Modeling Falling Column of Water

Remote Teaching Module

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Class simulation followed by discussion.

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Remote Teaching Modules on Teaching Differential Equations with Modeling.

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Registration is FREE.
Resources are Open Education Resources (OER).
All are downloadable and fully adaptable.

SIMIODE is funded by the National Science Foundation Division of Undergraduate Education.

Send questions to Director@simiode.org.

During this session we simulate a class activity on modeling with a differential equation. At the end we will conduct a discussion on issues, both pedagogical and technical.

All the resources used will be available at **SIMIODE's website**.

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SIMIODE A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

Modeling Falling Column of Water

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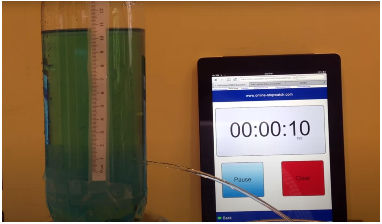
Modeling Falling Column of Water

We collect data on a falling column of water and model the height using first principles from physics with a differential equation.

1. Video, data, qualitative behavior, empirical model
2. First principles analytic model, Torricelli's Law
3. Differential equation, estimate parameters, validate model
4. Discussions

Source:
1-015-S-Torricelli **https://www.simiode.org/resources/488**
1-015-T-Torricelli **https://www.simiode.org/resources/463**

We use data taken from video at SIMIODE YouTube Channel



https://www.youtube.com/watch?v=NBr4DOj4OTE .

Cylindrical column (radius = 4.17 cm) of water empties through a hole (diameter = 11/16" = 0.218281 cm) in bottom of column. Exit hole at bottom of column - height is 0 cm.

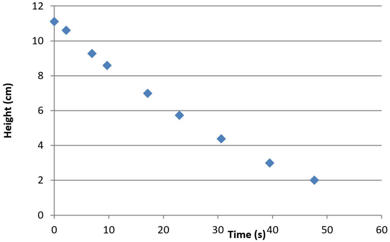
We seek to model $h(t)$, the height of the column of water.

Just visit. No run yet.

Here is data we collected. What do you see or notice?

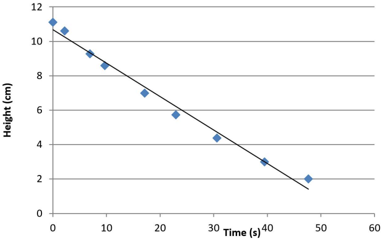
Make some observations now.

Time (s)	Height (cm)
0.0	11.1
2.187	10.6
6.933	9.3
9.717	8.6
17.102	7.0
22.968	5.75
30.603	4.4
39.503	3.0
47.663	2.0



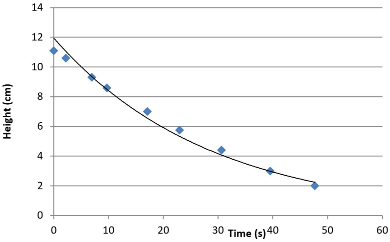
Linear Fit?

Time (s)	Height (cm)
0.0	11.1
2.187	10.6
6.933	9.3
9.717	8.6
17.102	7.0
22.968	5.75
30.603	4.4
39.503	3.0
47.663	2.0



Exponential Decay Fit?

Time (s)	Height (cm)
0.0	11.1
2.187	10.6
6.933	9.3
9.717	8.6
17.102	7.0
22.968	5.75
30.603	4.4
39.503	3.0
47.663	2.0



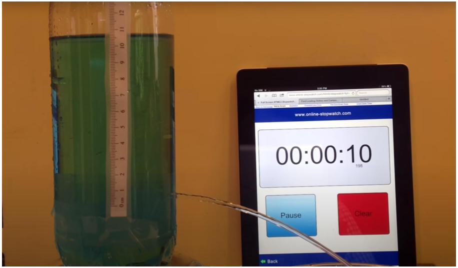
All are empirical fits with no understanding.

They just fit a function to data.

And neither line nor exponential are good.

Run video of falling column and observe.

https://www.youtube.com/watch?v=NBr4DOj4OTE



What happens to height $h(t)$?

How fast is column of water falling? Early and later?

From the video of the falling column what can we see?

For large $h(t)$ the column of water falls faster or slower or same ...

For small $h(t)$ falls faster or slower or same ...

Time (s)	Height (cm)
0.0	11.1
2.187	10.6
6.933	9.3
9.717	8.6
17.102	7.0
22.968	5.75
30.603	4.4
39.503	3.0
47.663	2.0

What is your conclusion about $\dfrac{dh(t)}{dt}$?

So might something like this be true?

$$\frac{dh(t)}{dt} = f(h(t)), \quad h(0) = h_0 .$$

Where for large $h(t)$ we have large $f(h(t))$

Where for small $h(t)$ we have small $f(h(t))$

And, of course, $f(h(t))$ is negative. Why?

Let's check out change in height over various intervals of time.

Time (s)	Height (cm)
0.0	11.1
2.187	10.6
6.933	9.3
9.717	8.6
17.102	7.0
22.968	5.75
30.603	4.4
39.503	3.0
47.663	2.0

Check out the average rate of falling of the height of the column of water in several intervals, say, $[0, 2.187]$,

$$\frac{10.6 - 11.1}{2.187 - 0} = -0.2286\,,$$

or in the interval $[39.503, 47.663]$,

$$\frac{2.0 - 3.0}{47.663 - 39.503} = -0.122549\,.$$

What do you see? What can you say about $h'(t)$?

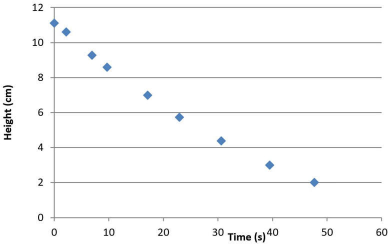
Let's find a model from some first principles.

This would be an analytic model.

NOT just fit a function to data.

NOT just "it looks like it falls faster or slower."

Time (s)	Height (cm)
0.0	11.1
2.187	10.6
6.933	9.3
9.717	8.6
17.102	7.0
22.968	5.75
30.603	4.4
39.503	3.0
47.663	2.0



Enter Evangelista Torricelli 1608–1647, an Italian physicist and mathematician, and a student of Galileo. Best known for his invention of the barometer.

Torricelli's Law to the rescue!

$$\frac{dh(t)}{dt} = -b\sqrt{g \cdot h(t)}, \quad h(0) = h_0 \quad b > 0.$$

Say it out loud in sentence form.

Explain to yourself what it means.

Does Torricelli's Law agree with observations?

For large $h(t)$ the column of water falls faster or slower ....

For small $h(t)$ the column of water falls faster or slower ....

Torricelli's Law

$$\frac{dh(t)}{dt} = -b\sqrt{g \cdot h(t)}, \quad h(0) = h_0 \quad b > 0.$$

Does Torricelli's Law agree with observations?

For large $h(t)$ falls **faster** or slower . . . YES!

For small $h(t)$ falls faster or **slower** . . . YES!

We build the model that IS Torricelli's Law from First Principles.

This will be an analytic model.

Given the cross sectional area of the cylinder of water as a function of height and the area of the tiny exit hole at the bottom of the cylinder can we model the outflow of the water from the cylinder?

We will apply this First Principle

The Law of Conservation of Energy.

We could ask you for a short statement of The Law of Conservation of Energy.

Thank you!

Basically, The Law of Conservation of Energy says that total energy is conserved.

We will apply it to a slab of water, first at the surface of the column of water and then at the bottom of the column ($h = 0$)

Total Energy is the the sum of the **potential energy** and the **kinetic energy** of a particle of mass m and this sum is constant at each instance in time, t.

Care to share formula for **potential energy** and for **kinetic energy**?

Go ahead and jump the gun!

Consider mass of water m initially atop a cylinder of water, some h meters above exit hole, this mass has

initial potential energy $PE_i = m \cdot g \cdot h$, where g is the acceleration due to gravity and

initial kinetic energy $KE_i = \frac{1}{2} m v_i^2$, where v_i is the initial velocity of the mass.

Thus we have initial total energy of TE_i when the mass of water is on the top of the cylinder of water:

$$TE_i = KE_i + PE_i = \frac{1}{2} m v_i^2 + mgh \,.$$

When this mass of water reaches the exit hole it has height 0 meters and a final velocity of v_f.

Hence, the total energy at the final time, TE_f, the mass reaches the exit hole where $h = 0$ is

$$TE_f = KE_f + PE_f = \frac{1}{2}mv_f^2 + mg \cdot h = \frac{1}{2}mv_f^2 + mg \cdot 0 = \frac{1}{2}mv_f^2 \,.$$

Now applying The Law Conservation of Energy, $TE_i = TE_f$, we build an equation - see the equal sign!!

Now by The Law Conservation of Energy,

$$TE_i = \frac{1}{2}mv_i^2 + mgh = \frac{1}{2}mv_f^2 = TE_f \ ,$$

Divide both sides by m and multiply by 2 - to solve for v_f:

$$v_f = \sqrt{2gh + v_i^2} \, .$$

Since $v_i = 0$ we have one classical form of Torricelli's Law

$$v_f = \sqrt{2gh} \, ,$$

where v_f is the speed of the water as it leaves the exit hole.

We employ an effective tool in modeling: **equate two ways of computing** loss in volume of water in tank at time t. **First**,

$$A(h(t))h'(t) \tag{1}$$

where $A(h(t))$ is cross sectional area of column at height $h(t)$, often constant, so $A(h(t)) = A$ in our case, and **second**

$$-v_f \cdot a \cdot \alpha = -\sqrt{2\,g\,h} \cdot a \cdot \alpha = -a \cdot \alpha \cdot \sqrt{2\,g\,h}\,. \tag{2}$$

- v_f is velocity of water exiting the column of water when the water is at height $h(t)$,
- a is the cross sectional area of the small bore hole through which the water exits, and
- α is an empirical number, percent of maximum flow rate through small hole, due to friction and constriction. α, is called *discharge or contraction coefficient*.

Now **equating** (1) and (2) - two different ways to calculate the rate at which the volume of water in the cylinder decreases - gives rise to a working version of Torricelli's Law for the height of a constant cross section column of water, $h(t)$, at time t:

$$A \cdot h'(t) = A(h(t)) \cdot h'(t) = -a\alpha\sqrt{2g \cdot h(t)}\,, \tag{3}$$

where $h(t)$ is the height of the column of water at time t.

In our case $A(h(t)) = A$ as cross section area of water is constant.

General form of Torricelli's Law in our case

$$A \cdot h'(t) = A(h(t)) \cdot h'(t) = -a\alpha\sqrt{2g \cdot h(t)}\,,$$

Solve (3) for $h'(t)$ and note that $A(h(t)) = A$. Gather all the constants except g into one big constant, b, we have

$$h'(t) = -b\sqrt{g \cdot h(t)}\,. \tag{4}$$

How could we use our water videos to validate this model.
Take your time, think before you share your ideas.

So we have an analytic model (differential equation!) for $h(t)$.

$$\frac{dh(t)}{dt} = -b\sqrt{g \cdot h(t)}\,, \quad h(0) = h_0\,.$$

We solve this differential equation for $h(t)$ to realize a model.

What strategy/technique can we employ? What technology?

We use this solution and our data to estimate parameter b and validate our model by comparing model predictions to data.

$$\frac{dh(t)}{dt} = -b\sqrt{g \cdot h(t)} = -b\sqrt{g} \cdot (h(t))^{1/2}.$$

Separate the variables

$$(h(t))^{-1/2} \cdot \frac{dh(t)}{dt} = -b\sqrt{g}.$$

OR

$$(h(t))^{-1/2} \cdot dh = -b\sqrt{g} \cdot dt.$$

Integrate both sides. (What is C?)

$$\int (h(t))^{-1/2} \cdot \frac{dh(t)}{dt}\, dt = \int -b\sqrt{g}\, dt + C\,,$$

$$2(h(t))^{1/2} = -b\sqrt{g} \cdot t + C.$$

Now to find C using Initial Conditions:

$$2(h(t))^{1/2} = -b\sqrt{g} \cdot t + C.$$

$$2(h(0))^{1/2} = -b\sqrt{g} \cdot 0 + C = C.$$

Thus we have

$$2(h(t))^{1/2} = -b\sqrt{g} \cdot t + 2(h(0))^{1/2}.$$

Divide both sides by 2 and then square both sides yields:

$$h(t) = \left(-\frac{b\sqrt{g}}{2} \cdot t + (h(0))^{1/2}\right)^2. \tag{5}$$

This is model for height of the column of water, $h(t)$, at time t.

What do we know and what do we need to estimate b in (5)?

$h(0) = 11.1$ cm and $g = 980$ cm/s^2

Thus from $h(0) = 11.1$ cm and $g = 980$ cm/s^2

$$h(t) = \left(-\frac{b\sqrt{g}}{2} \cdot t + (h(0))^{1/2}\right)^2$$

becomes

$$h(t) = \left(-\frac{b\sqrt{980}}{2} \cdot t + (11.1)^{1/2}\right)^2 ,$$

and expanded in decimals we have

$$h(t) = (-15.6525 \cdot b \cdot t + 3.33166)^2 . \tag{6}$$

We have arrived at our model and now we seek to determine b and validate our model.

We turn to our Excel spreadsheet and seek to determine the parameter b which minimizes the sum of the squared errors between our data (h_i) and our model $(h(t_i))$ over our data points.

$$SSE(b) = \sum_{i=1}^{9} (h_i - h(t_i))^2 \ .$$

Minimize as a function of the parameter b:

$$SSE(b) = \sum_{i=1}^{9} (h_i - h(t_i))^2 \ .$$

where

- t_i is the i^{th} time observation,
- h_i is the observed height at time t_i,
- $h(t_i)$ is our model's prediction of the height at time t_i, and
- $n = 9$ is the number of data points we have.

Model Analysis in Excel Using Solver

Data collected Friday, 5 August 2016 by Brian Winkel

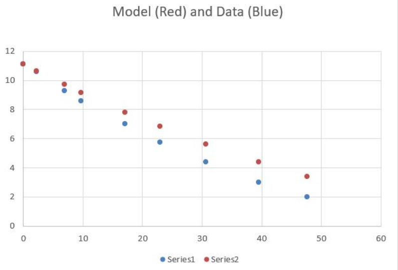
SOURCE for Data
https://www.youtube.com/watch?v=NBr4DOj4OTE
Radius of hole 11/64" = 0.218281 cm and radius of cylinder 4.17 cm

Model h'(t) = -b Sqrt(g h(t))

Model h(t) = (-b Sqrt(g)/2 + h(0)^(1/2))^2

b= 0.002

Time (s)	Zeroed Time	Actual Height (cm)	Model	SSE
8.679	0	11.1	11.09995836	1.73426E-09
10.866	2.187	10.6	10.64844791	0.0023472
15.612	6.933	9.3	9.700872913	0.160699092
18.396	9.717	8.6	9.165570453	0.319869938
25.781	17.102	7	7.819192408	0.671076202
31.647	22.968	5.75	6.825923093	1.157610501
39.282	30.603	4.4	5.634134022	1.523086785
48.182	39.503	3	4.389102871	1.929606788
56.342	47.663	2	3.384016994	1.915503039
			Total SEE	7.679799546



We can use Excel's Solver to minimize the TOTAL SEE or SSE which is currently 7.679799546 with parameter $b = 0.002$ by asking Solver to minimize SEE or SSE as a function of $b = 0.002$ cell.

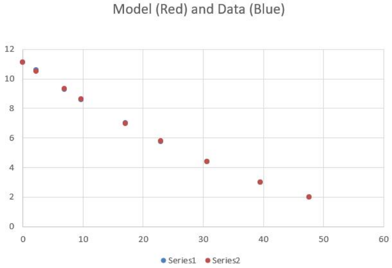
Parameter Estimation with Excel Solver - Results

Model h'(t) = -b Sqrt(g h(t))

Model h(t) = (-b Sqrt(g)/2 + h(0)^(1/2))^2

b= 0.002581

Time (s)	Zeroed Time	Actual Height (cm)	Model	SSE
8.679	0	11.1	11.09995836	1.73426E-09
10.866	2.187	10.6	10.51908638	0.006547014
15.612	6.933	9.3	9.312232219	0.000149627
18.396	9.717	8.6	8.638501405	0.001482358
25.781	17.102	7	6.973871293	0.000682709
31.647	22.968	5.75	5.778477118	0.000810946
39.282	30.603	4.4	4.390799244	8.46539E-05
48.182	39.503	3	3.013347567	0.000178158
56.342	47.663	2	1.977592125	0.000502113
			Total SEE	0.010437581



Note: TOTAL SEE was at 7.679799546.

Go live to Excel.

Assignment

1. Write an overview of the modeling process to obtain $h'(t)$ using first principles - not all details, just highlights. Arrive at the model

$$h(t) = \left(-\frac{b\sqrt{g}}{2} \cdot t + (h(0))^{1/2} \right)^2 .$$

2. Collect data for your team's cylinder from

   https://www.simiode.org/resources/488 .

   What if we took many data points? Few data points? Try both by taking a subset of ALL the data points you took for "few data" points and see how your parameter b fares.

In Excel we do these steps to determine best fit parameter b:

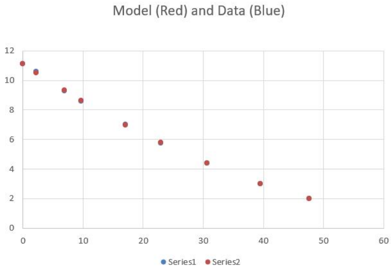
3. Compute our model value of the height $h(t_i)$ at time t_i.
4. Take the differences between actual data and model prediction, i.e. $h_i - h(t_i)$, and square these differences, $(h_i - h(t_i))^2$.
5. Sum these square errors to obtain $SSE(b)$.
6. Use Excel's Solver to minimize $SSE(b)$.
7. Read the value of b and put it in our model as best parameter estimate of b.
8. Plot our best model values on the same axes as our data and compare.
9. Collect the parameters b for the various outflow hole sizes from different videos selected by teams and see if there is any relationship between hole size and b.

Parameter Estimation with Excel Solver - Results

Model h'(t) = -b Sqrt(g h(t))

Model h(t) = (-b Sqrt(g)/2 + h(0)^(1/2))^2

				b=	0.002581
	Zeroed	**Actual**	**Model**	**SSE**	
Time (s)	**Time**	**Height (cm)**			
8.679	0	11.1	11.09995836	1.73426E-09	
10.866	2.187	10.6	10.51908638	0.006547014	
15.612	6.933	9.3	9.312232219	0.000149627	
18.396	9.717	8.6	8.638501405	0.001482358	
25.781	17.102	7	6.973871293	0.000682709	
31.647	22.968	5.75	5.778477118	0.000810946	
39.282	30.603	4.4	4.390799244	8.46539E-05	
48.182	39.503	3	3.013347567	0.000178158	
56.342	47.663	2	1.977592125	0.000502113	
			Total SEE	0.010437581	



CONGRATULATIONS!!!

We went from

- seeing and collecting data,
- to conjecturing empirical models,
- to building an analytical model from first principles,
- to realizing a differential equation model,
- to solving of the differential equation,
- to estimating our parameter,
- to comparing our model with the actual data.

Discussion, Questions, Ideas, please . . .

Special news and opportunities up next!

A word from one of our sponsors . . . .

https://www.simiode.org/scudem

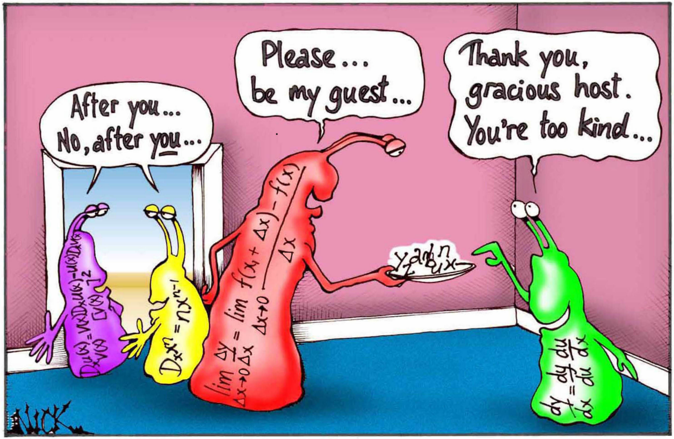
OR just search "SCUDEM" in Google.

Fall 2020 with Challenge Saturday due date 14 November 2020.

Features of SCUDEM V 2020 include

- teams of three high school, home school, individuals, or undergraduate students - undergraduate level or lower
- teams from same institution or assembled by SIMIODE from students around the world
- mentor/coach faculty engage with teams and fellow mentors/coaches
- mentoring period 1 September 2020 through 23 October 2020
- three problems from physics/engineering, chemistry/life sciences, and social sciences/humanities released on 23 October 2020
- students prepare 10 minute video presentation and upload to YouTube by Challenge Saturday, 14 November 2020
- faculty from around the world judge and give feedback
- Outstanding, Meritorious, and Successful awards
- SIMIODE posts ALL student team submissions and essay by problem poser on student submissions.

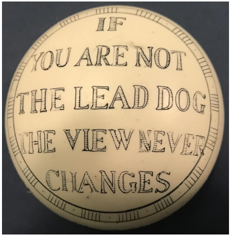
Discussions and Questions



Deferential equations.

Join us at SIMIODE and

take the lead for modeling differential equations.



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