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STUDENT VERSION
Bifurcations in language dynamics

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SCENARIO DESCRIPTION

This modeling project stems from a model of language death by Strogatz and Abrams [1]. We will use this model to explore the idea of bifurcations and the dependence of long-term dynamics on parameter values.

Consider a population that has two main languages that can be spoken by its members. Let $x(t)$ be the fraction of the population that speaks language X and $y(t)$ the fraction of the population that speaks language Y.
Note some modeling assumptions:

- The number of people in the total population is constant and the only movement occurs between the two classes of people speaking each language.
- All members of the population speak either language X or Y, and any person can speak only one of the two languages at any given time (nobody can speak both at the same time).
- The last assumption leads to the following relationship: $x(t) + y(t) = 1$. Thus, we can simply model the change in the fraction of people speaking language X and can always calculate $y(t)$ by subtracting from 1.
- The probability of switching from speaking language X to Y depends on the number of people currently speaking language X and its perceived status within a population, $s > 0$, which can be thought of as some socio-economic benefit of speaking language X. This implicitly makes the status of language Y to be $1 - s$.

These assumptions lead to the following differential equation for the rate of change of the fraction of people speaking language X:

$$\frac{dx}{dt} = (1-x)P_{yx} - xP_{xy} \tag{1}$$

where P_{yx} is the likelihood of switching from speaking language Y to X and P_{xy} is the reverse; see Figure 1 below.

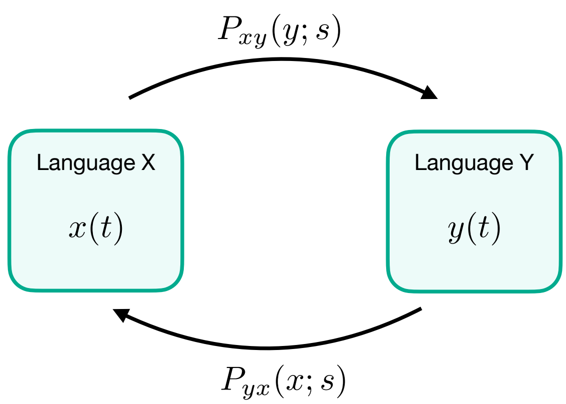

Figure 1. Schematic of language dynamics.

Following our assumptions, we write the rate of switching from speaking language Y to X as the product of the fraction of people speaking language X times the status, s

$$P_{yx} = xs.$$

Analogously, we can define the rate of switching from speaking language X to Y as

$$P_{xy} = (1-x)(1-s).$$

Assignment

Answer the following questions. Your responses to each question should include complete sentences, easy-to-follow mathematics, and discussion as appropriate. The report should be well-written and all mathematics should be typed in an equation editor. All graphs should include titles, legends, labels, and captions.

1. Explain how the equations for the likelihood of switching, P_{xy} and P_{yx}, satisfy the last model assumption.
2. Consider (1) and assume that nobody in the population speaks language X (i.e. $x = 0$). Explain what this differential equation predicts for the change in the fraction of people speaking language X. What if $x = 1$ or $x = \frac{1}{2}$? What do you notice about the direction of motion between the two populations in Figure 1 depending on s?

3. Plug the definitions for the likelihood of switching into (1) and show that you can write the model as

$$\frac{dx}{dt} = (2s-1)x - (2s-1)x^2 \tag{2}$$

4. Plot $\frac{dx}{dt}$ vs x. You should consider three cases: $s < \frac{1}{2}$, $s > \frac{1}{2}$, and $s = \frac{1}{2}$.
5. For each case of s above, sketch the phase portrait x vs t. Discuss all possible long-term behaviors. Explain why these results make sense considering the corresponding values of s. Recall that $0 \leq x \leq 1$.
6. Data suggests that the number of people speaking a language may not have a linear relationship with the probability of switching [1], as we previously considered. This leads to new functions for the likelihood of switching: $P_{xy} = x^a s$ and $P_{yx} = (1-x)^a(1-s)$. Our new model differential equation is now

$$\frac{dx}{dt} = (1-x)x^a s - x(1-x)^a(1-s) \tag{3}$$

Show that the equilibrium solutions to (3) are

$$x_1^* = 0, \quad x_2^* = \frac{(1-s)^{\frac{1}{a-1}}}{s^{\frac{1}{a-1}} + (1-s)^{\frac{1}{a-1}}}, \quad x_3^* = 1.$$

For $a > 1$, explain how the equilibrium solution x_2^* changes as s goes from 0 to 1. How would this change if $a < 1^*$?

7. Let $a = 2$ in (3). Plot x' vs x for $s = 0,\ 0.3,\ 0.5,\ 0.7,\ 1$, all on the same plot. Make sure to include a legend.
8. Use your plot from question 5 to sketch (by hand) five phase portraits, one for each value of s. Make sure to indicate the stability of each equilibrium solution. Explain what is changing in the system as s increases. What does this correspond to in the model? Why do your results make sense?
9. Sketch the bifurcation diagram of x^* vs s. Use solid lines to indicate stable fixed points and dashed lines to indicate unstable equilibrium solutions. What is the equation of the unstable branch? It might be useful to think about how you would find this for any value of a.
10. Explain, using your bifurcation diagram, why this model may be called "competition exclusive."
11. Now, let $a = \frac{1}{2}$ and repeat questions 5-7 for this new value of a. For question 7, plot the bifurcation diagram. Explain, using your new bifurcation diagram, why this model may be called "stable coexistence."
12. **Reflection** Write a paragraph describing what you learned by completing this project. Here are some things to consider while writing:
    - Did you learn a new concept or gain more intuition or understanding of a concept we went over in class? If so, what was it? How has your understanding changed?

*You may want to think about $\lim_{s\to 0} x_2^*$ and $\lim_{s\to 1} x_2^*$, since x_2^* is undefined at those values when $a < 1$.

- Did you enjoy seeing differential equations used in a modeling application? What did you learn by applying differential equations tools and techniques to a modeling problem?
- Were you surprised by anything you found or learned in this project?
- Which parts of this project were easy for you? Which parts were more difficult?
- If you were to give a future differential equations student some advice on this project, what would it be?

REFERENCES

[1] Abrams, D. M. and S. H. Strogatz. 2003. Modelling the dynamics of language death. *Nature*. 424: 900. https://www.nature.com/articles/424900a.