CHAPTER 1: MODELING AND SYSTEMS ANALYSIS

1 Overview

The fundamental step in performing systems analysis and control design in energy systems is *mathematical modeling*. That is, we seek to write the ordinary differential equations (ODEs) that describe the physics of the particular energy system of interest. This process is highly non-trivial, and requires a careful combination of first principles (e.g. physics, chemistry, thermodynamics), experience, and creativity. Once these equations are properly derived, we then formulate the system dynamics into a so-called "state-space" form. The state-space form is the canonical template for analysis and control. State-space models can be divided into linear and nonlinear systems. We next focus on linear systems, and how they can be derived from nonlinear systems. The next and final fundamental concept is "stability". Stability, in rough terms, means the energy system does not "blow up" in some sense. In summary, this chapter is organized as follows:

1. Mathematical modeling of dynamic systems
2. State-space representations
3. Linear Systems
4. Stability

2 Mathematical Modeling of Dynamic Systems

Energy systems convert and store energy from a variety of physical domains, such as mechanical (e.g. flywheel), electrical (e.g. ultracapacitor), hydraulic (e.g. accumulator), chemical (e.g. gasoline), thermal (e.g. ice storage), economic (e.g. bank account) and more. As such, engineers and scientists require a common framework for describing and analyzing energy systems. This common framework is mathematics, and we refer to our description of the dynamic energy systems as a *mathematical model*. One must understand that a mathematical model is, at best, a surrogate for the physical system, whose precision is subject to the assumptions and requirements made by the energy systems engineer. To quote eminent statistician Dr. George E. P. Box (1919 – 2013): *"Essentially, all models are wrong, but some are useful."*

2.1 First Principles Modeling

The process of translating an unstructured technical (or non-technical) energy system into a precise and clearly defined mathematical model is far from trivial. In many cases, a well-established set of models already exist. In other cases, a model must be generated for our purpose. Therefore, we desire a methodology for model generation.

Unfortunately, such a methodology cannot be "algorithmic" in the sense that it would offer a recipe that, when exactly followed, is guaranteed to produce the best possible model. The process of abstraction from a complex energy system to a mathematical model is simply not amenable to such a high degree of formalization. It is, in many instances, an "art form" that requires experience and intuition. Nevertheless, this section will provide some useful concepts for model generation.

The focus below will be on "control-oriented models," i.e., models which capture a system's main static and dynamic phenomena without creating an excessive computational burden. The main reason for this requirement is that the models are assumed to be used in real-time loops (for instance, in feedback control systems) or repeatedly in numerical computations (for instance, when optimizing the system parameters or feedforward control signals).

A contrasting modeling paradigm is often pursued by scientists. That is, a scientist often studies the natural or man-made world to theorize mathematical relations that explain her observations. The more accurate and detailed the model, the better. In this course, we follow Albert Einstein's advice: *"Everything Should Be Made as Simple as Possible, But Not Simpler."* That is, the mathematical model should be sufficiently detailed to suit our specific energy management goals.

Energy system model synthesis is based on physical first principles, for instance, the first and second laws of thermodynamics, the Lagrange equations in mechanics, or the Navier-Stokes law in fluid mechanics. Compared to data-driven methods (for instance, correlation methods), this approach has at least four major benefits:

- The models obtained are *explanatory*. In contrast to data-driven models, which seek to match input-output predictions to data, first-principle models explain the internal dynamics. This provides enhanced engineering insight into the energy system.
- The models obtained in principle are able to *extrapolate* the system behavior, i.e., they can be useful beyond the operating conditions used in model validation.
- The models can be formulated even if the real system is not available (system still in planning phase or too dangerous / expensive to be used for experiments).
- Once such a mathematical model exists for a first system, the adaptation of that model to minor system modifications is relatively easy. Subsequent controller designs, which are based on the system model, can then be carried out (almost) automatically. This time-saving approach is critical for real-world application.

Figure 1 provides a block-diagram schematic of a generic dynamic system model. The system, denoted by Σ, is characterized by a set of state variables x. The state variables are influenced by the input variables u, that represent the (controlled or uncontrolled) action of the system's environment on the system. The output variables y represent the observable or measurable aspects of the system's response.

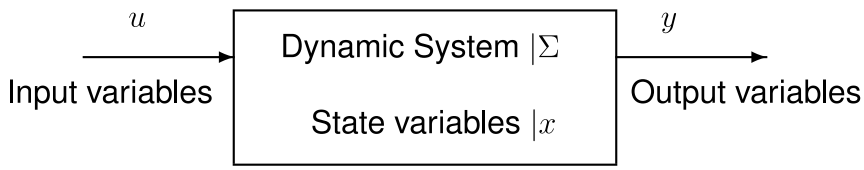

Figure 1: General dynamic system model.

In this chapter, we shall refine the definition of this dynamical model further. This definition, however, is sufficient to outline five potential uses of a mathematical model:

1. *Analysis.* Given a future trajectory of u, x at the present, and the system model Σ, predict the future of y. This use case is most commonly called "simulation". CH1 focuses on this topic.

2. *Model Identification.* Given time histories u and y, usually obtained from experimental data, determine a model Σ and its parameter values that are consistent with u and y. Clearly, a "good" model is one that is consistent with a variety of data sets u and y. This is often called "system identification" in the control literature or simply "modeling" in machine learning. CH2 focuses on this topic.

3. *State Estimation.* Given a system Σ with time histories u and y, find x that is consistent with Σ, u, y. This is the monitoring problem. That is, you cannot measure every state, yet you wish to monitor every state. You wish to synthesize an algorithm that fuses the model and measurable states to produce the unmeasurable states. CH3 focuses on this topic.

4. *System Design.* Given u and some desired y, find Σ such that u acting on Σ will produce y. Most engineering disciplines deal with design synthesis. Traditionally, one might create various physical prototypes to synthesize a desired system. This can, however, be very time-consuming and expensive. A mathematical model helps automate this process by virtually synthesizing designs. CH4 focuses on this topic.

5. *Control Synthesis.* Given a system Σ with current state x and some desired y, find u such that Σ will produce y. This is, put simply, the energy management problem. The idea is to construct a series of decisions that produce a desirable flow of energy through the system. CH5 focuses on this topic.

2.2 Reservoir-Based Approach

When modeling any physical system there are two main classes of objects that have to be taken into account:

- "reservoirs," for instance, thermal energy, kinetic energy, or information;
- "flows," for instance, heat, mass, etc. transferring between the reservoirs.

The notion of a reservoir is fundamental in system modeling, and only systems including one or more reservoirs exhibit dynamic behavior. To each reservoir there is an associated "level" variable that is a function of the reservoir's content (the term "state variable" is used for this quantity in the systems and control literature). The flows are typically driven by the differences in the reservoir levels. Several examples will be given below.

Armed with the reservoir concept, we present general guidelines to formulate a control-oriented model. This encompasses (at least) the following seven steps:

1. precisely define the modeling objective;
2. define the system boundaries (what inputs, what outputs, . . .);
3. identify the relevant reservoirs (of mass, energy, information,...) and the corresponding level (i.e. state) variables;
4. formulate the differential equations (conservation laws) for all relevant reservoirs
$$\frac{d}{dt}(\text{reservoir level}) = \sum \text{inflows} - \sum \text{outflows}; \tag{1}$$
5. formulate the (usually nonlinear) algebraic relations that express the flows between the reservoirs as functions of the level variable;
6. identify the unknown system parameters from experimental data; and
7. validate the model against experimental data that was not used for model identification.

Remark 2.1. *The most common mistake in modeling is skipping Step 1 above. Without a precise objective, one has no criteria in which to evaluate which modeling features are necessary and which modeling features are superfluous. TIP: Write down your objective. Make it precise.*

Remark 2.2. *The second most common mistake is poorly executing Step 2 above, in the following sense. Within the system boundaries, you seek to formulate mathematical equations that describe the system's evolution. Outside the system boundaries is the environment. You do not seek to formulate equations that describe the environment. Instead, you seek to understand the environment's impact on the system. TIP: Ask yourself this question: What is within my system boundaries? What do I consider the environment?*

Example 2.1 (Room Thermal Dynamics in Buildings)**.** *Suppose you are an HVAC engineer and you wish to consider our lecture room as a dynamic thermal system (see Fig. 2). The goal is to regulate temperature in the room within a comfortable range, while minimizing electrical costs, subject to time-varying changes in ambient temperature. Perform Steps 1-5 of the modeling formulation procedure. How would the model be expanded to consider multiple lecture halls? What role does a thermostatically controlled HVAC unit play?*

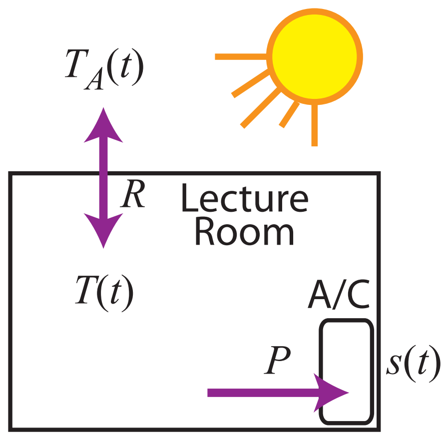

Figure 2: Schematic of Room Thermal Dynamics in Buildings.

Sample solution:

1. **Modeling Objective:** Our objective is to formulate a mathematical model that predicts how room temperature evolves, given interactions with the outside environment and an air conditioning (A/C) unit. This model will be used to design a controller that intelligently regulates room temperature to comfortable levels, while minimizing electricity costs.

2. **System Boundaries:** The system outputs include room temperature $T(t)$ and thermal power removed by the A/C unit $P(t)$. The system inputs include ambient environmental temperature $T_A(t)$, and the A/C mode $s(t) \in \{0, 1\}$, corresponding to cooling or off.

3. **Reservoir(s):** The reservoir is thermal energy stored in the room. The level or state variable, indicating the amount of stored thermal energy, is $T(t)$.

4. **Conservation Law(s):** The room temperature $T(t)$ changes due to heat transfer with the outside environment and with the A/C unit. The First Law of Thermodynamics gives us:

$$C\frac{d}{dt}T(t) = \frac{1}{R}\left[T_A(t) - T(t)\right] - s(t)P \tag{2}$$

where parameter C is the room's thermal capacitance (kWh/°C) and R is the thermal resistance between the room and outside (°C/kW). Consider the units. Note the left-hand side gives the time derivative of thermal energy inside the room. The right-hand side gives thermal power flowing in/out of the room.

5. **Relation between Flows and Level Variable:** These relations have already been expressed within the first and second terms on the right-hand side of (2). We can additionally compute the *electrical* power consumed by the A/C unit as:

$$P_{elec}(t) = s(t)P/\eta \tag{3}$$

where η represents the coefficient of performance (i.e. efficiency of electric-to-thermal power conversion).

- **Multiple Rooms:** We would consider heat transfer between the rooms and between the rooms and environment.
- **Thermostatically Controlled HVAC:** A control law is an algorithm that actuates the HVAC unit through control signal $s(t)$, using measurements of room temperature $T(t)$. A typical thermostatic control law (assuming A/C only) is given mathematically by:

$$s(t) = \begin{cases} 0 & \text{if } T(t) \leq T_{sp} - \frac{\Delta}{2} \\ 1 & \text{if } T(t) \geq T_{sp} + \frac{\Delta}{2} \\ s(t-1) & \text{otherwise} \end{cases} \tag{4}$$

where T_{sp} is the set-point temperature and Δ is a deadband around the set-point.

Example 2.2 (Vehicle Dynamics)**.** *Suppose you are a transportation engineer, and you wish to analyze the energy consumption dynamics of a moving vehicle. Consider a vehicle (Fig. 3) of mass m, receiving traction force $F(t)$ from the engine, and a resistive force $k_0 + k_1 v^2(t)$ that includes rolling friction forces and viscous aerodynamic drag. Perform Steps 1-5 of the modeling formulation procedure.*

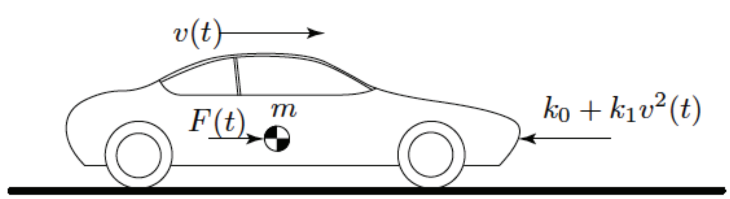

Figure 3: Free-body diagram of a moving vehicle (focused on longitudinal dynamics).

Sample solution:

1. **Modeling Objective:** Our objective is to predict energy consumption in a vehicle, given its velocity trajectory $v(t)$.

2. **System Boundaries:** The system outputs include traction force supplied by the engine $F(t)$, which we can map to fuel consumption given the appropriate relation. The system inputs include its velocity trajectory $v(t)$.

3. **Reservoir(s):** The reservoir is the kinetic energy of the vehicle. The level or state variable, indicating the amount of stored kinetic energy, is $v(t)$.

4. **Conservation Law(s):** The velocity $v(t)$ changes due to forces from the engine, rolling friction, and air drag. Newton's second law gives us:

$$m\frac{d}{dt}v(t) = F(t) - \left[k_0 + k_1 v^2(t)\right] \tag{5}$$

5. **Relation between Flows and Level Variable:** These relations have already been expressed on the right-hand side of (5).

2.3 System Theoretic Framework

After the ODEs have been written, usually based on first principles, we now think carefully about the unique roles of each variable or element, as detailed by the symbols in Table 4. All symbols may be scalar or interpreted as a column vector.

Figure 4: Mathematical Model Elements

Symbol	Description
x	State variable
u	Controllable inputs
w	Uncontrollable inputs
y	Process outputs
θ	Parameters

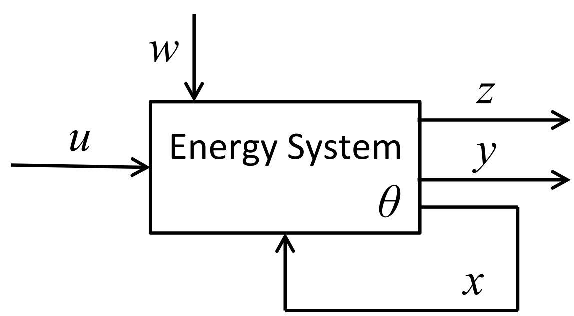

Figure 5: Block diagram of dynamic energy system.

The relationships between these variables are illustrated in Fig. 5. These five categories of variables serve distinct purposes in the dynamic energy system.

- The *state* x represents the dynamic condition of the energy system. Examples include battery state-of-charge, flywheel speed, power system frequency, room temperature, or financial account balance.

- The *controllable inputs* u represent a series of decisions we make to manage the energy system in a desirable way. These might include driver throttle position, thermostat set points, or wind turbine blade angles.

- The *uncontrollable inputs* w represent a series of signals entering the energy system, which is not under our control. Examples include wind speed, solar irradiation, and traffic flow.
- The *measured outputs* y represent physical quantities that we measure with sensors. These might include voltage, vehicle speed, or classroom humidity.
- The *performance outputs* z represent quantities that we monitor, but do not measure directly with sensors. These might include fuel costs, time, or resources consumed.
- The *parameters* θ are a vector of scalar quantities which encapsulate physical properties of the system that usually do not evolve with time. Examples include vehicle mass, thermal resistance of walls, or inductances of power transmission lines. These arise naturally from the physics of the system.

Example 2.3 (Flywheel Energy Storage)**.** Consider the flywheel shown in Fig. 6. The basic element of a flywheel is a rotating inertial mass connected to an electrical machine. Depending on which direction the machine applies torque, it converts energy between the electric and mechanical (kinetic energy) domains. Let us denote the flywheel inertia as I, the angular velocity as $\omega(t)$, a coefficient of friction as b, and the electric machine torque as $T(t)$. Then the equation of motion is governed by Euler's rotation equation:

$$I\dot{\omega}(t) = -b\omega(t) + T(t), \tag{6}$$

where $\dot{(\cdot)}$ denotes the derivative with respect to time. Note that $\omega(t)$ plays the role of state, $T(t)$ plays the role of controllable input, and the pair (I, b) play the role of parameters. If, in addition, we measure angular velocity then $\omega(t)$ is also a measurable output: $y(t) = \omega(t)$.

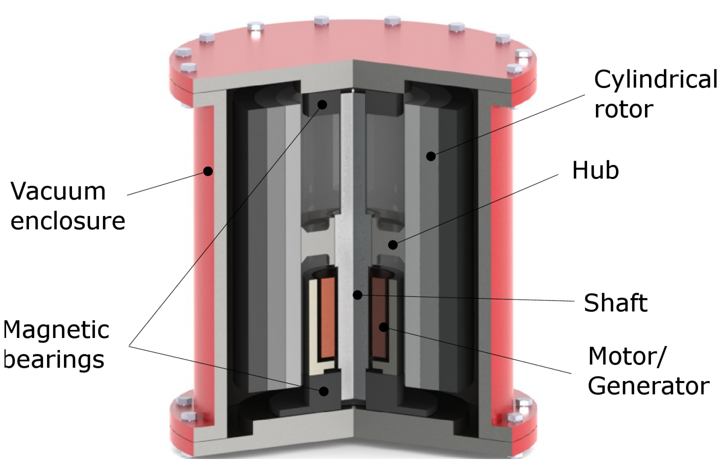

Figure 6: Main components of a typical flywheel.

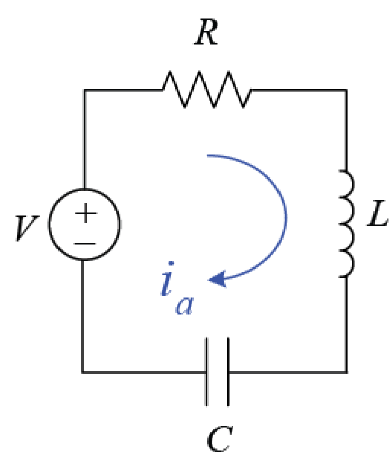

Figure 7: RLC circuit

Example 2.4 (RLC Circuit)**.** We will now consider a simple series combination of three passive electrical elements: a resistor R, an inductor L, and a capacitor C, in series with a controllable voltage source V. Suppose we measure voltage across the resistor. This electrical system is known as an RLC circuit.

Since this circuit is a single loop, application of Kirchoff's current law (KCL) shows that the current is the same throughout the circuit at any given time, $i(t)$. Applying Kirchoff's voltage law (KVL) around the loop and using the sign conventions indicated in the diagram, we arrive at the following governing equation.

$$V(t) - L\frac{di}{dt}(t) - Ri(t) - \frac{1}{C}\int_0^t i(\tau)d\tau = 0. \tag{7}$$

Suppose we define $q(t)$ to be the capacitor charge, such that $q(t) = \int_0^t i(\tau)d\tau$. Then (7) can be written as,

$$V(t) - L\ddot{q}(t) - R\dot{q}(t) - \frac{1}{C}q(t) = 0. \tag{8}$$

Note that (8) contains a double derivative in time. We call this a "second-order" system. In contrast, (6) from the flywheel energy storage example is a "first-order" system. In the present example, the pair $(q, \dot{q})$ is the state, V plays the role of controllable input, and triple (R, L, C) play the role of parameters. The measurable output is voltage across the resistor, given mathematically as:

$$y(t) = Ri(t) = R\dot{q}(t). \tag{9}$$

Next we discuss the canonical format for dynamical energy systems of any finite order, called the "state space representation".

3 State-Space Representation

The state-space representation is a convenient and compact way to write the dynamics of energy systems. Put simply, it consists of a finite number of coupled first-order ordinary differential equations (ODEs)

$$\begin{aligned}
\dot{x}_1 &= f_1\left(t, x_1, ..., x_n, u_1, ..., u_p\right) \\
\dot{x}_2 &= f_2\left(t, x_1, ..., x_n, u_1, ..., u_p\right) \\
\vdots \quad & \quad \vdots \\
\dot{x}_n &= f_n\left(t, x_1, ..., x_n, u_1, ..., u_p\right)
\end{aligned} \tag{10}$$

where n represents the number of states, and p represents the number of controllable inputs. The uncontrollable input, w, can be added as well, but we suppress this for simplicity at this moment.

We often use vector notation to write these equations in a compact form. Define

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}, \quad f(t,x,u) = \begin{bmatrix} f_1(t,x,u) \\ f_2(t,x,u) \\ \vdots \\ f_n(t,x,u) \end{bmatrix} \tag{11}$$

Then we can re-write n first-order differential equations as a single $n-$dimensional first-order vector differential equation

$$\dot{x} = f(t,x,u) \tag{12}$$

When q measurements are present, then we include another $q-$dimensional equation

$$y = h(t,x,u) \tag{13}$$

where y represents the q measured process outputs. In summary, we refer to (12) as the state equation and (13) as the output equation. When the dynamical system does not have any control input, i.e. it runs *autonomously*, then we refer to it as an *autonomous system*. Mathematically, this is written,

$$\dot{x} = f(t,x) \tag{14}$$

An important concept in dynamical systems is *equilibria*. Roughly, the equilibrium of a dynamical system is the point in the state-space where the states remain constant, i.e. steady-state. It is defined mathematically as follows:

Definition 3.1. *(Equilibrium) Consider an autonomous dynamical system $\dot{x} = f(t,x)$. The state x^{eq} is called the equilibrium when it satisfies the following equation,*

$$0 = f(t, x^{eq}) \tag{15}$$

For non-autonomous systems, i.e. $\dot{x} = f(t,x,u)$, consider a fixed input u^{eq}. The state x^{eq} is called the equilibrium with respect to (w.r.t.) input u^{eq} when it satisfies the following equation,

$$0 = f(t, x^{eq}, u^{eq}) \tag{16}$$

Remark 3.1. Energy systems do not always present themselves directly as a mathematical model in state space form with their equilibria explicitly defined. More often, we arrive at a collection of ODEs and algebraic equations from first principles. However, we can usually carefully select the state variables to form a state-space representation and compute equilibria, as shown by the following example.

Example 3.1 (RLC Circuit Revisited)**.** Recall the RLC circuit studied in Example 2.4. The dynamics are described by the following second-order differential equation,

$$V(t) - L\ddot{q}(t) - R\dot{q}(t) - \frac{1}{C}q(t) = 0. \tag{17}$$

where we defined $\dot{q}(t) = i(t)$. Consequently, we can re-write these second-order dynamics in state-space form

$$\begin{aligned} \dot{q}(t) &= i(t) & (18)\\ \dot{i}(t) &= -\frac{1}{LC}q(t) - \frac{R}{L}i(t) + \frac{1}{L}V(t) & (19) \end{aligned}$$

Note that the state vector is $x(t) = [q(t), i(t)]^T$ and the control input is $u(t) = V(t)$. Consequently, we can re-write the above equations in vector form

$$\frac{d}{dt}\begin{bmatrix} q(t) \\ i(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} q(t) \\ i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} V(t) \tag{20}$$

or

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{21}$$

with matrices $A \in \mathbb{R}^{2\times 2}$ and $B \in \mathbb{R}^{2\times 1}$ appropriately defined. Now consider a fixed voltage input $V(t) = V^{eq}$. Then the equilibrium x^{eq} by definition must satisfy

$$0 = Ax^{eq} + BV^{eq} \tag{22}$$

Assuming A is invertible, $x^{eq} = -A^{-1}BV^{eq}$.

Remark 3.2. Equation (21) is an extremely important and special case of (12). It is called a "linear system." The significance of (21) is that, when energy systems can be described by this format, there exists a very large set of tools for system analysis and control design. When (21) cannot appropriately capture the energy system dynamics, then the set of available tools decreases dramatically. In the next section we formally introduce linear systems. Before discussing linear systems, let us explore another example of equilibria for dynamic systems.

Example 3.2 (Pendulum, Section 1.2.1 of [1]). Consider the simple pendulum shown in Fig. 8, where L denotes the length of the rigid massless rod, m denotes the mass of the bob, and θ represents the rod's angle with the vertical axis. The pendulum is free to swing within a vertical two-dimensional plane. There are three forces acting on the bob. First, gravity acts in the downward vertical direction. Second, a tension force acts along the rod, toward the frictionless pivot. Third, a friction force resists the bob's motion through the medium, which we assume to be proportional to the bob's speed with coefficient of fiction k. It is straight-forward to derive the equation of motion by (i) drawing a Cartesian coordinate system centered on the bob, (ii) rotating the axes to be

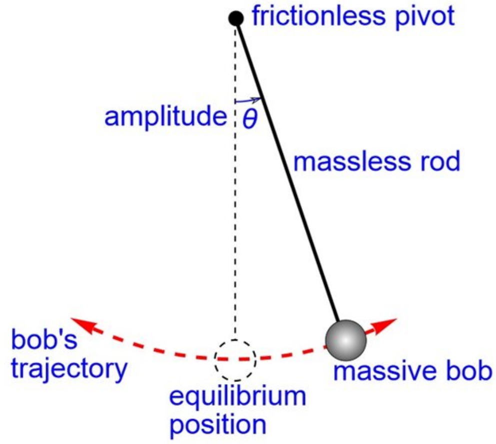

Figure 8: Pendulum.

parallel and perpendicular to the rod, (iii) drawing a free-body diagram with forces aligned with the coordinate axis, (iv) applying Newton's second law:

$$mL\ddot{\theta}(t) = -mg\sin\theta(t) - kL\dot{\theta}(t) \tag{23}$$

This is a second-order system, similar to Example 3.1. Defining the state vector as $x(t) = [x_1(t), x_2(t)]^T = [\theta(t), \dot{\theta}(t)]^T$, we can re-write the second-order dynamics into a state-space from

$$\dot{x}_1(t) = x_2(t), \tag{24}$$
$$\dot{x}_2(t) = -\frac{g}{L}\sin x_1(t) - \frac{k}{m}x_2(t). \tag{25}$$

Unlike Example 3.1, this system is nonlinear due to the sinusoidal term in (25). To find the equilibrium points, we set $\dot{x}_1, \dot{x}_2 = 0$ and solve for x_1, x_2:

$$0 = x_2^{eq}, \tag{26}$$
$$0 = -\frac{g}{L}\sin x_1^{eq} - \frac{k}{m}x_2^{eq}. \tag{27}$$

The equilibrium points are located at

$$x_1^{eq} = n\pi, \quad \text{for } n = 0, \pm 1, \pm 2, \cdots, \qquad x_2^{eq} = 0. \tag{28}$$

From the physical description, the pendulum has only two unique equilibrium points $(0, 0)$ and $(\pi, 0)$, which correspond to the downward and upward positions, respectively, with zero velocity. All other points are repetitions of these positions modulo an integer number of full swings from the reference position. Nevertheless there are, mathematically, an infinite number of equilibrium points. This example hints at the general fact that a nonlinear system may exhibit, one, multiple, or no equilibrium points.

Physical intuition suggests that equilibrium points $(0,0)$ and $(\pi,0)$ are quite distinct from each other. Namely, the pendulum can indeed rest at the $(0,0)$ equilibrium position. However, it can hardly remain at the $(\pi,0)$ position since any infinitesimally small disturbance will take the pendulum away from this position. The difference between these two equilibrium points is in their stability properties, a topic of enormous importance in practice and theory, which we study in Section 5.

4 Linear Systems

Linear systems are the foundation of systems and control engineering. In this section, we precisely define a linear system.

A *linear time-invariant (LTI)* system is described by the following state-space model:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{29}$$

$$y(t) = Cx(t) + Du(t) \tag{30}$$

where the state, control, and output have dimensions $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, and $y \in \mathbb{R}^q$, respectively. The matrices A, B, C, D are $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times p}$, $\mathbb{R}^{q\times n}$, and $\mathbb{R}^{q\times p}$, respectively. The system is said to be "time-invariant" when the matrices A, B, C, D are constant in time, e.g. Example 3.1

4.1 Linearization

No energy system is exactly linear and time-invariant. They are, in general, nonlinear and can be described by the nonlinear ODEs (12)-(13). Nonlinear system analysis is beyond the scope of this chapter (see [1]). However, linear approximations are often suitable for analysis and control design. Moreover, a wealth of tools exist for linear systems. Next, we describe how to approximate a nonlinear system by an LTI system. Recall Taylor's theorem:

Theorem 4.1. *(Taylor's Theorem) Let $k \geq 1$ be an integer. Let f be a nonlinear function that maps scalars to scalars, i.e. $f : \mathbb{R} \to \mathbb{R}$, and is k times differentiable at the value $x = a$. Consider the power series expansion of $f(x)$ around $x = a$,*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(k)}(a)}{k!}(x-a)^k + R_{k+1}(x) \tag{31}$$

where $R_{k+1}(x)$ is called the remainder term. Then infinite series converges to $f(x)$,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \tag{32}$$

if and only if $R_{k+1}(x) \to 0$ *as* $k \to \infty$.

In our case, the utility of Taylor's theorem is a systematic method to approximate nonlinear systems by an LTI system. Consider the nonlinear ODE

$$\dot{x}(t) = f(x(t), u(t)), \tag{33}$$

and the equilibrium x^{eq} corresponding to u^{eq}. Then, we can define the *perturbation* from the equilibrium as $\tilde{x}(t) = x(t) - x^{eq}$ and $\tilde{u}(t) = u(t) - u^{eq}$. Graphically, we have simply translated the coordinate system so the origin is at (x^{eq}, u^{eq}). The ODE for $\tilde{x}(t)$ is then

$$\dot{\tilde{x}}(t) = \dot{x}(t) - \dot{x}^{eq} = f\left(\tilde{x}(t) + x^{eq}, \tilde{u}(t) + u^{eq}\right) - 0 \tag{34}$$

Now we apply Taylor series expansion, where the role of x in (31) is played by the pair $(\tilde{x}(t) + x^{eq}, \tilde{u}(t) + u^{eq})$ and the role of a in (31) is played by the pair (x^{eq}, u^{eq}),

$$\begin{aligned} \dot{\tilde{x}} &= f(x^{eq}, u^{eq}) + \frac{\partial f}{\partial x}(x^{eq}, u^{eq})\left(\tilde{x}(t) + x^{eq} - x^{eq}\right) + \frac{\partial f}{\partial u}(x^{eq}, u^{eq})\left(\tilde{u}(t) + u^{eq} - u^{eq}\right) && (35) \\ &\quad + R_2(\tilde{x}(t) + x^{eq}, \tilde{u}(t) + u^{eq}) && (36) \end{aligned}$$

Note that the first term satisfies the definition of an equilibrium and is therefore zero. Truncating the Taylor series expansion to remove the second order and higher remainder terms results in

$$\dot{\tilde{x}}(t) = \frac{\partial f}{\partial x}(x^{eq}, u^{eq})\tilde{x}(t) + \frac{\partial f}{\partial u}(x^{eq}, u^{eq})\tilde{u}(t) \tag{37}$$

or

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t) \tag{38}$$

where $A = \frac{\partial f}{\partial x}(x^{eq}, u^{eq})$ and $B = \frac{\partial f}{\partial u}(x^{eq}, u^{eq})$. Matrices A and B are called the *Jacobians*.

To summarize, equation (38) represents the linearized dynamics of the nonlinear dynamic system (33), around equilibrium point (x^{eq}, u^{eq}). Very often, the linearized dynamics are sufficient to study energy systems, particularly around some desired operating point.

Example 4.1 (Magnetic Levitation, Example 4-11 of [2]). Figure 9 shows the diagram of a magnetic-ball suspension system. The objective of the system is to control the position of the steel ball by adjusting the current in the electromagnet through the input voltage $e(t)$. The differential equations for the system are given by

$$\begin{aligned} M\frac{d^2y(t)}{dt^2} &= Mg - \frac{i^2(t)}{y(t)}, && (39) \\ e(t) &= Ri(t) + L\frac{di(t)}{dt}. && (40) \end{aligned}$$

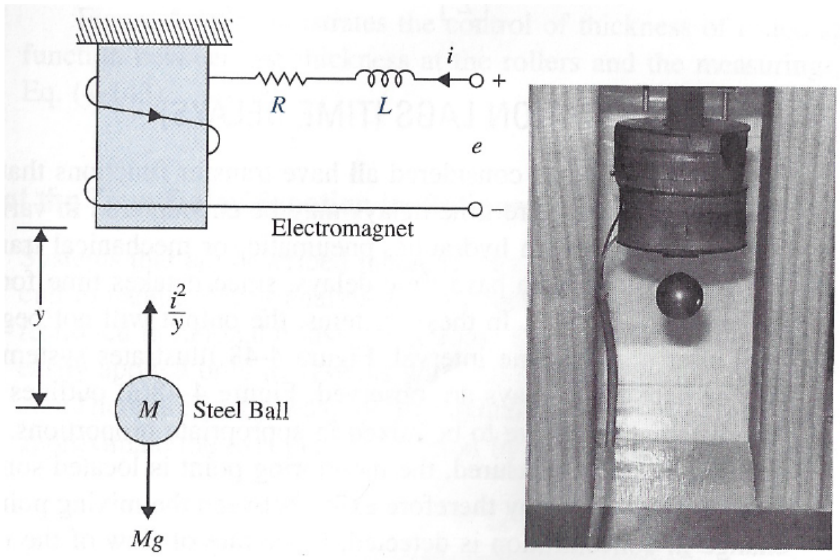

Figure 9: Magnetic ball suspension system.

where $e(t)$ is the input voltage, $y(t)$ is the ball position, $i(t)$ is the winding current, M is the ball's mass, g is gravitational acceleration, R is the winding resistance, and L is the winding inductance. Let us define the state variables as $x_1(t) = y(t)$, $x_2(t) = dy(t)/dt$, and $x_3(t) = i(t)$. The state equations of the system are

$$\frac{dx_1(t)}{dt} = x_2(t), \tag{41}$$
$$\frac{dx_2(t)}{dt} = g - \frac{1}{M}\frac{x_3^2(t)}{x_1(t)}, \tag{42}$$
$$\frac{dx_3(t)}{dt} = -\frac{R}{L}x_3(t) + \frac{1}{L}e(t). \tag{43}$$

Let us linearize the system about the equilibrium position $y^{eq} = x_1^{eq} =$ constant. The equilibrium values of the remaining states are,

$$x_2^{eq} = \frac{dx_1^{eq}}{dt} = 0, \qquad \frac{d^2x_1^{eq}}{dt^2} = 0 \tag{44}$$

The equilibrium value of $i(t)$ is determined by substituting the second expression of (44) into (39). Thus,

$$i^{eq} = x_3^{eq} = \sqrt{Mg \cdot x_1^{eq}} \tag{45}$$

Linearized around this equilibrium, it is straight forward to show the resulting system takes the form of (38), i.e. $\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t)$, with system matrices A and B defined as,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{(x_3^{eq})^2}{M(x_1^{eq})^2} & 0 & \frac{-2x_3^{eq}}{Mx_1^{eq}} \\ 0 & 0 & -\frac{R}{L} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{g}{x_1^{eq}} & 0 & -2\left(\frac{g}{Mx_1^{eq}}\right)^{1/2} \\ 0 & 0 & -\frac{R}{L} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix}. \tag{46}$$

Using mathematical model $\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t)$, with A, B defined above, we can study the magnetic ball dynamics around equilibrium position y^{eq}. From here, we can use this LTI system description to, for example:

- Design control signal $u(t)$ to move the ball from one position to another;
- Estimate states x_1, x_2, x_3 from measurements of just x_1;
- Learn model parameters M, R, L.

Exercise 1. *Consider the pendulum dynamics from Example 3.2. Also consider a counter clockwise input torque $\tau(t)$ acting at the frictionless pivot. Also, assume a sensor measures the pendulum angle θ. Linearize the pendulum dynamics around the two equilibrium points $(0,0)$, $(\pi,0)$. What are the A, B, C, D matrices for each equilibrium?*

5 Stability

Stability plays a central role in systems theory and engineering energy systems. In rough terms, stability means the energy system does not "blow up" in some sense. Think about thermal runaway in lithium-ion batteries, pressure buildup in steam generators, vibrations in wind turbines, or voltages in electrical distribution circuits. Clearly, guaranteeing stability is a desirable design criterion in the study of energy management systems. The notion of stability goes beyond certifying against catastrophe. As shown in CH2 and CH3, it's also the critical concept for guaranteeing convergence of algorithms.

Within the dynamical systems literature, many different kinds of stability exist. In this chapter, we focus on the stability of equilibria in linear systems.

5.1 Definitions of Stability

Consider an autonomous LTI system, that is, a linear system with no control input,

$$\dot{x}(t) = Ax(t), \tag{47}$$

with an initial condition $x(0) = x_0$. We are now positioned to formally define stability.

Definition 5.1. *(Stability) The autonomous LTI system $\dot{x}(t) = Ax(t)$ is marginally stable or stable in the sense of Lyapunov if the solution, $x(t)$, to the ODE is bounded, i.e.* $\max_t |x(t)| < \infty$

for all initial conditions x_0. The LTI system is asymptotically stable if the solution $x(t)$ converges asymptotically to zero, that is $\lim_{t \to \infty} x(t) = 0$ for all initial conditions x_0.

Example 5.1. Consider the scalar autonomous LTI system and initial condition

$$\dot{x}(t) = 0, \qquad x_0 = 5 \tag{48}$$

The solution $x(t)$ is provided in Fig. 10. The solution remains bounded, indicating the system is *stable in the sense of Lyapunov* or *marginally stable*.

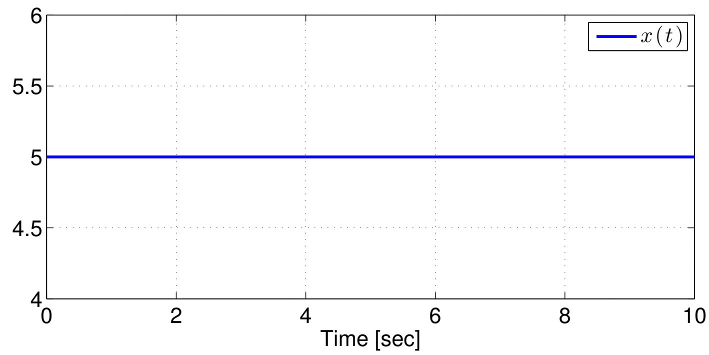

Figure 10: Solution to $\dot{x}(t) = 0, x_0 = 5$.

Example 5.2. Consider the second-order autonomous LTI system and initial condition

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad x_0 = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \tag{49}$$

The solution $(x_1(t), x_2(t))$ is provided in Fig. 11. The solution oscillates, but remains bounded, indicating the system is *stable in the sense of Lyapunov* or *marginally stable*.

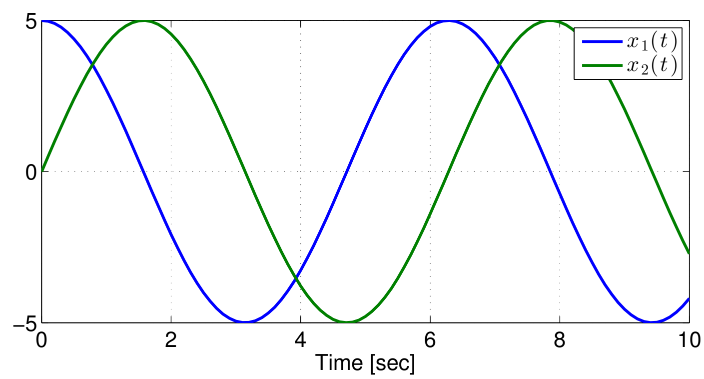

Figure 11: Solution to (49).

Example 5.3. Consider the slightly modified second-order autonomous LTI system and initial condition

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad x_0 = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \tag{50}$$

The solution $(x_1(t), x_2(t))$ is provided in Fig. 12. In this case, the states clearly converge toward zero, indicating the system is *asymptotically stable*.

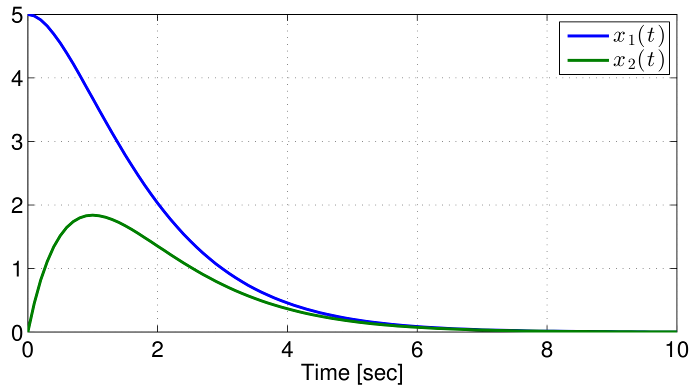

Figure 12: Solution to (50).

5.2 Tests for Stability

Although the previous definitions of stability are straightforward, it is undesirable to determine stability by solving the ODE directly. In practice, it would be more useful if a simple condition can be checked, without explicitly solving the ODE. For autonomous LTI systems, such a simple condition exists. It involves the eigenvalues of system matrix A. That is, denote λ_i, where $i = 1, ..., n$ the set of eigenvalues for A.

Theorem 5.1. *(LTI System Stability)*

1. *The LTI system $\dot{x}(t) = Ax$ is marginally stable or stable in the sense of Lyapunov if and only if all the eigenvalues of A contain zero or negative real parts, that is Re$[\lambda_i] \leq 0, \forall i$.*
2. *The LTI system $\dot{x}(t) = Ax$ is asymptotically stable if and only if all the eigenvalues of A have strictly negative real parts, that is Re$[\lambda_i] < 0, \forall i$.*

Let us revisit the previous three examples, to demonstrate this condition.

Example 5.4. Consider the scalar autonomous LTI system and initial condition

$$\dot{x}(t) = 0, \qquad x_0 = 5 \tag{51}$$

The eigenvalues of the system matrix, which is trivially zero in this case, is zero. Consequently the system is marginally stable.

Example 5.5. Consider the second-order autonomous LTI system and initial condition

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad x_0 = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \tag{52}$$

The eigenvalues of the system matrix are $\pm 1j$. The real parts are zero, indicating the system is marginally stable.

Example 5.6. Consider the slightly modified second-order autonomous LTI system and initial condition

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad x_0 = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \tag{53}$$

The eigenvalues of the system matrix are $-1, -1$, implying the system is asymptotically stable.

Exercise 2. *Consider the pendulum dynamics from Example 3.2 and the two equilibrium points* $(0,0)$, $(\pi,0)$. *Linearize the dynamics around each equilibrium point. Determine the system matrix* A *for each linearized system. Determine the stability properties. What are the eigenvalues?*

Exercise 3. *Consider the Room Thermal Dynamics Example 2.1. What are the equilibria when the HVAC unit is on* $s = 1$ *and off* $s = 0$*, for fixed ambient temperature* T_A^{eq}*? What are the eigenvalues in each case? Determine the stability properties.*

Exercise 4. *Consider the the Magnetically Levitated Ball from Example 4.1 with parameter values* $M = 1$*kg,* $L = 100$*mH,* R *= 10 m*Ω*. Determine a condition for the ball position* x_1^{eq} *in which the dynamics are asymptotically stable.*

Exercise 5. *Consider the RLC circuit from Example 2.4. Derive a condition on the parameter* R*, in terms of parameters* L, C*, that ensure asymptotic stability of the equilibrium corresponding to constant voltage input* V^{eq}*.*

6 Notes

Modeling dynamical systems is an extraordinarily rich topic – a critical area of study for anyone interested in systems and control. Several textbooks provide excellent expositions, including [3] and [4]. Textbook [3] in particular discusses a general theory for modeling dynamic systems based on a concept known as bond graphs. Reference [4] is rich with examples that are worth reading to build intuition. Readers interested in a deeper understanding of linear systems theory may consult [2,5–7]. These textbooks are typical required references for classical first-year introductory control systems courses. Readers interested in nonlinear systems and stability should consult [1] – the classical reference in this area.

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