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STUDENT VERSION

Linear Approximation, Always a Reliable Technique?

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STATEMENT

A mathematical model is a symbolic approximation of a quantifiable event with the goal to accurately predict future and past behaviour. An experiment is an idealized observation of the event where real time data may be collected. Both the model and the experiment are inherently prone to errors. Experimental setup and observational errors occur in every physical experiment. A perfect symbolic model is nearly impossible to create because there are way too many parameters that should be considered. Assumptions must be made to limit the number of parameters in order to create a reasonable mathematical model.

To test a mathematical model one needs to compare symbolic solutions to experimental observations. One way to test the model against the experiment is to compute the sum of square errors, the sum of the squares of the differences between the computed model and observed data. If the error is too large the symbolic model must be revised.

In order to create a stronger, more realistic model, fewer and fewer assumptions are made. As a result the mathematical model becomes more complicated. Often the more realistic models are impossible to solve in a closed form solution. One common technique used to analyze complex mathematical models (nonlinear) is to make first order linear approximations. These approximations are often easier to solve (by hand) and lend themselves to nicer qualitative analysis. The question now becomes, how well does the linear approximation reflect the actual physical event?

In this activity you will carryout a physical experiment, derive and solve a mathematical model, and compare the two results using the sum of square errors. Then you will make a first order approximation to the mathematical model, and solve and compare all three results.

1 Collecting Data

Two options are provided to collect data for this scenario. The first is a lab experiment with materials provided by your instructor. The second is a SIMIODE YouTube video that allows you to collect the data while watching the video. Whichever method is used, it is important you carefully collect and record the required data.

1.1 Lab Exercise: Emptying the Tank

This experiment will record time intervals as a cylindrical tank of liquid is emptied through a small hole near the bottom of a tank.

1. Record the initial height of the water in the tank, $h_0 =$
2. Calculate and record the area of the top of the open tank, $A =$
   Calculate and record the area of the opening of the spigot, $a =$
   Compute and record the ratio of the two areas, $\dfrac{a}{A} =$
3. Fill tank with water. Mark depths at 10% intervals.
4. Simultaneously open spigot at bottom of tank and start stopwatch.
5. Record time at each 10% mark in Table 1.
6. Plot height vs time.
7. What type of function (exponential, linear, quadratic, logistic, ...) do you think best fits this data?
8. List at least 5 opportunities where experimental error may have occurred in collecting the data.

1.2 SIMIODE YouTube

If you are unable, or do not have the time, to set up your own experiment, consider one of several SIMIODE YouTube videos on falling column of water [1].

Note the following with respect to the reference video:

- For the referenced video, the diameter (not radius) is 7/64 inches.
- Adjust Table 1 to record the time at every one inch mark.

2 Derive, Solve and Analyze the Mathematical Model

2.1 Derive a Model for the Rate of Change of Height

Torricelli's Law states
Water in an open tank will flow out through a small hole in the bottom with the velocity it would acquire in falling freely from the water level to the hole.

Assuming Torricelli's Law is true, verify that the differential equation for rate of change of height is

$$h'(t) = -\frac{a}{A}\sqrt{2gh}\,, \tag{1}$$

where A is the constant surface area of the tank, a is the area of the spigot opening and g is the constant acceleration due to gravity.

2.2 Solve for the Height Function

Equation (1) is a separable first order nonlinear ordinary differential equation (ODE) with parameters a, A and g. Letting $k = \frac{a}{A}\sqrt{2g}$, the ODE can be written

$$h'(t) = -k\sqrt{h}\,.$$

1. With initial height h_0, solve the initial value problem (IVP)

   $$h'(t) = -k\sqrt{h}, \quad h(0) = h_0\,. \tag{2}$$

   Note: Solving with the parameters k and h_0 will lead to a more general solution to your ODE.
2. Using the data compiled in Section 1, compute and record the value for k.

   $$k = \frac{a}{A}\sqrt{2g} = \underline{\hspace{3cm}}\,.$$
3. Finally substitute the values for k and h_0 into your solution to (2) and record the result below

   $$h(t) = \underline{\hspace{3cm}}\,.$$

2.3 Compare Experimental Date to the Model

Now it is time to compare the data collected from the original experiment with the symbolic solution to the model.

1. At each time t_i in Table 1 compute the height, $h(t_i)$. Plot the points on the same graph with the experimental data. Compute and record the difference $e_i = h_i - h(t_i)$ at each time.
2. Square Error: Compute $E = \sum_{i=1}^{9}(h_i - h(t_i))^2 = \underline{\hspace{3cm}}\,.$

   Note: Do NOT use the last data point in your computation.

3 Revisit the Model with a Linear Approximation

Even though an exact solution to this model was found, this is not always possible in more complicated applications. One technique to find a closed form solution to the first order ODE, $y' = f(y)$, is to find the linear approximation of $f(y)$.

Recall the linear approximation of $f(y)$ at $y = y_0$ is

$$f(y) \approx f(y_0) + f'(y_0)(y - y_0)\,.$$

time, t_i	height, h_i	$h(t_i)$	Error, $e_i = h_i - h(t_i)$	e_i^2
0	h_0			
	0			

Table 1. Tank Data/Model Comparison

3.1 Solve the Linear Approximation IVP

1. Verify the linear approximation of $\sqrt{h}$ at $h = h_0$ is

$$\sqrt{h} \approx \sqrt{h_0} + \frac{1}{2\sqrt{h_0}} h, \tag{3}$$

2. Substituting the approximation (3) into the IVP (2) yields

$$h'(t) = -k\left(\sqrt{h_0} + \frac{1}{2\sqrt{h_0}} h\right), \quad h(0) = h_0\,. \tag{4}$$

   Notice the new differential equation is both linear and separable. While you may select either solution method, it is good practice to use several methods for solving linear first order ODEs. Which ever method you choose, solve (4). Record your result below.

$$h_L = \underline{\hspace{3cm}}\,.$$

   Note: The subscript "L" will remind us this is the Linear Approximation solution.

3.2 Compare Experimental Date to the Approximated Model

Using the same time values from Table 1 compare the data collected to the solution of the approximated IVP.

1. For each time t_i in Table 1 use the solution to (4) to compute the height, $h_L(t_i)$. Plot these points on your graph. (At this time three should be three sets of points for each t_i.) Compute the difference at each time in Table 2.

time, t_i	height, h_i	$h_L(t_i)$	Error, $e_i = h_i - h_L(t_i)$	e_i^2
0	h_0			
	0			

Table 2. Tank Data/Approximation Comparison

2. Square Error for Approximation: Compute $E_L = \sum_{i=1}^{9}(h_i - h_L(t_i))^2 =$ ____________ .

   Note: Remember, do NOT use the last data point in your computation.

3. What assumptions were made in your liner approximation model that may have contributed to the large error?

4 Comparing Solutions

Often experimental data is not available, yet one still wants to compare the results of the model with its linear approximation. For this last bit of analysis all of the information is already collected.

1. Transfer the data from the previous tables to fill in the first three columns of Table 3.
2. Square Error for Model and Approximation: Compute $E_A = \sum_{i=1}^{9}(h(t_i) - h_L(t_i))^2 =$ __________ .

   Note: Once again, do NOT use the last data point in your computation.

3. Based on your graph of both solutions to the ODEs and the square error, is Linear Approximation a good technique for this model? Revisiting Toricell's Law explain why one must be very careful using Linear Approximation as a technique to solve complicated nonlinear ODEs.

REFERENCES

[1] SIMIODE. 2019. SIMIODE Torricelli's Law 7 Over 64 Inch Diameter Small Hole Collection Video. `https://www.youtube.com/watch?v=k7TSAqRfHgg`. Accessed 17 August 2019.

time, t_i	$h(t_i)$	$h_L(t_i)$	Error, $e_i = h(t_i) - h_L(t_i)$	e_i^2
0	h_0			
	0			

Table 3. Model/Approximation Comparison