SIMIODE EXPO: SCUDEM V Birds and Bicycles

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Introduction: Outline

We seek to

- Explain the problem clearly and concisely
- Apply natural assumptions
- Show how the general model follows from those assumptions
- Reduce the model to an idealized case for sanity checking
- Demonstrate the model with some simulations
- Conclude with a discussion on initial conditions

A viral video shows a bird perched on a bicycle wheel able to move itself so the wheel spins, our task is to model the phenomena with a small apparatus attached to a wheel able to move a mass to generate the motion.

Problem: Our Apparatus

Our apparatus would

- Consist of a small piston capable of moving a mass radially outward and inward
- May be imparted with initial angular displacement or velocity from some initial lateral movement
- Comes from the bird leaning to start the motion
- Satisfies initial conditions to be used later of initial angular position and angular velocity

Problem: Simulation Example

Problem: Equations of Motion

From physics we know:

- Linear velocity v:

$$v = r(\theta)\dot{\theta}$$

- Angular position (recall measured from the vertical) just $\theta(t)$
- Coordinate of center of mass of apparatus

$$x(\theta) = r(\theta)\sin(\theta), \quad y(\theta) = r(\theta)\cos(\theta)$$

- Here $r(\theta)$ is the distance from the axis of rotation (the axle) to the center of mass of the whole system.

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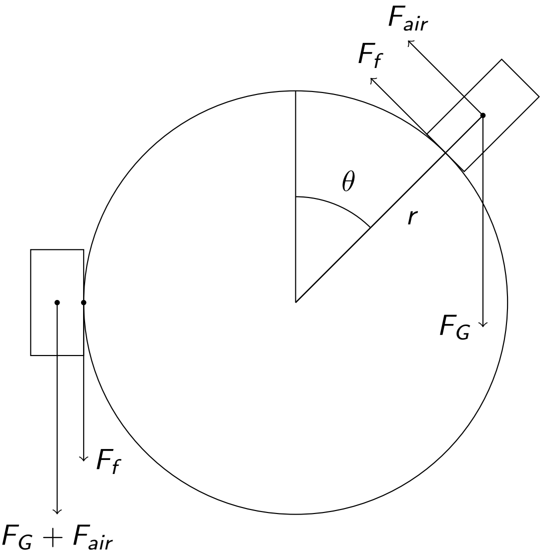
- Our bicycle wheel and apparatus represents a rigid body
- Our apparatus impacts motion by moving close or away from the bicycle wheel
- When we think rotational motion the analogy of Newton's second law becomes:

$$\sum \tau_i = I\alpha \iff \sum \tau_i - I\alpha = 0$$

  Where τ_i represent torques and I is the moment of inertia, and α is the angular acceleration $\ddot{\theta}$ wrt time t. And the 0 represents the zero vector since it arises from a cross product.

- The time our piston moves between states has negligible impact on the motion


F_{air}
F_f
θ
r
F_G
F_f
$F_G + F_{air}$

Model Creation: Development

Following the previous slide we provide some detail to the torque equation

- So our apparatus generates a torque by decreasing or increasing it's distance from the axis of rotation (center of wheel)
- The principal forces which generate torque are the apparatus (by gravity), air resistance, and friction
- These forces act on close enough points of contact to be treated the same
- When forces are tangential their torque component will be (in terms of the radius r)

$$\tau = r \times F \implies |\tau| = |F||r|\sin(\theta) = |F||r|$$

Model Creation: Development

Reducing the complexity

- Our main equation

$$\sum |\tau_i|\mathrm{sgn}(\tau_i) - I|\alpha| = 0$$

The sgn term represents the direction of the torque

- Torque by air resistance is proportional to velocity squared

$$\tau_{air} = rF = rk(v)^2 = kr(r\dot{\theta})^2 = kr^3(\dot{\theta})^2$$

- Torque by friction is proportional to normal force

$$\tau_{friction} = \mu rF = \mu mv^2 = m\mu r^2(\dot{\theta})^2$$

- Torque by gravity is standard

$$\tau_g = mgr\sin(\theta)$$

- Moment of inertia term for a uniform wheel

$$I(\ddot{\theta}) = mr^2(\theta)\ddot{\theta}$$

- Combining all terms to get

$$-mr^2(\theta)\frac{d^2\theta}{dt^2} - (kr^3(\theta) + m\mu r^2(\theta))\left(\frac{d\theta}{dt}\right)^2 + mgr(\theta)\sin(\theta) = 0$$

- Since $r(\theta) \neq 0$ our equation reduces to

$$-mr(\theta)\frac{d^2\theta}{dt^2} - (kr^2(\theta) + m\mu r(\theta))\left(\frac{d\theta}{dt}\right)^2 + mg\sin(\theta) = 0$$

Model Creation: Equation

Defining $r(\theta)$

$$r(\theta) = r_1 + r_2 H(\sin(\theta))$$

- $H-$Heaviside step function

$$r(\theta) = r_1 + r_2 H(\sin(\theta)) = \begin{cases} r_1 + r_2, & \sin(\theta) \in (0, 1] \\ r_1, & \sin(\theta) \in [-1, 0] \end{cases}$$

Model Creation: Ideal Case

Reduction to ideal case by

- Lubricate axle eliminates frictional torque
- Apparatus is quite small eliminates air resistance
- With no dampening the velocity should increase without bound

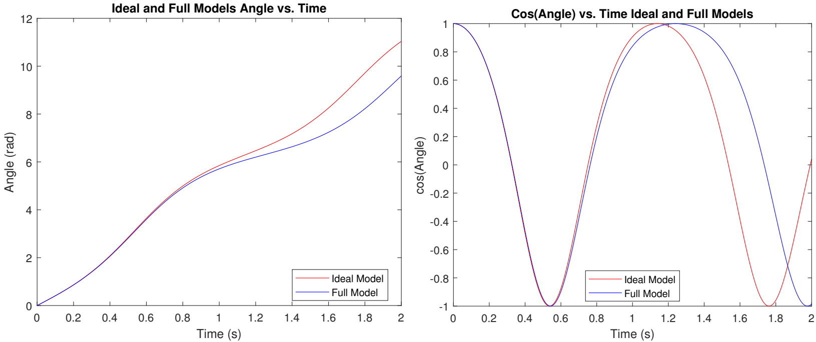
$$-mr(\theta)\frac{d^2\theta}{dt^2} + mg\sin(\theta) = 0$$

Alternatively:

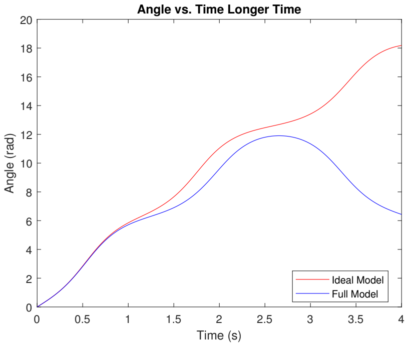
$$r(\theta)\frac{d^2\theta}{dt^2} = g\sin(\theta)$$

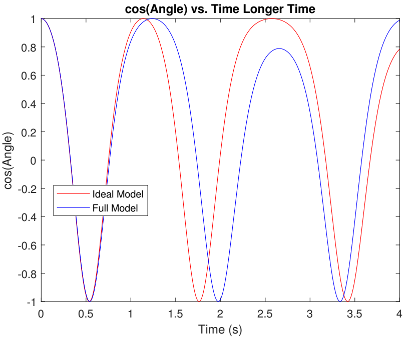
- For clarity we used $t \in [0, 2]$ s, and some approximate initial conditions:
  - $r_1 = 0.7\,\mathrm{m}$
  - $r_2 = 0.1\,\mathrm{m}$
  - $m = 5\,\mathrm{kg}$
  - $g = 9.8\,\mathrm{m\,s^{-2}}$
  - $k = 0.001\,\mathrm{kg\,m^{-1}}$
  - $\mu = .01$
  - $\theta(0) = 0\,\mathrm{rad}$
  - $\dot{\theta}(0) = 4\,\mathrm{rad\,s^{-1}}$
- Conditions can be changed from measurements as these were just estimated
- Applied a fourth order Runge-Kutta scheme for second order ODEs because of the incredible variance from initial conditions
- A longer time frame will be shown at the end of the analysis for further review

The models overlayed look like:



Running t from $t \in [0, 4]$ interval





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- Real case might lag slightly behind the ideal case and may even begin to fall back
- This provides a sanity check since air resistance and friction go against the motion of the apparatus
- The model is very difficult to predict and is very sensitive to changes in initial conditions
- Predictions for longer range times become more inaccurate meaning the longer time difference from our numeric scheme

From calculus we know a maximum value for speed can occur either on the boundary of your domain or at a value where the derivative is 0, so setting $\ddot{\theta} = 0$ we can get:

$$-(kr^2(\theta) + m\mu r(\theta)) \left(\frac{d\theta}{dt}\right)^2 + mg\sin(\theta) = 0$$

Hence our angular velocity becomes

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{mg\sin(\theta)}{kr^2(\theta) + m\mu r(\theta)}$$

$$\frac{d\theta}{dt} = \sqrt{\frac{mg\sin(\theta)}{kr^2(\theta) + m\mu r(\theta)}}$$

Maximal Speed: Critical Value

This only gives the maximum angular velocity so get the linear velocity we need a factor of radius meaning:

$$v = r(\theta)\frac{d\theta}{dt} = r(\theta)\sqrt{\frac{mg\sin(\theta)}{kr^2(\theta) + m\mu r(\theta)}} = \sqrt{\frac{mgr^2(\theta)\sin(\theta)}{kr^2(\theta) + m\mu r(\theta)}}$$

With some factoring:

$$v = \sqrt{\frac{mg\sin(\theta)}{k + \frac{m\mu}{r(\theta)}}}$$

Maximal Speed: Final Expression and Boundary

Maximized when $\sin(\theta) = 1$ evaluating to:

$$v_{max} = \sqrt{\frac{mg}{k + \frac{m\mu}{r_1 + r_2}}} = 16.8575\,\mathrm{m/s}$$

Note about the boundary and value

- Initial conditions change the left endpoint so nothing definite to say
- Unbounded right side so speed could increase without bound
- This max velocity assumes it actually occurs, but if we speed up without bound this quantity is obviously false and moreover nonexistent

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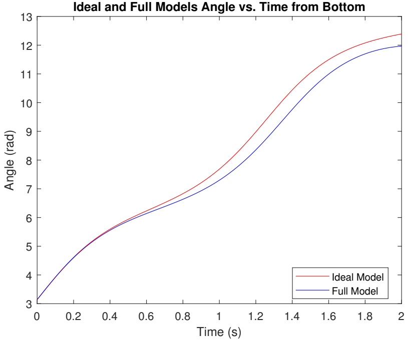
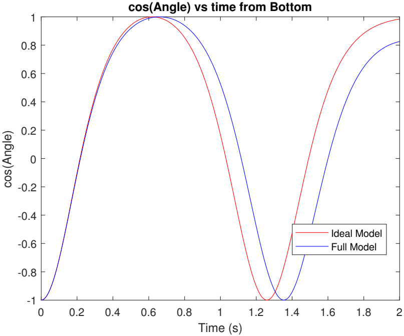
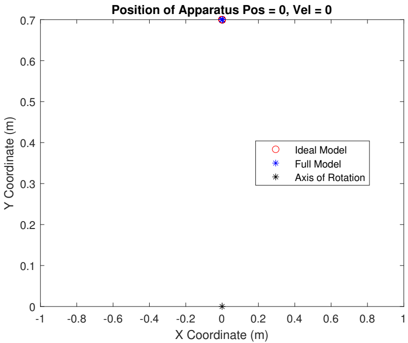
Can we still get motion if we start at the bottom? Of course with a high enough initial kick ($\dot{\theta}(0) = 8\,\mathrm{rad\,s^{-1}}$), here's what motion would look like in the ideal and full models:

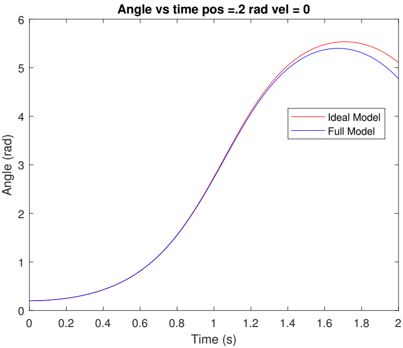




Initial Conditions: No initial velocity

With no initial velocity we can see no motion takes place, but if we change the position slightly we can still recover motion:





This also passes a sanity check since with no motion we don't turn and with no initial velocity it takes longer to get moving so we move less in a single time interval.

Initial Conditions: Limitations

- At our speeds air resistance and rotational friction are nearly meaningless
- At longer intervals our numerical estimates become more unreliable (RK-4)
- We rely on knowing the center of gravity for the apparatus
- Heaviside function is not a perfect fit only a simplification

Conclusion: Summary

To summarize:

- Set out to model the motion of a bird on a bicycle wheel
- Found with a given radius function we can keep spinning with increasing speed
- Depending on initial conditions we may not be able to even complete a rotation (e.g. lacking rotational velocity)
- Found an expression for the maximum speed
- Sanity checked our model with some numerical simulations
- Considered the limitations of our model

Conclusion: Our SIMIODE Experience

Working together with everyone

The epiphany moment

Applications in modern science

Conclusion: Thank You

A special thank you to Anthony Stefan for getting us interested and to SIMIODE/SCUDEM for taking the time to watch, review, and judge our presentation!