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STUDENT VERSION
Modeling One-Dimensional Groundwater Flow

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STATEMENT

In the mid 19th century, the city of Dijon, France decided to improve and enlarge the city's public water works. Henry Darcy, the local Inspector General of Bridges and Roads, was responsible for this civil engineering program. One of the challenges faced by Darcy was how to design sand filters for the city's public fountains. Sand is an example of material found in an *aquifer*, which is a water bearing porous medium, through which water flows easily, [4, 6].

Since there weren't any models for the flow of fluid through a porous medium, Darcy had to create one. To do so, he performed his own experiments on the flow of water through sand. His results, including what we now know as Darcy's Law, were published in 1856 as an appendix to his 647-page report on the project: *Les Fontaines Publiques de la Ville de Dijon*, [3, 7].

Using an experimental setup similar to that shown in Figure 1, Darcy determined that for one-dimensional flow, the *volumetric flow rate* of water, Q, varies jointly with respect to the *hydraulic gradient* i and cross sectional area A, via

$$Q = KiA. \tag{1}$$

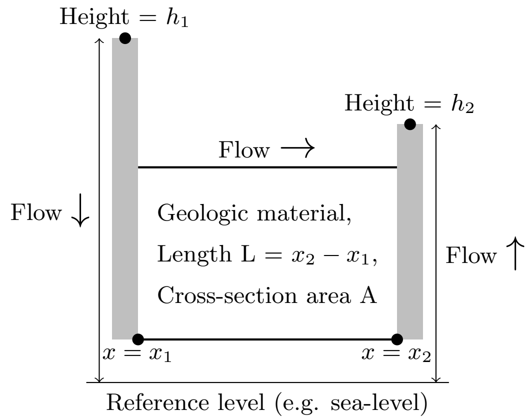

Figure 1: Experimental Set-up for Darcy's Law

This is known as *Darcy's Law*. The hydraulic gradient measures the loss in *hydraulic head* h (loosely defined as the height of the water level, relative to a reference point) over a given distance L. Referring to Figure 1, we write

$$i = \frac{\Delta h}{L} = \frac{h_1 - h_2}{x_2 - x_1},$$

[7]. Relabeling $x_1 = x$ and $x_2 = x + \Delta x$, and considering head drops over shorter and shorter intervals (i.e. letting $\Delta x = L \to 0$), the hydraulic gradient can be written in differential form as

$$i = \lim_{\Delta x \to 0} -\frac{h(x + \Delta x) - h(x)}{\Delta x} = -\frac{dh}{dx},$$

[15]. The proportionality constant K, called the *hydraulic conductivity*, depends on the geologic material through which the water is flowing. Since units of Q are volume per unit time and the hydraulic gradient is dimensionless, it follows that units of K are $[K]$ = length/time, [7].

One way to illustrate groundwater flow is via a physical *Groundwater Flow Model*. A Groundwater Flow Model or *sand tank*, such as the one pictured in Figure 2, from Ball State University's Department of Geological Sciences, "is an educational device constructed of sturdy layered sand lenses to represent a sliced section of earth. ... Through the use of water tinted with food coloring or grape Kool-Aid, it is possible to observe a wide range of groundwater movements, [9]." The sand tank is designed so that water can flow between narrow rectangular columns on either end into or out of the aquifer portion in the middle via small holes at the base of each column. For details on how a typical sand tank is set up, see [14].

Figure 3 shows a conceptual representation of the sand tank. The x coordinate is set from the left column of water at $x = 0$ to the right column at $x = a$. The boundary conditions H_0 and H_1 are positioned to represent observed static water levels at each end of the tank.

Using Darcy's Law (1) and the idea of continuity (conservation of mass), one can derive a partial differential equation that describes hydraulic head level h at any point in an aquifer. A simple flow

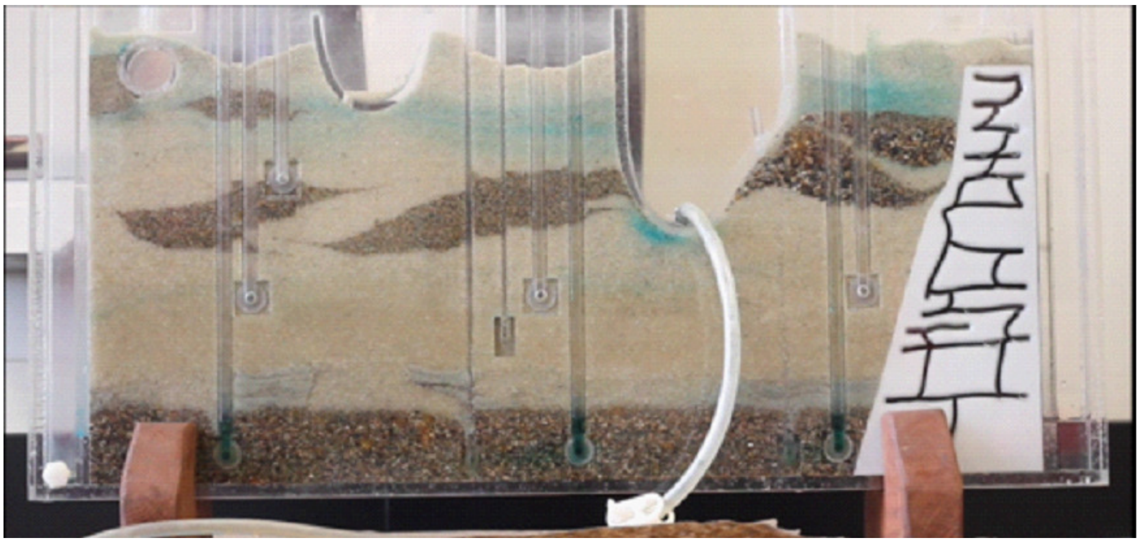

Figure 2: Groundwater Flow Model (Sand Tank)

scenario occurs when we assume we have a homogeneous aquifer of constant hydraulic conductivity K, in which flow is one dimensional (horizontal). By homogeneous, we mean that the medium has constant physical properties, such as hydraulic conductivity K and specific storage S_s (defined further below). The diagram in Figure 3 shows water flowing from right to left through such a homogeneous medium of constant thickness, along with a small representative element of width Δx and cross sectional area A, that is the focus of our calculations.

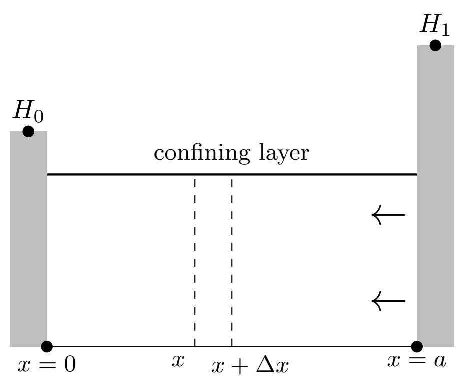

Figure 3: One dimensional confined groundwater flow.

There are four possible contributions to flow through this representative element: (1) flow across the right face, $-Q(x + \Delta x, t)$; (2) flow across the left face, $-Q(x, t)$, (3) flow introduced from an external source, $R(x,t)A\Delta x$; and (4) flow released from - or contributing to - storage within the medium itself, $S_s A\Delta x(\Delta h/\Delta t)$. As defined above, $Q(x,t)$ is the volumetric flow rate, i.e. volume of water per unit time, with $[Q]$ = volume/time.[1] The term $R(x,t)$ is often called *recharge*, and in groundwater flow scenarios, it can be related to precipitation or leakage into (or out of) the medium from layers above or below; it describes the volume of water added per unit time per unit volume of the aquifer, with $[R]$ = volume/(time $\times$ volume). The *specific storage* S_s is the volume of water added to, or released from, storage per unit volume of aquifer per unit change in hydraulic head, with $[S_s]$ = volume/(volume $\times$ length).

The Law of Conservation of Mass tailored to our scenario says that "inflow must equal outflow", or *the flow into the right face plus flow introduced from an external source must be equal to flow across the left face plus any change in flow due to storage in the medium itself.* [7, 15]. Mathematically, we can write this as

$$-Q(x+\Delta x,t)+R(x,t)A\Delta x=-Q(x,t)+S_s A\Delta x\frac{\Delta h}{\Delta t} \tag{2}$$

or

$$-Q(x+\Delta x,t)+Q(x,t)=S_s A\Delta x\frac{\Delta h}{\Delta t}-R(x,t)A\Delta x \tag{3}$$

Using Darcy's Law (1), with $i=-\dfrac{\partial h}{\partial x}$, we can update our flow equation (3) as

$$KA\frac{\partial h(x+\Delta x,t)}{\partial x}-KA\frac{\partial h(x,t)}{\partial x}=S_s A\Delta x\frac{\Delta h}{\Delta t}-R(x,t)A\Delta x \tag{4}$$

After dividing (4) by $KA\Delta x$, we get

$$\frac{\frac{\partial h(x+\Delta x,t)}{\partial x}-\frac{\partial h(x,t)}{\partial x}}{\Delta x}=\frac{S_s}{K}\frac{\Delta h}{\Delta t}-\frac{R(x,t)}{K}. \tag{5}$$

Letting $\Delta x\to 0$ and $\Delta t\to 0$ in (5) leads to the *one-dimensional groundwater flow equation*

$$\frac{\partial^2 h}{\partial x^2}=\frac{1}{k}\frac{\partial h}{\partial t}-\frac{R(x,t)}{K},\quad \text{for}\quad 0<x<a, t>0, \tag{6}$$

where we have introduced the *hydraulic diffusivity* k via $k := K/S_s$. Table 1 provides some values for k for sand, [5].

While not specifically designed to measure well head levels, using a video camera (cell phone), along with Tracker software [2], we were able to collect well head levels in three wells over time within the available sand tank. Pouring green food-colored water at a fixed rate into the column at the right end of the sand tank and allowing the water to flow from the aquifer portion to the left column and drain out through a flexible tube connected to the drain hole at the left end of the sand tank, we were able to physically simulate one-dimensional groundwater flow through an aquifer with fixed head levels at each boundary.

[1]Figure 3 is designed to show flow from right to left, as that will match our laboratory scenario. Typically, flow $Q(x,t)>0$ indicates flow in the positive $x-$direction, so we introduce the negative sign to indicate the groundwater is flowing in the negative $x-$direction.

	min (m/sec)	max (m/sec)
Hydraulic Conductivity K for sand estimates*	0.0000009	0.0002
Specific Storage S_s for various sand types*	0.000127953	0.00101706
	min (m^(-1))	max (m^(-1))
Hydraulic Diffusivity Estimates ($k = K/S_s$)	0.000884903	1.563076923
	min (m^2/sec)	max (m^2/sec)
*using measured values provided by [5]		

Table 1: Hydraulic Diffusivity Ranges for Sand

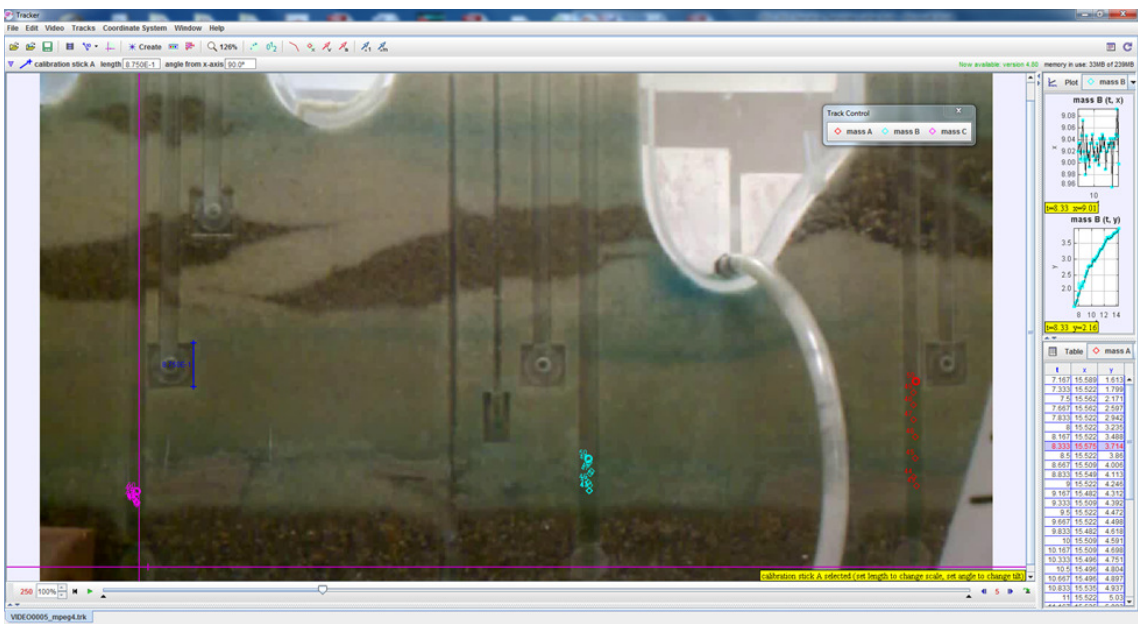

Figure 4: Collecting Well Head Data via Tracker

Figure 4 shows a screenshot from TRACKER, being used to analyze a video of the sand tank, with the three wells (left, middle, and right, indicated by respective colors pink, green, and red), located at $x_L = 3.75$ in, $x_M = 12$ in, and $x_R = 18.5$ in, respectively. This is done within TRACKER by establishing a coordinate system with axes, origin, and scale that is calibrated to a known distance in the sand tank. In Figure 4 we see the origin at the base (round circle) of the left well and the blue calibration stick (length 0.875 inches) at the base of the well just to the right of the left well. Head levels can be measured for each of the three wells for each frame, using TRACKER'S point mass tool, treating the top of the water column in each well as a "point mass", and marking the location of each "point mass" (i.e. top of the water columns) frame by frame. These marks appear in Figure

4 as the colored symbols in each of the three wells. We found that it is easier to see changes in water levels in each well by setting TRACKER to advance the video five frames at a time. The time (in seconds) and coordinates of each "point mass" (in inches) are recorded in TRACKER and can be imported into an Excel file. The y-coordinate of a "point mass" at a given time corresponds to the height of a water column in one of the wells.

The horizontal length of the aquifer is $a = 23.75$ in, with fixed head levels $H_0 = 0.5625$ in and $H_1 = 9.6875$ in at the left and right boundaries, $x = 0$ and $x = a$, respectively.[2] Well head data collected at each of these wells over time (43 measurements for each well, taken once every 5/30 second) as well as additional sand tank measurements are provided in an accompanying Excel file (headdata.xlsx) and PDF file (sandtankdimensions.pdf). NOTE: This data may be collected as a separate project via a provided AVI video file (Video0005_mpeg4.avi) or a sand tank and video camera, if you have these available, along with TRACKER. In the video, the water level starts to rise at 7 seconds and continues to do so, slowing down as time increases. The video file can be opened in TRACKER and head levels collected via the technique described above. We collected data from 7 seconds until about 14.167 seconds, corresponding to frames 220 to 430, until we had sufficient data to compare to a mathematical model.

Student Activities

1. COMPARISON OF DERIVATIONS

   Compare the groundwater flow equation (6) to the equation for one-dimensional heat flow through a rod of conducting material – a derivation of this equation can be found in standard partial differential equations texts, such as [11]. A key to the derivation of equation (6) is the application of the Law of Conservation of Mass to groundwater flowing through a small rectangular volume element in an aquifer. What analogous ideas and conservation law are used to derive the heat equation? In groundwater flow, Darcy's Law (1) says that flow is proportional to hydraulic gradient. What is the analogous principle in the derivation of the heat equation?

2. BOUNDARY AND INITIAL CONDITIONS

   A mathematical model for one-dimensional groundwater flow consists of the groundwater flow equation (6) along with appropriate boundary conditions and an initial condition. Set up boundary conditions for your groundwater flow model if there is a fixed head level at each end of an aquifer, say H_0 at the left end and H_1 at the right end for all times $t > 0$. Also, choose an appropriate initial head level distribution, given that we know the head level is $f(x)$ for $0 < x < a$, at time $t = 0$. What assumptions should be made about this function? Why?

3. SOLVE THE CONCEPTUAL MODEL

   Using the ideas developed for solving the heat equation by means of separation of variables,

[2] To get the measured height of the water at the right end of the sand tank, we poured water into the right column at a fixed rate to keep the water level at a specific measured height. For the left end of the sand tank, we measured the height of the hole at the base of the left column and used this for the measured height of the water at this end.

solve the initial value boundary value problem (IVBVP) consisting of equation (6) derived above along with the initial condition and boundary conditions chosen in Question 2. To do so, assume there is no recharge in the aquifer, i.e. $R(x,t) = 0$, and write the solution function as $h(x,t) = H(x) + w(x,t)$, where

(a) $H(x)$ solves the steady-state problem associated with the IVBVP.

(b) $w(x,t)$ contains all transient information associated with the IVBVP.

Hint: Find $H(x)$ first, then employ $H(x)$ in a solution for $w(x,t)$ and so also $h(x,t)$.

4. Model Implementation

   Using Mathematica, Maple, MATLAB, Excel, or other appropriate software, implement your model in a computer setting. If possible, include a mechanism for modifying parameters in your model dynamically. This can be done, for instance in Mathematica via the 'Manipulate' command or Excel via Form Controls. Determine an appropriate number of terms in your series solution.

5. Comparison of Model to Data

   Develop a method for comparing your mathematical model to the collected data. Do they agree satisfactorily? Justify your answer.

6. Revise the Model

   If the model and measured data considered in Question 5 do NOT agree, what changes (if any, such as including a recharge term) could be made to your model to account for these discrepancies? Justify your suggested changes.

7. Revise the Physical Process?

   (If you have access to a sand tank) instead of the mathematical model, could anything related to the process or methods used to collect the data be modified (for example, type of material in tank)? Justify your suggested changes.

8. Next Iteration

   Implement the changes suggested in Questions 6 and 7. Then repeat Questions 3 – 5 as appropriate.

Acknowledgements

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