www.simiode.org
SIMIODE A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

STUDENT VERSION
Bifurcations in fish harvesting

Jennifer Crodelle
Department of Mathematics
Middlebury College
Middlebury VT USA

SCENARIO DESCRIPTION

This modeling scenario introduces the idea of a **bifurcation**, or a qualitative change in dynamics. Many differential equations have a **bifurcation point**, or a parameter value for which the differential equation exhibits a qualitative change in dynamics. This can be a change in the stability of an equilibrium solution or the number of equilibrium solutions. We will learn how to study bifurcations using a population model.

Suppose there is a stream with salmon nearby to a family of bears. In the absence of the bears (when they are on vacation, for example), the salmon grow according to logistic growth at a rate $r > 0$ to a carrying capacity $K > 0$. Suppose that the bears can remove a constant amount of salmon, $R > 0$, from the stream at any time t. We can write the following differential equation to model the number of salmon in a stream, $S(t)$, at any given time t:

$$\frac{dS}{dt} = rS\left(1 - \frac{S}{K}\right) - R, \tag{1}$$

We will use qualitative analysis to help the bears understand how the number of salmon in the stream change depending on their rate of removal R.

Assignment

1. First, set $K = 20$ and $r = 4$ in (1). What do these values say about the growth of the salmon population?
2. Plot $\frac{dS}{dt}$ vs S for $R = 0$. Note that this is the typical logistic growth equation we've seen before, without any harvesting. From this plot, sketch the plot of S vs t, making sure to indicate

stability of the equilibrium solutions. Discuss all possible long-term behaviors of the salmon population given any initial condition.

3. Now let our harvest rate $R = 15$ salmon per unit time. Plot $\frac{dS}{dt}$ vs S for this case. How many equilibrium solutions are there? Sketch the corresponding plot of S vs t, including the stability of the equilibrium solutions. Discuss the long-term behavior.
4. Find the value of R for which there is only one equilibrium solution. What would this correspond to physically? Plot $\frac{dS}{dt}$ vs S for this case and sketch the corresponding S vs t plot. Explain why this value of R is a **bifurcation point**.
5. We can draw a **bifurcation diagram** to illustrate how the dynamics change with continuous changes in R. To do this, we sketch the equilibrium solutions, S^*, on one axis and R on the other. A solid line indicates that the equilibrium point is stable and a dashed line indicates that it's unstable. See Figure 1 below for the bifurcation diagram of (1).

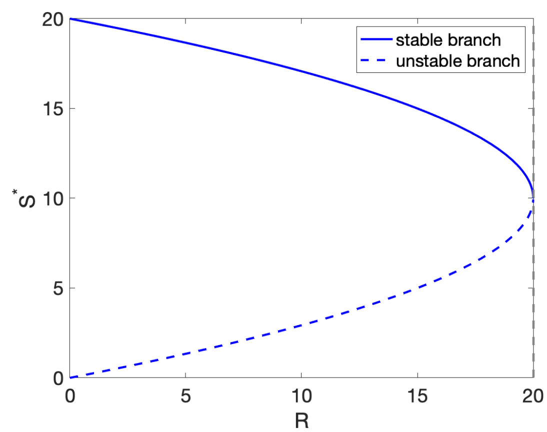

Figure 1. Bifurcation diagram for (1) with $r = 4$ and $K = 20$ as we change the rate of removal, R.

Use your answers from questions 2 - 4 to explain this bifurcation diagram. Imagine that the nearby family of bears would like to use this bifurcation diagram to figure out the rate at which they can take salmon from the stream. Explain to the bears how this rate of removal, R, is related to the initial number of salmon and what the long-term consequences of their removal will be.