SPECTRAL ESTIMATION: BASIC THEORY

The spectral content, $Y(\omega)$, of the process which produces a time dependent (observed) signal, $y(t)$, is mathematically defined as:

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\ e^{-j\ \omega\ t}$$

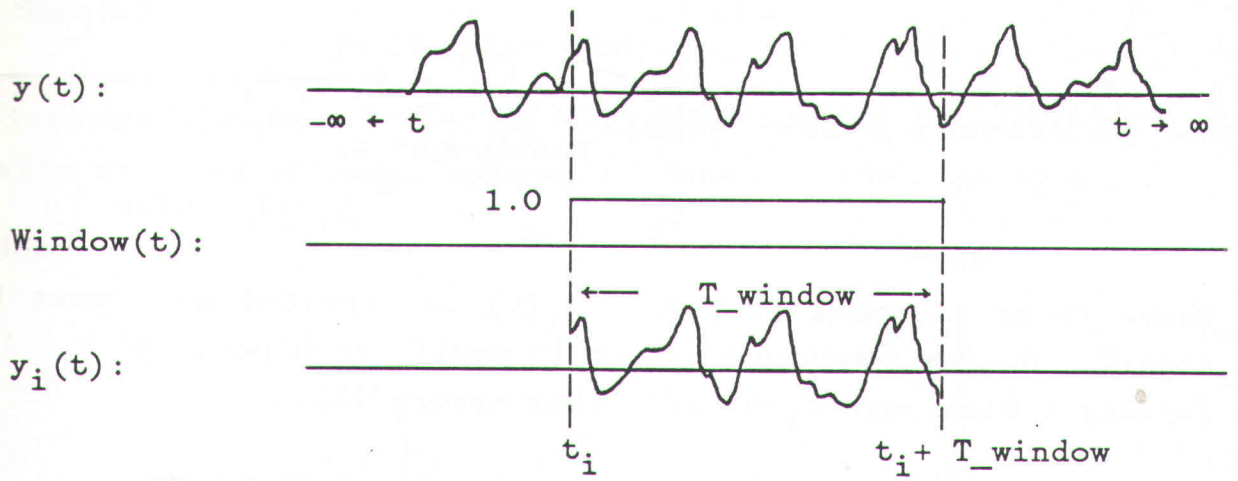
Obviously not all time data, from the beginning to the end of time, is available for calculating the spectral content, $Y(\omega)$. A time window (T_window) segment of $y(t)$ is used for a computer spectral estimation (e.g. Fourier Transform) calculation:

$$y_i(t) = y(t) \bullet \text{Window}(t)$$

where:

$$\text{Window}(t) = \begin{cases} 0 & \text{if } -\infty \le t < t_i \\ 1 & \text{if } t_i \le t \le t_i + \text{T_window} \\ 0 & \text{if } t_i + \text{T_window} < t \le \infty \end{cases}$$

For illustration:

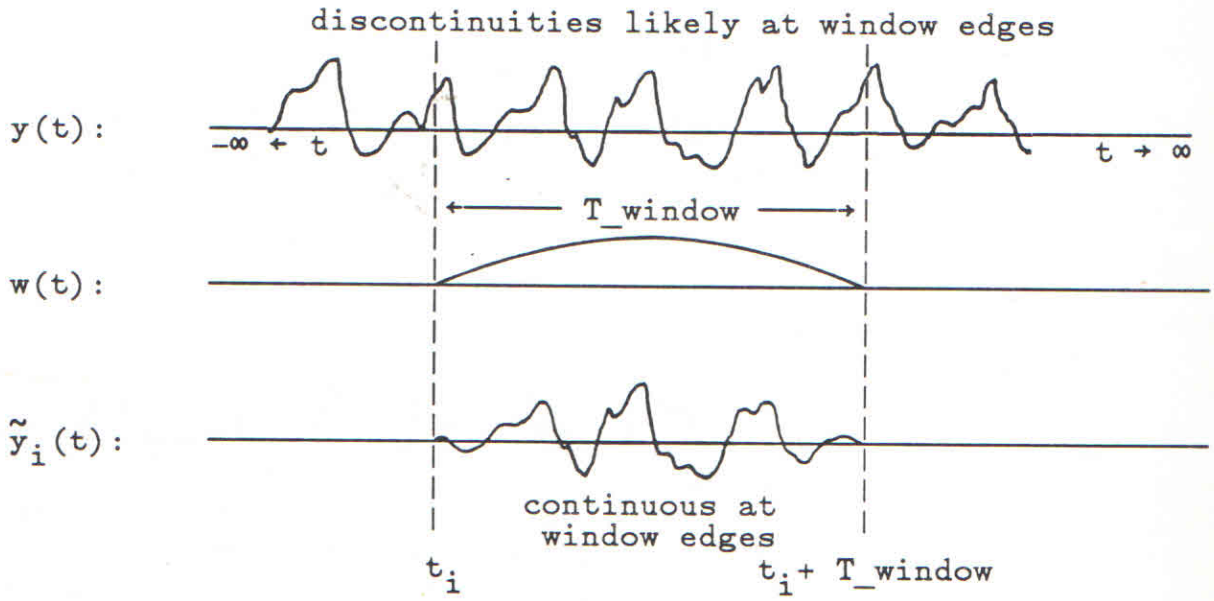


Thus the spectral content would be defined as:

$$Y_i(\omega) = \int_{-\infty}^{\infty} y_i(t)\ e^{-j\ \omega\ t}\ dt = \int_{t_i}^{t_i + \text{T_window}} y_i(t)\ e^{-j\ \omega\ t}\ dt$$

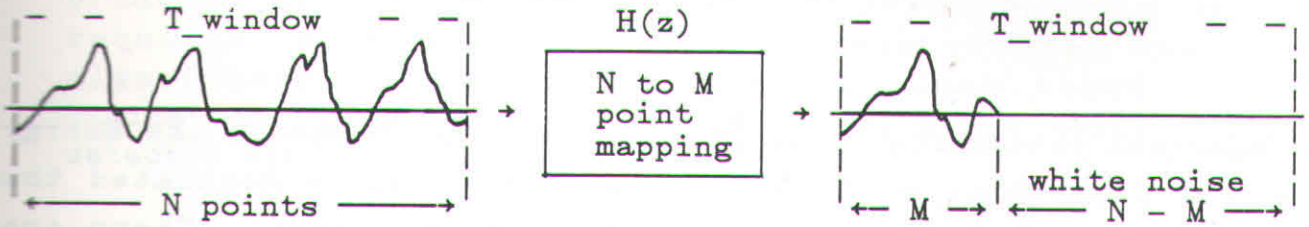
The Fast Fourier Transform (FFT) algorithm was the favorite computer method during the 1970's and 1980's for approximating $Y_i(\omega)$. The FFT assumes $y_i(t)$ is one period of data from the periodic $y(t)$ and that $Y_i(\omega)$ is band-limited. Discontinuities at the beginning and ending of the data window caused $Y_i(\omega)$ to be convolved with a Sinc function. Attempts to remove the Sinc function caused users to investigate the world of windows [3, 4]. The idea was to multiply $y(t)$ by a window function, $w(t)$, that would eliminate the discontinuities (or better yet, the effects thereof) at the window edges. To illustrate:

$$\tilde{y}_i(t) = y(t) \cdot w(t)$$



There is no discontinuity when $\tilde{y}_i(t)$ is repeated and connected to itself. Window functions basically modify each point of $y_i(t)$, thus forcing a bias upon $\tilde{Y}_i(\omega)$ and other errors [5].

Modern spectral estimation algorithms map an input data sequence of N points into a pseudo-sequence process estimate containing only M terms (points). The pseudo-sequence mapping tries to retain the same spectral content as the original N points of data. To do this N - M points are filtered to remove the detected spectrum, leaving no spectral content (i.e. a white noise signal).

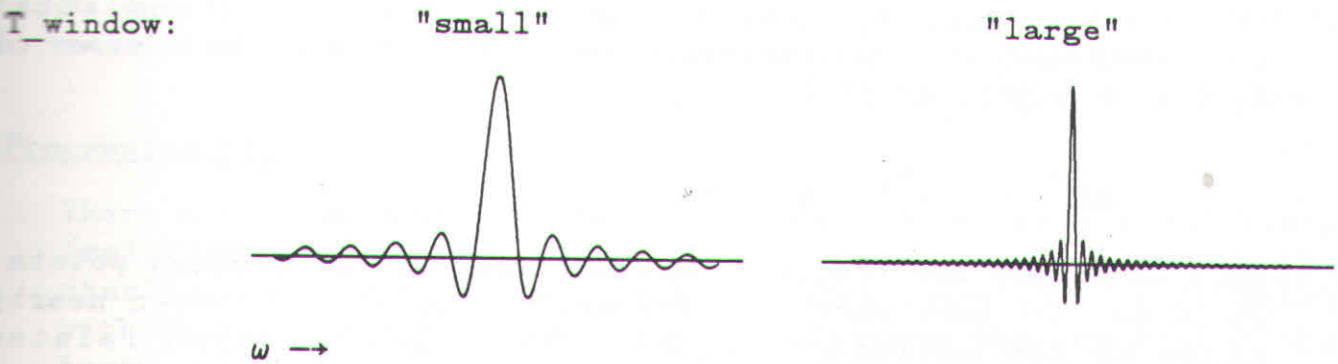


The M points form in effect a shorter "effective T_window" since the N - M points are discarded. This smaller window produces a Sinc function with a lower frequency than if all N points had been used. At this lower frequency the Sinc function sidelobe peaks are farther apart and thus there are fewer of them within the Nyquist frequency limit. This reduces the number of false peaks (i.e. spectral artifacts) which aids in determining the true spectral content, $Y(\omega)$. ("M" is defined as the sum of the parameters NPOLES and NZEROS.)

In order to illustrate the affect on a Sinc function of varying the data window length, note that the Sinc function has the form:

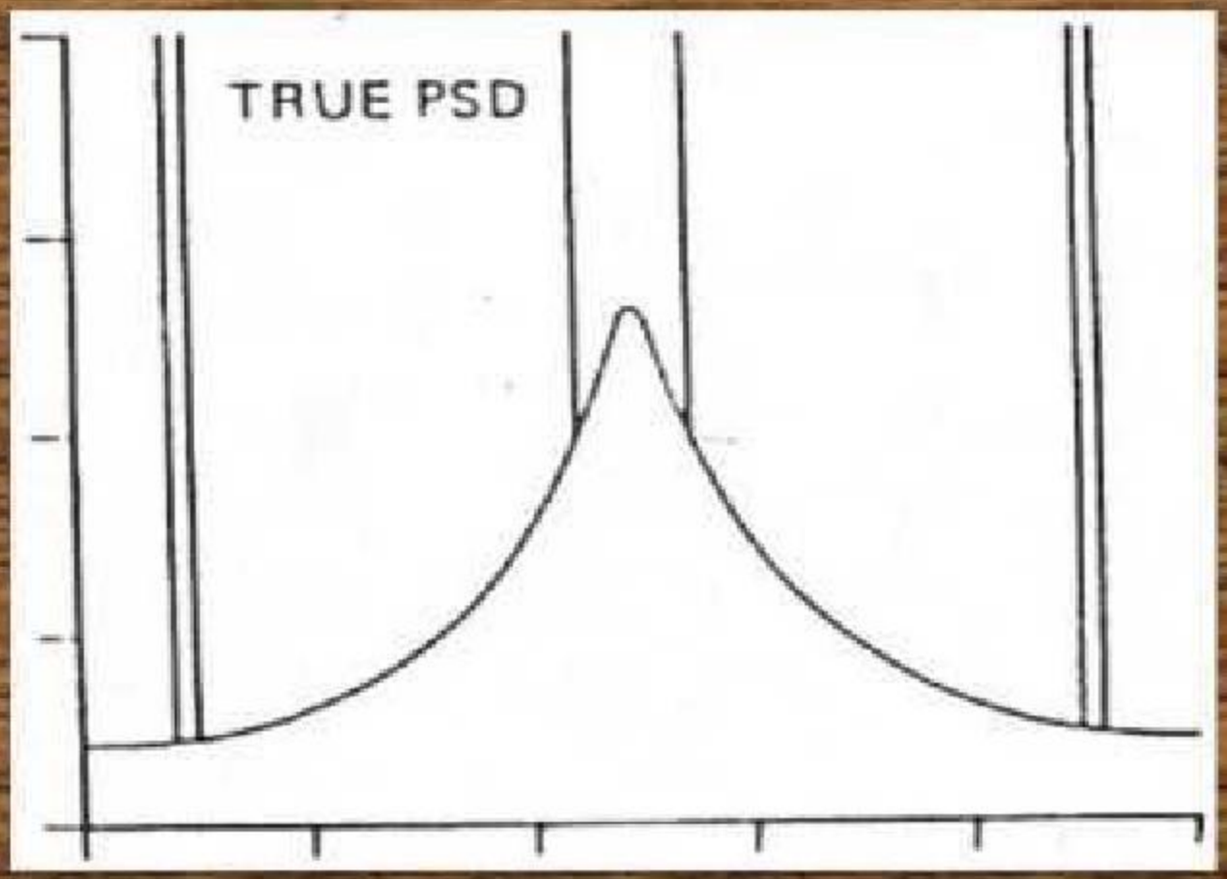
$$\text{Sinc}(\omega) = \frac{\text{Sin}(\omega/\text{effective T_window})}{\omega/\text{effective T_window}}$$

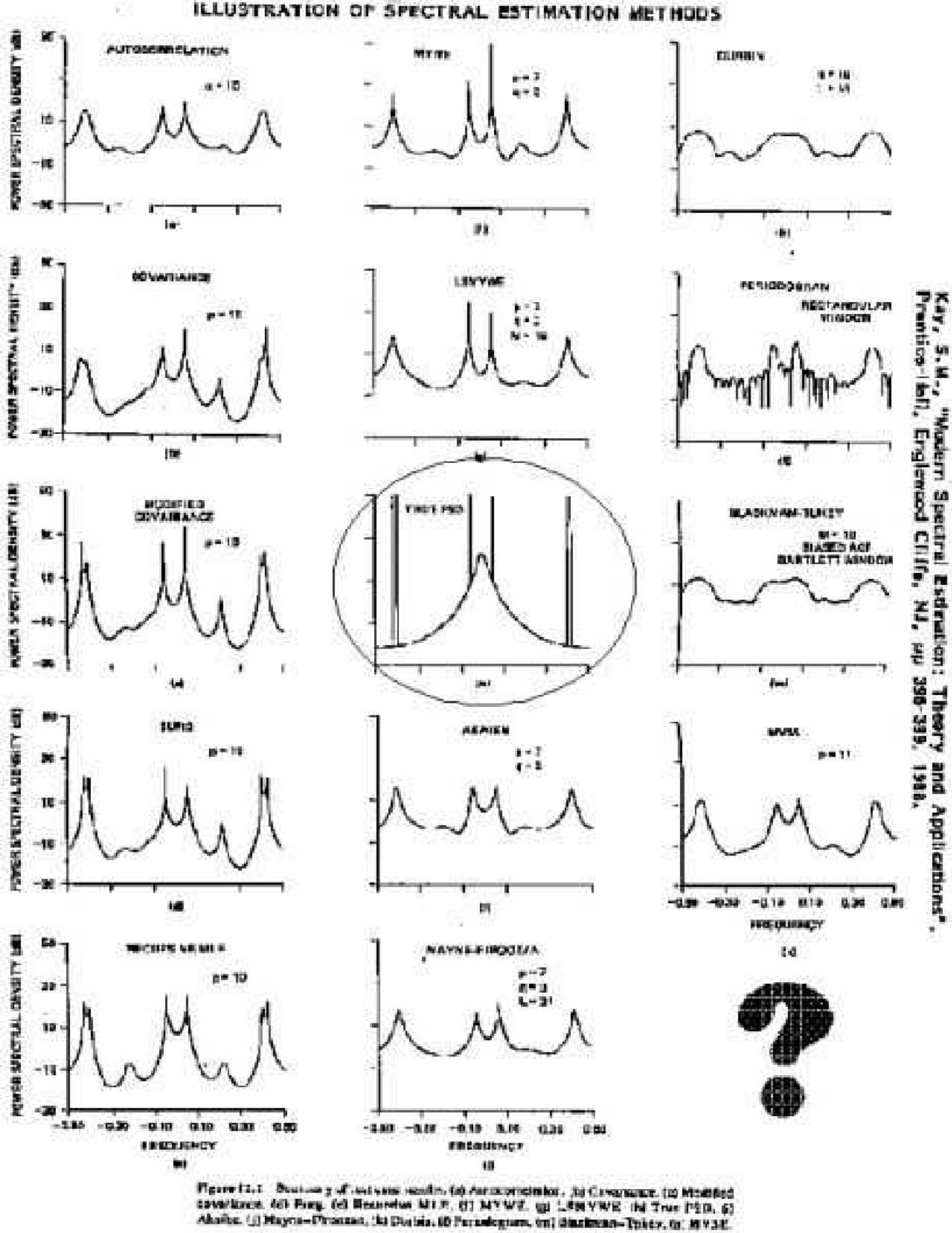
Thus increasing the effective T_window length results in the following relative changes in the Sinc function for a given ω.



PSO

True PSD for Comparison





Kay, S.M., "Modern Spectral Estimation: Theory and Applications", Englewood Cliffs, NJ, pp 368-369, 1988.