Acknowledgment. Many thanks to Russ Monson, Peter Blanken, PI at Niwot Ridge, Ricardo Valentini, PI at Collelongo, Brian Amiro, and PI at Black Spruce for use of their data. Thanks to Jazmine Darden and Nana Owusu for their hard work during a undergraduate summer research project investigating net carbon uptake.

This work used open access NEE data acquired by the FLUXNET community. The policy and acknowledgment for this dataset can be found at: http://www.fluxdata.org/Shared%20Documents/Policy_Opened_Final.pdf.

Summary. Using the fundamental theorem of calculus and numerical integration, we investigate carbon absorption of ecosystems with measurements from a global database. The results illustrate the dynamic nature of ecosystems and their ability to absorb atmospheric carbon.

References


Climate Modeling in the Calculus and Differential Equations Classroom

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We introduce here the basic principles of climate science for a one-dimensional Energy Balance Model (EBM), as first proposed by Peter Imkeller [2]. Calculus and differential equations students can use it as a basis for further research into the climate system, particularly the ice-albedo feedback mechanism.

http://dx.doi.org/10.4169/college.math.j.44.5.424
MSC: 34A34, 34D20, 97M10

THE MATHEMATICAL ASSOCIATION OF AMERICA

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An energy balance model postulates that the rate of change in the Earth’s temperature is proportional to the difference between the incoming $R_{\text{in}}$ and outgoing $R_{\text{out}}$ radiation. In symbols,

$$k \frac{dT}{dt} = R_{\text{in}} - R_{\text{out}}.$$  \hspace{1cm} (1)\]

The constant of proportionality $k$ is the *heat capacity*, a measure of how long it takes to change the earth’s temperature dependent on the density of the atmosphere, its chemical composition, and other factors; precise values vary in climate models, and here we use $k = 10^{23} \frac{J}{K}$ ([4]). In addition, we take temperature in °Kelvin, time in seconds, and radiation in Watts.

**Absorbed energy.** The amount of energy from the sun that reaches the Earth ($R_{\text{in}}$) changes according to the Earth’s orbit and cycles in the Sun’s activity itself. For the sake of simplicity, energy balance models often take solar energy as a constant, $Q = 1365 \text{ W/m}^2$ ([3]), representing the average amount of energy from the Sun reaching a square meter of the Earth. For the area of the Earth receiving solar energy, imagine a disc hovering in space in front of the Earth, exactly covering it, and absorbing the solar rays that would otherwise reach it. This disc would have a radius equal to that of the Earth, $6.3781 \times 10^6 \text{ m}$ ([7]), and so would have an area of $A_E = \pi (6.3781 \times 10^6)^2 \approx 1.278 \times 10^{14} \text{ m}^2$. Some proportion of this energy, however, is reflected away from the earth immediately, like light off a mirror. This phenomenon is known as *albedo*. Let $\alpha$ be the proportion of lost energy. The Earth’s absorbed energy is then

$$R_{\text{in}} = A_E Q (1 - \alpha).$$  \hspace{1cm} (2)\]

At the Earth’s surface, albedo varies according to the ground type: Generally, darker surfaces absorb most of the solar energy they receive, while lighter surfaces reflect most of it away. Some average values for albedo are in Table 1.

<table>
<thead>
<tr>
<th>Surface</th>
<th>Snow</th>
<th>Ocean ice</th>
<th>Desert</th>
<th>Meadow</th>
<th>Soil</th>
<th>Road</th>
<th>Ocean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average albedo</td>
<td>.75–.95</td>
<td>.30–.40</td>
<td>.24–.30</td>
<td>.10–.20</td>
<td>.05–.15</td>
<td>.05–.10</td>
<td>.07</td>
</tr>
</tbody>
</table>

The albedo of cold surfaces, like snow and ice, is usually higher than that of warm surfaces. A simple way to express this relationship at a global scale is to assume a constant high albedo if the earth is covered in ice ($\alpha$ at temperature $T$) and a constant low albedo if the earth is ice-free ($\bar{\alpha}$ at temperature $\bar{T}$); between these two temperatures, we assume that albedo changes linearly according to the latitude of the polar ice caps (see, for example, [2]). This leads to

$$\alpha(T) = \begin{cases} 
\bar{\alpha}, & \text{if } T \leq T; \\
\frac{\alpha - \bar{\alpha}}{\bar{T} - T} (T - \bar{T}) + \bar{\alpha}, & \text{if } \bar{T} < T < \bar{T}; \\
\bar{\alpha}, & \text{if } T \geq \bar{T}.
\end{cases}$$  \hspace{1cm} (3)\]
Radiated energy. To model the energy the earth radiates ($R_{\text{out}}$), we use the Stefan–Boltzmann Law, which postulates that an object radiates energy at a rate proportional to the fourth power of its temperature. For ideal black bodies, objects that radiate perfectly at a quantum level, this proportion is the Stefan–Boltzmann constant $[1]$:

$$\sigma = 5.6704 \times 10^{-8} \frac{\text{W}}{\text{m}^2\text{K}^4}.$$  

The Earth, however, does not radiate energy perfectly; the planet’s chemical composition affects the transfer of energy in significant ways. To account for this, many EMBs incorporate a multiplicative factor called emissivity $\varepsilon \in (0, 1)$. A climate factor with significant impact on emissivity is the greenhouse effect. Increased amounts of carbon and other particulate matters decrease the Earth’s emissivity, and thereby allow less energy to leave the Earth’s atmosphere $[3]$. The current emissivity of the Earth, although changing, is generally taken to be near $\varepsilon = 0.622$ $[5]$.

If we assume again that the Earth is a perfect sphere of radius $6.3781 \times 10^6$ m, its surface area is $S_E = 4\pi(6.3781 \times 10^6)^2 \approx 5.11 \times 10^{14}$ m$^2$. Then, the total energy emitted by the Earth is

$$R_{\text{out}} = S_E \sigma \varepsilon T^4. \quad (4)$$

Energy balance model. Combining equations (1), (2), (3), and (4), we obtain our version of the standard EBM:

$$k \frac{dT}{dt} = A_E Q(1 - \alpha(T)) - S_E \sigma \varepsilon T^4, \quad (5)$$

where the parameters are $k = 10^{23} \text{W/K}$, $A_E = 1.278 \times 10^{14}$ m$^2$, $Q = 1365$ W/m$^2$, $S_E = 5.11 \times 10^{14}$ m$^2$, $\sigma = 5.6704 \times 10^{-8}$ W/m$^2$K$^4$, and $\varepsilon = 0.622$. We also assume that the albedo $\alpha(T)$ is of the form (3) with $T = 255$K, $\alpha = 0.6$, $T = 290$K, and $\overline{\alpha} = 0.2$. Therefore,

$$\alpha(T) = \begin{cases} 0.6, & \text{if } T \leq 255K \\ \frac{-2}{175}T + \frac{123}{35}, & \text{if } 255K < T < 290K \\ 0.2, & \text{if } T \geq 290K. \end{cases} \quad (6)$$

For this model, note that the interval $T \leq 255$K contains an equilibrium value of 249.42K, and the interval $T \geq 290$K contains an equilibrium value of 296.61K. For the interval $255K < T < 290K$, there are two equilibrium values, 263.7K and 338.03K, but the latter is not practical, being outside the interval. So, our three practical equilibria to this model are 249.42K, 263.71K, and 296.6K (see Figure 1). Numerical simulation indicates that the former and latter equilibria are stable, while the middle one is unstable.

Conclusion. This model is not a scientifically precise reflection of the global climate system; among other problems, the Earth does not inevitably gravitate toward only an ice-covered (249.42K) or an ice-free (296.6K) stable condition. Many more complicated factors that affect climate changes are at play. The purpose of energy balance models is less to give a totally accurate picture of the Earth’s climate than to simplify the system radically in order to isolate certain phenomena, therefore enabling a deeper understanding of the processes at work or a more relevant comparison with observations’ ([$3$], p. 113). Energy balance models vary based on the assumptions and goals
of the model, and a more detailed example is discussed in the paper [6] by Walsh and McGehee in this issue, starting on page 350.

Our model is designed to illustrate the ice-albedo feedback mechanism. Since warmer global temperatures create geophysical changes (such as the retreat of the polar ice caps) that in turn cause lower albedo and therefore higher absorption of energy from the sun, the effect of the planet’s albedo in any climate model is to amplify warming trends over time. For parallel reasons, albedo also amplifies cooling trends over time. Our unstable equilibrium at 263.71K is an approximate tipping point between these self-reinforcing warming and cooling trends.

We encourage students to use their own research to alter the parameters of this model or include new ones, and explore how these changes affect the climate system.

Summary. Students in college-level mathematics classes can build the differential equations of an energy balance model of the Earth’s climate themselves, from a basic understanding of the background science. Here we use variable albedo and qualitative analysis to find stable and unstable equilibria of such a model, providing a problem or perhaps a research project for students interested in environmental studies and mathematics.

References