www.simiode.org
SIMIODE A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

STUDENT VERSION

Measuring the Quality of Insulated Water Bottles

Brody Johnson, Elodie Pozzi
Department of Mathematics & Statistics
Saint Louis University
Saint Louis MO USA

SCENARIO DESCRIPTION

The goal of this project is to compare the effectiveness of two insulated bottles to keep a drink cold over a long period of time. If the two bottles are filled with water of equal temperature and kept in the same environment, this comparison can be accomplished by measuring the temperatures of the contents at any later time. The bottle with the colder contents must be better. But what if the contents do not start at the same temperature or are not kept in the same environment? In order to make a fair comparison when these variables are not controlled, a mathematical model is needed. The basis for this mathematical model comes from work carried out over three hundred years ago by the famous scholar Isaac Newton. Newton studied the evolution of the temperature of a small body over time and his experiments showed that the temperature of the body changed at a rate proportional to the temperature difference between the body and the surrounding environment. More precisely, Newton discovered that

$$\frac{dT}{dt} = k\left(T_E - T\right), \tag{1}$$

where $T(t)$ represents the temperature of the small body at time t, T_E is the temperature of the surrounding environment, and $k > 0$ is a constant of proportionality. The first-order differential equation (1) is now known as *Newton's Law of Cooling*, although this law applies to warming equally well.

The basic outline for this modeling activity is as follows. Temperature data will be collected for two water bottles over a period of at least 4 hours in an environment of more or less constant temperature. Newton's Law of Cooling will be used to predict the future temperature of the contents from the starting temperature for any values of the constants k and T_E. The constants will then be estimated through a trial-and-error procedure in which the predicted values are made as close as possible to the observed temperatures. The empirical estimates of k and T_E for each experiment will then be used to determine which bottle provides the better protection for a cold drink. This general framework allows for the examination of a wide variety of interesting questions.

- Are modern vacuum-insulated steel bottles better than the glass-walled thermoses that were popular 30–40 years ago? (See [1] for an excellent treatment.)
- Does a large bottle offer any advantage over a small bottle of similar construction?
- Is a brand-name insulated water bottle worth the cost? Or, could a cheaper knock-off bottle offer comparable performance?
- How does a typical plastic water bottle from a vending machine stack up against a vacuum-insulated water bottle?

MATERIALS

The following materials are needed in order to conduct the experiment:

- two insulated water bottles;
- a pitcher or other large container;
- a waterproof food thermometer;
- a clock;
- cold water.

It is recommended that bottles of similar capacity be used unless the intent of the experiment is to specifically examine the effect of bottle size and, in this case, the two bottles should be of the same construction/brand. Also note that the probe on the thermometer must be small enough to be inserted into the mouth of the water bottle and long enough to reach past the neck of the bottle.

1 Data Collection

Data collection will take a minimum of 4 hours, so it is recommended that this be done outside of class. It may make sense for each student to collect data for one bottle and share their findings (as well as a description of their bottle) with the rest of the class/group. This will likely facilitate numerous interesting comparisons.

Student Task: Carefully follow the procedure below, noting any deviations.

- **Cool a large pitcher of water overnight.**

  The actual temperature of the cool water is not terribly important as long as the water remains a liquid and will be cooler than the ambient temperature where the experiment is to be conducted.

- **Devise a means to insert the thermometer probe into the water bottle.**

  Many insulated water bottles have a drinking spout that will allow for insertion of the probe for the food thermometer. Make sure that each temperature reading can be made relatively quickly (≈ 15 seconds), so the cap is open for as little time as possible.

- **Fill the bottle(s) with cool water to within one inch of the top.**

  It is important to make sure the temperature probe will easily reach the water. Also, less air in the bottle will mean less opportunity for heat exchange when the cap is opened and closed.

- **Place the bottle(s) in an environment where the surrounding temperature will be relatively constant.**

  The mathematical model (1) assumes a constant environmental temperature, so large changes in the temperature of the surroundings will be problematic. Good locations will be indoors and away from windows and heat/cooling vents. If it is possible to measure/estimate the environmental temperature, do so.

- **Record the water temperature at regular intervals for at least 4 hours.**

  A high quality insulated water bottle can keep water cool for several hours. It would be a good idea to keep the bottle(s) close by as you work on homework or binge watch a new streaming series. Ideally, the temperature of the water should be recorded every 30 minutes for 3 or 4 hours. Record the temperature every 15 minutes if the insulating properties of the bottle are not expected to be good, e.g., for a clear plastic water bottle or single-walled steel bottle.

For those who are unable to perform their own experiment, data sets for three different water bottles (of comparable size) are provided in Table 1.

2 Qualitative Analysis

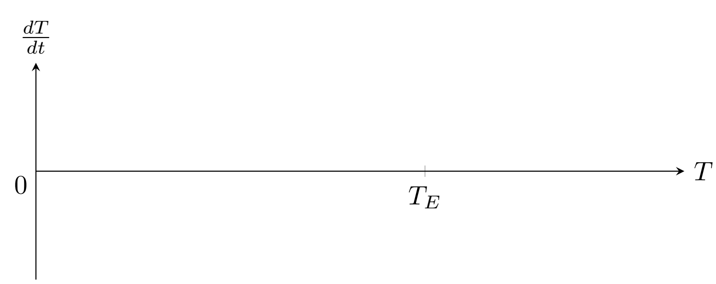
Qualitative analysis leads to the understanding of certain characteristics of the solutions of a differential equation (what is going to happen) without explicitly solving the differential equation (no formula is used). This is especially important when the differential equation cannot be solved using available techniques. Notice that the differential equation (1) is *autonomous*, which means the slope $\frac{dT}{dt}$ at any point (t, T) is dependent only on T. Recall that the *phase line* of an autonomous differential equation $\frac{dT}{dt} = f(T)$ is essentially an annotated graph of $f(T)$ over the T-axis. Positive values of $f(T)$ correspond to positive values of $\frac{dT}{dt}$, which means T will increase (move to the right)

Table 1. Temperature data for three water bottles.

Time (minutes)	Bottle 1 Temperature (°F)	Bottle 2 Temperature (°F)	Bottle 3 Temperature (°F)
0	39.0	47.5	39.2
30	42.8	48.2	49.5
60	44.9	49.5	55.2
90	46.2	50.2	59.9
120	48.3	51.1	63.3
150	49.9	52.2	65.5
180	51.1	52.9	67.5
210	52.7	54.0	69.3
240	54.7	54.5	70.7
270	56.3	55.2	71.6
300	57.6	55.9	72.5
330	58.1	56.5	73.0
360	59.2	57.0	73.6
390	60.8	57.4	73.9
420	62.1	58.1	74.3

as t increases. Similarly, negative values of $f(T)$ correspond to negative values of $\frac{dT}{dt}$, which means T will decrease (move to the left) as t increases. These directions are noted on the T-axis using arrows to the right or left. If $f(T) = 0$, however, then $\frac{dT}{dt} = 0$ and $T(t)$ will remain constant. Such a point T is called an *equilibrium point* of the differential equation. If nearby solutions tend towards the equilibrium point it is said to be *stable*. Otherwise, the equilibrium point is said to be *unstable*.

Student Task: Construct a phase line for the differential equation (1) in the space provided below. Assume that both k and T_E are positive.



Questions:

2.1 What are the equilibrium points of this model? Describe the stability of each equilibrium point.

2.2 What can you say about $T(t)$ for large values of t? Does your answer depend on $T(0)$? Explain.

2.3 How does the parameter T_E affect the phase line? Explain.

2.4 How does the parameter k affect the phase line? Explain.

3 Model Solution

In order to validate the model using real data, it will be necessary to predict the temperatures $T(t)$ for $t > 0$ for any values of k and T_E from a given initial condition, $T(0)$. In other words, a solution of (1) must be found. Since this differential equation is separable, it is possible to find an explicit solution via integration. Notice that (1) can be written as

$$\frac{dT}{dt} = -k(T - T_E),$$

which leads to the integral equation

$$\int \frac{1}{T - T_E}\, dT = \int -k\, dt. \tag{2}$$

Student Task: Solve the integral equation (2) and find an expression for $T(t)$, $t > 0$, in terms of the initial condition, $T(0)$. Be careful not to forget the constant of the integration!

Questions:

3.1 What happens to $T(t)$ as $t \to \infty$? Does your answer agree with the findings of qualitative analysis?

3.2 Assume $T(0) = 39°\text{F}$ and $T_E = 75°\text{F}$. Find the expression for $T(t)$, which will still contain the constant k.

3.3 Graph $T(t)$ for $k = 0.1$ and $k = 0.2$ for $0 \leq t \leq 60$, again using $T(0) = 39°\text{F}$ and $T_E = 75°\text{F}$. What effect does k have on the solution curve?

3.4 How do the constants k and T_E relate to the quality of the insulated water bottle? How would a well-insulated bottle be distinguished from a poorly insulated bottle using these constants?

4 Model Implementation

The final step in the project is to incorporate real data in order to estimate the constants k and T_E for each experiment. This will be accomplished using a spreadsheet.

Student Task: Create a spreadsheet to compare the mathematical model with the observed temperature data. Adjust the constants k and T_E so that the sum of the squared error (SSE) is as small as possible. Let T_n represent the temperature predicted by the mathematical model at time t_n and let $\widehat{T}_n$ represent the experimental temperature observed at time t_n. The SSE can be computed as

$$\text{SSE} = \sum_{n=1}^{N} \left(T_n - \widehat{T}_n\right)^2 .$$

Tips for the Spreadsheet: (see Figure 1)

	A	B	C	D	E
1	T(0)	k	T_E	A	SSE
2	39.2	0.100000	72.00	-32.80	1021.29
3					
4		Temperature	Temperature		
5	Time	Observed	Predicted	Sq. Error	
6	(minutes)	(Fahrenheit)	(Fahrenheit)		
7	0	39.2	39.20	0.00	
8	30	49.5	70.37	435.43	

Figure 1. Screenshot of a sample spreadsheet.

- Use the following columns: (A) Time t_n, (B) Temperature T_n (predicted), (C) Temperature $\widehat{T}_n$ (observed), (D) Squared Error.
- Create cells for k, T_E, $T(0)$, and SSE.
- Enter the value of $T(0)$.
- Create a scatter plot to compare T_n (predicted) with $\widehat{T}_n$ (observed).
- Start with $k = 0.1$ and $T_E = 72$. (If you measured the ambient temperature you may want to start with this value instead of $T_E = 72$.) Make small adjustments to k and T_E while watching the graph and the sum of the squared error. Think about how the change will affect the graph of T_n (predicted) before implementing it.
- Optional: If the spreadsheet is equipped with a solver, use the solver to optimize the values of k and T_E so that the sum of the squared errors is minimized. The value of T_E should be constrained to a limited range, say, 68°F to 80°F, in most cases.
- Repeat these steps for each experiment.

Questions:

4.1 Does the model provide a good approximation of the data for each experiment? Are there any data points that stand out? Explain.

4.2 What were the optimal values of k and T_E for each experiment? Do the values for T_E make sense based on the location where the experiment was conducted?

4.3 Define the *half-life*, $t_{0.5}$, of a water bottle as the time for which

$$T(t_{0.5}) = \frac{T_E + T(0)}{2}.$$

Use the general solution of (1) found in Section 3 to derive a formula for $t_{0.5}$ depending only on k.

4.4 Calculate the half-life for each bottle studied. Does one bottle have superior insulating properties? If so, is the difference in half-life substantial?

REFERENCES

[1] Sorensen, Rick. 2021. The Thermodynamics of Vacuum-Insulated Bottles: An Investigation. `https://www.vernier.com/2020/03/10/the-thermodynamics-of-vacuum-insulated-bottles-an-investigation/`. Accessed 17 June 2021.