Let our time variable, \( t \), be measured in seconds (s), and the displacement, \( y \) in meters (m), with mass \( M \) in kilograms (kg).

Each force in the DE is measured in Newtons. (1 Newton = 1 kg m/s^2.)

A compact cross-over sport utility vehicle might be 3,400 lbs, with a mass on one wheel of 386 kg; a basic full-size pick-up truck might be 5,000 lbs, with a mass on one wheel of 568 kg.

Thus, \( k \) is in Newtons per meter (N/m).

Spring constant values varied between 16,462 and 27,8452 Newtons per meter (N/m) on one website but was over 57,250 N/m on another.

We measure \( c \) in N s/m.

---

### General Model

1. The characteristic equation for the DE, and its roots

\[
\begin{align*}
\text{In[1]=} & \quad \text{Clear}[M, c, k, r, \text{charEqn}] \\
\text{charEqn} & = M r^2 + c r + k = 0 \\
\text{Solve[charEqn, r]} \\
\text{Out[2]=} & \quad k - c r + M r^2 = 0 \\
\text{Out[3]=} & \quad \left\{ \left\{ r \rightarrow \frac{-c - \sqrt{c^2 - 4 k M}}{2 M} \right\}, \left\{ r \rightarrow \frac{-c + \sqrt{c^2 - 4 k M}}{2 M} \right\} \right\}
\end{align*}
\]

2. Why would the suspension system with a shock absorber give a more comfortable ride? The differential equation with no damping:

\( c = 0 \), roots of the characteristic equation are purely imaginary, solutions are sinusoidal with constant amplitude.
More than was specifically asked in the module

\textbf{overdamped: } \quad c^2 - 4 k M > 0 . \\
\text{ solutions } e^\frac{-c t}{2 M} c_1 + e\frac{-c t}{2 M} t c_2 \\
oscillate but with decreasing amplitude

\textbf{undamped: } \quad c = 0 , \quad \text{roots of the characteristic equation are purely imaginary, solutions are sinusoidal with constant amplitude.}

\textbf{critically damped: } \quad c^2 - 4 k M = 0 , \quad r = - c/(2M) , \quad \text{single repeated root of the characteristic equation, solutions are } e^\frac{ct}{2M} c_1 + e^\frac{-ct}{2M} t c_2 \text{ and die down rapidly}

\textbf{underdamped: } \quad c^2 - 4 k M < 0 , \quad r = - c/(2M) , \quad \text{roots of the characteristic equation are complex with negative real portion, solutions are oscillatory with decreasing amplitude.}
UNDAMPED

m1 = 390
k1 = 44 000
c1 = 0
y01 = 0.2
de1 = m1 y''[t] + c1 y'[t] + k1 y[t] = 0;
ivp = {de1, y[0] = y01, y'[0] = 0}
sol = DSolve[ivp, y[t], t]

Plot[sol[[1, 1, 2]], {t, 0, 5}, PlotRange -> All]

a qualitative image, if desired
Determine Damping

We use the following values: the mass on one wheel 390 kg, the spring constant 44,000 N/m, the initial position is 0.2 m above the static equilibrium.

4. Relate these values to the constants and variables in the DE: \( M=390, \ k=44000, \ y_0=0.2. \)

5. Determine the values of \( c \) for an overdamped system. Graph the resulting displacement. Is this a smooth or bumpy ride?
CRITICALLY DAMPED

In[1]:= Clear[m1, c1, k1, y01, y, t, de1, ivp, sol]

\[m1 = 390.\]
\[k1 = 44000.\]
\[c1 = 2 Sqrt[m1 k1]\]
\[y01 = 0.2\]
\[de1 = m1 y''[t] + c1 y'[t] + k1 y[t] = 0;\]
\[ivp = \{de1, y[0] = y01, y'[0] = 0\}\]
\[sol = DSolve[ivp, y[t], t]\]

\[Plot[sol[[1, 1, 2]], \{t, 0, 5\}, PlotRange \rightarrow All]\]

Out[3]= 8284.93
Out[4]= 0.2
Out[5]= \{44000. y[t] + 8284.93 y'[t] + 390. y''[t] = 0, y[0] = 0.2, y'[0] = 0\}
Out[6]= \{\{y[t] \rightarrow 2.12434 e^{-10.6217 t} (0.0941469 + 1. t)\}\}
This is quite a smooth ride, with no oscillations or bounces.

**MEETING TOLERANCE**

6. How long does it take for the critically damped system to stay within a tolerance of 0.1 m of the static equilibrium? Illustrate with a graph.

Because our solution is decreasing from a positive displacement, to stay within a tolerance of 0.1 m from the static equilibrium, we find the first time at which our solution has a displacement of 0.1 m.

```
In[•]:= Clear[sol3, tval]
sol3 = Solve[sol[[1, 1, 2]] == .1, PositiveReals]
tval = sol3[[1, 1, 2]]
Out[•]= {{t -> 0.158011}}
```

It takes 0.158 seconds to reach the tolerance. This is faster than the blink of an eye, which is 1/3 of a second. We illustrate this in a graph.
Changing the Ride Quality

We begin with the following values: the mass on one wheel 450 kg, spring constant 16,000 N/m, damping coefficient 1600 N-s/M, with initial displacement 0.2.

USING THESE VALUES

7. (a) Graph the resulting displacement.
\begin{verbatim}
In[ ] := Clear[m1, c1, k1, y01, y, t, de1, ivp, sol]
m1 = 450.
k1 = 16000.
c1 = 1600.
y01 = 0.2
de1 = m1 y''[t] + c1 y'[t] + k1 y[t] == 0;
ivp = {de1, y[0] == y01, y'[0] == 0};
sol = DSolve[ivp, y[t], t];
Plot[sol[[1, 1, 2]], {t, 0, 5}, PlotRange -> All]
\end{verbatim}
7. (b) Is this a smooth or bumpy ride? Is this system underdamped or overdamped?

This underdamped ride is more bumpy than the previous, critically damped, one but would barely be felt, if the oscillations are felt at all.

evidence of underdamping: The image shows decreasing oscillations, the DE solution indicates decreasing oscillations, and the discriminant of the characteristic equation is slightly negative.

7. (c) How long does it take for this system to stay within a tolerance of 0.1 m of the static equilibrium? Illustrate with a graph.

Because our system oscillates, we check both above and below equilibrium. We see from the graph that we still can rely upon the first crossing.

It takes 0.198097 seconds to reach the tolerance. This system takes a bit longer to reach the tolerance than the previous set-up, but it is still faster than the blink of an eye.
8. How does the ride change in this situation if the total mass of the car is increased by 1,000 lbs? Show a graph for both rides, and give a brief description.
In[1]:= Clear[m1, c1, k1, y01, y, t, de1, ivp, sol, m2, de2, sol2]

m1 = 450.
m2 = 1000 / 2.2 / 4
k1 = 16000.
c1 = 1600.
y01 = 0.2
de1 = m1 y''[t] + c1 y'[t] + k1 y[t] == 0;
ivp = {de1, y[0] == y01, y'[0] == 0}
sol = DSolve[ivp, y[t], t]
de2 = (m1 + m2) y''[t] + c1 y'[t] + k1 y[t] == 0;
ivp2 = {de2, y[0] == y01, y'[0] == 0}
sol2 = DSolve[ivp2, y[t], t]

Out[5]= 0.2

Out[6]= {16000. y[t] + 1600. y'[t] + 450. y''[t] == 0, y[0] == 0.2, y'[0] == 0}

Out[7]= {{y[t] -> 0.2 e^(-1.77778 t) (1. Cos[5.69167 t] + 0.312348 Sin[5.69167 t])}}

Out[8]= {16000. y[t] + 1600. y'[t] + 563.636 y''[t] == 0, y[0] == 0.2, y'[0] == 0}

Out[9]= {{y[t] -> 0.2 e^(-1.41935 t) (1. Cos[5.13542 t] + 0.276385 Sin[5.13542 t])}}

In[10]:= Plot[{{sol[[1, 1, 2]], sol2[[1, 1, 2]]}, {t, 0, 2.5},
PlotStyle -> {{Blue}, {Black, Dashed}}, PlotRange -> All]
The oscillations take a bit longer to diminish than the previous ride, but still faster than the blink of an eye. The oscillations also are a bit larger. The graph illustrates these differences.

\[ \text{SPRING CORRECTION FACTOR BASED ON INSTALLATION ANGLE} \]

9. Consider installing the spring at an angle of \( \theta \leq \pi/4 \) from vertical by rotating the top of the spring. (a) Determine the spring rate correction factor in general, meaning the ratio of the spring rate to equivalent vertical portion. Determine the new spring constant that we would need to compensate for the angle of installation.

We see that the spring rate correction factor is \( \cos(\theta) = (k_v y)/(k y) = k_v / k. \)

(b) Determine the new spring constant \( k \) that we would need to compensate for the angles of installation of \( \theta = \pi/16, \pi/8, \pi/4 \) if we wish an effective spring constant, \( k_v \) of 16,000 N/m.
We must choose a spring with spring constant \( k = k_v / \cos(\theta) \).

```
In[4]:= Clear[kv, theta]
kv = 16000;
TableForm@Table[{theta, theta*1., kv/Cos[theta*1.]}, {theta, \[Pi]/16, \[Pi]/4, \[Pi]/16}] // TableForm
```

```
\begin{array}{ll}
\text{\[Pi]/16} & 0.019635 \\
\pi & 0.0392699 \\
3 \pi & 0.0589049 \\
\pi/4 & 0.0785398 \\
\end{array}
```

(c) Compare the rides without additional mass using a spring installed at angles of 20° and 40°. Show a graph for both rides, and give a brief description.

```
In[12]:= Clear[m1, c1, k1, y01, y, t, de1, ivp, sol, ang1, ang2, de2, sol2]
m1 = 450.
k1 = 16000.
c1 = 1600.
y01 = 0.2
ang1 = 20 * \[Pi]/180
k1Cos[ang1]
ang2 = 40 * \[Pi]/180
k1Cos[ang2]
de1 = m1 y''[t] + c1 y'[t] + k1Cos[ang1] y[t] == 0;
ivp = {de1, y[0] == y01, y'[0] == 0}
sol = DSolve[ivp, y[t], t]
de2 = m1 y''[t] + c1 y'[t] + k1Cos[ang2] y[t] == 0;
ivp2 = {de2, y[0] == y01, y'[0] == 0}
sol2 = DSolve[ivp2, y[t], t]
```

```
Out[12]= 0.2
Out[12]= \[Pi]/9
Out[12]= 15035.1
Out[12]= 2 \[Pi]/9
Out[12]= 12256.7
Out[12]= {15035.1 y[t] + 1600. y'[t] + 450. y''[t] == 0, y[0] == 0.2, y'[0] == 0}
Out[12]= {{y[t] \[RightArrow] 0.2 e^{-1.77778 t} (1. \cos[5.50007 t] + 0.323232 \sin[5.50007 t])}}
Out[12]= {12256.7 y[t] + 1600. y'[t] + 450. y''[t] == 0, y[0] == 0.2, y'[0] == 0}
Out[12]= {{y[t] \[RightArrow] 0.2 e^{-1.77778 t} (1. \cos[4.9068 t] + 0.362309 \sin[4.9068 t])}}
```
As we see in the graph, the oscillations for a larger installation angle are a bit larger and last longer. The smoother ride with the more effective spring makes sense. At an angle of \( \frac{\pi}{9} \), the effective spring constant is 15,035 N/m, whereas the angle of \( \frac{2\pi}{9} \) results in an effective spring constant of nearly 12,257 N/m.