity, a positive value indicating that the strongest gradients occur near the boundaries [5]. E. N. Lorenz noted [5] that equations (1), (2), and (3) may give realistic results when the Rayleigh number is slightly supercritical, $b > 1$.

ASSIGNMENT-2: Analyzing Some Important Features of the Lorenz System of Differential Equations

You are asked to identify some important basic features of the Lorenz system (1), (2), and (3), including the following ones:

- o   Is this system autonomous? ("Autonomous" means that time does not explicitly appear on the right hand side of the equations).
- o   Does the system evolution depend only on the values of x, y, and z at the time? HINT: The equations involve only first order time derivatives.
- o   Is the system of equations linear or non-linear? HINT: Consider the terms xz and xy in (2) and (3), correspondingly.
- o   Is the system dissipative or not?
  HINTS: By definition, the system is dissipative when the following inequality holds:
  $$\mathrm{div} f = \partial \dot{x}/\partial x + \partial \dot{y}/\partial y + \partial \dot{z}/\partial z < 0$$

Here a dot denotes a derivative with respect to the time. Also, the parameters a, b, and c in (1) – (3), denoting the physical characteristics of the air flow, are positive. The solutions of a dissipative system are bounded.

- o   Is the system symmetric with respect to the z axis? HINT: Check, if the system is invariant for the coordinate transformation:
  $$(x,\ y,\ z) \rightarrow (-x,\ -y,\ z)$$

Read more about the features of the Lorenz system in:

[R9] Moghtadaei, M., and Hashemi Golpayegani, M. R. "Complex Dynamic Behaviors of the Complex Lorenz System", *Scientia Iranica*, 2012, 19(3): 733-738. DOI: 10.1016/j.scient.2010.11.001.
https://www.sciencedirect.com/science/article/pii/S1026309811002513.

ASSIGNMENT-3: Apply the Linear Theory to Study the Stability of a Solution of the Lorenz System

Following the analysis in [5] (pp. 135-136), you are encouraged to study the stability of a solution $(x(t), y(t),$ and $z(t))$ of the Lorenz System (1)-(3) by using the linear transformation:
$$x' = x, \; y' = y, \; z' = z - b - a .$$

Find the linearized equations (in the matrix form) [5] (p. 135) for small superposed perturbations $x_0(t), y_0(t),$ and $z_0(t)$, which can be used for the analysis of the stability of a solution $(x(t), y(t),$ and $z(t))$.

Estimate the diagonal sum of the matrix coefficients. Compare it with the calculated value of the divergence:
$$\mathrm{div} f = \partial \dot{x}/\partial x + \partial \dot{y}/\partial y + \partial \dot{z}/\partial z = - (a + c + 1) < 0.$$

Note that equations (1) – (3) possess the steady-state solution $x = y = z = 0$, representing the state of no convection [5]. With this basic solution, prove that the characteristic equation of the matrix of the linearized equations is in the following form [5]:

$$[\lambda + c][\lambda^2 + (a + 1)\lambda + a(1 - b)] = 0.$$

Study the three real roots of this characteristic equation when $b > 0$, and prove that the criterion for the onset of convection is $b = 1$ or $R_a = R_c$, in agreement with Rayleigh's fundamental result [5, R7].

When $b > 1$, prove that (1)-(3) possess two additional steady-state solutions $x = y = \pm[c(b\text{-}1)]^{1/2}$, $z = b - 1$. For either of these solutions, the characteristic equation of the matrix of the linearized equations can be written in the following form [5]:

$$\lambda^3 + (a + c + 1)\lambda^2 + (b + a)c\lambda + 2ac(b - 1) = 0.$$

This equation possesses one real negative root and two complex conjugate roots when $b > 1$. The complex conjugate roots are pure imaginary if the product of the coefficients of λ^2 and λ equals the constant term [5], or

$$b = a(a + c + 3)/(a - c - 1).$$

And this is the criterion (see above) for estimating the critical value of b for the instability of steady convection. Prove that if $a < c + 1$, no positive value of r satisfies this criterion, and steady convection is always stable, but if $a > c + 1$, steady convection is unstable for sufficiently high Rayleigh numbers characterized by the parameter b. Lorenz mentioned [5] that the presence of complex roots (see the criterion above) shows that if unstable steady convection is disturbed, the motion will oscillate in intensity. The linear theory is not applied for studies when the disturbances become large. To investigate the finite-amplitude convection, the numerical integration of the Lorenz system (1) – (3) should be performed.

ASSIGNMENT-4: Optionally, the Advanced Analytical Studies [6] Can Be Performed

The general assumption is that $a, b, c > 0$; $a = 10$, and b is varied [5]. The system exhibits chaotic behavior for $b = 28$, but displays knotted periodic orbits for other values of b. A saddle-node bifurcation occurs at $c(b - 1) = 0$. When $a \neq 0$ and $c(b - 1) \geq 0$, the equations generate three critical points. The critical points at $(0,0,0)$ correspond to no convection, and the critical points at $(\pm[c(b - 1)]^{1/2}, \pm[c(b - 1)]^{1/2}, b - 1)$ correspond to steady convection. This pair is stable only if $b < a(a + c + 3)/(a - c - 1)$. When $a = 10$, $b = 28$, $c = 8/3$, the Lorenz system has chaotic solutions, but not all solutions are chaotic [6].

Numerical Solutions of the Lorenz System for Different Values of *b*

There were numerous attempts to create computer codes using MATLAB (see Appendix A), Mathematica simulation [6], Python [6], and other programming languages.

ASSIGNMENT-5: Watching the Video "Simulating the Lorenz System in MATLAB"

You are encouraged to watch the video (length=00:15.08) [R10] available from YouTube:
> [R10] Brunton, Steve. "Simulating the Lorenz System in MATLAB". June 12, 2018, YouTube video. https://www.youtube.com/watch?v=EnsB1wP3LFM.

This video shows how simple it is to simulate dynamical systems, such as the Lorenz system, in MATLAB, using ode45 subroutine. The similar approach was used by the author of this case study (see Appendix A).

Optionally, you can watched the video (length=00:21:20) [R11] also available from YouTube:
> [R11] Shiffman, Daniel. "Coding Challenge #12: The Lorenz Attractor in Processing". May 6, 2016, YouTube video. https://www.youtube.com/watch?v=f0lkz2gSsIk.

It shows how to create a visualization of the Lorenz Attractor using the Java code processing.

ASSIGNMENT-6: Exploring Properties of Numerical Solutions of the Lorenz System

Using the MATLAB code (see Appendix A) or creating their own code programs (in any programming language), you are encouraged to explore the properties of numerical solutions of the Lorenz system for different values of key parameters *a*, *b*, and *c*. See Figure 1, as an example.
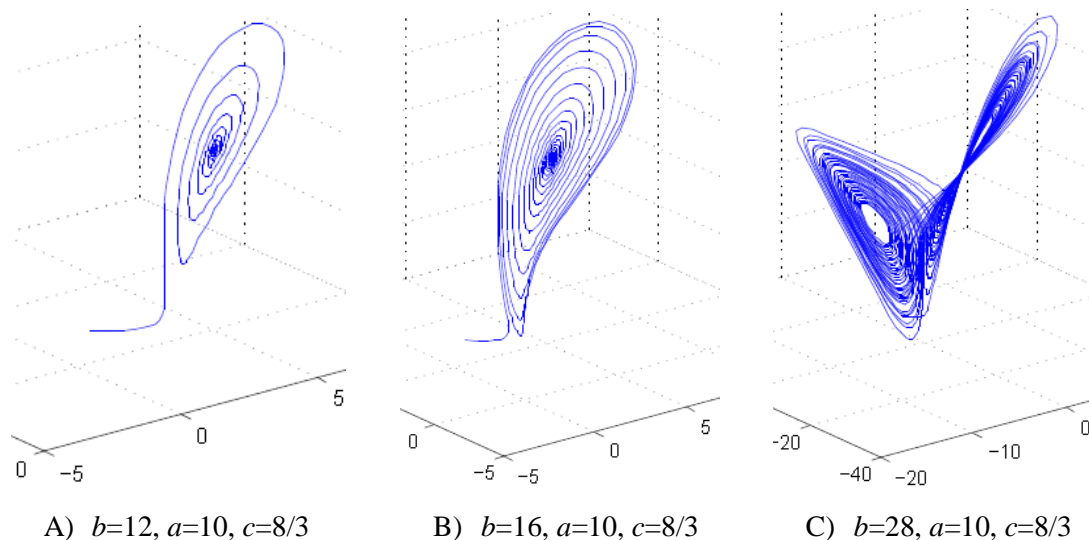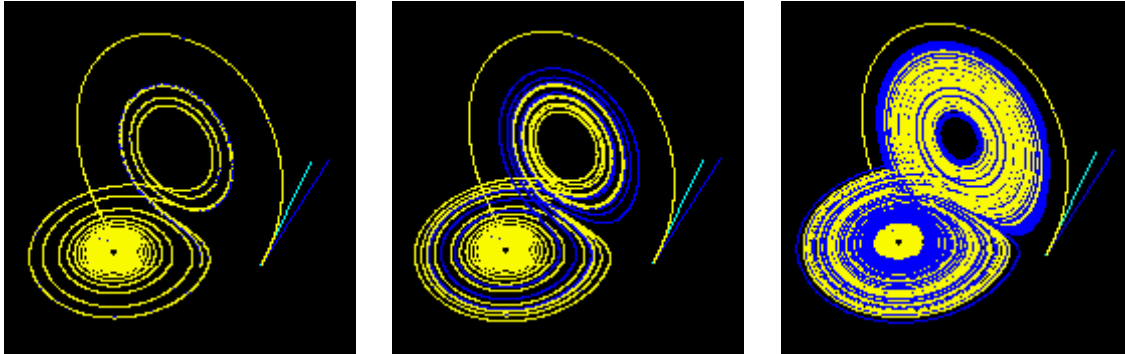

A) *b*=12, *a*=10, *c*=8/3   B) *b*=16, *a*=10, *c*=8/3   C) *b*=28, *a*=10, *c*=8/3

Figure 1. Solutions of the Lorenz system ((1), (2), and (3)) for different values of *b* with fixed *a* and *c* [9].

The MATLAB calculations show the system evolution for different values of *b* (see Figure 1 [9]). For small values of *b*, the system is stable and evolves to one of two fixed point attractors. When *b* is larger than 24.28, the fixed points become repulsors and the trajectory is repelled by them in a very complex way, evolving without ever crossing itself [8].

Run numerical simulations by fixing two parameters, $a = 10$, $c = 8/3$, and varying the third parameter, *b*, with the values of 1, 23.9, 69.8, 71.5, 92.2, and 100. Use the same initial conditions in all these simulations: $x_0 = 0.1$, $y_0 = 0.3$, and $z_0 = 1$. For every parameter b, determine the principle

behavior of the Lorenz system: Fixed point, Transition chaos, Chaotic, or Periodic. Compare your findings with the results found in [R9].

The Lorenz System: Sensitive dependence on the initial condition

| a)   Time t=1 | b)   Time t=2 | c)   Time t=3 |

Figure 2. Sensitive dependence of the solution on initial condition at a=10, b=28, c=8/3 [6, 7].

Three time segments (taken from the Java animation [7] and shown in Figure 2 [6]) illustrate the 3-D evolution of two trajectories (one in blue and the other in yellow) in the Lorenz attractor starting at two initial points that differ only by 10^{-5} in the x-coordinate. Initially, these two trajectories seem coincident (only the yellow one can be seen, as it is drawn over the blue one) but, after some time, the divergence is obvious [6].

ASSIGNMENT-7: Experimenting with Different Initial Conditions

In this exercise, you will study the sensitive dependence of the solutions on the initial condition.

Run numerical simulations by fixing two parameters, $a = 10$, $c = 8/3$, and varying the third parameter, b, with the values of 1, 23.9, 69.8, and 100. Use the new initial conditions in all these simulations: $x_0 = 0.15$, $y_0 = 0.3$, and $z_0 = 1$. Compare the results with the data previously calculated at the same parameters a, c, and b, but with "old" initial conditions: $x_0 = 0.1$, $y_0 = 0.3$, and $z_0 = 1$.

Part II: Physical Model Simulating the Lorenz System

There were several attempts to create physical models to simulate the behavior of the Lorenz system.

ASSIGNMENT-8: Modeling the Lorenz System Behavior with an Analog Circuit

You are invited to watch the Prof. Paul Horowitz's video presentation [R12] on a design of an analog circuit (shown in Figure 3) that behaves exactly like the Lorenz system attractor:
    [R12] Horowitz, Paul. "An analog circuit that behaves like an attractor". M4V video
    [Size=64.5 MB, length=00:08:59]. Available from
    https://drive.google.com/file/d/1Gn2EfTQIAYLOCsPFgwH5_zbf1mZtb1tR/view?usp=sharin
    g (it will redirect you to
    https://drive.google.com/file/d/1Gn2EfTQIAYLOCsPFgwH5_zbf1mZtb1tR/view, and
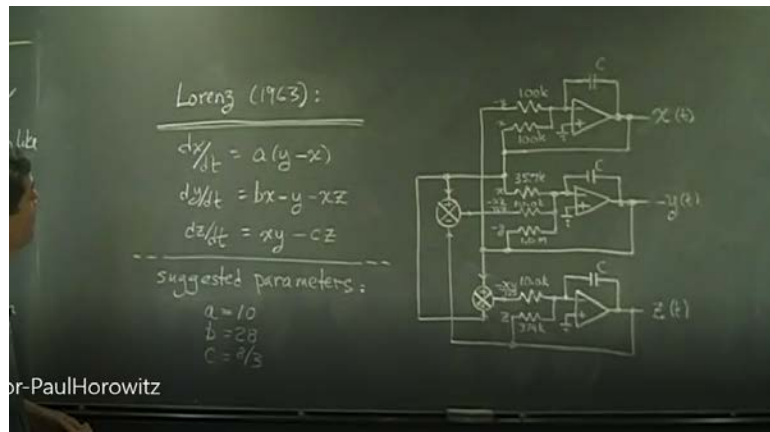   LorenzAttractor-PaulHorowitz.m4v file can be downloaded).

Figure 3. A fragment of Prof. Paul Horowitz's video presentation [R12] on a design of an analog circuit that behaves like the Lorenz system attractor.

You should confirm that the deterministic chaotic trajectories found in these experiments with the analog circuit match the results of the numerical simulations discussed in previous section.

"The Waterwheel Model" of Simulating the Lorenz System

A physical model for simulating the Lorenz equations has been attributed to Willem Malkus and Lou Howard around 1970 [6, 8]. The model replaces the circular tube with a waterwheel (see Figure 4 below) in the following setting: cups are arranged around the wheel and are filled with water when they pass under a faucet located above the wheel (which corresponds to the heating of the fluid at the bottom of the tube). The water leaks from the cups as they move around the wheel (which corresponds to the cooling of the fluid). The heavier cups will rotate to the bottom of the wheel, just as the heated fluid rises to the top of the tube. Under different input flow rates you should be able to convince yourself that under just the right flow rate the wheel will spin one way and then the other chaotically [8].
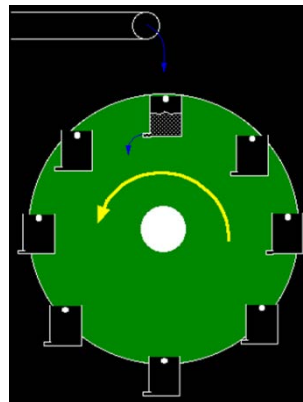

Figure 4. A waterwheel model of simulating the Lorenz system [6, 8].

Figure 5. The waterwheel demo [6].

The waterwheel (see Figure 5 [6]) was actually built by Planeten Paultje for the Dutch Annual Physics Teacher Conference "Woudschotenconferentie Natuurkunde 2005" in December 2005.

The chaotic behavior of the Malkus–Lorenz waterwheel system was analyzed in [R13] and [R14]. The Harvard Natural Sciences Lecture Demonstrations site [R15] posts a video that demonstrates a waterwheel with leaky buckets that undergoes chaotic motion (Figure 6).

Figure 6. The Chaotic Waterwheel settings made by the Harvard Natural Sciences Lecture Demonstrations group [R15].

ASSIGNMENT-9: Examining the Chaotic Behavior of the Leaking Waterwheel System

You are invited to observe the experiments with the Chaotic Waterwheel settings (see Figure 6) performed by the Harvard Natural Sciences Lecture Demonstrations group [R15].

[R15] "Chaotic Waterwheel". Harvard Natural Sciences Lecture Demonstrations. YouTube (length=00:05:19), May 13, 2020. https://www.youtube.com/watch?v=Lx8gMBJBlP8.

A vertical waterwheel with eight leaky buckets (started operating in neutral equilibrium and free to rotate in either direction) undergoes chaotic motion. The wheel is about 1 meter in diameter and was fabricated with wood in the Harvard University shop. The little buckets are citronella candle holders with ¼" holes drilled out of the bottom. Fixed directly above the center of the wheel is a faucet connected to a pump. The sump pump was purchased from the local hardware store. A ball

valve at the faucet regulates the water flow. The wheel and pump both sit in a concrete mixing tub. In nature, chaotic behavior readily occurs in turbulent flows and in large-scale weather patterns, but scaling these systems to a laboratory or classroom setting is far from trivial. The idea of building a waterwheel as a discrete, mechanical example of a chaotic system was proposed and realized by Willem Malkus, Louis Howard, and Ruby Krishnamurti in the early 1970s. According to Edward Lorenz, their original design "was a precision instrument, suitable for controlled laboratory experiments." The Harvard design is simpler and geared more towards pedagogical impact than experimental fidelity, but it is considered as charming and instructive nonetheless. For more details on the setup, including links to download some of the video clips of the wheel in motion, see https://sciencedemonstrations.fas.harvard.edu/presentations/chaotic-waterwheel and a video (length=00:05:19) on YouTube https://youtu.be/Lx8gMBJBlP8.

When the pump is turned on, a stream of water flows downwards from the faucet, with some of it inundating the buckets, filling them faster than they can drain out. Eventually the entire wheel will become unbalanced and start to rotate accordingly: the side with the fuller, heavier buckets will get pulled down, and the relatively empty buckets on the other side will swing up so that they can get refilled under the faucet. The situation is further complicated by buckets taking water from the ones directly above them, either via the drainage holes or because of spillage over the top.

As you could observe, after several seconds of the wheel spinning steadily in one direction, it may suddenly speed up; it may slow down and start to rotate in the opposite direction; it may sometimes oscillate between clockwise and counterclockwise rotation; it may go through periods when motion in either direction is barely noticeable. Whatever the state of motion, it never survives for long. The motion is clearly non-periodic and unpredictable.

Another puzzling feature of the water wheel is observed when the demonstrator pauses the pump and allows all the buckets to drain completely. With the wheel in very nearly the same angular orientation as before, the water flow was resumed. What you can find is that, over the course of a minute or two, the complete motion of the wheel will be dramatically different than during the previous run. This apparent sensitivity to the initial conditions of the system, along with the wheel's non-periodic motion, are hallmarks of chaotic behavior [6].

To make the demo a little more quantitative, students may try to keep count of revolutions and oscillations in sequential order to verify the non-periodic behavior. Four video clips of the wheel in action https://drive.google.com/drive/folders/1vFTQEL04PfAW5nAQi3nBGsDSYIxgkfKq so have been made available that students may download them and use motion-tracking software to analyze the wheel's motion.

Numerical Simulation of the Waterwheel System Behavior

ASSIGNMENT-10: Exploring the Waterwheel System Behavior with the Java code

Use the Java code [10] (see Appendix B) to confirm the deterministic chaotic behavior of the Waterwheel System briefly described in the article [10]. For example, you could follow these exercises:

By using the Java program [10] (see Appendix B) that simulates the waterwheel performance, the angular acceleration of the wheel is calculated according to the laws of physics based on the changing mass of each cup, and this acceleration modifies the angular velocity on each iteration of the code.

The angular velocity of the simulated waterwheel over time is shown in Figure 7 below.
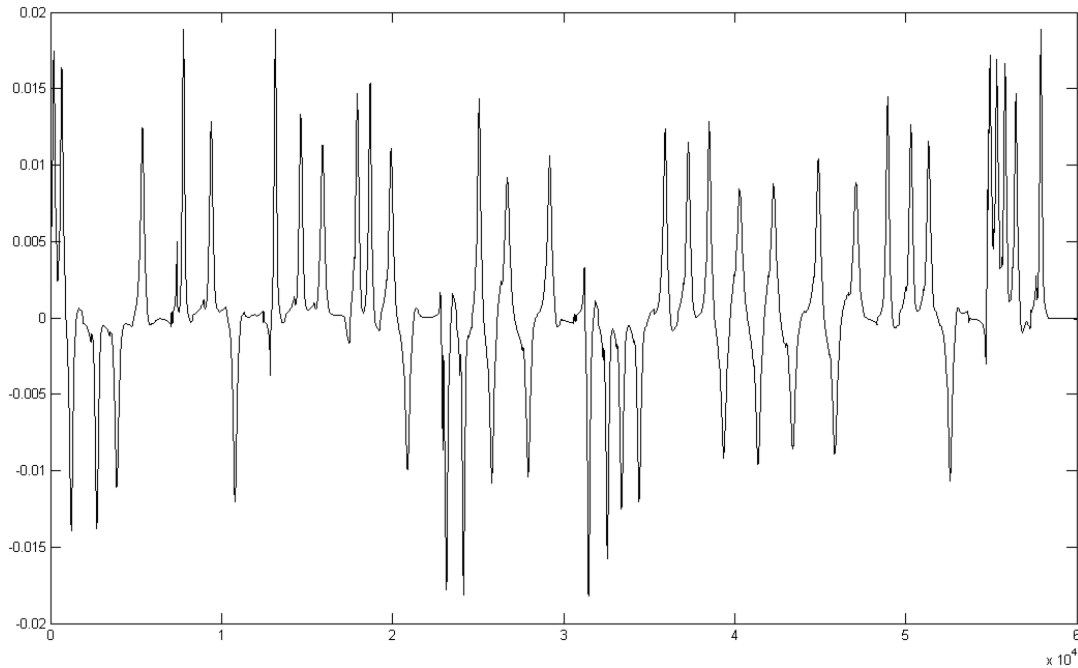
Figure 7. Angular velocity as a function of time [10].

In Figure 7, positive values of angular velocity represent rotation of the wheel in the positive direction, while negative values represent rotation in the negative direction. Consequently, changes between positive and negative values represent changes in the direction of the wheel's rotation.
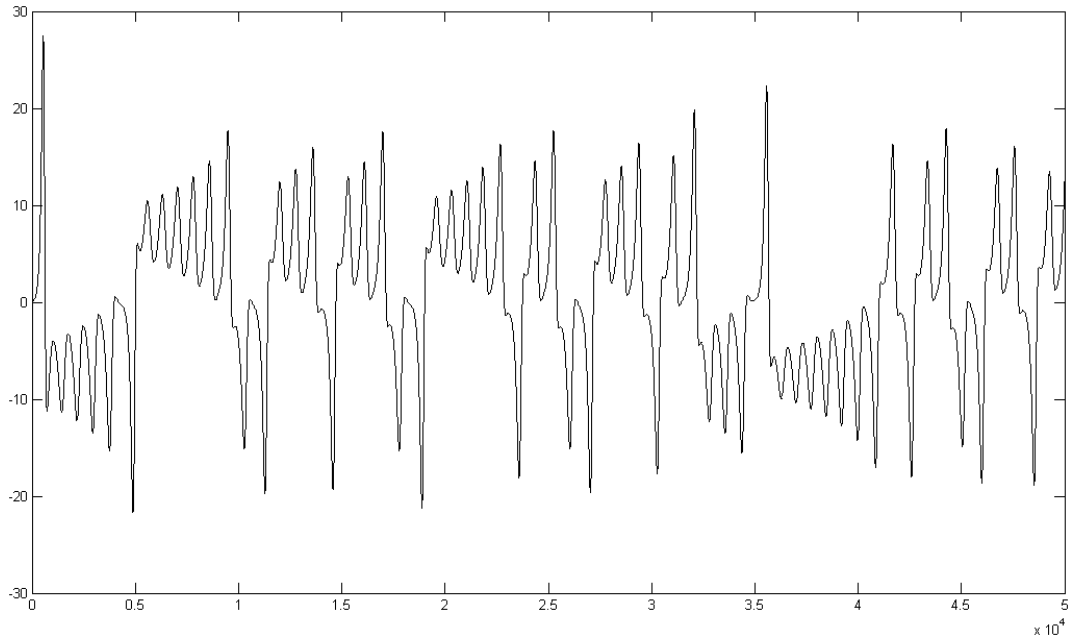

Figure 8. Variable y of the Lorenz system as a function of time [10].

The graph in Figure 8 shows variable y of the Lorenz system as it changes over time t. Figure 9 (see below) shows the same process as the one in Figure 8 with y on the horizontal axis and z on the vertical axis.
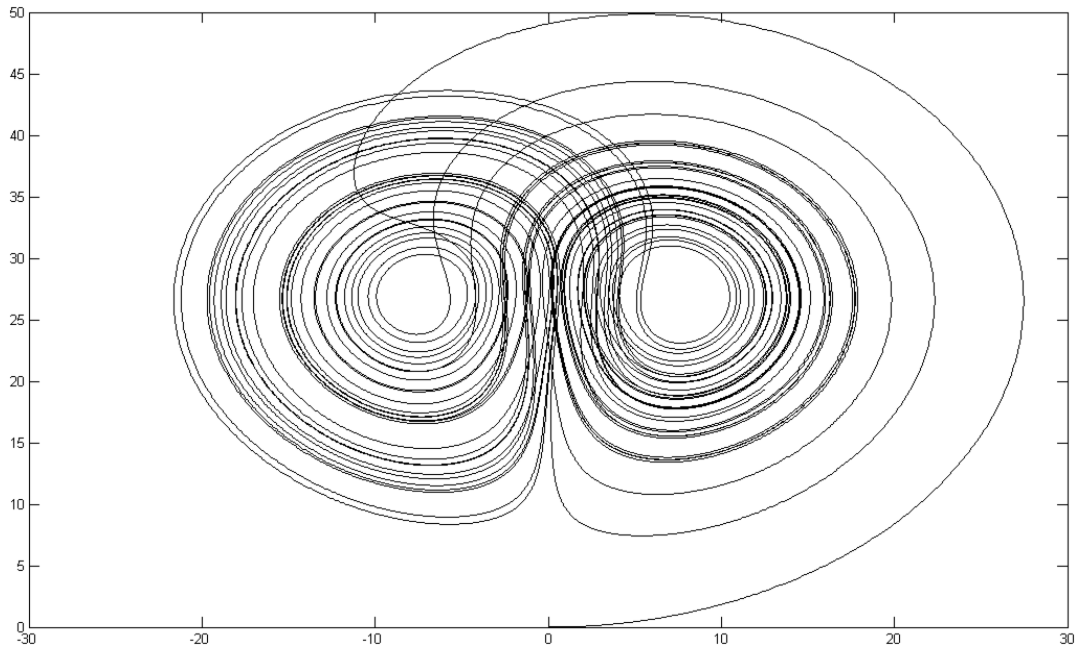
Figure 9. Relation between *y* and *z* coordinates in the Lorenz system [10].

Positive values of *y* in Figure 8 correspond to the right attractor shown in Figure 9, while negative values of *y* in Figure 8 correspond to the left attractor (see Figure 9). Changes between positive and negative values of *y* in Figure 8 represent transfers between the two attractors. Thus, the two attractors shown in Figure 9 correspond to the two directions of rotation of the waterwheel. The chaotic nature of the Lorenz system makes the transfers between the two attractors, and, consequently, the changes in the rotational direction of the waterwheel it models, unpredictable. This animation demonstrates chaotic behavior of the numerical solution of the Lorenz' system of nonlinear ordinary differential equations.

Supplements

The case-study supplementary materials (computer codes and files listed in Appendix C) will be available for students and instructors from the website: https://www.simiode.org/.

Concluding Remarks on Students' Involvement

After in-class discussions of the considered case studies, each student can continue working on a selected case analyzing in-depth mathematical models, creating computer codes (in MATLAB, C/C++, Java, Mathematica, or Python), running the codes with various parameters, comparing computations with experimental data, and presenting the findings to classmates.

SUBMITTING YOUR FEEDBACK COMMENTS

After finishing your work on this case study, you may submit your feedback comments to the course instructor, including the following:

1) At the beginning of case-study activities, you were introduced (in historical retrospective [R4, R5]) to the concept of deterministic chaos [1, 5] (see Assignment-1), mathematical properties of a simplified system of nonlinear ordinary differential equations modeling atmospheric dynamics (the Lorenz attractor) [1, 5, 6], and their numerical solutions using the MATLAB code (see Assignment-6 and Appendix A). Have you been satisfied with these activities? Was it difficult to you to run the programming code in MATLAB environment?

2) Did you identify the different patterns of the Lorenz system evolution at different values of the parameter *b* (1, 23.9, 69.8, 71.5, 92.2, and 100) and the fixed values of two other parameters, *a* = 10 and

$c = 8/3$ (see Assignment-6)? Why the system bifurcations appear at $b \geq 28$? In the Assignment-3, you applied the linear theory to study the stability of a solution of the Lorenz system. Was this exercise helpful for you to understand the nature of bifurcations?

3) Did you find the dependence of the solutions of the Lorenz equations (1) – (3) on the initial conditions (see Assignment-7), especially in the presence of bifurcations? Can this sensitive dependence be classified as the "butterfly effect" [1, 5]?

4) In the Assignment-8, you watched the video recording of Prof. Paul Horowitz's experiments [R12] with an analog circuit that behaves like the Lorenz system attractor. Can you confirm that the deterministic chaotic trajectories found in these experiments with the analog circuit match the results of your numerical simulations using the MATLAB code (Appendix B)?

5) Finally, in Assignments 9 and 10, you examined the chaotic behavior of the Malkus–Lorenz leaking waterwheel system [R13-R15]. Have your numerical solutions received with the Java code for waterwheel animation (Appendix C) confirmed the deterministic chaotic system behavior that was found in experiments performed by the Harvard Natural Sciences Lecture Demonstrations group [R15]? Have you been satisfied with these activities? Was it difficult to you to run the Java programming code available in Appendix C?

Would you be interested in utilizing the acquired knowledge and analytical and programming skills in your new potential project that includes other challenging cases (e.g., complex economic dynamics [4], models of lasers and dynamos [2], etc.)?

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Appendix A: The Source Code to Simulate the Lorenz System with MATLAB

```
% A simple Lorenz Solver in MATLAB code
function dxdt=fLorenz(t,x)
% The RHS of the Lorenz attractor
% Save this function in a separate file 'fLorenz.m'
a = 10;
b = 28;
c = 8/3;
dxdt=[a*(x(2)-x(1));
b*x(1)-x(2)-x(1)*x(3);
x(1)*x(2)-c*x(3)];
end
%% Main program: Save the program in a separate .m file
%% and run it.
clear all; % clear all variables
t=linspace(0,50,3000)'; % specified time variables
y0=[-1;3;4]; % Initial conditions should be specified
[t,Y] = ode45(@fLorenz,t,y0); %Invoking the solver 'ode45'
plot3(Y(:,1),Y(:,2),Y(:,3)); % Plot results
grid on;
```

Appendix B: Java Code for Waterwheel Animation [10]

NOTE: This Java code [10] is an adaptation of the C code from [8].

```java
import javax.swing.*;
import java.awt.*;
import java.io.*;

class LorenzSimulation{
    public static void main(String[] args){
        JFrame frame = new JFrame("The Lorenz System");
        frame.setSize(600,760);
        frame.setDefaultCloseOperation(JFrame.EXIT_ON_CLOSE);
        Panel pane = new Panel();
        pane.setBackground(Color.black);
        frame.setContentPane(pane);
        frame.setVisible(true);
    }
}

class Panel extends JPanel{
    Waterwheel wheel;
    protected Panel(){
        wheel = new Waterwheel(new Point(300,400), 200, 12);
    }
    protected void paintComponent(Graphics g){
        super.paintComponent(g);
        g.setColor(Color.lightGray);
        g.fillRect(0,20,285,30);      //horizontal pipe
        g.fillRect(285,50,30,40);     //vertical pipe
        g.fillArc(255,20,60,60,0,90); //curve in pipe
        g.setColor(new Color(150,150,250));
        g.fillRect(295,90,10,110);    //stream of water
        wheel.draw(g);
        try{Thread.sleep(2);}
        catch(InterruptedException e){}
        wheel.calculate();
        repaint();
    }
}

class Waterwheel{
    final double FILL_SPEED = .4;  //.4 for chaotic motion; .9 for stable motion
    final double DRAIN_SPEED = .02;
    final double V_COEFFICIENT = 1/50.0;   //coefficient of rotational velocity
    Point center;                          //added to angle in each iteration
    int r; //radius
    double a; //angle of shift from original position
    double v; //velocity of rotation
    int numCups;
    Cup[] cups;
    PrintWriter outFile;
    public Waterwheel(Point c, int rIn, int n){
        center = c;                            //Waterwheel constructor
        r = rIn;
```

```java
      a = 0.0;
      v = 0.01;
      numCups = n;
      cups = new Cup[numCups];                      //array of Cups
      for(int i=0; i<numCups; i++) cups[i] = new Cup(i*(2*Math.PI/numCups));
      try{outFile = new PrintWriter("Waterwheel_Output.txt");}
      catch(IOException ignore){ }
      outFile.printf("%10s%10s%n","Angle","Velocity");
   }
   public void calculate(){ //recalculate position and conditions of Waterwheel
      double m = 0;
      for(int i=0; i<numCups; i++){
         if(Math.abs(Math.cos(a+cups[i].a)) <.1 && Math.sin(a+cups[i].a) > 0
           && cups[i].volume < 50) cups[i].volume += FILL_SPEED;
         if(cups[i].volume > 0) cups[i].volume -= DRAIN_SPEED;
         v -= (cups[i].volume) * Math.cos(a+cups[i].a) * V_COEFFICIENT;
         m += cups[i].volume;
      }
      v /= m;
      a += v;
      outFile.printf("%10.5f%10.5f;%n",a,v);
   }
   public void draw(Graphics g){                     //draw Waterwheel
      g.setColor(Color.gray);
      g.drawOval(center.x - r, center.y - r, 2*r, 2*r); //draw large circle
      g.setColor(new Color(50,50,200));//new Color(0,0,100));
      for(int i=0; i<numCups; i++) cups[i].draw(center, a, g); //draw cups
   }

   private class Cup{                                //internal Cup class
      private double volume; //radius of cup circle
      private double a;      //position on wheel
      private Cup(double aIn){                        //Cup constructor
         a = aIn;
         volume = 0;
      }
      //draw individual cup:
      private void draw(Point center, double aWheel, Graphics g){
         g.fillOval((int)(center.x + r*Math.cos(a+aWheel) - volume - 3),
                 (int)(center.y - r*Math.sin(a+aWheel) - volume - 3),
                 2*(int)volume + 6, 2*(int)volume + 6);
      }
   }
}

import java.io.*;
class LorenzEquations{
   public static void main(String[] args) throws IOException{
      double h = .001;
      double a = 10;
      double b = 28;
      double c = 8/3;
      double x0 = 0.1;
      double y0 = 0;
      double z0 = 0;
```

```
    double x1,y1,z1;
    PrintWriter outFile = new PrintWriter("Equations_Output.txt");
    outFile.printf("%10c%10c%10c%n",'x','y','z');
    for(int i=0; i<50000; i++){
       x1 = x0 + h*a*(y0-x0);
       y1 = y0 + h*(x0*(b-z0) - y0);
       z1 = z0 + h*(x0*y0 -c*z0);
       x0 = x1;
       y0 = y1;
       z0 = z1;
       outFile.printf("%10.5f%10.5f%10.5f;%n",x0,y0,z0);
    }
    outFile.close();
  }
}
```

NOTE: Figures 7, 8, and 9 [10] (see above) were generated from output of the Java programs using MATLAB.

Appendix C: Supplements

The following case-study supplementary materials (computer codes and PDF files) will be available for students and instructors from the website: https://www.simiode.org/:

- MATLAB code files:
  - attrMain.m – Main program for solving the Lorenz System equations with MATLAB's ode45 program;
  - fLorenz.m – MATLAB code for representation of the Lorenz System equations.
- Java code files:
  - LorenzEquations.java – Java program (created by M. Sukharev-Chuyan [10]) for solving the Lorenz System equations;
  - Lorenz.java – Main Java source file (created by M. Sukharev-Chuyan [10]) for Waterwheel system simulation;
  - Waterwheel.java – Secondary Java source file (created by M. Sukharev-Chuyan [10]) for visualization of the Waterwheel system simulation.
- PDF files with texts from books and articles recommended for reading:
  - Lorenz_1963.pdf – The copy of [5] Lorenz, Edward N. Deterministic Nonperiodic Flow. *Journal of Atmospheric Sciences*. 1963, 20(2): 130–141. DOI: 10.1175/1520-0469(1963)020<0130:DNF>2.0.CO;2. Free access is available from: https://journals.ametsoc.org/view/journals/atsc/20/2/1520-0469_1963_020_0130_dnf_2_0_co_2.xml.
  - Lorenz_1993.pdf – The copy of [1] Lorenz, Edward N. *The Essence of Chaos*. Seattle, WA: University of Washington Press, 1993.
  - LorenzAttractor-PaulHorowitz.m4v – The M4V video file (size=64.5 MB, length=00:08:59) [R12] Horowitz, Paul. "An analog circuit that behaves like an attractor". https://drive.google.com/file/d/1Gn2EfTQIAYLOCsPFgwH5_zbf1mZtb1tR/view?usp=sharing (it will redirect to https://drive.google.com/file/d/1Gn2EfTQIAYLOCsPFgwH5_zbf1mZtb1tR/view, and LorenzAttractor-PaulHorowitz.m4v file can be downloaded).
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