Sometimes Predicting the Future is Easier than Deciphering the Past and Other Aspects of …

MODELING THE DRAINING OF A BOTTLE

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JMM 2020: AMS Special Session 7A: Wall to Wall Modeling Activities in DE Courses

Saturday January 18th, 2020

OUTLINE

- Overview of Activity
- Major Pedagogical Elements of the Activity
- Pre-Activity Coursework and Class Characteristics
- Walk through data collection and data-driven model
- Work through differential equations model
- Post-Activity Coursework
- Questions

WHAT'S IN THE ACTIVITY?

- Goal: to model the height of a liquid column in a bottle as it drains through a small hole.
- Steps:
  - Collect Data from Video
  - Create a regression model
  - Derive a differential equations model from physical law/reasoning
  - Solve differential equation
  - Create and solve initial value problem (IVP)
  - Model verification
- Adapted from 1-15-T-Torricelli, SIMIODE modeling scenario by Dr. Brian Winkel, Professor Emeritus, USMA, Director SIMIODE.

PEDAGOGICAL ELEMENTS

- This activity requires students to:
  - Compare and contrast data-driven, regression model with physically-reasoned, differential equations model.
  - Review/practice using the limit definition of derivative.
  - Derive a differential equation from physical reasoning.
  - Give physical meaning to an IVP.
  - Practice solving a separable equation/IVP (with atypical notation)
  - Give physical meaning to existence and uniqueness of solutions.
  - Give physical meaning to differentiable, piecewise-defined functions.

CLASS CHARACTERISTICS

- ~150-200 students
- Overwhelmingly sophomores
  - Prerequisites: Calculus I and II
- Linear Algebra is part of the course
- Whole-class lectures 3x per week
- ~20 student recitations 2x per week
- Modeling Scenarios started in lecture, finished in recitation
- 4-5 Modeling Scenarios per semester

PRE-ACTIVITY COURSEWORK

- Characterizing and solving separable equations.

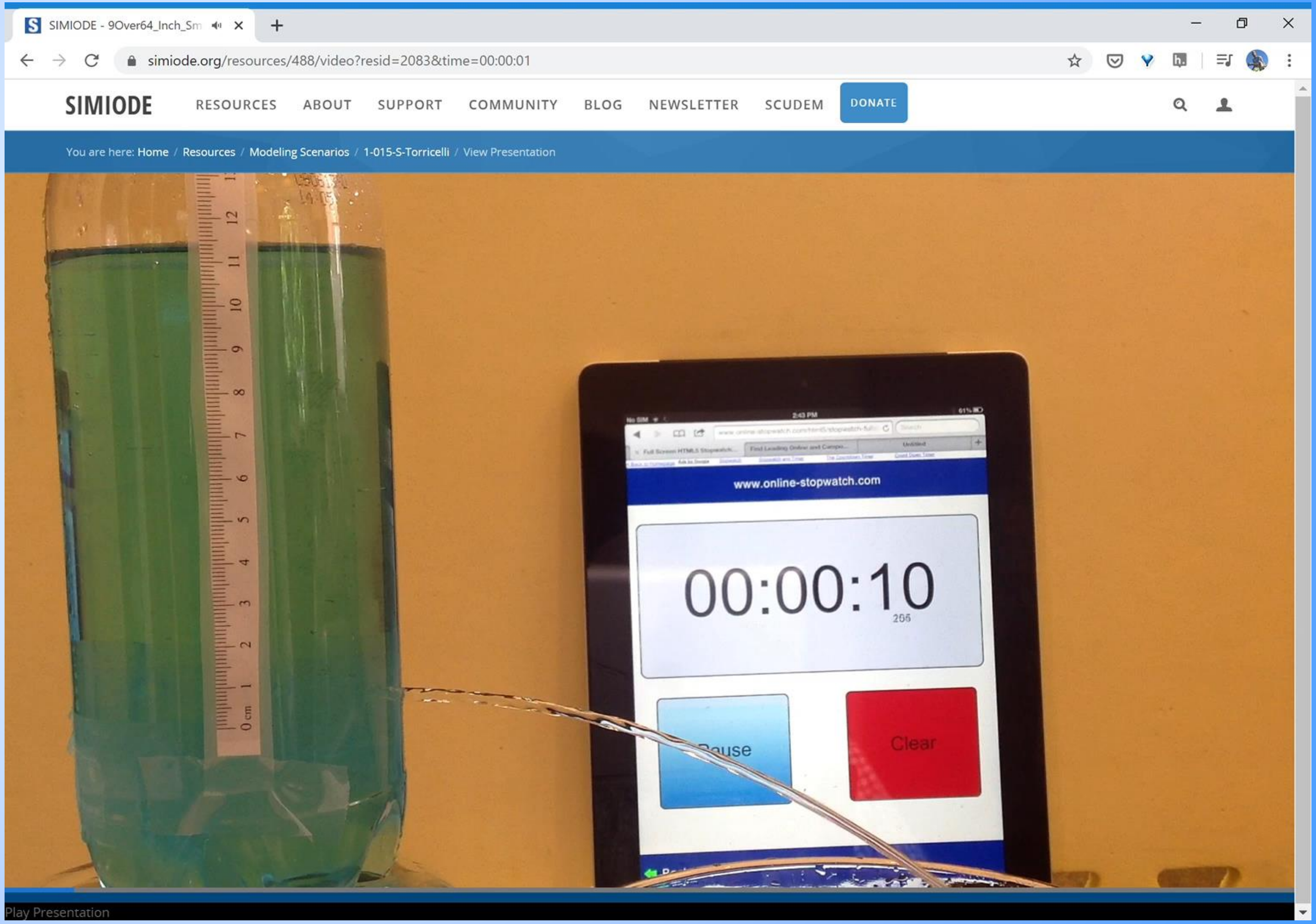
$$y' = f(x)g(y)$$

- Defining initial conditions and initial value problems.
- Perhaps, reviewing the limit definition of derivative and Δ notation.

$$y' = \lim_{\Delta x \to 0} \frac{\Delta \mathrm{y}}{\Delta x}$$

DATA-DRIVEN, REGRESSION MODEL

- Data collection video
- From SIMIODE:
- Brian Winkel (2015), "1-015-S-Torricelli," https://www.simiode.org/resources/488.
- Go to Modeling Scenario worksheet.



1 Model 1: A Data-driven Function Model

We are interested in describing the height of the fluid above the hole over time.

1. What do you notice about the height over time? Describe your expectations for the graph of the height over time. Increasing or decreasing; concave up or concave down?

   The function should be decreasing, and its graph will be concave up because it seems to be draining more slowly over time.

2. On what types of things might this function depend?

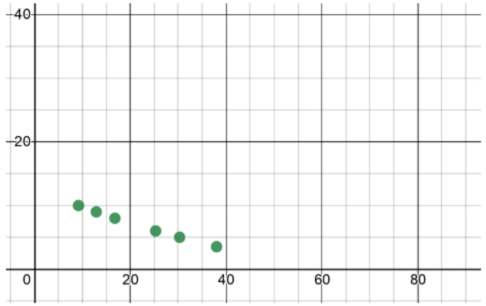
   - Size of the bottle (cross-sectional area)]item Size of the hole
   - Shape of the hole
   - Viscosity of fluid
   - etc.

3. Now in the table below, record the values of the height at various times.

HEIGHT OF WATER COLUMN

Time (s)	9.2	12.9	16.8	25.3	30.3	38.0
Height above hole (cm)	10.0	9.0	8.0	6.0	5.0	3.5

4. Plot these data points below.



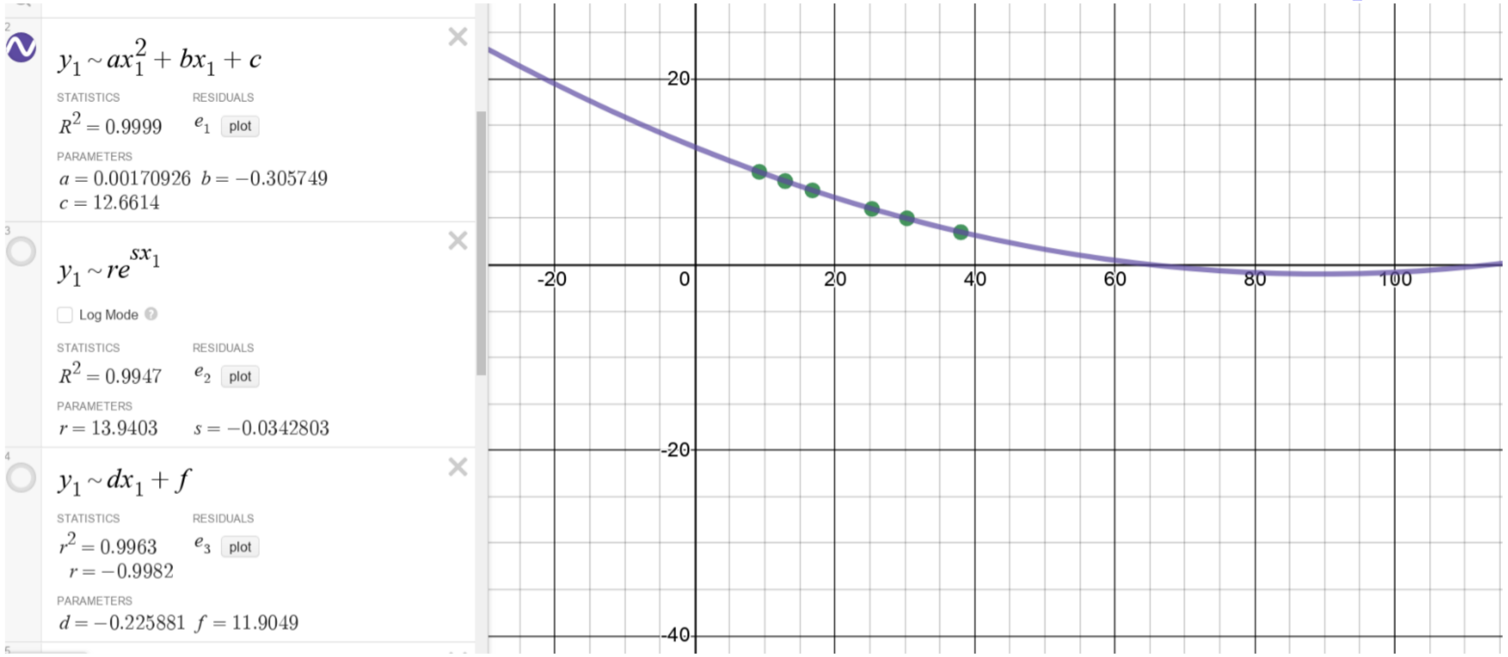
Now we attempt to build a mathematical model of this situation, by creating a function that models the height. Let $h(t)$ be the height above the hole at time t.

5. Can you come up with a guess for the functional form of $h(t)$?

   - Linear?
   - Quadratic?
   - Exponential decay?

6. We will use regression analysis (a type of curve fitting) to produce a function that approximates our data points. To do this I will use desmos.com. Jot down a few of the commands that I use: enough so that you can perform some quick regression analysis if you ever need to.

   Click on + and create a table. Fill in table with data. Type $y_1 \sim ax_1^2 + bx_1 + c$.



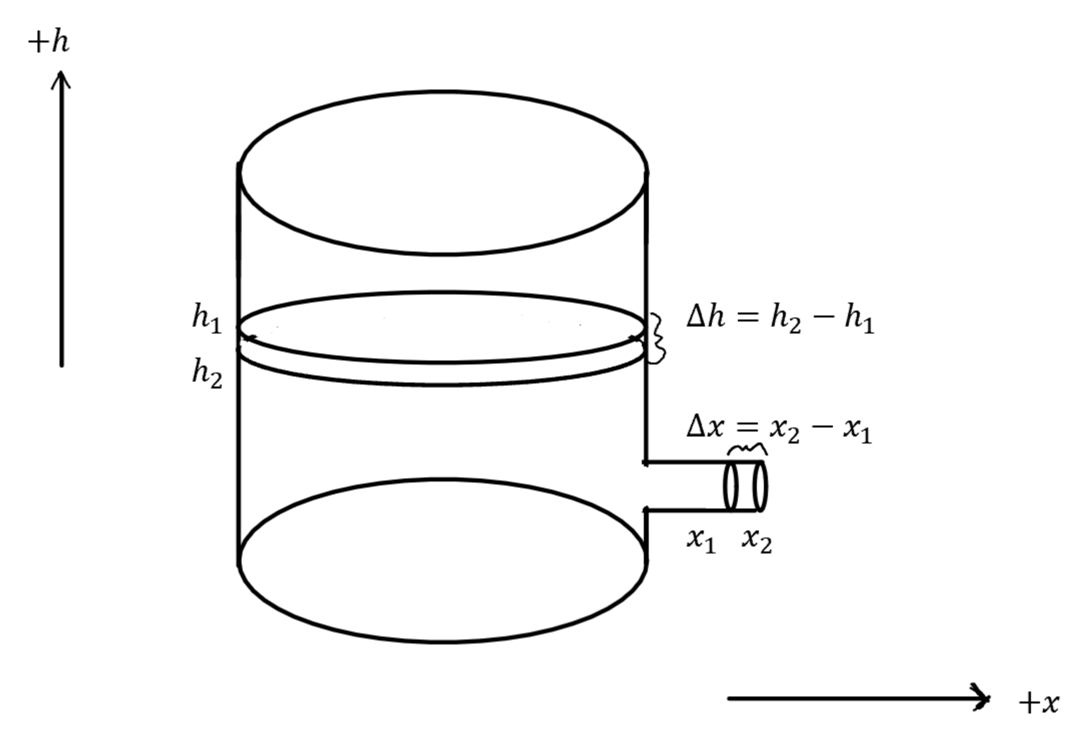
7. How reasonable is our model?

   It's good for some times, but it predicts negative values at some times (and an increasing height after some time).

8. Let's say that I told you the radius of this bottle and the radius of the drainage hole. Could you use this model to predict the height over time for draining from a different size hole or different bottle? Why or why not?

   No, one would need more data to perform another regression. This method only works for this one specific system.

PHYSICAL, DIFFERENTIAL EQUATIONS MODEL

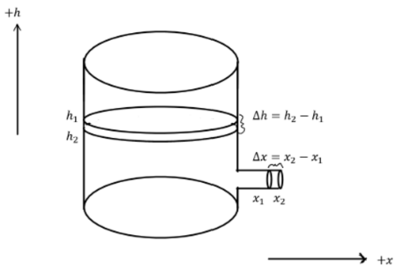


- Torricelli's Law
- Equating the change in fluid volume at the top to the change of fluid volume at the hole.
- Work on Section 2, #9 and #10.

Our drainage situation is quite different from free fall in kinematics, as we guessed in #1: the height decreases, but more slowly over time. A falling object falls faster over time. It turns out, however, that a physical law, known as Torricelli's Law, will help us. Torricelli's Law states that the velocity of the fluid exiting the drainage hole depends on the height of the column of fluid as

$$v_e(h) = \sqrt{2gh} \tag{1}$$

where g is the acceleration due to gravity. This law can be derived from data and regression or from physical principles, but we'll take it as given for now and use it to derive a differential equation.



9. We do this by computing the rate of change of the volume in the bottle in two ways and setting them equal. Let A be the cross-sectional area of the bottle. Let a be the cross-sectional area of the draining hole. (See diagram on previous page.)

   (a) Compute the rate of change of the volume in the bottle using A and $h'(t)$. (Hint: $\Delta V = A\Delta h$, so what is $\lim_{\Delta t \to 0} \Delta V/\Delta t$?)

   (b) Compute the rate of change of the volume in the bottle using a and v_e, then use Torricelli's Law to write this in terms of $h(t)$. (Hint: $\Delta V = -a\Delta x$, so what is $\lim_{\Delta t \to 0} \Delta V/\Delta t$? Think about the sign; v_e is positive.)

   (c) Equate these two rates of change to give a differential equation for $h(t)$. (And rearrange so that $h'(t)$ is alone on one side.)

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   $V'(t) = \lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t} = \lim_{\Delta t \to 0} A\frac{\Delta h}{\Delta t} = Ah'(t)$

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   $V'(t) = \lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t} = -a \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = -ax'(t) = -av_e = -a\sqrt{2gh}$

   (c) Equate these two rates of change to give a differential equation for $h(t)$. (And rearrange so that $h'(t)$ is alone on one side.)
   $Ah'(t) = -a\sqrt{2gh(t)}$. Solving for $h'(t)$, $h'(t) = -\frac{a}{A}\sqrt{2gh}$.

(d) Simplify the equation by using a new constant, k, which we define as a combination of the other parameters by $k = \frac{a\sqrt{2g}}{A}$. (Simply rewrite your equation using k.)

$h' = -k\sqrt{h}$.

10. Now, we'll solve this ODE, create and solve the appropriate initial value problem, and use the solution to determine how the model behaves and what it predicts about the physical system.

(a) Solve this equation (leaving k as a parameter in your solution).

$$h^{-1/2}h' = -k$$

$$2h^{1/2} = -kt + C, (-kt + C > 0)$$

$$h(t) = \left(C - \frac{kt}{2}\right)^2, \left(-\frac{kt}{2} + C > 0\right)$$

(b) If the bottle or tank is a different size or the drainage hole is a different size, how does your work above change?

It would just change the value of k, so nothing changes up to this point.

(c) The bottle used in the video has a radius of approximately 4.17 cm and a hole of radius approximately 0.218 cm. And, in fact, researchers have found that an effective area of the small hole of about $0.7a$ works better than just a in this model, due to the interaction of water with the edge of the hole. This means that for our system

$$k \approx \frac{0.7a}{A}\sqrt{2g} \approx \frac{0.7\pi(0.00218)^2}{\pi(0.0417)^2}\sqrt{2 \cdot 9.8} \approx 0.00847\,(\sqrt{m}/s).$$

(d) Use this k and our data from the video to create an initial value problem.

$$h' = -0.00847\sqrt{h}, h(t_0) = h_0$$

$$\text{e.g, } h' = -0.00847\sqrt{h}, h(12.9) = 0.09\,m$$

(e) Solve the initial value problem. (You already have a family of solutions from part (a); just replace k with the numerical value and find the particular solution that satisfies the initial condition.)

$$h(t) = \left(C - \frac{0.00847t}{2}\right)^2$$

$$h(12.9) = (C - (0.004235)(12.9))^2 = 0.09\,m$$

$$C \approx 0.0355\,(\sqrt{m})$$

(f) What was the height of the water at time $t = 0$?

$h(0) \approx (0.0355)^2 \approx 0.126\,m.$

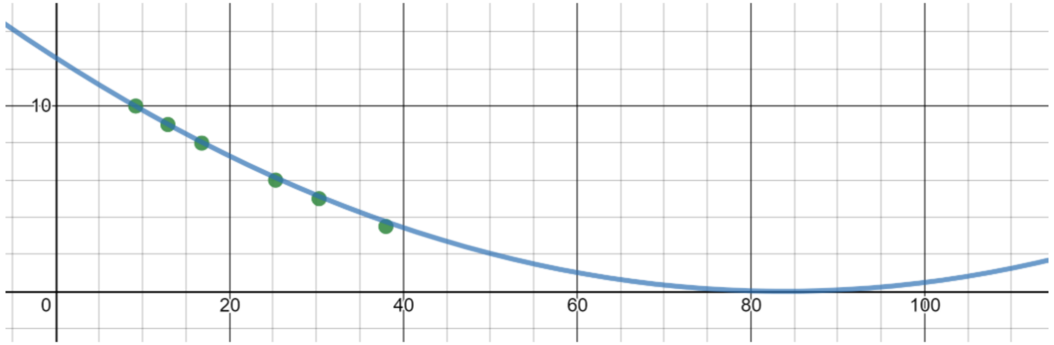
(g) How long does it takes for a bottle to drain all the way to the height of the hole?

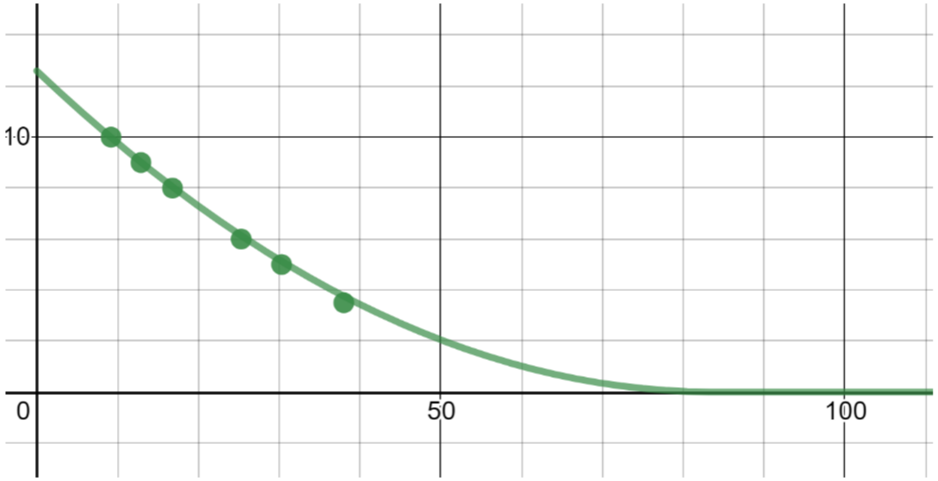
$$h(t_f) = (0.0355 - 0.004235t)^2 = 0$$

$$t_f = \frac{0.0355}{0.004235} \approx 83.8\,s.$$

(h) What does your model predict after the bottle empties, down to the height of the hole? (This depends on how careful you were in part (a).) Can you adjust it to do better? What should the graph of the solution look like?

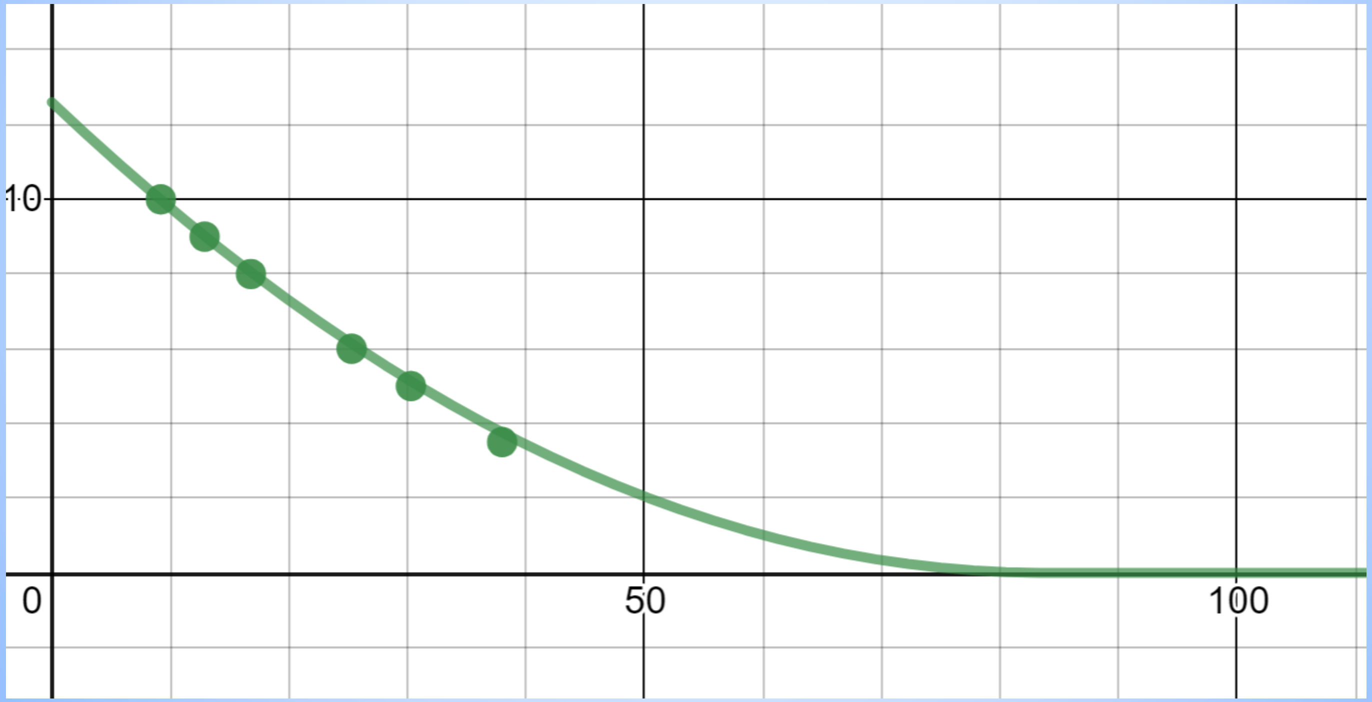
If we ignore domain issues, the solution that we found predicts that the bottle fills back up after it empties. However, in fact, our solution method was only valid as long as $h \neq 0$. So, our solution is only valid before the point of emptying. But, at the emptying point $h = 0$, of course, and also for all later times. In fact, a piecewise defined function that is our function up to the emptying time ($\approx 83.8\,s$) and 0 beyond.





POST-ACTIVITY COURSEWORK

- More model verification (Desmos.com)



$$h(t) = \{0 \le t < 83.8 : (3.55 - 0.0424t)^2, t \ge 83.73 : 0\}$$

POST-ACTIVITY COURSEWORK

- More model verification
  - Desmos plot
- Existence and Uniqueness (Predicting the future vs deciphering the past)

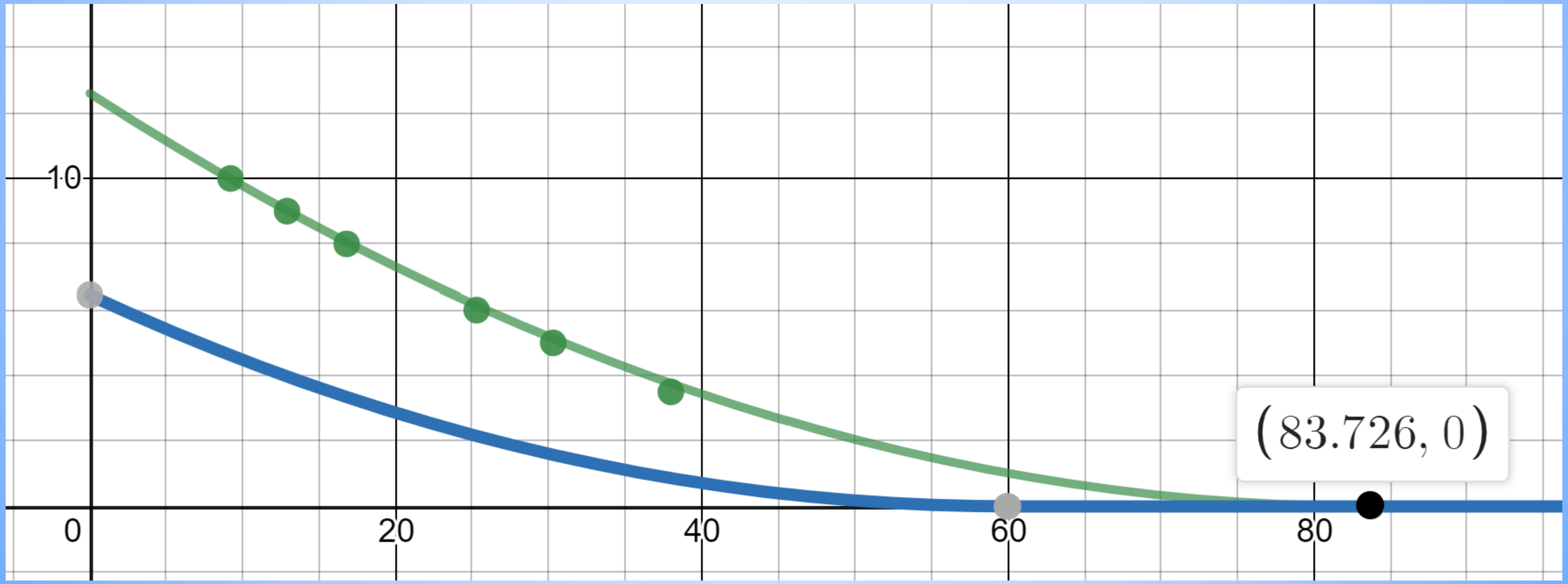
Theorem 1.2.1 (Picard's theorem on existence and uniqueness). *If $f(x, y)$ is continuous (as a function of two variables) and $\frac{\partial f}{\partial y}$ exists and is continuous near some (x_0, y_0), then a solution to*

$$y' = f(x, y), \qquad y(x_0) = y_0,$$

exists (at least for some small interval of x's) and is unique.

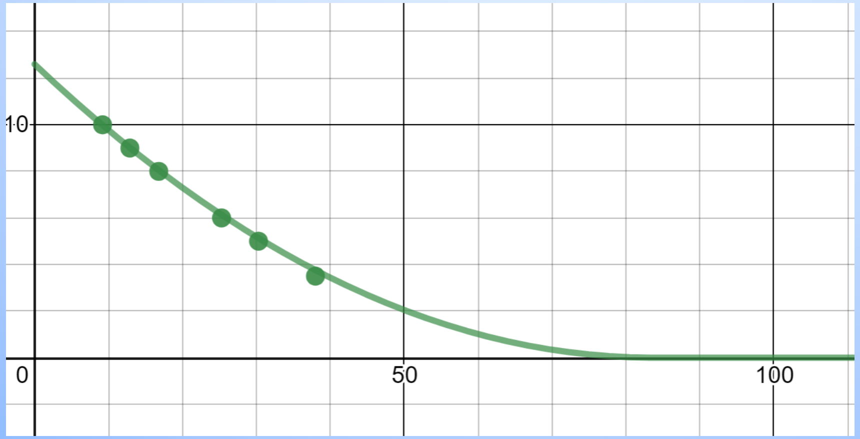
- If initial condition is $h(t_0) = h_0 > 0$, then a unique solution is guaranteed by Picard's Theorem.
  - If we know the height at some time when bottle is not empty, our model explains past and predicts future.
- If initial condition is $h(t_0) = 0$, Picard's Theorem (and most others) does not apply.
  - If we know a time when the bottle is empty, we can predict the future, but have no knowledge of the past.

Non-uniqueness: If the bottle is empty at $\approx 84\ s$, it could have emptied any time before.



POST-ACTIVITY COURSEWORK

- Homework problem asking students to show that the piecewise solution is differentiable and is a solution.
  - One-sided derivatives or limits involved.
  - Very difficult for students.
  - Many do not grasp the significance.



POST-ACTIVITY COURSEWORK

- Final Exam Extra Credit Problem
  - Completely unprompted; draining a bottle not included on exam topic list
  - "Give an example of a physical system that can be modeled by an initial value problem which has multiple solutions. (In other words, while solutions exist, there is not a unique solution.) Describe the physical system, the initial condition, and at least two solutions."
  - ~13 of 184 described draining a bottle.

THANKS!

- Questions?