

## STUDENT VERSION

### Modeling Car Suspensions

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**Abstract:** We examine the spring-mass-dashpot that is part of a car suspension, how the ride is related to parameter values, and the effect of changing the angle of installation. We model a “quarter car”, meaning a single wheel.

**Keywords:** spring mass dashpot, spring constant, underdamping, design

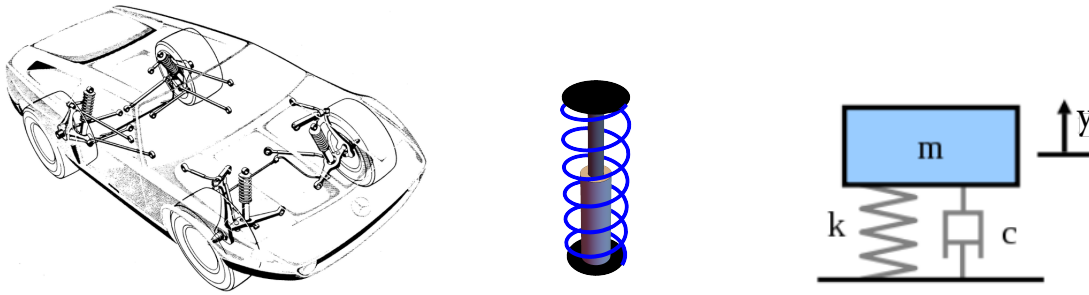
**Tags:** second order, linear, constant coefficient, homogeneous, ordinary differential equation, static equilibrium, automotive, car suspension, suspension tolerance, Mathematica, angle, engineering, directed, long, develop model

#### STATEMENT

Some of us may be familiar with car suspensions, others of us clueless, and many of us in between. We can use the interplay between math/physics and cars to improve understanding of all by examining different combinations of suspension components. A car suspension acts as a spring-mass-dashpot system on each wheel. This allows the car to stay on the road and enable good handling of the car under normal driving conditions. We examine a “quarter car” model that involves a second order differential equation for one wheel. Students might refer to other sources about solutions to a second order, linear, homogeneous differential equation with constant coefficients.

#### GENERAL MODEL: Relating Car Suspensions and Spring-Mass-Dashpot

A car suspension is reasonably complicated, with multiple parts and various types, and has evolved over at least five hundred years of human transportation when the passenger part of a horse-drawn carriage was “slung from leather straps attached to four posts of a chassis that looked like an upturned table. Because the carriage body was suspended from the chassis, the system came to be known as a ‘suspension’ - a term still used today.” [2] Today, a car suspension includes coil springs with shock absorbers, and more, as in Figure 1(a). (For instance, consider [1], [2], [3]).



**Figure 1.** (a) Suspension (b) Spring Assembly (c) Diagram

A stylized version of an uninstalled spring assembly is shown in Figure 1(b). A simplified version of the spring and shock absorber that directly relates to our differential equation is shown in Figure 1(c).

We model the vertical displacement  $y(t)$  from its equilibrium position, meaning when the vehicle is at rest. The differential equation, Equation 1, for the displacement in a standard spring-mass-dashpot situation shows a sum of the three forces in Figure 1(c). Our righthand side is zero for no external forces, such as for a car on a smooth road.

$$M \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = 0, \quad y(0) = y_0, \quad y'(0) = 0. \quad (1)$$

Here are the parameters of our system:

- $M$  is the mass on one wheel. Let us consider one quarter of the mass of a car.
- $k$  is the spring constant. A spring restores motion proportional to the amount of displacement. Here is a statement of Hooke's Law in a car context: the more a car spring is stretched or compressed, the more it acts to return to an equilibrium position of rest.
- $c$  is the damping coefficient for the reduction in movement by the shock absorber. Here is a statement of the velocity relationship of damping in a car context: a shock absorber resists motion more when the suspension moves faster. "All modern shock absorbers are velocity-sensitive - the faster the suspension moves, the more resistance the shock absorber provides. This enables shocks to adjust to road conditions and to control all of the unwanted motions that can occur in a moving vehicle, including bounce, sway, brake dive and acceleration squat." [2]
- $y(0) = y_0, \quad y'(0) = 0$ . We begin our clock,  $t_0 = 0$ , when the car is at a small displacement,  $y_0$ , with initial velocity zero. That could be from a person hopping onto the hood of a car and compressing the spring for a negative initial displacement, or at some instant in a lab test.

Throughout, we could consider differences between front and rear suspensions. For the examples in [8], the rear axle bears between 48% and 53% of the total car mass. Some automotive sites, such as [5] and [4] have specific information about front and rear car springs. Front and rear dampers are studied in [7]. Students might enjoy a scavenger hunt to gather information on specific vehicles, and some might share their expert knowledge.

**Analyze General Model**

1. State the characteristic equation for Equation 1 and its roots.
2. Why would the suspension system with a shock absorber give a more comfortable ride? State the differential equation if we had no shock absorber.
3. Relate the mass, damping coefficient, and spring constant to a critically damped situation. Recall that “critical damping” means the combination of equation parameters that first results in no oscillations.

**SPECIFIC SITUATIONS**

Let our time variable,  $t$ , be measured in seconds ( $s$ ), and the displacement,  $y$  in meters ( $m$ ), with mass  $M$  in kilograms ( $kg$ ), so that each of the three forces that are summed in Equation 1 is measured in Newtons. (Recall that  $1 \text{ Newton} = 1 \text{ kg} \cdot m/s^2$ .) A compact cross-over sport utility vehicle might be  $3,400 \text{ lbs}$ , with a mass on one wheel of  $386 \text{ kg}$ ; a basic full-size pick-up truck might be  $5,000 \text{ lbs}$ , with a mass on one wheel of  $568 \text{ kg}$ . We measure  $k$  in Newtons per meter ( $N/m$ ), with reasonable values between  $16,000$  and  $80,000$  Newtons per meter ( $N/m$ ). We measure  $c$  in  $N \cdot s/m$ , with reasonable values between  $1,000$  and  $21,000$ .

Spring constant values varied a lot in our searches: between  $16,000$  and  $28,000$  Newtons per meter ( $N/m$ ) on [4]; over  $57,000 \text{ N/m}$  in the example in [6];  $64,000$  and  $73,000 \text{ N/m}$  on [5]; between  $16,000$  and  $73,000 \text{ N/m}$  in some portions of [8]. As with most US sites, this required conversions, such as  $1 \text{ lb/in} = 175.13 \text{ N/m}$ .) Values given in [7] for front dampers range between  $1,600$  and  $21,000$ ; for rear dampers, between  $870$  and  $5,100$ .

**Determine Damping**

We use the following values: the mass on one wheel  $390 \text{ kg}$ , the spring constant  $44,000 \text{ N/m}$ , and the initial position is  $0.2 \text{ m}$  above the static equilibrium.

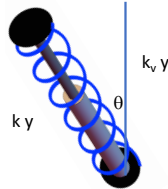
4. Relate these values to the constants and variables in Equation 1.
5. Determine the values of  $c$  for a critically damped system. Graph the resulting displacement.
6. How long does it take for the critically damped system to stay within a tolerance of  $0.1 \text{ m}$  from the static equilibrium? Illustrate with a graph.

**Changing the Ride Quality**

We begin with the following values: the mass on one wheel  $450 \text{ kg}$ , the spring constant  $16,000 \text{ N/m}$ , the damping coefficient  $1600 \text{ N} \cdot s/m$ , and the initial position is  $0.2 \text{ m}$  above the static equilibrium.

7. (a) Graph the resulting displacement.
  - (b) Is this a smooth or bumpy ride? Is this system underdamped, critically damped, or overdamped?
  - (c) How long does it take for this system to stay within a tolerance of  $0.1 \text{ m}$  from the static equilibrium? Illustrate with a graph.

8. How does the ride change in this situation if the total mass of the car is increased by 1,000 lbs? Show a graph for both rides, and give a brief description.
9. Consider installing the spring at an angle of  $\theta$  from vertical by rotating the top of the spring, as in in Figure 2, up to  $\pi/4$ .



**Figure 2.** Angle of Installation

- (a) Determine the spring rate correction factor in general, meaning the ratio of the spring constant to equivalent vertical portion.
- (b) Determine the new spring constant  $k$  that we would need to compensate for the angles of installation of  $\pi/16, \pi/8, \pi/4$ , if we wish an effective spring constant of  $k_v = 16,000 N/m$ .
- (c) Compare the rides without additional mass using a spring installed at angles of  $20^\circ$  and  $40^\circ$ . Show a graph for both rides, and give a brief description.

### Conclusion

10. Write a short description of how differential equations can model a simple car suspension system.

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