# Dice Activities That Motivate Differential Equation Models

Teaching Contemporary Mathematics
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This document includes descriptions of four activities that are appropriate for a calculus or differential equations class, all using dice to motivate a differential equation model for a real-world scenario. Each can be used either to introduce a new idea or to provide a real-world application for principles students have already studied. The activities can be used with classes of different sizes, although they tend to work best when the class is between about 15 and 25 students.

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**Activity 1: Fishing from a stocked pond**

Needed: about 80 dice.

Scenario: A community has a pond that is stocked with 10,000 fish annually. People fish throughout the year, catching fish at a rate proportional to how many fish are in the pond, with the proportionality constant being about 1/6 per year. At time $t = 0$ there are 15,000 fish in the pond.

Activity: Discuss with students what they think how the fish population will behave, without confirming or denying any of their ideas. Then use the simulation described below to help students get a "feel" for what's happening. Finally, have the students write a differential equation model representing the rate at which the fish population is changing. Ideally, if they superimpose a graph of their solution on a plot of the simulated data, the fit will be good.

Simulation: Let each die represent 1,000 fish. (However, in this description each die is sometimes referred to as "a fish", because it seems clearer.) Have the students stand in a circle around the "pond". Put 15 dice in the middle to represent the initial fish population. On the blackboard prepare a data table with two columns labeled "$t$ (years)" and "$F$ (thousands of fish)". Then complete the first row of the table with 0 for $t$ and 15 for $F$.

Now have the students pick up the dice (they can take more than one die, but shouldn't get grabby!) and roll them. Any die that rolls a 6 represents a fish that got caught and removed from the population. Take those dice back from the students, counting them as you go. Announce that the students have caught so many fish, and that you're now adding an additional 10, to represent the stocked fish that are added annually. Distribute 10 dice among the students, and add the next row to the table on the blackboard, with $t = 1$ and the new count of fish $F$, after having removed the caught fish and added the stocked fish.

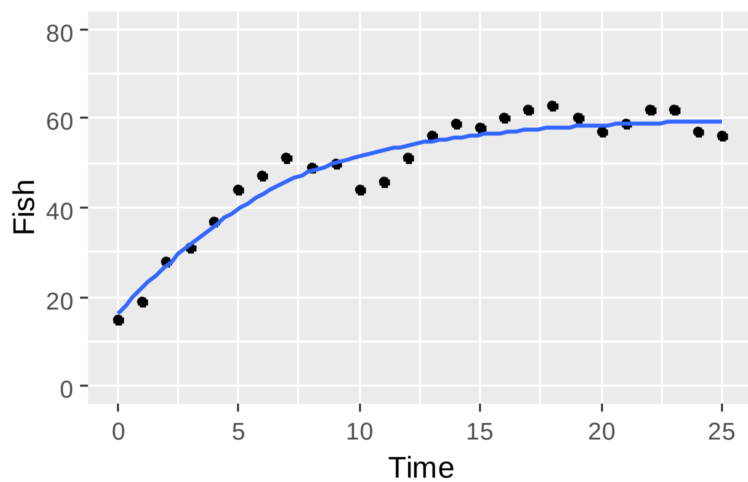
Repeat this long enough that the fish population comes near to its rough equilibrium of 60 fish. Perhaps go a little longer if you think the students don't yet see that they've reached an equilibrium point.

Now use technology (calculator or graphing program) to plot the data. Perhaps discuss with the students whether this was what they expected.

Calculus: Have the students write a differential equation model, including initial conditions, for the rate of change of the fish population (in thousands) as a function of time. Before continuing, be sure they all understand the following model:

$$\frac{dF}{dt} = -\frac{1}{6}F + 10 \text{, with } F = 15 \text{ when } t = 0.$$

The solution to this differential equation is $F = -45e^{\frac{-1}{6}t} + 60$. The randomness inherent in the dice rolling means that this function will not fit the classroom data perfectly, but it should be reasonably good. Below is a graph of a typical outcome.

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**Activity 2: Spread of a Disease**

Needed: 2 dice for each student, preferably of different colors. (In this description they're called white and green, but they can be any two colors, or simply all white.)

Scenario: In a small enclosed community, a person returns from a trip with a contagious disease. Each day everyone in the community mingles with some others in the community. If two people mingle and one has the disease and the other doesn't, then there's a chance that the uninfected person will get sick.

Activity: Discuss with students how they think the disease will spread. Under what conditions will is spread quickly? How long do you think it will take until everyone gets sick? Perform the simulation described below so that everyone can get a "feel" for how the disease spreads. Then have the students write and solve a differential equation modeling how fast the disease spreads. Ideally, the solution curve will fit the simulated data reasonably well.

Simulation: At the blackboard write a table with two columns labeled "$t$ (days)" and "$S$ (sick people)". Fill in the first row of the table with $t = 0$ and $S = 1$.

Each student gets a white die. Select a single student to represent the initial sick person, and give her a green die in addition to the white die. The healthy (uninfected) people line up somewhere, and then the sick person walks down the row of her classmates, one by one. For each of them, she and they simultaneously roll their white dice. If the numbers match, then those two people mingled that day. In that case, the sick person then rolls her green die. If it shows a 1*, then the disease has spread to the uninfected person. That person remains in place until the sick person has completed the entire row of classmates.

Now update the table of simulated data with $t = 1$ and $S =$ however many people are now sick. Give green dice to all the newly infected people. They form a row parallel to the row of uninfected people, and the process from before is repeated, only now with potentially more sick people and fewer uninfected people.

Keep repeating this until everyone is sick and plot the data. Discuss with students whether the graph matches their initial expectations.

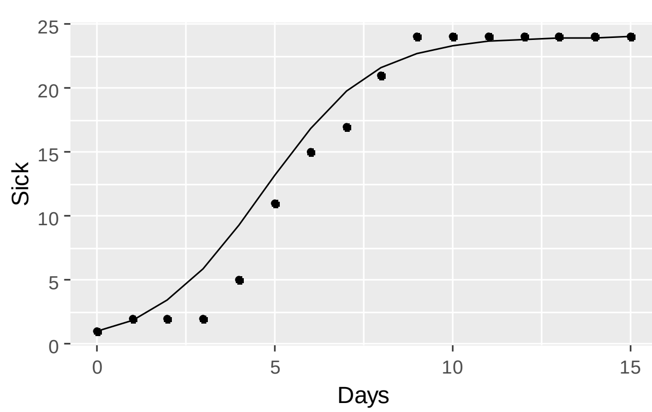
Calculus: Have the students write a differential equation model, including initial conditions, for the rate at which the disease spreads. Before continuing, be sure they all understand the following model. (I've assumed a class size of 24 students.)

$$\frac{dS}{dt} = \frac{1}{6} \cdot \frac{1}{6} \cdot S(24 - S) \text{, with } S = 1 \text{ when } t = 0.$$

The solution to this differential equation is:

$$S = \frac{24e^{\frac{2}{3}t}}{23 + e^{\frac{2}{3}t}}.$$

As before, the randomness in dice mean that the curve will not fit the data perfectly, but it should fit reasonably well. Below is a graph of a typical outcome.




* With a smallish class (say, with 15 or fewer students), the simulation as described may take overly long. You may want to let two or three faces on the green die represent disease transmission, rather than just 1.

**Activity 3: Osmosis**

Needed: About 20 times as many dice as there are students.

Scenario: A tank containing fresh water is next to a tank containing salty water. (There is a total of 20 pounds of salt in the water.) The two tanks are separated by a film through which salt can pass by osmosis. The water in both tanks is continually churning. Each salt molecule in either tank has one chance in six, per hour, of passing through the separating film.

Activity: Let $S$ be the salt content, in pounds, of the amount of salt in the tank that began as salty water. Discuss with students how they think $S$ will change over time. With salt able to go back and forth, will $S$ be periodic? Perform the simulation activity described below so that everyone can get a "feel" for how the osmosis works. Then have the students write and solve a differential equation modeling how fast the salt osmoses. Ideally, the solution curve will fit the simulated data reasonably well.

Simulation: Students do this activity in pairs. One student in each pair gets 20 dice, each representing a pound of salt. The other student gets no dice, representing fresh water. Have the students keep a table of $t$ (hours) and $S$, with $t = 0$ and $S = 20$ being the initial table values.

The "salt student" rolls all 20 dice. Any that roll a 6 osmose to the other student. The students record the new value fo $S$.

This is now repeated several times, except that once the "fresh water" student has some dice, then he also rolls them and passes back to the first student any that show a 6. The students keep track only of $S$, and repeat it until they think the process has nothing more to reveal. (That is when the dice are split about evenly between the students, ten dice each.)

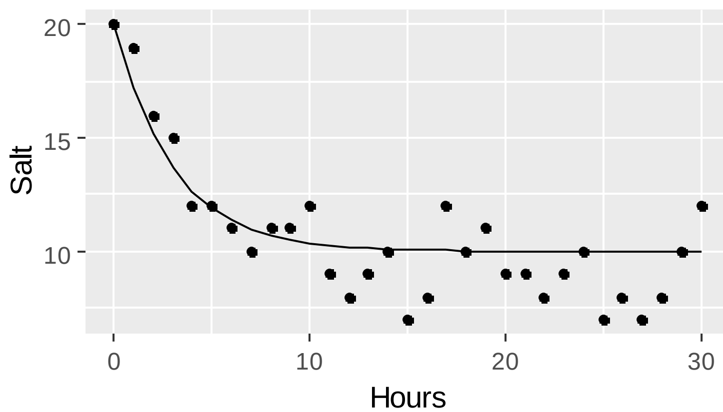
Calculus: Have the students write a differential equation model, including initial conditions, for the rate at which the salt osmoses. Before continuing, be sure they all understand the following model. (I've assumed a class size of 24 students.)

$$\frac{dS}{dt} = -\frac{1}{6}S + \frac{1}{6}(20 - S)\text{, with } S = 20 \text{ when } t = 0.$$

The solution to this differential equation is:

$$S = 10e^{-\frac{t}{3}} + 10.$$

As before, the randomness in dice mean that the curve will not fit the data perfectly, but it should fit reasonably well. Below is a graph of a typical outcome.



**Activity 4: Combat**

Needed: 1 die for each student.

Scenario: Two armies are fighting one another. One is twice as large as the other, but the smaller army has more skill with their weapons. (Each soldier is twice as likely to his her target as is a soldier in the larger army.) Soldiers keep firing their weapons until one army is killed.

Activity: Count your students off by threes: 1, 2, 3, 1, 2, 3, etc. The 1's and 2's form army A and the 3's form army B. Give each student a die which represents her weapon. For soldiers in army A, it takes a 1 to kill her target. For soldiers in army B, it takes a 1 or a 2.

Let $A$ and $B$ be, respectively, the sizes of army A and army B. Write a data table on the board with three columns: $t$ (fighting rounds), $A$, and $B$. Write in the initial values in the table. For the rest of this activity description, we'll assume a class size of 24 students, so the first row of the table would be $t = 0, A = 16, B = 8$.

Discuss with students who they think will win: the larger army or the one with greater weapon accuracy? Why?

Then do the activity described below and record the data on the blackboard.

Afterwards, have the students write a system of *coupled* differential equations: one for $\frac{dA}{dt}$ and one for $\frac{dB}{dt}$. They won't be able to solve the system without advanced techniques, but the can use the chain rule to combine their equations into a single differential equation for $\frac{dB}{dA}$. Have them develop that equation, write the initial conditions, and then solve the differential equation.

The resulting curve, expressing $B$ in terms of $A$ is called a "phase plot": time is not represented. Be sure the students understand what the phase plot means. Then superimpose the phase plot and the data. Hopefully the fit will be reasonably good!

Simulation: Have the students form two lines facing one another. Ask them each to secretly select a student in the opposing army to "aim" at. Once every student has selected a target, say, "Fire!", and the students all roll their dice. Students then announce (hopefully truthfully!) any targets who were hit, and they leave the battlefield. (If a student who leaves the battlefield herself hit a target during that round, the both students leave.)

Repeat until one army is destroyed, keeping track on the blackboard of the army sizes at the conclusion of each round.

Calculus: The initial conditions are $t = 0$, $A = 16$, and $B = 8$. (for a class of 24 students.) The system of differential equations (this may require careful thought) is:

$$\begin{cases} \frac{dA}{dt} & = & -\frac{2}{6}B \\ \frac{dB}{dt} & = & -\frac{1}{6}A \end{cases}$$

Using the chain rule, that leads to the following differential equation:

$$\frac{dB}{dA} = \frac{dB/dt}{dA/dt} = \frac{A}{2B}$$

Its solution is:

$$B = \sqrt{\frac{A^2}{2} - 64}$$

The following graph shows a typical outcome from the simulation, along with the solution curve above.