www.simiode.org
SIMIODE A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

STUDENT VERSION
Temperature Distribution in a Uniform Slender Bar

Yuxin Zhang
School of Engineering and Applied Sciences
Washington State University
Richland WA USA

SCENARIO DESCRIPTION

Conduction and convection are two important modes of heat transfer. To understand the mechanisms of these two types of heat transfer processes, we study the temperature distribution in uniform slender bars though experiments, modeling, and numerical simulations. Various boundary conditions are investigated to demonstrate the effects of conduction and convection on the temperature distribution. Some of the related engineering applications of this study include the design of a heat exchanger with fins or a thermal radiator.

1 Experiments

We investigate the steady-state temperature distribution within a uniform slender bar, such as a thin rod or a slender rectangular bar.

1.1 Preparation

We use the following experimental equipment and materials: 1m-long, thin metal bars (circular and rectangular), thermocouple, infrared thermometer, electric heater, water bath, insulation layer, and tape. To account for the effect of thermal conductivities [1] on conductive heat transfer, we use bars made of three different materials, i.e. aluminum, steel and copper. In addition, two ambient conditions - air and water - will be considered in order to investigate the effect of convective heat transfer on the temperature distribution.

In the following experiments, we will set and maintain the left end of the metal bar at a constant high temperature T_b, for example, $T_b = 80°$C, while the lateral surface and the right tip of the bar are subject to varying conditions. We will measure the temperature distribution along the bar at different locations $x = 0.2, 0.4, 0.6, 0.8$ and 1m.

1.2 Conduction Only

To understand heat transfer due to conduction, we will start with a simple scenario.

(i) We take a copper circular rod, attach wired thermocouples to the lateral surface of the rod at prescribed locations, wrap the lateral surface with an insulation layer, and maintain the right tip of the rod at a fixed temperature, for example $T_{tip} = 40°$C such that $T_{tip} < T_b$. Then we measure the steady-state temperature distribution along the bar at prescribed locations.

Record and plot the steady-state temperature distribution along the rod. Find out whether the temperature has a linear distribution. Explain the mode of heat transfer in this case.

(ii) For the same setting, replace the rod with other materials - aluminum and steel, and observe temperature changes due to various thermal conductivities.

1.3 Conduction and Convection

Now we remove the insulation layer around the lateral surface of the bar (circular rod) and expose it to the air. In this case, in addition to conductive heat transfer within the bar, there is convective heat transfer between the metal bar and the surrounding air flow. Again, we will measure and study the temperature distribution. There are several experimental conditions we can set up including replacing the ambient fluids from air to water, changing the tip conditions (prescribed temperature, prescribed flux or convective condition) of the bar, etc. Please conduct the following experiments:

(i) Keep the same tip condition as in the previous experiment, for example at a constant temperature $T_{tip} = 40°$C, and investigate the temperature distribution due to different ambient fluid flows - air and water.

(ii) When the lateral surface is exposed to air, investigate the effects of the following tip conditions on the temperature distribution within the bar.

- A prescribed temperature: keep the tip at a constant temperature.
- An insulated or adiabatic tip (zero flux): wrap the tip with an insulation layer.
- A convective condition: expose the tip to air.

(iii) Replace the circular rod by a slender rectangular bar made of the same material. See how the distribution of temperature changes due to varying geometries.

Record and tabulate the above experimental results from §1.2 - §1.3 in tables. The data will be used for further analysis and comparison.

2 Modeling

The above heat transfer process is governed by two physical laws: Fourier's law and Newton's law of cooling [1]. Assume the uniform slender bar has a length of L and a constant cross-sectional area

A. The temperature at the left end ($x = 0$) of the bar is fixed at $T(0) = T_b$. The right end ($x = L$) of the bar is subject to varying boundary conditions. The lateral surface of the bar is exposed to a moving fluid at temperature T_{fluid} with a convective heat transfer coefficient h. If we take a finite element of length dx from the bar, this small slice of material undergoes conduction along the axial direction and convective cooling/heating on the external surface. Assume the conductive heat transfer rates on the two sides of the element are q_x and q_{x+dx} and the differential convection rate through the side surface is dq_s (see Figure 1).

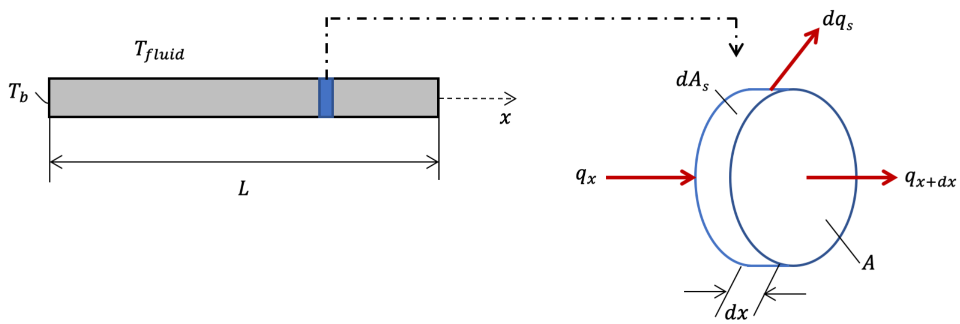

Figure 1. Schematic of energy balance for an element of the bar

Applying the principle of conservation of energy to the finite element gives:

$$q_x = q_{x+dx} + dq_s. \tag{1}$$

From Fourier's law, we know that $q_x = -kA\frac{dT}{dx}$ where k is the thermal conductivity of the material. The heat rate at the right side of the element can be approximated using a 1st-order Taylor series expansion $q_{x+dx} = q_x + \frac{dq_x}{dx}dx$. The differential heat transfer rate dq_s due to convection is determined by the Newton's law of cooling $dq_s = hdA_s(T - T_{fluid})$ where dA_s is the lateral surface area of the finite element.

Substituting the above quantities into (1), we can derive the governing equation for temperature distribution $T(x)$ along a uniform slender bar as follows:

$$\frac{d^2T}{dx^2} - \frac{hp}{kA}(T - T_{fluid}) = 0, \tag{2}$$

where p is the perimeter of the cross section and $p = \frac{dA_s}{dx}$. The temperature distribution $T(x)$ is affected by the thermal conductivity k of the material, the perimeter p and area A of the cross section, the ambient fluid temperature T_{fluid}, and the convective heat transfer coefficient h.

If we define a new temperature variable $\Phi(x) \equiv T(x) - T_{fluid}$, (2) reduces to

$$\frac{d^2\Phi}{dx^2} - C^2\Phi = 0, \tag{3}$$

where $C^2 = \frac{hp}{kA}$. It can be verified that the solution of (3) takes the form of $\Phi(x) = Ae^{Cx} + Be^{-Cx}$. Here the coefficients A and B can be evaluated using the boundary conditions at the two ends of the bar.

Now we need to specify the boundary conditions. According to the experiments, the temperature on the left ($x = 0$) is fixed by $T(0) = T_b$. Thus, $\Phi(0) \equiv T(0) - T_{fluid} = T_b - T_{fluid} \equiv \Phi_b$. In accordance with the experimental setups, the boundary condition on the right ($x = L$) can be specified by one of the following:

- A prescribed temperature $\Phi(L) = \Phi_L$.
- A prescribed flux $\frac{d\Phi}{dx}_{|x=L} = q_f$. If $q_f = 0$, it is an insulated (adiabatic) tip.
- A convective condition $-k\frac{d\Phi}{dx}_{|x=L} = h\Phi(L)$.

Then, the temperature distribution $\Phi(x)$ can be obtained from solving (3) with boundary conditions.

2.1 Example

Consider a slender, finite-length rod with a convective tip. Derive the expression of the temperature distribution along the rod.

From the above analysis, we know that the solution can be represented by $\Phi(x) = Ae^{Cx} + Be^{-Cx}$ with unknown coefficients A and B. The two boundary conditions are $\Phi(0) = \Phi_b$ and $-k\frac{d\Phi}{dx}_{|x=L} = h\Phi(L)$. Substituting $\Phi(x)$ into boundary conditions gives:

$$\begin{aligned} \Phi_b &= A + B \\ -kC(Ae^{CL} - Be^{-CL}) &= h(Ae^{CL} + Be^{-CL}). \end{aligned} \tag{4}$$

The coefficients A and B can be determined by solving the equation system (4). Then, the temperature distribution $\Phi(x)$ along the rod is

$$\Phi(x) = \frac{\cosh C(L-x) + \frac{h}{Ck}\sinh C(L-x)}{\cosh CL + \frac{h}{Ck}\sinh CL}\Phi_b. \tag{5}$$

A plot of the temperature $T(x) = T_{fluid} + \Phi(x)$ along a rod with specified geometric and thermophysical conditions can be found by the curve of the analytical solution in §3.

2.2 Exercises

(i) Solve the temperature distribution along the rod when the tip temperature is prescribed by $T(L) = T_{tip}$.

(ii) Solve the temperature distribution along the rod when the tip is insulated, i.e., $\frac{d\Phi}{dx}_{|x=L} = 0$.

(iii) Replace the above circular rod by a rectangular bar with the same cross-sectional area. Discuss how this shape change affects the temperature distribution.

3 Numerical Simulations

The steady-state equation (2) with boundary conditions is solved numerically using the finite volume method [2]. The computational domain is divided into N node-centered control volumes or cells, as shown in Figure 2. The grid has a uniform mesh size of $\Delta x = \frac{L}{N}$. Integration of (2) over a control volume gives

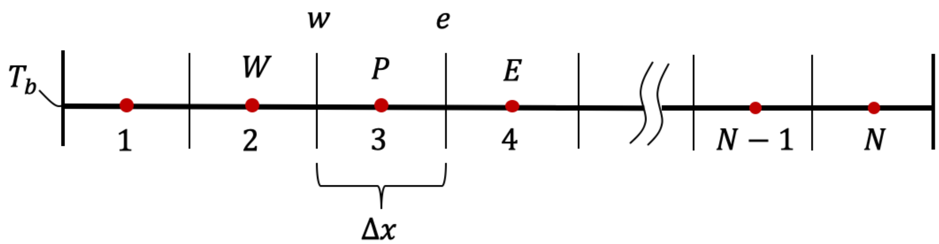

Figure 2. The grid

$$\int_S \frac{dT}{dx}\vec{n}\,dS - \int_{\mathcal{V}} C^2(T - T_{fluid})d\mathcal{V} = 0, \tag{6}$$

where the integrals in (6) are evaluated locally within each control volume. The first surface integral represents the heat flux through the surfaces of a control volume. The second volume integral can be regarded as a source term within the control volume. For example, evaluating (6) over control volume 3 gives

$$(\frac{dT}{dx})_e - (\frac{dT}{dx})_w - C^2(T_P - T_{fluid})\Delta x = 0. \tag{7}$$

Here the subscripts e and w refer to the east and west surfaces of the control volume. P represents the center of the control volume. The two temperature gradients in (7) can be evaluated using linear approximations with temperatures at neighboring cells 2 and 4. Therefore, for internal cells from 2 to $N-1$, replacing the temperature gradients in (7) gives

$$\frac{T_E - T_P}{\Delta x} - \frac{T_P - T_W}{\Delta x} - C^2(T_P - T_{fluid})\Delta x = 0, \tag{8}$$

where the subscripts E and W refer to the center nodes of the east and west cells around the P cell. For boundary cells 1 and N, the temperature gradients in (7) need to be evaluated using the given boundary conditions. Then, we can establish a system of N discretized equations. The temperature distribution $T(x)$ at each cell node can be calculated by solving the system of linear equations.

Figure 3 shows an interactive user interface for simulating the temperature distribution along a bar with varying tip conditions. On the left panel, the dimensions (length L, cross-sectional area A and perimeter p) of the bar and the thermophysical properties (thermal conductivity k and convective heat transfer coefficient h) can be entered. The ambient temperature condition is determined by the fluid temperature (Fluid T). For boundary conditions, the temperature on the left is prescribed at a base temperature (Base T). The condition on the right can be set up from the dropdown list and the corresponding tip temperature (Tip T) or flux (Tip Flux) is to be entered. On the right panel, users can input data from experiments. The middle panel shows the result of temperature distribution along the bar. The top plot shows the comparison of results from simulation (blue line with square markers), analytical solution (red line), and experiment (yellow circles). The error indicates the difference between the numerical solution and the exact solution. The bottom scales present the temperature distribution from the simulation.

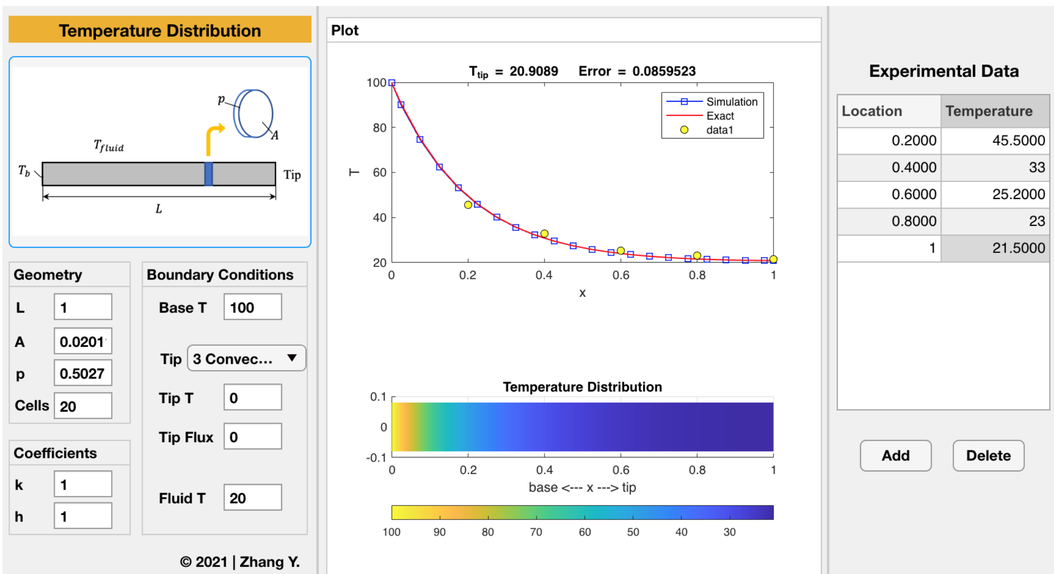

Figure 3. Temperature distribution MATLAB App (downloadable on SIMIODE website)

3.1 Numerical Exercises

(i) Investigate the interaction between conduction and convection by changing the values of thermal conductivity k and convective heat transfer coefficient h. If $k \gg h$, what is the temperature distribution when the tip is maintained at a constant room temperature? Explain the corresponding physical condition for this case. What happens if $k \ll h$? How is heat transferred in the bar?

(ii) Assume that the ambient fluid temperature remains constant. If a circular rod is under the convective tip condition, how could one increase the heat transfer rate by changing other parameters? Then, replace the above circular rod by a square bar with the same cross-sectional area. Discuss how this shape change affects the temperature distribution, especially the tip temperature.

(iii) What is the difference between the adiabatic and the convective tip conditions? To reach the same tip temperature, how are these two conditions related to the length of the bar?

(iv) How could one improve the accuracy of a numerical solution, especially when sharp gradients exist in the solution?

4 Comparison of Results

We investigate the steady-state temperature distribution along slender metal bars through experimental, analytical, and numerical approaches. Now we want to compare the analytical and numerical solutions with the temperature data collected from selected experiments. Use the MATLAB App discussed in §3 to compare results in the following scenarios:

- §1.2 (i) The temperature distribution in a copper rod due to conduction.
- §1.3 (ii) The temperature distribution in a rod with varying tip conditions.
- §1.3 (iii) The change of temperature distribution due to varying geometries.

The comparison of results will provide more insights on the heat transfer process and show correlation between the modeling and the experiments.

REFERENCES

[1] Bergman, T. L., F.P. Incropera, D.P. DeWitt, and A.S. Lavine. 2011. *Fundamentals of heat and mass transfer*. New York: John Wiley & Sons.

[2] Ferziger, J. H., M. Perić, and R.L. Street. 2002. *Computational methods for fluid dynamics.* New York: Springer.