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STUDENT VERSION

Modeling the Deflection of a Cantilever Beam

Brody Dylan Johnson
Department of Mathematics & Statistics
Saint Louis University
Saint Louis MO USA

SCENARIO DESCRIPTION

The study of beams and their deflection under distributed loads is of great importance in engineering and led to the development of skyscrapers and truss bridges. This modeling scenario involves hands-on experiments with a beam, subject to two different types of distributed loads. The data from these experiments will then be used to validate the solutions of the corresponding mathematical models.

The Euler-Bernoulli theory of beams, developed in the 18th century, showed that small deflections of a beam subject to transverse loads are governed by the fourth order, linear differential equation

$$EI\frac{d^4w}{dx^4} = q(x), \qquad 0 \le x \le \ell, \tag{1}$$

where $w(x)$ is the transverse deflection of the beam at coordinate x, $q(x)$ is the transverse load at coordinate x, E is the *Young's modulus* of the material, and I is the area moment of inertia for the cross-section of the beam [1]. This modeling scenario restricts attention to a *cantilever* beam, depicted in Figure 1, which is fixed at $x = 0$ and free at $x = \ell$.

The Young's modulus (or *modulus of elasticity*) of the beam is dependent on the material from which it is constructed and quantifies how much the material will tense or compress under an axial load. Transverse loads tend to bend the beam, which causes both tension and compression within a single cross-section of the beam. The Young's modulus and density for some common materials are provided in Table 1.

The area moment of inertia for a beam is a property of its geometry and is independent of the material. It will be assumed throughout this modeling scenario that the cross-section of the beams considered do not change with x. Consequently, the cross-section can be thought of as a subset

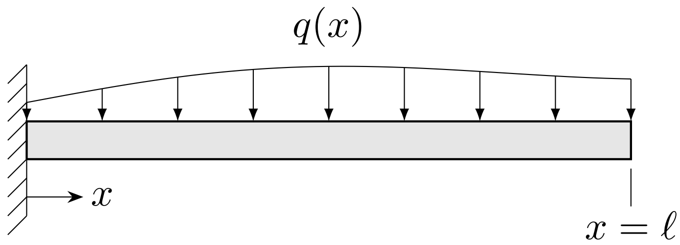

Figure 1. A cantilever beam subject to a distributed load.

Material	Steel	White Pine	PVC
Young's Modulus (N/cm^2)	2.0×10^7	1×10^6	3.1×10^5
Density (kg/cm^3)	7.8×10^{-3}	3.9×10^{-4}	1.44×10^{-3}

Table 1. Common materials and their structural properties.

R of the uv-plane, where u represents the horizontal direction and v represents the vertical (or transverse) direction. (See Figure 2.) The area moment of inertia for the cross-section is given by the area integral

$$I = \int_R v^2 \, dA.$$

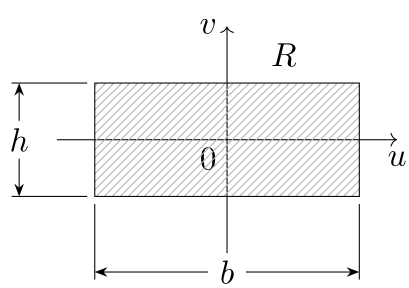

Figure 2. A rectangular cross-section.

Here, dA represents an area integral, which can often be computed as an iterated double integral. In particular, for the rectangular cross-section of Figure 2, the area moment of inertia is easily calculated as

$$I = \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} v^2 \, dv \, du = \frac{bh^3}{12}.$$

Notice how doubling the height h will increase I by a factor of eight, while doubling b will only increase I by a factor of two. In general, the area moment of inertia quantifies how difficult it is to bend a beam of a given cross-section in the transverse direction. The rectangular cross-section illustrates a general principle in that thin and wide beams are easier to bend than tall and narrow beams under transverse loads. This principle motivated the development of modern I-beams, which concentrate the mass of the beam as far away from the horizontal axis as possible in order to increase the area moment of inertia.

With this modest background on the subject of beams it is possible to begin the task of gathering data and using it to validate the model provided by (1).

Materials

In order to carry out experiments, the following materials are required:

- A flat steel bar;
- Two strong clamps;
- A work table on a level surface;
- A metric measuring stick;
- A black permanent marker;
- A ball of string;
- A pair of scissors;
- A small, recyclable plastic water bottle.
- A drill with a $\frac{1}{8}$-inch bit.

The included data used a flat steel bar with a rectangular cross-section ($b = 0.5$ inches, $h = 0.125$ inches). The overall length of the bar was about 42 inches.

1 Experimental Setup

The following procedure should be used to construct the cantilever beam.

1. **Decide on the length ℓ for the cantilever beam.**

   Choose a length that is a multiple of 5 `cm` and leaves at least 10 `cm` with which the beam can be clamped to the work table.

2. **Use the marker to create lines across the top surface of the beam every 5 `cm`.**

   Start at the far end of the beam and work backwards towards zero, labeling the distances as you go. This should leave at least 10 `cm` on the other side of the zero mark.

3. **Use the drill and $\frac{1}{8}$-inch bit to make a hole in the center of the beam, 5 `cm` from the end.**

   The hole will allow you to hang a water bottle from the beam using the string in a later part of the project.

4. **Use the two clamps to secure the beam to the work table.**

   Be sure to align the zero mark with the edge of the table and attach one of the clamps directly over this point. The area where the clamp is in contact with the beam should be at and behind the zero mark, not ahead of it. (See Figure 3.) The second clamp can be used 5–6 centimeters behind the first clamp and will help keep the beam in place.

Figure 3. Picture of clamped cantilever beam, which extends to the right out of the frame.

2 Data Collection

With the beam secured to the worktable, it is now possible to collect data.

2.1 Distributed Load

A close inspection of the beam should reveal the fact that it is already bending under its own weight, which acts on the beam as a uniformly distributed load. This deflection will be the focus of the first experiment. If the beam does not seem to be bending under its own weight, it may be too stiff for this experiment because deflections smaller than 1 `cm` will be difficult to measure with any precision.

Student Task: Use the procedure below to determine the vertical deflection of the beam at each marked point due to its own weight.

1. Use the measuring stick to measure the vertical distance (in `cm`) between the top surface of the beam and the floor at each of the marked points along the beam. Make sure the measuring stick is completely vertical as each distance is measured.
2. Let $u(x_n)$, $0 \le n \le N$, represent the vertical distances measured in the previous step. Calculate the beam deflection using the formula

$$w(x_j) = u(x_0) - u(x_j), \qquad 0 \le j \le N.$$

The data given in Table 2 corresponds to a thin steel beam with a rectangular cross-section ($b = 1.27$ `cm` inches, $h = 0.3175$ cm), 85 `cm` in length.

n	0	1	2	3	4	5	6	7	8
x_n	0	5	10	15	20	25	30	35	40
$u(x_n)$	75.6	75.5	75.4	75.3	75.1	75.0	74.8	74.6	74.5
n	9	10	11	12	13	14	15	16	17
x_n	45	50	55	60	65	70	75	80	85
$u(x_n)$	74.3	74.1	73.9	73.8	73.5	73.3	73.0	72.8	72.7

Table 2. Beam deflection (in `cm`) under a uniformly distributed load.

2.2 Concentrated Load

The second experiment will add a transverse force 5 `cm` from the end of the beam. In order to isolate the effect of this concentrated load, the deflection due to the weight of the beam itself will be removed.

Student Task: Use the procedure below to determine the vertical deflection of the beam at each marked point due to the concentrated load.

Procedure:

- Fill the water bottle and replace the cap, noting the capacity of the bottle (in `ml`).
- Use the string to hang the water bottle from the hole that is 5 `cm` from the end of the beam. Make sure the string is short enough that the bottle will not touch the ground after it is suspended from the beam.
- Use the measuring stick to measure the vertical distance (in `cm`) between the top surface of the beam and the floor at each of the marked points along the beam. Make sure the measuring stick is completely vertical as each distance is measured.
- Let $v(x_n)$, $0 \leq n \leq N$, represent the vertical distances measured in the previous step. Calculate the beam deflection using the formula

  $$w(x_j) = u(x_j) - v(x_j), \qquad 0 \leq j \leq N.$$

  Notice that this removes the deflection due to distributed weight of the beam.

The data given in Table 3 corresponds to the same beam used to produce Table 2, but with a water bottle containing 500 `ml` of water suspended at $x = 80$.

3 Mathematical Model

The Laplace transform will be used to solve (1), leading to a mathematical model for the deflection of the cantilever beam. The usual convention is to define the Laplace transform for functions of t, but, for obvious reasons, t will be replaced by x here. The Laplace transform of $f(x)$ is defined by

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty f(x)e^{-sx}\,dx, \tag{2}$$

n	0	1	2	3	4	5	6	7	8
x_n	0	5	10	15	20	25	30	35	40
$v(x_n)$	75.6	75.3	75.0	74.5	73.9	73.3	72.4	71.5	70.5

n	9	10	11	12	13	14	15	16	17
x_n	45	50	55	60	65	70	75	80	85
$v(x_n)$	69.5	68.4	67.3	66.0	64.8	63.4	62.1	61.1	59.6

Table 3. Beam deflection (in `cm`) under a concentrated load.

for all s such that the integral converges. One of the key properties of the Laplace transform stems from the way it interacts with derivatives. Recall that

$$\mathcal{L}\left\{f'(x)\right\} = sF(s) - f(0), \tag{3}$$

which allows for the conversion of differential equations into algebraic equations. The idea is to solve the algebraic equation for $F(s)$ from which $f(x)$ can be determined using a knowledge of Laplace transform pairs. Table 4 contains a list of Laplace transform pairs that will be helpful for this modeling scenario.

1. $\mathcal{L}\left\{\delta_a(x)\right\} = e^{-as}$
2. $\mathcal{L}\left\{u_a(x)\right\} = \dfrac{e^{-as}}{s}$
3. $\mathcal{L}\left\{u_a(x)f(x-a)\right\} = e^{-as}F(s)$
4. $\mathcal{L}\left\{x^n\right\} = \dfrac{n!}{s^{n+1}}$
5. $\mathcal{L}\left\{u_a(x)(x-a)^n\right\} = e^{-as}\dfrac{n!}{s^{n+1}}$

Table 4. Laplace Transform Pairs

Notice that the beam equation (1) involves the fourth derivative of $w(x)$. As the reader will verify below, it can be shown that

$$\mathcal{L}\left\{\frac{d^4w}{dx^4}\right\} = s^4W(s) - s^3w(0) - s^2w'(0) - sw''(0) - w^{(3)}(0). \tag{4}$$

Thus, in order to determine $w(x)$, each of the *initial values* $w(0)$, $w'(0)$, $w''(0)$, and $w^{(3)}(0)$ must be known. Meanwhile, the type of support (or lack thereof) at the ends of the beam correspond to specific *boundary conditions* at $x = 0$ and $x = \ell$. The cantilever beam is said to be *fixed* at $x = 0$ due to the clamps, leading to the boundary conditions

$$w(0) = 0 \qquad \text{and} \qquad w'(0) = 0.$$

The second and third derivatives of $w(x)$ correspond, respectively, to bending and shear stresses within the beam. The clamp is certain to exert some combination of a moment and shear force at

$x = 0$, which will affect the values of $w''(0)$ and $w^{(3)}(0)$, respectively. At $x = \ell$ there is no external support for the beam, so the beam is said to be *free* at $x = \ell$. The lack of external support means the beam cannot experience any bending stress or shear stress at $x = \ell$, leading to the boundary conditions

$$w''(\ell) = 0 \qquad \text{and} \qquad w^{(3)}(\ell) = 0.$$

However, At $x = \ell$, there is nothing to prevent the beam from having a nonzero deflection or slope, which correspond to $w(\ell)$ and $w'(\ell)$ being nonzero.

The following strategy can be used to solve (1) for a distributed load $q(x)$ subject to these boundary conditions.

1. Apply the Laplace transform to (1).
2. Use the boundary conditions $w(0) = 0$ and $w'(0) = 0$ to eliminate two terms from the equation.
3. Solve for $W(s)$ and use Laplace transform pairs to determine $w(x)$.
4. Compute the derivatives $w''(x)$ and $w^{(3)}(x)$ and use the boundary conditions at $x = \ell$ to determine $w''(0)$ and $w^{(3)}(0)$.

Student Task: Complete each of the problems below.

1. Use (3) to derive (4). Make sure to show all of the steps involved.
2. Solve the beam equation (1) subject to the distributed load

   $$q(x) = W_0(1 - u_\ell(x))$$

   using the boundary conditions described above for the cantilever beam. The constant W_0 represents the weight per unit length of the beam in `N/cm`.
3. Solve the beam equation (1) subject to the concentrated load

   $$q(x) = W_1 \delta_a(x)$$

   using the boundary conditions described above for the cantilever beam. The constant W_1 represents the weight of the mass suspended from the beam at $x = a$.

4 Model Validation

The final task in the modeling scenario is to compare the experimental data with the findings of the mathematical model.

Student Task: Complete each of the problems below.

1. Determine the appropriate physical properties (Young's modulus and density) for the beam used in your experiments. You may need to search online or look up the material in a textbook on the mechanics of materials [2].

2. Calculate the area moment of inertia for the beam used in your experiments.
3. Calculate the constant W_0 for the beam used in your first experiment.
4. Calculate the constant W_1 for the weight suspended from the beam in your second experiment given that the density of water is 0.001 `kg/ml`.
5. Plot the observed and predicted deflections for the beam subject to the distributed load due to the weight of the beam. How well does the model fit the data?
6. Plot the observed and predicted deflections for the beam subject to the concentrated load due to the weight suspended from the beam. How well does the model fit the data?

REFERENCES

[1] Wikipedia . 2021. Euler–Bernoulli beam theory. https://en.wikipedia.org/wiki/Euler-Bernoulli_beam_theory. Accessed 25 June 2021.

[2] Beer, F., R. Johnston, J. DeWolf, and D. Mazurek. 2014. *Mechanics of Materials, 7th Edition*. New York: McGraw-Hill.