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STUDENT VERSION
Unearthing the Truth

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INTRODUCTION

Prior to the Second World War, the city of Vilnius in Lithuania had over 200,000 residents, about 80,000 of whom were Jewish. Vilnius was major center for European and Jewish culture and had been described as the "Jerusalem of the North." When war broke out in 1939, Vilnius fell under the control of the Soviet Union, then briefly Lithuania, and again the Soviets. Nazi Germany captured the city in 1941 and by July of that year began the systematic extermination of the Jewish population, months before they began their genocide against the Jews elsewhere in Europe. By the end of 1941 it is estimated that about 50,000 Jews had been executed, and more followed in later years. In total it is estimated that 60,000-70,000 Jews and 30,000 others were executed by the Nazis or Lithuanian collaborators in or near Vilnius during the war, in what became known as the "Ponary Massacre" [6, 7].

Most of the victims were marched in large groups to the forest outside of Vilnius, stripped, and shot. Their bodies were then dumped into one of six large cylindrical pits, 12 to 30 meters in diameter and 5 to 8 meters deep, that had previously been dug for oil storage tanks by the Soviets, but never used. By 1943 the Soviet Union was threatening to recapture Vilnius, so the Nazis began an attempt to hide their atrocities. Approximately 80 Jewish prisoners were forced to enter and reside in the pits, exhuming and burning the remains of the bodies, sometimes finding the bodies of their own wives and children. The ashes were then mixed with sand and scattered in the forest, eliminating evidence of what the Nazis had done. The prisoners realized that as soon as their work was done, their ashes would also be added to the forest, and so they came up with an escape plan. Each night under the cover of darkness, over a period of two and a half months, they dug an escape tunnel from one of the pits, using only bare hands and implements recovered from the bodies. The tunnel ranged as deep as several meters, just wide enough for one person at a time, and led 30 meters to an opening away from the pit. On a moonless night in April of 1944 the prisoners quietly exited the pit through this tunnel, but not quietly enough. The guards were alerted and pursued the escapees with dogs and guns. Only 12 prisoners escaped into the forest. Of these, 11 survived the war to bear witness to what had happened in the forests outside of Vilnius [7].

The stories of these survivors and the escape tunnel they helped dig became a part of the history of the Holocaust. The pits and executions outside of Vilnius have been well-documented, but until recently no evidence or remains of the tunnel had been found. As this is considered hallowed ground, any type of excavation that would disturb victims' remains was out the question. But in 2016 a team of researchers were able to use new imaging methods to locate and map the remains of the tunnel below ground, without disturbing the area [4], and so contribute modern forensic evidence to the story of the survivors. The Public Broadcasting Service NOVA special [5] has a fascinating account of the work done to discover this tunnel, or see [1] for an interesting day-to-day blog of one of the geophysicists involved in the work.

One of the main methods used to image the remains of the tunnel was *electrical resistivity tomography* (ERT), a technique used in geophysical imaging. The goal of ERT is to produce an image of the subsurface structure of the earth, a sort of x-ray or CT scan of what lies below the surface in a localized area. However, ERT doesn't use x-rays, but rather, electrical measurements. It is closely related to *electrical impedance tomography* (EIT), a method that has been explored for medical diagnosis and nondestructive testing. In either technique we want to image the inaccessible interior of an object, for example, subsurface soil or a human body, using electrical current.

Refer to the left panel in Figure 1 for an illustration of how ERT works. One injects an electrical current into an object or region of interest. In Figure 1 a current is being injected into the earth at the leftmost electrode, and withdrawn at the rightmost electrode; the manner in which the current flows through the earth is affected by the subsurface electrical resistivity, and this in turn affects the voltage throughout the region, including the surface. By measuring this voltage using the other electrodes we can gather information about the subsurface electrical resistivity of the soil. This might be repeated by using other input/output electrodes, and with this information and appropriate mathematics we can build a "resistivity image" of the subsurface soil. A tunnel, even collapsed, might manifest its presence in altered resistivity of the disturbed soil.

In this project we'll look at a special case of ERT in a simplified setting, illustrated in the right panel of Figure 1, in which we seek to determine the radius of a circular nonconductive hole or void inside a bounded two-dimensional region, instead of an unbounded half-plane. In our setting the region will be the unit disk centered at the origin (the outer circle in the right panel) and the nonconductive void will be a concentric disk of unknown radius $R < 1$ (the black disk in the right panel). We have access only to the outer surface (the unit circle). We attach electrodes to the outer surface and pass electrical current through the region, illustrated by the red curves, which are flow lines for the current and must flow around the void. This alters the voltage throughout the conductive region, and in particular on the outer surface. By measuring this voltage on the outer

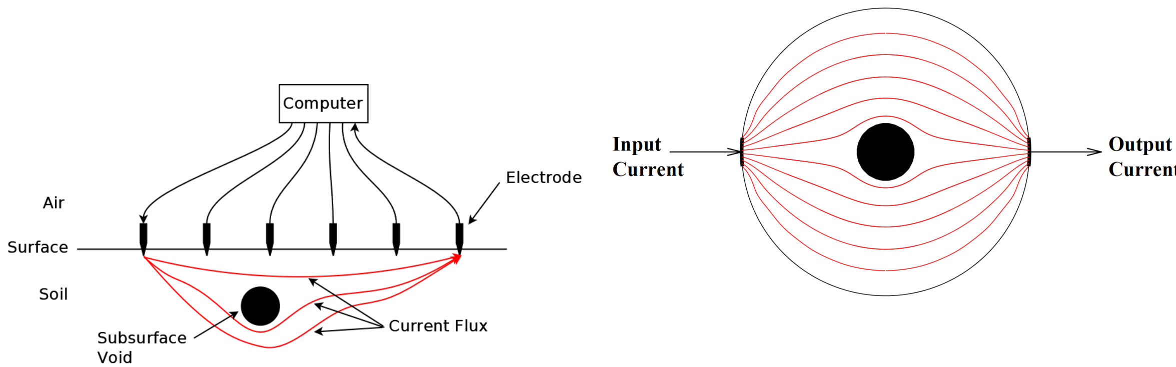

Figure 1. Left panel: Illustration of ERT for imaging subsurface structure. Right panel: Analogous problem, imaging a hole of unknown radius inside a circular region.

surface we attempt to determine the radius of the void.

To understand how this type of imaging is done, in this simplified setting or in general, we need to know a bit about how electrical current flows through a conductive material.

MODELING ELECTRICAL CONDUCTION

Current Flux

First, all of our models will be steady-state, so nothing will depend on time. There will be no magnetic fields, just electrostatic fields. We'll also stick to two dimensions; the extension to three dimensions is straightforward. Suppose that Ω denotes an electrically conductive object or region in two dimensions; Ω could be a half-space, if imaging subsurface features of the earth, or a bounded region, for example, a cross-section of a human body.

To describe the flow of electrical current through Ω we will use a vector field $\mathbf{J}(x, y)$, and denote the rectangular coordinate components of this vector field by $J_1(x, y)$ and $J_2(x, y)$, so $\mathbf{J}(x, y) = \langle J_1(x, y), J_2(x, y)\rangle$. At each point in Ω the current flux $\mathbf{J}$ has both a direction and a magnitude. The direction of $\mathbf{J}$ indicates the direction in which charge is flowing, and we'll take the conventional approach in which charge is positive, though it doesn't really matter. The magnitude $\|\mathbf{J}\| = \sqrt{J_1^2 + J_2^2}$ at a point $\mathbf{p} = (x, y)$ is defined to be the "intensity" of the current flowing past a neighborhood of $\mathbf{p}$. This "intensity" is quantified precisely in the following manner: imagine a hypothetical short line segment "L" centered at $\mathbf{p}$, with length $|L|$ and unit normal vector $\mathbf{n}$, oriented so that $\mathbf{n}$ is parallel to $\mathbf{J}$ at $\mathbf{p}$ (so L is at a right angle to $\mathbf{J}$). Refer to the left panel in Figure 2, with representative points $\mathbf{p} = \mathbf{p}_1$ or $\mathbf{p} = \mathbf{p}_2$. The red line segment in each case is a short segment L, with normal $\mathbf{n}$, oriented to be parallel to the local flow of $\mathbf{J}$. The magnitude $\|\mathbf{J}\|$ at $\mathbf{p}$ is defined so that the current flux over L (charge per unit time) is given approximately as

$$\text{current flux over } L \approx |L|\|\mathbf{J}\|,$$

at least if L is very short. More precisely, the magnitude $\|\mathbf{J}\|$ is defined as

$$\|\mathbf{J}\| = \lim_{|L|\to 0}\left(\frac{\text{current flux over } L}{|L|}\right). \tag{1}$$

This means $\mathbf{J}$ has units of charge per time per length.

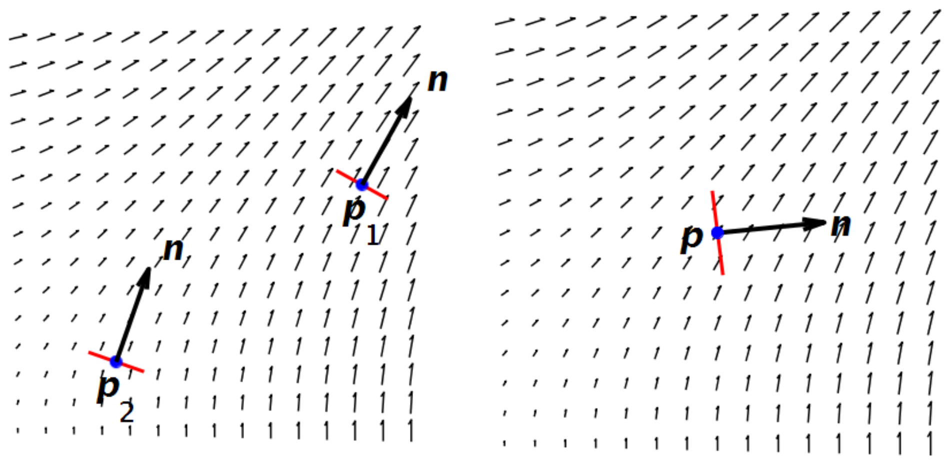

Figure 2. Left panel: Current flux field with points $\mathbf{p}_1, \mathbf{p}_2$ and short reference segments (red) with unit normal vectors in the same direction as the current flux. Right panel: point $\mathbf{p}$ with reference segment (red) and unit normal vector at arbitrary orientation.

On the right in Figure 2 we show the more general case with a point $\mathbf{p}$ and short line segment L that is not oriented at a right angle to the local current field $\mathbf{J}$. In this case the flux of $\mathbf{J}$ over L is equal to $\|\mathbf{J}\||L|\cos(\theta)$, where θ is the angle between the unit normal $\mathbf{n}$ and the current flux $\mathbf{J}$ at $\mathbf{p}$. If $\theta = 0$ this is just (1). See Problem 1 for some intuition about cases where $\theta > 0$. Since $\|\mathbf{n}\| = 1$ we can write the flux in the general case as $|L|\|\mathbf{J}\|\|\mathbf{n}\|\cos(\theta)$ or, using the basic properties of the dot product, as

$$\text{current flux over } L \approx |L|\|\mathbf{J}\|\|\mathbf{n}\|\cos(\theta) = |L|(\mathbf{J}\cdot\mathbf{n}). \tag{2}$$

Problem 1 Sketch a situation in which the current flux $\mathbf{J}$ at a point $\mathbf{p}$ cuts orthogonally across a short line segment L centered on $\mathbf{p}$ and $\mathbf{n}$ points in the same direction at $\mathbf{J}$. In this case $\theta = 0$. Repeat for the case in which $\mathbf{n}$ is orthogonal to $\mathbf{J}$ at $\mathbf{p}$. What is θ here, and why does (2) make physical sense in this case? Suppose that $\mathbf{n}$ is exactly opposed to $\mathbf{J}$—what is θ now? What does (2) give for the current flux over L, and how does it compare to the case $\theta = 0$?

Conservation of Charge

An important fact is that if electric charge is conserved then

$$\mathrm{div}(\mathbf{J}) = 0 \tag{3}$$

inside Ω, where $\mathrm{div}(\mathbf{J})$ is the *divergence of* $\mathbf{J}$. In rectangular coordinates $\mathrm{div}(\mathbf{J}) = \frac{\partial J_1}{\partial x} + \frac{\partial J_2}{\partial y}$.

To see why (3) is true, suppose that C is a simple smooth curve inside Ω, and let $\mathbf{n}$ be a unit normal vector on C, consistently oriented (so as we move along C the vector $\mathbf{n}$ changes continuously, and always points to the same "side" of C.) The total current flux of $\mathbf{J}$ over C in the direction of $\mathbf{n}$ is given by the integral

$$\text{flux of } \mathbf{J} \text{ over } C = \int_C \mathbf{J} \cdot \mathbf{n}\, ds \tag{4}$$

where ds is arc length on C. Equation (4) follows by chopping the curve C up into many short pieces of length "ds", applying (2) to each piece (yielding net flux $\mathbf{J} \cdot \mathbf{n}\, ds$ over each piece) and summing (integrating) the results.

If C is also a closed curve that bounds a region D and charge is conserved then, since we're in the steady-state case, it must be that the current flux over C is zero, for otherwise charge would be building up or being depleted in D, which is not steady-state. Thus

$$\int_C \mathbf{J} \cdot \mathbf{n}\, ds = 0. \tag{5}$$

If we apply the Divergence Theorem and use (5) we find

$$\int_D \mathrm{div}(\mathbf{J})\, dA = \int_C \mathbf{J} \cdot \mathbf{n}\, ds = 0 \tag{6}$$

where dA is the area element. But the curve C is an arbitrary simple closed curve, so D on the left in (6) could be any region with such a boundary, for example, any disk. It is a fact that if $f(x, y)$ is a continuous function defined in some two-dimensional region and

$$\int_D f(x, y)\, dA = 0 \tag{7}$$

for *every* disk D then f must be identically zero. (To see this, suppose $f(x_0, y_0) > 0$ at some point (x_0, y_0). Then $f(x, y) > 0$ for all (x, y) close enough to (x_0, y_0), say within a distance ϵ, since f is continuous. Take D to be a disk of radius ϵ centered at (x_0, y_0) and (7) would be violated since the integral of a positive function must be positive.) Using this reasoning we see that the integrand in the left integral of (6) is forced to be identically zero in Ω, and so (3) must hold inside the conductive region.

Problem 2 Use (3) to decide which of the following vector fields $\mathbf{J}$ could represent the flux for a steady-state current flow in some region Ω.

a. $\mathbf{J}(x, y) = < 1, 3 >$.

b. $\mathbf{J}(x, y) = < x, y >$.

c. $\mathbf{J}(x, y) = < x, -y >$.

d. $\mathbf{J}(x, y) = < x^2 y, -y >$.

e. $\mathbf{J}(x, y) = < xy, -y^2/2 >$.

Problem 3 Suppose that $\mathbf{J}(x, y) = < x + x^2 y, f(x, y) >$ represents the current flux of a steady-state flow. What can you deduce about $f(x, y)$?

Ohm's Law in Two (or Three) Dimensions

Recall Ohm's Law for a resistor: $V = IR$, where V is the voltage (potential) drop across the resistor, I is the current through the resistor in the direction of the voltage drop, and R is the *resistance* of the resistor. It is the drop from high potential at one end of the resistor to a lower potential at the other end that indicates the presence of an electric field in the resistor, which impels charge to move and current to flow. The rate at which current flows is (often) proportional to the voltage drop, that is, $I = V/R$, hence Ohm's Law. In this context $1/R$ is called the *conductivity* of the resistor.

We will generalize Ohm's Law to two (or three) dimensions. We now make a modeling assumption. Electrical current is just moving charge. Why would charge move? Charges move because they are being pushed by an electrical field. In the simplest case a positively charged particle will move in the direction of the electric field. Based on this reasoning, we are going to assume that inside whatever electrical conductor we're dealing with we have

$$\mathbf{J} = \gamma \mathbf{E} \tag{8}$$

where $\mathbf{E}$ is the electric field inside Ω and $\gamma \geq 0$ is called the *electrical conductivity* of Ω. The scalar quantity γ could be constant throughout the object, or it could be a function of position. It doesn't matter for now. Equation (8) will be the two-dimensional analogue of Ohm's Law. It says that charge flows in the direction it's being pushed by $\mathbf{E}$, in proportion to the magnitude of $\mathbf{E}$.

You should think about why γ is called the "conductivity." For a fixed electrical field $\mathbf{E}$, large values of γ (high conductivity) result in larger $\mathbf{J}$, while small γ results in little current flow. The case $\gamma \equiv 0$ corresponds to an insulator, in which no current flows at all, regardless of the electric field. On the other hand if "$\gamma = \infty$" (a perfect conductor) then we must have $\mathbf{E} = 0$, or else unlimited current would flow. This is the (possibly familiar) fact that the electric field inside a perfect conductor is zero. The quantity $1/\gamma$ is a called the *resistivity*, and of course may be a function of position.

The Equation for the Electric Potential

We will use $\phi(x, y)$ for the electric potential (voltage) inside Ω. Recall that potential is always "relative" to some fixed zero point, which we'll worry about later. Our ultimate goal is to derive a differential equation that ϕ must satisfy.

It is a fact that for an electrostatic field $\mathbf{E}$ we have $\mathbf{E} = -\nabla\phi$ where ϕ is the potential or voltage function; that is, $\mathbf{E}$ is a *conservative* vector field. From (8) this means that

$$\mathbf{J} = -\gamma\nabla\phi. \tag{9}$$

Combining this with (3) shows that ϕ must satisfy

$$\operatorname{div}(\gamma\nabla\phi) = 0 \tag{10}$$

inside Ω. This is a partial differential equation (PDE) for ϕ. Keep in mind that in two dimensions ϕ and γ are functions of x and y, while in three dimensions they're functions of x, y, and z. You can write out (10) explicitly in two dimensional rectangular coordinates, as

$$\gamma\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{\partial\phi}{\partial x}\frac{\partial\gamma}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\gamma}{\partial y} = 0. \tag{11}$$

In the special case in which $\gamma > 0$ is constant, (10) becomes $\operatorname{div}(\nabla\phi) = 0$, or in two dimensions with rectangular coordinates,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0. \tag{12}$$

Equation (12) is *Laplace's equation*, one of the most common and important equations of mathematical physics. A function $\phi(x, y)$ that satisfies (12) is said to be *harmonic*.

Given the modeling assumptions we have made, (10) must be satisfied at all points $(x, y) \in \Omega$. Thus if we knew the conductivity γ inside the object, we could try to solve the PDE (10) to find ϕ. But there still isn't enough information to nail down a unique solution for the potential $\phi(x, y)$. We must also specify boundary conditions, either the voltage ϕ or the input current, on the boundary of Ω.

Boundary Conditions—Injecting a Current Flux

We will use $\partial\Omega$ to denote the boundary of Ω. For example, if Ω is a two-dimensional half-space $\{(x, y); y < 0\}$ then $\partial\Omega$ consists of the x axis; if Ω is a disk, $\partial\Omega$ is the outer boundary, a circle. In our application we inject a specified current into $\partial\Omega$ and then measure the resulting potential (voltage) on $\partial\Omega$. Refer to Figure 3, in which a hypothetical current field is shown in a portion of Ω and a corresponding portion of $\partial\Omega$ is shown as a quarter circle. At the point $\mathbf{p}$ on $\partial\Omega$ the current flux is generally outward from Ω. A short portion of $\partial\Omega$ near $\mathbf{p}$ is shown in red; this short portion is, to good approximation, a line segment L, with indicated outward unit normal vector $\mathbf{n}$. As per equation (2) above, the rate at which current is leaving Ω over L is $|L|(\mathbf{J} \cdot \mathbf{n})$, or equivalently, the rate at which current is entering Ω over L is minus this, so

$$\text{rate current enters } \Omega \text{ across } L = -|L|(\mathbf{J} \cdot \mathbf{n}). \tag{13}$$

When we inject current into a region Ω at the boundary it is conventional to specify the amount of charge injected per unit time (current) *per length of the boundary* at any point on $\partial\Omega$. That is to say, we specify $-\mathbf{J}\cdot\mathbf{n}$ on $\partial\Omega$ (so $|L|$ is divided out in (13)). Let g be such a specified input current density, so that our boundary condition becomes $-\mathbf{J}\cdot\mathbf{n} = g$ on $\partial\Omega$. If we make use of (9) we obtain a boundary condition

$$\gamma\nabla\phi\cdot\mathbf{n} = g \tag{14}$$

for ϕ on $\partial\Omega$. A boundary condition of the form (14), in which we specify information related to $\nabla\phi\cdot\mathbf{n}$, the normal derivative of ϕ, is called a *Neumann boundary condition.*

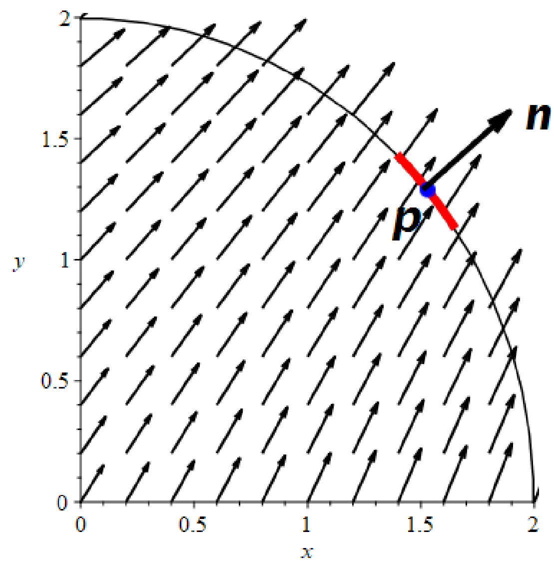

Figure 3. Current flux field on a portion of Ω, with a short "linear" piece of $\partial\Omega$ near $\mathbf{p}$, unit outward normal vector $\mathbf{n}$ Charge exits Ω at a rate of $|L|(\mathbf{J}(\mathbf{p})\cdot\mathbf{n})$ Coulombs per second.

The Forward Problem

For a given region Ω with boundary $\partial\Omega$, a specified conductivity $\gamma(x,y) > 0$ defined on Ω, and a specified input current flux function g defined on $\partial\Omega$, equation (10) and the boundary condition (14) constitute a *boundary value problem*, to be solved for the function $\phi(x,y)$. Physically, we inject a current flux g into a region Ω with internal conductivity γ and seek to find $\phi(x,y)$, the electrical potential or voltage at all points in Ω. This is the traditional thing to do with a differential equation and boundary data—find the solution. We refer to this as the *forward* or *direct* problem.

In the case that Ω is a bounded region it turns out that equation (10) with the boundary condition (14) always has a solution if g satisfies a certain condition (see Problem 7 below), but the solution is not quite unique; you can easily check that if $\phi(x,y)$ is any solution then so is $\phi(x,y)+c$ for any constant c. However, this is the only way solutions can fail to be unique: if ϕ_1 and ϕ_2 both satisfy (10) with the boundary condition (14) then $\phi_2 = \phi_1 + c$ for some constant c. Physically, this means that potentials are only unique up to an arbitrary additive constant (determined by the "ground" reference point.) If Ω is unbounded (e.g., a half plane) then we also need to specify that the solution ϕ is bounded as $(x, y) \to \infty$ to obtain a unique solution.

Problem 4 Suppose that Ω is the square $0 < x, y < 1$ in the plane. We input a current flux g equal to -1 on the left side of Ω (where $x = 0$), 0 on the top and bottom of Ω, and $g = 1$ on the right ($x = 1$) side of Ω. Physically, we are injecting a current of 1 ampere per meter of boundary on the right edge of Ω and withdrawing 1 ampere per meter of boundary on the left edge; the top and bottom have no current input or output, i.e., they are insulated. Verify that the function $\phi(x, y) = x + c$ satisfies equation (10) with $\gamma \equiv 1$, and that the boundary condition (14) holds, for any constant c. Hint: the unit outward normal vector $\mathbf{n}$ equals $< -1, 0 >$ on the left side of Ω and is equally easy to find on the other sides.

Problem 5 Suppose that Ω is the square $0 < x, y < 1$ in the plane, with conductivity $\gamma(x, y) = 1 + xy$. We input a current flux g given by

$$g = \begin{cases} 0, & \text{on } x = 0 \text{ (the left side of } \Omega) \\ -2x - 2, & \text{on } y = 1 \text{ (the top of } \Omega) \\ 2y + 2, & \text{on } x = 1 \text{ (the right side of } \Omega) \\ 0, & \text{on } y = 0 \text{ (the bottom of } \Omega) \end{cases}$$

Verify that the function $\phi(x, y) = x^2 - y^2 + c$ satisfies equation (10) with $\gamma(x, y) = 1 + xy$, and that the boundary condition (14) holds, for any constant c.

Problem 6 Show that if $\phi(x, y)$ is a solution to (10) with boundary condition (14) then so is $\phi(x, y) + c$ for any constant c.

Problem 7 Show that if $\operatorname{div}(\mathbf{J}) = 0$ holds in Ω then it must be the case that

$$\int_{\partial\Omega} \mathbf{J} \cdot \mathbf{n} \, ds = 0$$

where $\mathbf{n}$ is an outward unit normal vector on $\partial\Omega$. Hint: Look at (5). Use this to deduce that if (14) holds then it must be the case that

$$\int_{\partial\Omega} g \, ds = 0.$$

Thus equation (10) with the boundary condition (14) is solvable only if g integrates to zero around $\partial\Omega$. What is the physical interpretation of this condition? Hint: in a steady-state problem, what goes in must come out.

Solution to the Forward Problem on an Annulus

In general the solution to (10) can't be written out explicitly, except in special cases. One such important special case is that of constant conductivity and a simple geometry for Ω, for example, a rectangle, disk, or annulus. It is the last geometry that we'll consider, with conductivity $\gamma \equiv 1$. This simplified setting will serve to illustrate the essential idea of ERT: it is possible to determine something about the interior electrical properties or geometry of a region from input current/output voltage data.

So to narrow our focus, we suppose that Ω is an annulus. Given the circular symmetry of the annulus, it's most convenient to work in polar coordinates (r, θ). Let Ω have inner radius R and outer radius 1, so in polar coordinates with $r = 0$ as the center of the annulus we have

$$\Omega = \{(r, \theta); R < r < 1, -\pi < \theta \leq \pi\}. \tag{15}$$

Think of Ω as an object to which we have access only at $r = 1$, the outer surface. Our goal is to determine the radius R of the inside "hole" from data taken on the outside at $r = 1$. We will assume that Ω has constant conductivity 1. The hole $r < R$ is to be interpreted as a nonconductive void, through which electrical current cannot pass. One might think of it as a cross-section of a cylindrical tunnel.

To image the inside of Ω we apply an input current flux g at $r = 1$; the function g thus varies only with the angular variable θ, that is, $g = g(\theta)$. This causes current to flow through Ω, and an electric potential is induced throughout Ω. Let's write this potential as $\phi(r, \theta)$, a function of polar radius r and angle θ. In polar equations Laplace's equation, which ϕ must satisfy, becomes

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0. \tag{16}$$

The usual process of separating variables, as $\phi(r, \theta) = p(r)q(\theta)$ leads us to separable solutions to Laplace's equation of the form

$$1, \ln(r), r^k \sin(k\theta), r^k \cos(k\theta), r^{-k} \sin(k\theta), r^{-k} \cos(k\theta)$$

where $k \geq 1$ is an integer. Moreover, any linear combination of these basic functions is also a solution to Laplace's equation. Note that since $r > R$ in Ω, the function $\ln(r)$ and r^{-k} pieces are valid possibilities, since we cannot approach $r = 0$. We thus consider solutions that consist of a linear combination

$$\phi(r, \theta) = c_0 + d_0 \ln(r) + \sum_{k=1}^{\infty} \left((c_k r^k + c_{-k} r^{-k}) \cos(k\theta) + (d_k r^k + d_{-k} r^{-k}) \sin(k\theta) \right) \tag{17}$$

of the above functions, for appropriate constants c_k, d_k, $-\infty < k < \infty$. By adjusting the c_k and d_k for $-\infty < k < \infty$ we will be able to obtain the required boundary conditions.

On the outer boundary $r = 1$ it's easy to see that the unit normal vector $\mathbf{n}$ points radially outward; the quantity $\nabla\phi \cdot \mathbf{n}$ is thus just the directional derivative of ϕ in the radial direction, that is, $\nabla\phi \cdot \mathbf{n} = \frac{\partial \phi}{\partial r}$; we want this quantity to equal $g(\theta)$ on the outer boundary $r = 1$. By differentiating the right side of (17) term-by-term with respect to r we find that, from the power rule,

$$\frac{\partial \phi}{\partial r}(r, \theta) = d_0/r + \sum_{k=1}^{\infty} \left((kc_k r^{k-1} - kc_{-k} r^{-k-1}) \cos(k\theta) + (kd_k r^{k-1} - kd_{-k} r^{-k-1}) \sin(k\theta) \right). \tag{18}$$

(The constant c_0 drops out.) At $r = 1$ we find

$$\frac{\partial \phi}{\partial r}(1, \theta) = d_0 + \sum_{k=1}^{\infty} \left((kc_k - kc_{-k}) \cos(k\theta) + (kd_k - kd_{-k}) \sin(k\theta) \right). \tag{19}$$

We must choose the c_k and d_k so that the expression on the right equals $g(\theta)$. Now $g(\theta)$ itself has a Fourier sine/cosine expansion of the form

$$g(\theta) = a_0/2 + \sum_{k=1}^{\infty} (a_k \cos(k\theta) + b_k \sin(k\theta))$$

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\cos(k\theta)\,d\theta, \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\sin(k\theta)\,d\theta. \tag{20}$$

In the present case $a_0 = 0$ (recall Problem 7). In order for the right side of (19) to match $g(\theta)$ we must match the remaining Fourier sine/cosine coefficients, so that we need $d_0 = a_0/2 = 0$ and

$$k(c_k - c_{-k}) = a_k \tag{21}$$

$$k(d_k - d_{-k}) = b_k \tag{22}$$

for $k \geq 1$. For each integer $k \geq 1$ equation (21) is a single equation for two unknowns c_k and c_{-k}. The same holds true for equation (22) and unknowns d_k, d_{-k}. We need another equation for each pair of unknowns.

The additional equations come from a boundary condition on $r = R$. On this inside radius $r = R$ no electrical current can enter or exit Ω, since the region $r < R$ is nonconductive. On $r = R$ the outward unit normal vector $\mathbf{n}$ to $\partial\Omega$ points directly toward the origin, and in fact $\nabla\phi\cdot\mathbf{n} = -\frac{\partial\phi}{\partial r}(R,\theta)$. Since no current can cross the boundary $r = R$ we must have $-\frac{\partial\phi}{\partial r}(R,\theta) = 0$. Making use of (18) with $r = R$ leads us to

$$\frac{\partial\phi}{\partial r}(r,\theta) = \sum_{k=1}^{\infty}\left(k(c_k R^{k-1} - c_{-k}R^{-k-1})\cos(k\theta) + k(d_k R^{k-1} - d_{-k}R^{-k-1})\sin(k\theta)\right) = 0 \tag{23}$$

since we deduced that $d_0 = 0$ above. In order for the sum in (23) to be identically zero the Fourier sine/cosine coefficients must all be zero, so that

$$k(c_k R^{k-1} - c_{-k}R^{-k-1}) = 0 \tag{24}$$

$$k(d_k R^{k-1} - d_{-k}R^{-k-1}) = 0 \tag{25}$$

for each integer $k \geq 1$. Note that k can be divided out of each equation above.

Equations (21) and (24) are a pair of equations for c_k and c_{-k}, for each $k \geq 1$, with solution

$$c_k = \frac{R^{-k}}{k(R^{-k} - R^k)}a_k \tag{26}$$

$$c_{-k} = \frac{R^{k}}{k(R^{-k} - R^k)}a_k. \tag{27}$$

Similar reasoning shows that we need

$$d_k = \frac{R^{-k}}{k(R^{-k} - R^k)}b_k \tag{28}$$

$$d_{-k} = \frac{R^{k}}{k(R^{-k} - R^k)}b_k. \tag{29}$$

In summary, we can solve Laplace's equation (16) on the annulus Ω defined by (15) with boundary data $\frac{\partial u}{\partial r}(1,\theta) = g(\theta)$ (physically, a current flux injected into the outer boundary) and $\frac{\partial u}{\partial r}(R,\theta) = 0$ (a nonconductive void of radius R in the middle) with the following procedure:

1. Compute all Fourier sine/cosine coefficients a_k, b_k for $g(\theta)$ via (20) for $k \geq 1$ (note $a_0 = 0$).
2. Set $d_0 = 0$ and compute c_k, d_k for all integers $k \neq 0$ according to (26)-(29).
3. Form solution $\phi(r,\theta)$ via (17). Here c_0 is an arbitrary constant that is dictated by our reference or "ground" value for ϕ. It is common to chose c_0 so that $\displaystyle\int_{-\pi}^{\pi} \phi(1,\theta)\,d\theta = 0$ (the solution is normalized to have zero integral over the outer boundary $r = 1$). This dictates the value $c_0 = 0$.

Let's use the procedure above to work an example.

Example 1 Suppose that the input current flux is given by $g(\theta) = 2\cos(\theta)$ on $r = 1$ and that the inside hole has radius $R = 1/2$. Note that g integrates to zero over this curve, that is, $\int_{-\pi}^{\pi} g(\theta)\,d\theta = 0$, so the boundary value problem is solvable. In Step 1 above we find Fourier sine/cosine coefficients $a_1 = 2$ and all other $a_k = 0$, as well as all $b_k = 0$ (that is, happily, g has a finite Fourier sine/cosine expansion). In Step 2 using (26) shows that $c_1 = 8/3, c_{-1} = 2/3$, and all other $c_k = 0$, as well as $d_k = 0$ for all k. In Step 3 we make use of(17) to see that the solution to Laplace's equation with this outer boundary data is

$$\phi(r,\theta) = \frac{8}{3} r\cos(\theta) + \frac{2}{3r}\cos(\theta).$$

It's straightforward to check that $\frac{\partial \phi}{\partial r}(1,\theta) = 2\cos(\theta)$ and $\frac{\partial \phi}{\partial r}(1/2,\theta) = 0$.

Problem 8 Suppose $R = 1/3$, $\gamma \equiv 1$, and the input flux is $g(\theta) = 2\sin(4\theta)$. Use the procedure illustrated in Example 1) to compute $\phi(r,\theta)$.

THE INVERSE PROBLEM AND ERT

The recipe above and its application in Example 1 illustrates the classic task in differential equations: Given a domain, a differential equation, and boundary (and/or initial) data, find the solution. In ERT we're interested in a slight twist on this, the *inverse problem.* We are actually given the solution, or at least some information about the solution, and asked to find coefficients in the differential equation, or deduce something about the geometry of the domain that we can't see from the outside. In the present case we will be given the input current flux $g(\theta)$, the "measured" potential $\phi(1,\theta)$ on the outer boundary of Ω, and then asked to deduce the inner radius R, under the assumption that Ω has a constant conductivity $\gamma = 1$.

Example 2 Suppose that, as in Example 1, we are told that a current flux $g(\theta) = 2\cos(\theta)$ was injected into Ω and the resulting boundary potential was measured to be $\phi(1,\theta) = \frac{10}{3}\cos(\theta)$ (the solution from Example 1 at $r = 1$). The radius R of the void (tunnel?) is unknown; possibly even $R = 0$, so the void isn't present. Can we deduce the value of R?

The solution (17) with the associated recipe for the c_k and d_k is still perfectly valid. We simply don't know R. But we do know from $g(\theta) = 2\cos(\theta)$ that $a_1 = 2$ and all other $a_k = 0$, as well as all $b_k = 0$, just as in Example 1. The formulae (26)-(29) for the c_k and d_k then dictate that

$$c_1 = \frac{2R^{-1}}{(R^{-1} - R)}, \tag{30}$$

$$c_{-1} = \frac{2R}{(R^{-1} - R)} \tag{31}$$

with all other $c_k = 0$ and all $d_k = 0$. We might be able to use (30) or (31) and determine R if we know something about c_1 and/or c_{-1}, which we do! Equation (17) indicates that

$$\phi(r, \theta) = (c_1 r + c_{-1}/r)\cos(\theta).$$

When $r = 1$ as on the outer boundary of Ω this becomes

$$\phi(1, \theta) = (c_1 + c_{-1})\cos(\theta).$$

Since we "measured" $\phi(1, \theta) = \frac{10}{3}\cos(\theta)$ we conclude that

$$c_1 + c_{-1} = \frac{10}{3}$$

after cancelling the cosine terms. With (30) and (31) this becomes

$$\frac{2(R^{-1} + R)}{(R^{-1} - R)} = \frac{10}{3}, \tag{32}$$

an equation for R with unique nonnegative solution $R = 1/2$ ($R = -1/2$ also works, but our context demands $R \geq 0$). If it's there, we can find the radius of the void!

The natural questions to ask about this inverse problem are those one asks for any inverse problem. Among these are

Uniqueness: Does the given data uniquely determine the unknown quantity? If not, what else is needed?
Reconstruction: How can we efficiently find the unknown from the given data?
Stability: If the data are noisy or erroneous, how will this affect our estimate of the unknown?

In Example 2 above we saw that the value of R was uniquely determined by equation (32), and easy to reconstruct (it results in a quadratic equation for R). In Problems 10-11 below you can examine a more general case.

Problem 9 Suppose that the input current flux $g(\theta) = \cos(3\theta)$; the resulting boundary voltage is $\phi(1, \theta) = \frac{4097}{12285}\cos(3\theta)$ (no noise!) Determine R by emulating Example 2. A computer algebra system might help. Is $R \geq 0$ uniquely determined?

Problem 10 Suppose that the input current flux $g(\theta) = \cos(m\theta)$ for some integer $m \geq 1$; we measure the boundary voltage $\phi(1,\theta)$ for $-\pi < \theta \leq \pi$.

a. Suppose the true inner radius is R_0 (unknown, but $0 < R_0 < 1$). Solve the forward problem to show that the resulting boundary voltage is $\phi(1,\theta) = q_m \cos(m\theta)$ where

$$q_m = \frac{1 + R_0^{2m}}{m(1 - R_0^{2m})}.$$

Show that $q_m > 1/m$.

b. Show that to solve the inverse problem we must solve the equation

$$\frac{1 + R^{2m}}{m(1 - R^{2m})} = q_m$$

for unknown R, and that there is only one nonnegative solution for R if $q_m > 1/m$. Hint: Try substituting $T = R^{2m}$ so that the equation above becomes

$$\frac{1 + T}{m(1 - T)} = q_m. \tag{33}$$

When $T = 0$ the left side of (33) equals $1/m$; as $T \to 1^-$ the left side approaches ∞. Differentiate the left side of (33) with respect to T and show that $(1+T)/(1-T)$ is strictly increasing for $0 < T < 1$, and conclude that (33) has a unique solution for T. Since $R^{2m} = T$, what does this say about R?

Problem 11 Suppose the true inner radius is R_0. Show that if g is any input flux that is not identically zero (but that integrates to zero around $\partial\Omega$, of course) then the data $\phi(1,\theta)$ allows us to uniquely determine R_0. Hint: If g is not identically zero then at least one Fourier coefficient a_m or b_m is not zero.

EXTENSIONS AND CONCLUSION

We've considered a rather restricted case of ERT, but it does illustrate the essential idea—one can use input current/output voltage measurements on the boundary of a region to deduce information about the interior. In the present case we considered the unknown to be a nonconductive circular void centered at $r = 0$ in the unit disk, but this can be relaxed to allow the void to be centered elsewhere, or not a disk, or with some nonzero electrical conductivity. Of course Ω does not have to be the unit disk. Depending on how many quantities are considered unknown, it may take multiple input current/output boundary voltage measurements to find the unknowns, if it is even possible. Also, the solutions to the boundary value problem will not have a simple closed form, so more sophisticated or approximate methods may be needed. The most general extension allows the unknown interior conductivity to be a function of position, so $\gamma = \gamma(x, y)$ (or $\gamma(x, y, z)$ in three dimensions). In this case multiple input/output data sets will definitely be required to find any significant information about γ and form a resistivity image. See [3] for a survey of methods used for ERT, or [2] for an overview of the closely related EIT problem.

REFERENCES

[1] URL `http://www.paulbaumangeophysics.com/blog/previous/6`. Accessed 15 August 2019.

[2] Cheney M., D. Isaacson, and J. C. Newell. 1999. Electrical impedance tomography. *SIAM Review* 41(1): 85-101.

[3] Loke, M. and R. Barker. 1996. Practical techniques for 3D resistivity surveys and data inversion. *Geophysical Prospecting.* 44: 499-523.

[4] McClymont, A., R. Bauman, R. Freund, J. Seligman, J., H. Jol, K. Bensimon, K. and P. Reeder. 2017. The Forgotten Holocaust in Lithuania - Geophysical Investigations at the Ponary Extermination Site. In the *Proceedings of the 23*$^{\text{rd}}$ *European Meeting of Environmental and Engineering Geophysics*, Malmö. Sweden. 3-7 September.

[5] Public Broadcasting Service NOVA Special "Holocaust Escape Tunnel." 19 April 2017. `https://www.pbs.org/wgbh/nova/video/holocaust-escape-tunnel`, also freely available at `https://www.dailymotion.com/video/x5j57yg`.

[6] Sakowicz, K. and Y. Arad. 2005. *Ponary Diary, 1941-1943: A Bystander's Account of a Mass Murder*. (Translated by Laurence Weinbaum).New Haven: Yale University Press.

[7] Shaer, M. 2017. The Holocaust's Great Escape, *Smithsonian Magazine.* 47(10): 42-53.