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STUDENT VERSION

The Matrix Exponential

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MOTIVATION: EXPONENTIAL GROWTH

Exponential Solutions to Scalar Differential Equations

Probably the first nontrivial ordinary differential equation (ODE) that you ever encountered was

$$x'(t) = ax(t) \tag{1}$$

where a is a real number and $x(t)$ is an "unknown" function for which we must solve. The general solution to (1) can be found by using an integrating factor or by separation of variables, and is $x(t) = Ce^{at}$, where C is an arbitrary constant. If we are given an initial condition at $t = 0$, say

$$x(0) = x_0 \tag{2}$$

then the unique solution to (1) that satisfies (2) is

$$x(t) = e^{at}x_0 \tag{3}$$

(here x_0 is written on the right of the exponential, for reasons that will become clear).

Linear Systems of ODE's and Exponentiation

Consider a system of linear, constant coefficient, homogeneous ODE's. As an example let's take

$$\begin{aligned} x_1'(t) &= x_1(t) - 2x_2(t) \\ x_2'(t) &= 4x_1(t) - 5x_2(t) \end{aligned} \tag{4}$$

with initial conditions $x_1(0) = 1, x_2(0) = 2$. The system (4) can be written very conveniently in matrix-vector notation by defining a vector-valued function $\mathbf{x}(t)$, a matrix $\mathbf{A}$, and a constant vector $\mathbf{x}_0$, as

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} 1 & -2 \\ 4 & -5 \end{bmatrix}, \text{ and } \mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \tag{5}$$

With these definitions the system (4) can be expressed as

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) \tag{6}$$

with initial condition $\mathbf{x}(0) = \mathbf{x}_0$. Now compare equations (1) and (6): they look very similar, with a matrix $\mathbf{A}$ in (6) instead of the scalar a in (1), and the vector-valued function $\mathbf{x}(t)$ in place of the scalar function $x(t)$.

One might take a wild leap of faith and hope the solution to (6) can be expressed in a fashion similar to (3), namely

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}_0. \tag{7}$$

We will write $t\mathbf{A}$, since a scalar times a matrix is defined, whereas $\mathbf{A}t$ doesn't quite make sense. In this case since t is a scalar, the quantity $t\mathbf{A}$ is a 2×2 matrix, but what does $e^{t\mathbf{A}}$ mean? Given that $\mathbf{x}(t)$ and $\mathbf{x}_0$ in (7) are both two-dimensional vectors, it's clear that $e^{t\mathbf{A}}$ must be a 2×2 matrix, but how do we define it so that (7) really is a solution to the system (6)? More generally, can we do this for any $n \times n$ system of linear constant coefficient differential equations?

It shouldn't be too surprising that this can be done, and is extremely useful. For inspiration, recall that in elementary complex arithmetic we defined what it means to exponentiate a complex number, via Euler's identity, as

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{8}$$

where θ is any real number. There is no denying that Euler's identity and exponentiation of complex numbers are indispensable (and interesting!) so it may be worth pursuing this in seeking a definition for $e^{t\mathbf{A}}$.

THE MATRIX EXPONENTIAL

The Definition of the Matrix Exponential

Recall the Taylor's series for e^t, centered at $t = 0$, from elementary calculus:

$$\begin{aligned} e^t &= 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!}. \end{aligned} \tag{9}$$

The series on the right converges for any real number t, which you can prove with a Ratio Test. Euler's Identity (8) followed from (9) by plugging in $t = i\theta$ and regrouping terms. Perhaps we can make sense of $e^{t\mathbf{A}}$ in a similar fashion.

For the moment forget about exponentiating $t\mathbf{A}$ (which depends on t) and let's just consider how we might define $e^{\mathbf{B}}$ where $\mathbf{B}$ is some constant $n \times n$ matrix, and may have real or complex entries—it doesn't matter. If we naively plug "$t = \mathbf{B}$" into (9) we obtain

$$\begin{aligned} e^{\mathbf{B}} &= 1 + \mathbf{B} + \frac{\mathbf{B}^2}{2} + \frac{\mathbf{B}^3}{6} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{\mathbf{B}^k}{k!}. \end{aligned} \tag{10}$$

A few remarks are in order:

1. If $\mathbf{B}$ is an $n \times n$ matrix then so are $\mathbf{B}^2, \mathbf{B}^3$, etc. More generally, $\mathbf{B}^k$ is an $n \times n$ matrix for any exponent k.
2. As a consequence of point (1) above, each summand in (10) is an $n \times n$ matrix, so the sum of any finite number of terms in (10), say
$$\sum_{k=0}^{m} \frac{\mathbf{B}^k}{k!} \tag{11}$$
(the first $m + 1$ terms) makes sense as an $n \times n$ matrix.
3. It's not obvious that the sum on the right in (10) (the limit of (11) as $m \to \infty$) converges!

As remarked, the infinite sum on the right in (9) converges to some real number, and this number is e^t; in fact, we can take the limit of the sum on the right in (9) as the very *definition* of e^t. Perhaps we can similarly show that the sum on the right in (10) converges.

Let's try an example and see what happens.

Example 1 *Let*

$$\mathbf{B} = \begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}.$$

If we take $m = 0$ in (11) we just get $\mathbf{I}$, the 2×2 identity matrix, which doesn't even depend on $\mathbf{B}$. Using $m = 1$ produces

$$\sum_{k=0}^{1} \frac{\mathbf{B}^k}{k!} = \mathbf{I} + \mathbf{B} = \begin{bmatrix} 4 & -2 \\ 4 & -2 \end{bmatrix}.$$

Taking $m = 2$ produces

$$\sum_{k=0}^{2} \frac{\mathbf{B}^k}{k!} = \mathbf{I} + \mathbf{B} + \frac{\mathbf{B}^2}{2} = \begin{bmatrix} 9/2 & -2 \\ 4 & -3/2 \end{bmatrix}.$$

When $m = 5$ we find

$$\begin{aligned} \sum_{k=0}^{5} \frac{\mathbf{B}^k}{k!} &= \mathbf{I} + \mathbf{B} + \frac{\mathbf{B}^2}{2} + \cdots + \frac{\mathbf{B}^5}{120} \\ &= \begin{bmatrix} 76/15 & -2 \\ 4 & -3/2 \end{bmatrix} \\ &\approx \begin{bmatrix} 5.067 & -2.350 \\ 4.700 & -1.983 \end{bmatrix}. \end{aligned}$$

When $m = 10$ we find

$$\sum_{k=0}^{10} \frac{\mathbf{B}^k}{k!} \approx \begin{bmatrix} 5.069 & -2.350 \\ 4.700 & -1.983 \end{bmatrix}$$

to three significant figures. It seems that as m increases, the sum converges to "something." As in the scalar case (9), it seems like this convergence is facilitated by the rapid growth of $k!$ in the denominator.

The above example illustrates what happens in the general case, which we'll state as a Theorem. See the Appendix if you want a proof.

Theorem 1 *Let $\mathbf{B}$ be an $n \times n$ matrix with real (or complex) entries. Define $n \times n$ matrices $\mathbf{S}_m$ as*

$$\mathbf{S}_m = \sum_{k=0}^{m} \frac{\mathbf{B}^k}{k!}, \tag{12}$$

for $m \geq 0$, a partial sum of the series (10). Then the sequence $\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2, \ldots$ converges, in the sense that

$$\lim_{m \to \infty} (\mathbf{S}_m)_{i,j} \tag{13}$$

exists for each i, j with $1 \leq i, j \leq n$, where $(\mathbf{S}_m)_{i,j}$ denotes the row i, column j entry of $\mathbf{S}_m$.

The matrix with row i, column j entry equal to the limit in (13) is what we call $e^{\mathbf{B}}$. Let's state this as a definition.

Definition 1 *For an $n \times n$ matrix $\mathbf{B}$ we define $e^{\mathbf{B}}$ as that $n \times n$ matrix with row i, column j entries*

$$(e^{\mathbf{B}})_{i,j} = \lim_{m \to \infty} (\mathbf{S}_m)_{i,j}$$

where the $\mathbf{S}_m$ are defined by (12). (The limit above exists).

Exercise 1

Let

$$\mathbf{B} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}.$$

Compute $\mathbf{S}_m$ in (12) for $m = 0, 1, 5, 10, 50$ (you'll need appropriate software!). Does the sum stabilize as m increases?

Properties of the Matrix Exponential

Here are a few properties of the matrix exponential, identical to or closely related to those for exponentials of real or complex numbers.

1. $e^{\mathbf{0}} = \mathbf{I}$, where $\mathbf{0}$ denotes a square matrix of all 0's and $\mathbf{I}$ is the identity matrix.
2. If $\mathbf{AB} = \mathbf{BA}$ (that is, $\mathbf{A}$ and $\mathbf{B}$ commute) then $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}$. This is usually false if $\mathbf{AB} \neq \mathbf{BA}$.

3. For any square matrix $\mathbf{B}$ we have $e^{\mathbf{B}}e^{-\mathbf{B}} = \mathbf{I}$. In particular, $e^{\mathbf{B}}$ is always invertible with inverse $e^{-\mathbf{B}}$.
4. For any matrix $\mathbf{B}$ we have $\mathbf{B}e^{\mathbf{B}} = e^{\mathbf{B}}\mathbf{B}$.
5. The derivative of $e^{t\mathbf{A}}$ with respect to t is an $n \times n$ matrix and

$$\frac{d}{dt}(e^{t\mathbf{A}}) = \mathbf{A}e^{t\mathbf{A}}. \tag{14}$$

Property (1) is an easy exercise using (10). For a proof of Properties (2), (4) and (5) see the Appendix. Property (3) follows from Properties (1) and (2), since then $\mathbf{I} = e^{\mathbf{0}} = e^{\mathbf{B}+(-\mathbf{B})} = e^{\mathbf{B}}e^{-\mathbf{B}}$, noting that $\mathbf{B}$ and $-\mathbf{B}$ commute.

USING THE MATRIX EXPONENTIAL TO SOLVE SYSTEMS OF ODE'S

The Matrix Exponential Solution

Does $\mathbf{x}(t)$ as defined by (7) actually provide a solution to (6)? If we differentiate both sides of (7) with respect to t by using (14) (treat $\mathbf{x}_0$ as a constant) we obtain

$$\mathbf{x}'(t) = \mathbf{A}e^{t\mathbf{A}}\mathbf{x}_0 = \mathbf{A}(e^{t\mathbf{A}}\mathbf{x}_0) = \mathbf{A}\mathbf{x}(t)$$

so that (7) does indeed provide a solution to (6).

We seem to have found a very convenient and simple way to solve linear systems of constant coefficient ODE's, but there is one hitch: we need to compute $e^{t\mathbf{A}}$. For a specific value of t and given matrix $\mathbf{A}$ we could use (10) by summing a finite, but large, number of terms (as in Example 1), but then we only have $\mathbf{x}(t)$ for that particular value of t, and it's not clear how many terms to sum to get an accurate value. And if we want a symbolic solution, the series (10) looks quite hopeless.

We'll look at how to overcome these obstacles in the next sections, but let's pause to consider an example.

Example 2 *Let's return to the system (4), with* $\mathbf{A}$ *as defined in (5). Take the author's word for the moment that*

$$e^{t\mathbf{A}} = \begin{bmatrix} 2e^{-t} - e^{-3t} & -e^{-t} + e^{-3t} \\ 2e^{-t} - 2e^{-3t} & -e^{-t} + 2e^{-3t} \end{bmatrix}.$$

You will be able to verify this for yourself shortly. We then find that the solution with initial condition $\mathbf{x}_0 = \langle 1, 2 \rangle$ *is*

$$\mathbf{x}(t) = e^{t\mathbf{A}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} e^{-3t} \\ 2e^{-3t} \end{bmatrix}.$$

Computing The Matrix Exponential: The Diagonal Case

There is one case in which computing the matrix exponential is easy: when the matrix is diagonal. Suppose that $\mathbf{D}$ is an $n \times n$ diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_n$, so

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & \cdots & 0 & 0 \\ 0 & d_2 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & d_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & d_n \end{bmatrix}.$$

You can easily check that any power $\mathbf{D}^k$ is given by

$$\mathbf{D}^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 & 0 \\ 0 & d_2^k & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & d_{n-1}^k & 0 \\ 0 & 0 & \cdots & 0 & d_n^k \end{bmatrix}.$$

In this case we find that when we substitute $\mathbf{D}$ into the Taylor series (10) and sum the matrices component-by-component we obtain

$$e^{\mathbf{D}} = \sum_{k=0}^{\infty} \frac{\mathbf{D}^k}{k!} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{d_1^k}{k!} & 0 & \cdots & 0 & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{d_2^k}{k!} & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & \sum_{k=0}^{\infty} \frac{d_{n-1}^k}{k!} & 0 \\ 0 & 0 & \cdots & 0 & \sum_{k=0}^{\infty} \frac{d_n^k}{k!} \end{bmatrix}.$$

But the sums on the diagonals are just the Taylor series for e^{d_m}, $1 \leq m \leq n$. Thus

$$e^{\mathbf{D}} = \begin{bmatrix} e^{d_1} & 0 & \cdots & 0 & 0 \\ 0 & e^{d_2} & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & e^{d_{n-1}} & 0 \\ 0 & 0 & \cdots & 0 & e^{d_2} \end{bmatrix}. \tag{15}$$

Of course if we want to compute $e^{t\mathbf{D}}$ we simply note that $t\mathbf{D}$ is a diagonal matrix with diagonal entries $d_m t$, so from (15) $e^{t\mathbf{D}}$ is the diagonal matrix with diagonal entries $e^{d_m t}$.

Exercise 2

Formulate the system

$$x_1'(t) = 3x_1(t)$$
$$x_2'(t) = x_2(t)$$

as $\mathbf{x}'(t) = \mathbf{D}\mathbf{x}(t)$ (write out $\mathbf{D}$ explicitly). Use the matrix exponential (15) (with d_m replaced by $d_m t$) to solve the system with initial conditions $x_1(0) = 2, x_2(0) = 5$.

Computing The Matrix Exponential: The Diagonalizable Case

Let's look at an efficient method for computing $e^{\mathbf{B}}$ in a very common case; we can then apply the method with $\mathbf{B} = t\mathbf{A}$ to solve systems of ODE's. We will assume that the matrix $\mathbf{B}$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$; duplicates may occur, and eigenvalues may be complex. Let $\mathbf{v}_k$ be an eigenvector with eigenvalue λ_k, so

$$\mathbf{B}\mathbf{v}_k = \lambda_k \mathbf{v}_k \tag{16}$$

for $1 \leq k \leq n$. Most importantly, suppose that the $\mathbf{v}_k$ form a linearly independent set of vectors. If you are not familiar with the term "linearly independent," let us instead state an equivalent condition: We suppose that the $n \times n$ matrix

$$\mathbf{P} = [\mathbf{v}_1|\mathbf{v}_2|\cdots|\mathbf{v}_n] \tag{17}$$

with kth column given by $\mathbf{v}_k$, is invertible. This is not always the case, but it is quite common.

Let $\mathbf{D}$ denote the $n \times n$ diagonal matrix with the eigenvalue λ_k in the row k, column k position of $\mathbf{D}$. There is a special relationship between $\mathbf{A}, \mathbf{D}$, and $\mathbf{P}$. Specifically, we find that

$$\begin{aligned}
\mathbf{BP} &= \mathbf{B}[\mathbf{v}_1|\mathbf{v}_2|\cdots|\mathbf{v}_n] \\
&= [\mathbf{B}\mathbf{v}_1|\mathbf{B}\mathbf{v}_2|\cdots|\mathbf{B}\mathbf{v}_n] \\
&= [\lambda_1\mathbf{v}_1|\lambda_2\mathbf{v}_2|\cdots|\lambda_n\mathbf{v}_n] \\
&= [\mathbf{v}_1|\mathbf{v}_2|\cdots|\mathbf{v}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix} && (18) \\
&= \mathbf{PD}. && (19)
\end{aligned}$$

The transition from the first line to the second line follows from the definition of matrix multiplication, since the product $\mathbf{BP}$ is computed by multiplying each column of $\mathbf{P}$ by $\mathbf{B}$. The transition from the second line to the third line follows from (16). The transition from the third line to the fourth line also follows from the definition of matrix multiplication—write out a small example to convince yourself. We have thus established in (19) that $\mathbf{BP} = \mathbf{PD}$. Since we assumed that $\mathbf{P}$ is invertible we can multiply both sides of $\mathbf{BP} = \mathbf{PD}$ on the right by $\mathbf{P}^{-1}$ and obtain

$$\mathbf{B} = \mathbf{PDP}^{-1}. \tag{20}$$

This process is referred to as *diagonalizing* the matrix $\mathbf{B}$ where equation (20) (or its equivalent version, $\mathbf{D} = \mathbf{P}^{-1}\mathbf{BP}$, obtained by multiplying (20) on the left by $\mathbf{P}^{-1}$ and on the right by $\mathbf{P}$) is the *diagonalization* of $\mathbf{B}$. It requires that the $\mathbf{B}$ has eigenvectors such that the matrix $\mathbf{P}$ is invertible, in which case we say that $\mathbf{B}$ is *diagonalizable*. Not all matrices are diagonalizable! See Exercise 6.

If we can diagonalize a matrix, which requires finding its eigenvalues and eigenvectors, we can easily compute the matrix exponential. First note that if we have the diagonalization (20) of $\mathbf{B}$ we can easily compute $\mathbf{B}^k$ for any power k, since

$$\begin{aligned}
\mathbf{B}^k &= \underbrace{(\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})\cdots(\mathbf{PDP}^{-1})}_{k \text{ copies}} \\
&= \mathbf{PD}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}\cdots(\mathbf{P}^{-1}\mathbf{P})\mathbf{DP}^{-1} \\
&= \mathbf{PD}^k\mathbf{P}^{-1}. \qquad (21)
\end{aligned}$$

The second line follows from the first line via the associativity of matrix multiplication, so we may group the $\mathbf{PP}^{-1}$ factors together. In the transition to the third line we use $\mathbf{PP}^{-1} = \mathbf{I}$, so the product "telescopes." The intermediate terms collapse to $\mathbf{D}^k$. By using (21) we find that

$$\begin{aligned}
\sum_{k=0}^{m} \mathbf{B}^k &= \sum_{k=0}^{m} \mathbf{PD}^k\mathbf{P}^{-1} \\
&= \mathbf{P}\left(\sum_{k=0}^{m} \mathbf{D}^k\right)\mathbf{P}^{-1} \\
&= \mathbf{P}\begin{bmatrix}
\sum_{k=0}^{m} \frac{\lambda_1^k}{k!} & 0 & \cdots & 0 & 0 \\
0 & \sum_{k=0}^{m} \frac{\lambda_2^k}{k!} & \cdots & 0 & 0 \\
0 & 0 & \ddots & 0 & 0 \\
0 & 0 & \cdots & \sum_{k=0}^{m} \frac{\lambda_{n-1}^k}{k!} & 0 \\
0 & 0 & \cdots & 0 & \sum_{k=0}^{m} \frac{\lambda_n^k}{k!}
\end{bmatrix}\mathbf{P}^{-1}. \qquad (22)
\end{aligned}$$

As $m \to \infty$ the ith diagonal sum approaches e^{λ_i}. We conclude that

$$e^{\mathbf{B}} = \mathbf{P}e^{\mathbf{D}}\mathbf{P}^{-1} \qquad (23)$$

where $e^{\mathbf{D}}$ is the diagonal matrix with ith diagonal entry e^{λ_i}. We should note that if $\mathbf{v}_k$ is an eigenvalue for $\mathbf{B}$ with eigenvalue λ_k then any nonzero multiple of $\mathbf{v}_k$ is also an eigenvector. This changes the matrix $\mathbf{P}$, of course, but $\mathbf{P}^{-1}$ is also altered and (23) will remain valid.

In the case in which we want $e^{t\mathbf{A}}$, simply substitute $\mathbf{B} = t\mathbf{A}$ in (23) and note that $t\mathbf{A}$ is just t times $\mathbf{A}$, and so has the same eigenvectors as $\mathbf{A}$ and eigenvalues $\lambda_k t$. Then

$$e^{t\mathbf{A}} = \mathbf{P}e^{t\mathbf{D}}\mathbf{P}^{-1}. \qquad (24)$$

Example 3 *Let's use the matrix exponential to solve the linear system*

$$\begin{aligned}
x_1'(t) &= 7x_1(t) - 6x_2(t) \\
x_2'(t) &= 12x_1(t) - 10x_2(t)
\end{aligned}$$

with initial conditions $x_1(0) = 1, x_2(0) = 2$. *In matrix terms we have* $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ *with*

$$\mathbf{A} = \begin{bmatrix} 7 & -6 \\ 12 & -10 \end{bmatrix}.$$

The eigenvalues for $\mathbf{A}$ *are* $\lambda_1 = -2$ *and* $\lambda_2 = -1$ *with corresponding eigenvectors* $\mathbf{v}_1 = \langle 2, 3\rangle$ *and* $\mathbf{v}_2 = \langle 3, 4\rangle$. *Thus*

$$\mathbf{D} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } \mathbf{P} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}.$$

We find that

$$\begin{aligned} e^{t\mathbf{A}} &= \mathbf{P}e^{t\mathbf{D}}\mathbf{P}^{-1} \\ &= \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 9e^{-t} - 8e^{2t} & -6e^{-t} + 6e^{-2t} \\ 12e^{-t} - 12e^{-2t} & -8e^{-t} + 9e^{-2t} \end{bmatrix}. \end{aligned}$$

The solution to the system with the given initial conditions is then

$$\mathbf{x}(t) = e^{t\mathbf{A}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9e^{-t} - 8e^{2t} & -6e^{-t} + 6e^{-2t} \\ 12e^{-t} - 12e^{-2t} & -8e^{-t} + 9e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3e^{-t} + 4e^{-2t} \\ -4e^{-t} + 6e^{-2t} \end{bmatrix}.$$

Exercise 3

Verify that $e^{t\mathbf{A}}$ in Example 2 (with $\mathbf{A}$ as in (5)) is correct.

Exercise 4

Formulate the system

$$\begin{aligned} x_1'(t) &= x_1(t) - x_2(t) - x_3(t) \\ x_2'(t) &= x_1(t) + 3x_2(t) + x_3(t) \\ x_3'(t) &= -3x_1(t) + x_2(t) - x_3(t) \end{aligned}$$

as $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ and use the matrix exponential $e^{t\mathbf{A}}$ to solve with $x_1(0) = 1, x_2(0) = 0, x_3(0) = -1$. Hint: the eigenvalues and eigenvectors here are "simple," e.g., integer eigenvalues.

Exercise 5

You've seen how to solve a nonhomogeneous linear differentiable equation of the form $x'(t) = ax(t) + b(t)$, where a is a constant and $b(t)$ a function. This is usually done using the "integrating factor" technique. Generalize this to the case of n constant coefficient linear nonhomogeneous ODE's, of the form

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}(t)$$

where $\mathbf{b}(t) = \langle b_1(t), \ldots, b_n(t)\rangle$. In particular, solve the system

$$x_1'(t) = x_1(t) - 2x_2(t) + 1$$
$$x_2'(t) = 4x_1(t) - 5x_2(t) + t$$

with $x_1(0) = 1, x_2(0) = 2$.

Exercise 6

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

a. Try using the diagonalization procedure to compute $e^{t\mathbf{A}}$. What goes wrong?

b. We could try using (10) directly (generally not a good idea, but we'll try it here). Show that

$$\mathbf{A}^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

and so the exponential series for $t\mathbf{A}$ becomes

$$e^{t\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} t^k = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!} & \sum_{k=0}^{\infty} k\frac{t^k}{k!} \\ 0 & \sum_{k=0}^{\infty} \frac{t^k}{k!} \end{bmatrix}.$$

You should recognize the diagonal elements in $e^{t\mathbf{A}}$ as e^t. What does the row 1, column 2 series sum to? Hint:

$$\sum_{k=0}^{\infty} k\frac{t^k}{k!} = \sum_{k=1}^{\infty} \frac{t^k}{(k-1)!} = t\sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!}.$$

The General Case: Putzer's Algorithm

Exercise 6 illustrates what can go wrong with using diagonalization to compute the matrix exponential: not all matrices are diagonalizable. Nonetheless, the series (10) converges for any matrix, and so all matrices can be exponentiated. What do we do if a matrix cannot be diagonalized? *Putzer's Algorithm* provides a procedure for exponentiating any matrix, diagonalizable or not. It's really geared toward computing $e^{t\mathbf{A}}$ (t already included) so we'll examine it in that form. We present the algorithm and examples below. The reader interested in a proof that Putzer's algorithm produces $e^{t\mathbf{A}}$ can consult [1].

We begin by supposing that the $n \times n$ matrix $\mathbf{A}$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$; these eigenvalue need not be distinct. Putzer's Algorithm expresses $e^{t\mathbf{A}}$ in the form

$$e^{t\mathbf{A}} = \sum_{j=0}^{n-1} r_{j+1}(t)\mathbf{P}_j \tag{25}$$

where the $r_{j+1}(t)$ are polynomials in t and the $\mathbf{P}_j$ are certain matrices computed as follows. First, set $\mathbf{P}_0 = \mathbf{I}$ and

$$\mathbf{P}_j = \prod_{k=1}^{j} (\mathbf{A} - \lambda_k \mathbf{I}) \tag{26}$$

for $1 \leq j \leq n-1$. The $r_j(t)$ are the scalar components $r_1(t), \ldots, r_n(t)$ of the vector $\mathbf{r}(t)$ that solves

$$\mathbf{p}'(t) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 1 & \lambda_2 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 1 & \lambda_n \end{bmatrix} \mathbf{p}(t) \tag{27}$$

with initial condition $\mathbf{p}(t) = \langle 1, 0, 0, \ldots, 0 \rangle$.

Example 4 *Let* $\mathbf{A}$ *be the matrix in (5), so we can use* $e^{t\mathbf{A}}$ *to solve the system (4), as was already done in Example 2. Here* $n = 2$ *and the eigenvalues of this matrix are* $\lambda_1 = -1, \lambda_2 = -3$*. We find that*

$$\mathbf{P}_0 = \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{P}_1 = \prod_{k=1}^{1} (\mathbf{A} - \lambda_k \mathbf{I}) = (\mathbf{A} + \mathbf{I}) = \begin{bmatrix} 2 & -2 \\ 4 & -4 \end{bmatrix}.$$

Equation (27) becomes

$$\begin{bmatrix} r_1'(t) \\ r_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}$$

with $r_1(0) = 1, r_2(0) = 0$*. The first equation,* $r_1'(t) = -r_1(t)$ *with* $r_1(0) = 1$ *is decoupled from the second and has solution* $r_1(t) = e^{-t}$*. The second equation for* $r_2(t)$ *then becomes* $r_2'(t) = e^{-t} + 3r_2(t)$ *with* $r_2(0) = 0$*. This is a scalar constant coefficient linear ODE, easy to solve via the integrating factor approach. We find* $r_2(t) = e^{-t}/2 - e^{-3t}/2$*. From (25) we have*

$$\begin{aligned} e^{t\mathbf{A}} &= \sum_{j=0}^{1} r_{j+1}(t) \mathbf{P}_j \\ &= r_1(t)\mathbf{P}_0 + r_2(t)\mathbf{P}_1 \\ &= e^{-t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(\frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \right) \begin{bmatrix} 2 & -2 \\ 4 & -4 \end{bmatrix} \\ &= \begin{bmatrix} 2e^{-t} - e^{-3t} & -e^{-t} + e^{-3t} \\ 2e^{-t} - 2e^{-3t} & -e^{-t} + 2e^{-3t} \end{bmatrix} \end{aligned}$$

just as Example 2.

Example 5 *Let's compute $e^{t\mathbf{A}}$ using $\mathbf{A}$ as in Exercise 6. We find eigenvalues $\lambda_1 = 1, \lambda_2 = 1$ (duplicates) and we have $n = 2$. Then*

$$\mathbf{P}_0 = \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{P}_1 = \prod_{k=1}^{1} (\mathbf{A} - \lambda_k \mathbf{I}) = (\mathbf{A} - \mathbf{I}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Equation (27) becomes

$$\begin{bmatrix} r_1'(t) \\ r_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}$$

with $r_1(0) = 1, r_2(0) = 0$. As in the last example, the first equation, $r_1'(t) = r_1(t)$ with $r_1(0) = 1$ is decoupled from the second, with solution $r_1(t) = e^t$. The second equation for $r_2(t)$ then becomes $r_2'(t) = e^t + r_2(t)$ with $r_2(0) = 0$. This is a scalar constant coefficient linear ODE, easy to solve via the integrating factor approach. We find $r_2(t) = te^t$. From (25) we have

$$\begin{aligned} e^{t\mathbf{A}} &= \sum_{j=0}^{1} r_{j+1}(t)\mathbf{P}_j \\ &= r_1(t)\mathbf{P}_0 + r_2(t)\mathbf{P}_1 \\ &= e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}. \end{aligned}$$

Example 6 *Let*

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ -3 & -1 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

This matrix is not diagonalizable. The eigenvalues are $\lambda_1 = -1, \lambda_2 = 2$, and $\lambda_3 = 2$ (the order doesn't matter though). There is an eigenvector $\langle 0, 1, 0 \rangle$ for λ_1, but the second and third eigenvalues have only the eigenvector $\langle -1, 1, 0 \rangle$ (or multiples thereof) between the two of them. The diagonalization approach to computing $e^{t\mathbf{A}}$ won't work, but Putzer's algorithm will. With $n = 3$ we find

$$\mathbf{P}_0 = \mathbf{I}$$

$$\mathbf{P}_1 = (\mathbf{A} + \mathbf{I}) = \begin{bmatrix} 3 & 0 & 1 \\ -3 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\mathbf{P}_2 = (\mathbf{A} + \mathbf{I})(\mathbf{A} - 2\mathbf{I}) = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Equation (27) becomes

$$\begin{bmatrix} r_1'(t) \\ r_2'(t) \\ r_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{bmatrix}$$

with $r_1(0) = 1, r_2(0) = 0, r_3(0) = 0$. As in the previous examples, the first equation, $r_1'(t) = -r_1(t)$ with $r_1(0) = 1$ is decoupled from the second, with solution $r_1(t) = e^{-t}$. The second equation for $r_2(t)$ then becomes $r_2'(t) = e^{-t} + 2r_2(t)$ with $r_2(0) = 0$. This is a scalar constant coefficient linear ODE, easy to solve via the integrating factor approach. We find $r_2(t) = e^{2t}/3 - e^{-t}/3$. With $r_2(t)$ in hand the third equation becomes $r_3'(t) = e^{2t}/3 - e^{-t}/3 + 2r_3(t)$ with $r_3(0) = 0$. Again, this is a scalar constant coefficient linear ODE, easy to solve via the integrating factor approach. We find $r_3(t) = (3t-1)e^{2t}/9 + e^{-t}/9$. From (25) we have

$$\begin{aligned} e^{t\mathbf{A}} &= \sum_{j=0}^{2} r_{j+1}(t)\mathbf{P}_j \\ &= r_1(t)\mathbf{P}_0 + r_2(t)\mathbf{P}_1 + r_3(t)\mathbf{P}_2 \\ &= e^{-t} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{e^{2t} - e^{-t}}{3} \begin{bmatrix} 3 & 0 & 1 \\ -3 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix} + \frac{(3t-1)e^{2t} + e^{-t}}{9} \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} e^{2t} & 0 & te^{2t} \\ e^{-t} - e^{2t} & e^{-t} & e^{2t} - e^{-t} + te^{-2t} \\ 0 & 0 & e^{2t} \end{bmatrix}. \end{aligned}$$

In each example above notice how we can find the $r_j(t)$ one at a time, first $r_1(t)$ in isolation from the other $r_j(t)$, then $r_2(t)$ (using knowledge of r_1), then $r_3(t)$, and so on. In each case we can compute $r_j(t)$ as the solution to a scalar ODE by using an integrating factor approach.

Exercise 7

Let

$$\mathbf{A} = \begin{bmatrix} 2 & -6 \\ 2 & -5 \end{bmatrix}.$$

Use both diagonalization and Putzer's algorithm to compute $e^{t\mathbf{A}}$. Use this to solve $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ with $x_1(0) = 1, x_2(0) = 2$.

Exercise 8

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}.$$

Use Putzer's algorithm to compute $e^{t\mathbf{A}}$. Use this to solve $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ with $x_1(0) = 1, x_2(0) = 2$.

Exercise 9

Let

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & -1 \end{bmatrix}$$

Use Putzer's algorithm to compute $e^{t\mathbf{A}}$. Use this to solve $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ with $x_1(0) = 1, x_2(0) = 0, x_3(0) = -1$.

A FINAL REMARK

The solution (3) can be thought of as a prescription for how the initial value x_0 evolves in time by the ODE (1). If $a > 0$ then x_0 is multiplied by a factor that grows in time, while $a < 0$ means x_0 is diminished toward zero. The matrix exponential embodies the same idea. The initial condition vector $\mathbf{x}_0 \in \mathbb{R}^n$ evolves in time under the action of the matrix $e^{t\mathbf{A}}$. For any fixed t we can think of $e^{t\mathbf{A}}$ as an "operator" that maps n-dimensional vectors (like $\mathbf{x}_0$) to new n-dimensional vectors (the solution $\mathbf{x}(t)$ at a particular time.) This framework of an operator that evolves the solution to a differential equation forward in time is very useful in more sophisticated and general situations, e.g., partial differential equations or "evolution equations," and plays a large role in many more advanced areas of mathematics and physics.

APPENDIX

In this appendix we'll provide a little more analysis of the definition and properties of the matrix exponential, for the more demanding reader. The main goals are to

- Establish that the series (10) for the matrix exponential converges in an appropriate sense, for any matrix.

- Establish some of the algebraic properties of the matrix exponential.
- Establish that the derivative of $e^{t\mathbf{A}}$ with respect to t is $\mathbf{A}e^{t\mathbf{A}}$.

We do all of this using elementary facts you've seen in a calculus course, and basic matrix algebra.

A Matrix Norm

Recall from basic calculus that when studying whether a sequence or series of real numbers converges to a limit, we need a way to quantify the "distance" between two real numbers x and y. This distance is usually defined using the absolute value, as $|x-y|$, and x is considered to be "close" to y if $|x-y|$ is sufficiently close to zero. This allows us to define what it means for a sequence to converge: we say that x_n converges to a limit L when $|x_n - L|$ converges to 0 as $n \to \infty$. The situation is similar for matrices. In order to study the convergence of a series like (10), we need a way to measure the distance between matrices. The first step is to quantify the size of a matrix; this allows us to say that two matrices $\mathbf{A}$ and $\mathbf{B}$ are close to each other if the difference $\mathbf{A} - \mathbf{B}$ is suitably small. We can then make a more detailed examination of what it means for a sequence or series of matrices to converge.

Let $\mathbf{A}$ be an $n \times n$ matrix with real or complex entries. In what follows we'll use the notation $(\mathbf{A})_{jk}$ for the row j, column k entry of a matrix $\mathbf{A}$. There are many ways to define the size of $\mathbf{A}$, but one simple approach is to use the quantity $\|\mathbf{A}\|_{\max}$ where

$$\|\mathbf{A}\|_{\max} = \max_{1 \le j,k \le n} |(\mathbf{A})_{jk}|. \tag{28}$$

The quantity $|(\mathbf{A})_{jk}|$ is just the usual absolute value or magnitude of the number $(\mathbf{A})_{jk}$, whether $(\mathbf{A})_{jk}$ is real or complex. We'll call $\|\mathbf{A}\|_{\max}$ the "maximum norm" of $\mathbf{A}$.

It's straightforward to check that

1. $\|\mathbf{A}\|_{\max} \ge 0$ and $\|\mathbf{A}\|_{\max} = 0$ if and only if $\mathbf{A} = \mathbf{0}$, where "$\mathbf{0}$" here means the $n \times n$ matrix of all zeroes.
2. If k is any scalar then $\|k\mathbf{A}\|_{\max} = |k|\|\mathbf{A}\|_{\max}$.
3. For any two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ we have $\|\mathbf{A} + \mathbf{B}\|_{\max} \le \|\mathbf{A}\|_{\max} + \|\mathbf{B}\|_{\max}$. This follows from the triangle inequality $|x + y| \le |x| + |y|$ for real or complex numbers, applied to $\mathbf{A} + \mathbf{B}$ component-by-component. In fact, we'll also call this the "triangle inequality."
4. If ϵ is any nonnegative real number then $\|\mathbf{A}\|_{\max} \le \epsilon$ if and only if each component $(\mathbf{A})_{jk}$ of $\mathbf{A}$ satisfies $|(\mathbf{A})_{jk}| \le \epsilon$. This follows immediately from the very definition of $\|\mathbf{A}\|_{\max}$.
5. Given a sequence of matrices $\mathbf{A}_m$, $m = 1, 2, \ldots$ and a matrix $\mathbf{A}$,

$$\lim_{m \to \infty} \|\mathbf{A}_m - \mathbf{A}\|_{\max} = 0 \tag{29}$$

if and only if for each j and k we have

$$\lim_{m \to \infty} |(\mathbf{A}_m)_{jk} - (\mathbf{A})_{jk}| = 0. \tag{30}$$

Thus the convergence of a sequence $\mathbf{A}_m$ to a limit $\mathbf{A}$, as measured in the maximum norm, is exactly equivalent to each individual sequence $(\mathbf{A})_{jk}$ of real or complex numbers converging to $(\mathbf{A})_{jk}$.

One other useful fact is this:

Lemma 1 *If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices with real or complex entries then*

$$\|\mathbf{AB}\|_{\max} \leq n\|\mathbf{A}\|_{\max}\|\mathbf{B}\|_{\max}. \tag{31}$$

Proof: This can be proved using the definition of matrix multiplication and the triangle inequality for real or complex numbers. First, from the definition of matrix multiplication

$$(\mathbf{AB})_{jk} = \sum_{m=1}^{n} (\mathbf{A})_{jm}(\mathbf{B})_{mk}.$$

Take the magnitude of both sides above and apply the triangle inequality to the right side (in the form $|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$) to find

$$\begin{aligned} |(\mathbf{AB})_{jk}| &= \left| \sum_{m=1}^{n} (\mathbf{A})_{jm}(\mathbf{B})_{mk} \right| \\ &\leq \sum_{m=1}^{n} |(\mathbf{A})_{jm}(\mathbf{B})_{mk}| \\ &= \sum_{m=1}^{n} |(\mathbf{A})_{jm}||(\mathbf{B})_{mk}|. \end{aligned} \tag{32}$$

But $|(\mathbf{A})_{jm}| \leq \|\mathbf{A}\|_{\max}$ and $|(\mathbf{B})_{mk}| \leq \|\mathbf{B}\|_{\max}$ (this follows from the definition of the maximum norm) so

$$\sum_{m=1}^{n} |(\mathbf{A})_{jm}||(\mathbf{B})_{mk}| \leq \sum_{m=1}^{n} \|\mathbf{A}\|_{\max}\|\mathbf{B}\|_{\max} = n\|\mathbf{A}\|_{\max}\|\mathbf{B}\|_{\max}.$$

Using this last bound in (32) shows that $|(\mathbf{AB})_{jk}| \leq n\|\mathbf{A}\|_{\max}\|\mathbf{B}\|_{\max}$ for each j and k. From this and the definition of the maximum norm, (31) follows.

A consequence of Lemma 1 is

Lemma 2 *If $\mathbf{A}$ is an $n \times n$ matrix then*

$$\|\mathbf{A}^k\|_{\max} \leq n^{k-1}\|\mathbf{A}\|_{\max}^{k+1}$$

Proof: Note that Lemma 1 with $\mathbf{A} = \mathbf{B}$ yields

$$\|\mathbf{A}^2\|_{\max} \leq n\|\mathbf{A}\|_{\max}^2.$$

Then (using the above inequality)

$$\|\mathbf{A}^3\|_{\max} = \|\mathbf{A}^2\mathbf{A}\|_{\max} \leq n\|\mathbf{A}^2\|_{\max}\|\mathbf{A}\|_{\max} \leq n^2\|\mathbf{A}\|_{\max}^3.$$

An obvious induction proof shows that if we have established $\|\mathbf{A}^k\|_{\max} \leq n^{k-1}\|\mathbf{A}\|_{\max}^k$ then

$$\begin{aligned}\|\mathbf{A}^{k+1}\|_{\max} &= \|\mathbf{A}^k\mathbf{A}\|_{\max} \\ &\leq n\|\mathbf{A}^k\|_{\max}\|\mathbf{A}\|_{\max} \\ &\leq n^k\|\mathbf{A}\|_{\max}^{k+1}\end{aligned}$$

which establishes the lemma.

The Series For The Matrix Exponential Converges

Let's now show that the series (10) converges in a precise sense; the limit is what we will call $e^{\mathbf{B}}$. Given an $n \times n$ matrix $\mathbf{B}$, define the partial sum $\mathbf{S}_m$ according to (12). We will show that the sequence $\mathbf{S}_m$ approaches some limit as $m \to \infty$, or equivalently (by (29)-(30)), that each component of $\mathbf{S}_m$ approaches a limit.

Consider the row i, column j entry in $\mathbf{S}_m$, which is the sum of the row i, column j entries of $\mathbf{B}^k/k!$ for $0 \leq k \leq m$ (since matrix addition is done component-by-component). That is,

$$(\mathbf{S}_m)_{ij} = \sum_{k=0}^{m} \frac{(\mathbf{B}^k)_{ij}}{k!}. \tag{33}$$

The sum on the right in (33) above is just a sum of real (or complex) numbers. Now from Lemma 2 and the definition of $\|\mathbf{B}\|_{\max}$ we see that

$$|(\mathbf{B}^k)_{ij}| \leq \|\mathbf{B}^k\|_{\max} \leq n^{k-1}\|\mathbf{B}\|_{\max}^k. \tag{34}$$

Consider the series

$$\sum_{k=0}^{m} \frac{n^{k-1}\|\mathbf{B}\|_{\max}^k}{k!}, \tag{35}$$

an ordinary series of real numbers. This series converges, for an application of the Ratio Test shows that

$$\rho = \lim_{k\to\infty} \frac{n^k\|\mathbf{B}\|_{\max}^{k+1}/(k+1)!}{n^{k-1}\|\mathbf{B}\|_{\max}^k/k!} = \lim_{k\to\infty} \frac{n\|\mathbf{B}\|_{\max}}{k+1} = 0$$

since the numerator $n\|\mathbf{B}\|_{\max}$ doesn't depend on k. Since (35) converges and the inequality (34) holds, it follows that the series (33) converges absolutely, and so converges. In other words, $\lim_{m\to\infty}(\mathbf{S}_m)_{ij}$ exists, for each i and j. Define a matrix $\mathbf{E}$ with components

$$(\mathbf{E})_{ij} = \lim_{m\to\infty}(\mathbf{S}_m)_{ij}.$$

From Property (5) above (equations (29)-(30)) this component-by-component convergence means that the sequence of matrices $\mathbf{S}_m$ converges to $\mathbf{E}$ as $m \to \infty$, or from (33),

$$\lim_{m\to\infty} \left\| \mathbf{E} - \sum_{k=0}^{m} \frac{\mathbf{B}^k}{k!} \right\|_{\max} = 0$$

which is the definition of

$$\sum_{k=0}^{\infty} \frac{\mathbf{B}^k}{k!} = \mathbf{E}.$$

This establishes that the series (10) converges. The matrix $\mathbf{E}$ is what we take as the definition for $e^{\mathbf{B}}$.

Proof that $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}$ when A and B Commute

First note that for any two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ we have

$$\begin{aligned}(\mathbf{A}+\mathbf{B})^2 &= (\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B}) \\ &= \mathbf{AA} + \mathbf{AB} + \mathbf{BA} + \mathbf{BB} \\ &= \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2.\end{aligned}$$

In general, that's as far as things can be simplified, but if $\mathbf{AB} = \mathbf{BA}$ then $\mathbf{AB} + \mathbf{BA} = 2\mathbf{AB}$ and we have

$$(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2,$$

just as we have the familiar $(x+y)^2 = x^2 + 2xy + y^2$ for real or complex numbers.

We can generalize the above to higher powers of $\mathbf{A}+\mathbf{B}$. The Binomial Theorem

$$(x+y)^m = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} x^k y^{m-k}$$

that holds for real or complex x and y also holds for matrices $\mathbf{A}$ and $\mathbf{B}$, *if* $\mathbf{A}$ *and* $\mathbf{B}$ *commute.* That is,

$$(\mathbf{A}+\mathbf{B})^m = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \mathbf{A}^k \mathbf{B}^{m-k}. \tag{36}$$

You can demonstrate this by looking at a proof of the standard Binomial Theorem with x and y replaced by $\mathbf{A}$ and $\mathbf{B}$, and noting that any product of $\mathbf{A}$'s and $\mathbf{B}$'s that contains k copies of $\mathbf{A}$ and $m-k$ copies of $\mathbf{B}$ can be collapsed to $\mathbf{A}^k\mathbf{B}^{m-k}$, if $\mathbf{A}$ and $\mathbf{B}$ commute.

With this fact in hand, we can show that $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}$ when $\mathbf{A}$ and $\mathbf{B}$ commute. Start with

$$\begin{aligned} e^{\mathbf{A}} &= \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \cdots \frac{\mathbf{A}^j}{j!} + \cdots \\ e^{\mathbf{B}} &= \mathbf{I} + \mathbf{B} + \frac{\mathbf{B}^2}{2!} + \cdots \frac{\mathbf{B}^k}{k!} + \cdots \end{aligned} \tag{37}$$

When we compute the product of the right sides above (which will be $e^{\mathbf{A}}e^{\mathbf{B}}$) we obtain a sum of the form

$$\sum_{j,k=0}^{\infty} \frac{\mathbf{A}^k}{k!} \frac{\mathbf{B}^j}{j!} \tag{38}$$

of all pairwise products $\frac{\mathbf{A}^k}{k!}\frac{\mathbf{B}^j}{j!}$ as j and k both range independently from 0 to ∞. Consider rearranging the sum (38) by grouping together all such terms in which $j+k=m$, where m is any fixed nonnegative integer, so j ranges from 0 to m, while k ranges from m to 0. We find then that

$$(\mathbf{I}+\mathbf{A}+\frac{\mathbf{A}^2}{2!}+\cdots\frac{\mathbf{A}^j}{j!}+\cdots)(\mathbf{I}+\mathbf{B}+\frac{\mathbf{B}^2}{2!}+\cdots\frac{\mathbf{B}^k}{k!}+\cdots)=\sum_{m=0}^{\infty}\sum_{j+k=m}\frac{\mathbf{A}^k}{k!}\frac{\mathbf{B}^j}{j!}. \tag{39}$$

We can rewrite the inner sum $\displaystyle\sum_{j+k=m}\frac{\mathbf{A}^k}{k!}\frac{\mathbf{B}^j}{j!}$ on the right by letting $j=m-k$ and summing from $k=0$ to $k=m$. This inner sum becomes

$$\sum_{j+k=m}\frac{\mathbf{A}^k}{k!}\frac{\mathbf{B}^j}{j!}=\sum_{k=0}^{m}\frac{\mathbf{A}^k}{k!}\frac{\mathbf{B}^{m-k}}{(m-k)!}. \tag{40}$$

Multiply both sides of (40) by $m!$ and note that by (36), the right side of (40) becomes just $(\mathbf{A}+\mathbf{B})^m$, so

$$m!\sum_{j+k=m}\frac{\mathbf{A}^k}{k!}\frac{\mathbf{B}^j}{j!}=(\mathbf{A}+\mathbf{B})^m. \tag{41}$$

With (41) and (39) we find that (37) can be written as

$$e^{\mathbf{A}}e^{\mathbf{B}}=\sum_{m=0}^{\infty}\frac{(\mathbf{A}+\mathbf{B})^m}{m!}. \tag{42}$$

But the right side of (42) is, from (10), exactly the definition of $e^{\mathbf{A}+\mathbf{B}}$, so

$$e^{\mathbf{A}+\mathbf{B}}=e^{\mathbf{A}}e^{\mathbf{B}} \tag{43}$$

when $\mathbf{A}$ and $\mathbf{B}$ commute.

Another useful fact is that for any square matrix $\mathbf{B}$

$$\mathbf{B}e^{\mathbf{B}}=e^{\mathbf{B}}\mathbf{B}. \tag{44}$$

The proof is left as an exercise, an easy consequence of the series expansion for $e^{\mathbf{B}}$.

The Derivative of $e^{t\mathbf{A}}$

The quantity $e^{t\mathbf{A}}$ is an $n\times n$ matrix. Its derivative with respect to t is defined by

$$\frac{d}{dt}(e^{t\mathbf{A}})=\lim_{\delta t\to 0}\left(\frac{e^{(t+\delta t)\mathbf{A}}-e^{t\mathbf{A}}}{\delta t}\right). \tag{45}$$

The quantity under the limit on the right in (45) is an $n\times n$ matrix, and after passage to the limit we ought to obtain an $n\times n$ matrix for the derivative of $e^{t\mathbf{A}}$. In fact if we use the chain rule as we did in Calc I and treat $\mathbf{A}$ as a scalar it seems we should have

$$\frac{d}{dt}(e^{t\mathbf{A}})=\mathbf{A}e^{t\mathbf{A}}. \tag{46}$$

And given that $\mathbf{A}$ is an $n \times n$ matrix, the product on the right in (46) is defined and is an $n \times n$ matrix, so (46) seems very plausible. But is it correct?

The answer is yes, but we can give a solid justification for this. One easy approach is to use the Taylor series expansion (10) and a fact from introductory calculus. First, the i, j entry in $e^{t\mathbf{A}}$ is the infinite sum of the i, j entries of $(t\mathbf{A})^m/m!$, that is,

$$(e^{t\mathbf{A}})_{ij} = \sum_{m=0}^{\infty} \frac{(\mathbf{A}^m)_{ij}}{k!} t^m \tag{47}$$

using $(t\mathbf{A})^m/m! = (\mathbf{A})^m t^m/m!$. The right side of (47) is a power series in t as you have encountered in calculus, and we already established that this series converges for all real t. Recall from elementary calculus that power series may be differentiated term-by-term wherever they converge. We conclude that

$$\frac{d((e^{t\mathbf{A}})_{ij})}{dt} = \sum_{m=1}^{\infty} m \frac{(\mathbf{A}^m)_{ij}}{m!} t^{m-1} = \sum_{m=1}^{\infty} \frac{(\mathbf{A}^m)_{ij}}{(m-1)!} t^{m-1} \tag{48}$$

(the $m = 0$ term drops out). If we make a change of summation index $k = m - 1$ in the last sum on the right in (48), we find

$$\frac{d((e^{t\mathbf{A}})_{ij})}{dt} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}^{k+1})_{ij}}{k!} t^k \tag{49}$$

Equation (49) is just the component-by-component statement that

$$\frac{d(e^{t\mathbf{A}})}{dt} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^{k+1}}{k!} t^k. \tag{50}$$

Now consider the quantity $\mathbf{A}e^{t\mathbf{A}}$. If we use the Taylor series (10) for $e^{t\mathbf{A}}$ and multiply term-by-term by $\mathbf{A}$ we have

$$\mathbf{A}e^{t\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^{k+1}}{k!} t^k. \tag{51}$$

The right side of (50) and (51) are identical, and this establishes that (46) is indeed correct.

REFERENCES

[1] Putzer, E. J. 1966. Avoiding the Jordan Canonical Form in the Discussion of Linear Systems with Constant Coefficients. *The American Mathematical Monthly.* 73(1): 2-7.