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STUDENT VERSION
AN INTERACTIVE INTRODUCTION TO REGULAR PERTURBATION FOR ORDINARY DIFFERENTIAL EQUATIONS

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STATEMENT

The purpose of this paper is to introduce a mathematical method known as *regular perturbation*. Regular perturbation is a tool for obtaining analytical approximations to problems for which closed-form solutions do not exist or are very difficult to obtain. But why the need for another method? Would not numerical methods fill the gaps? As is often the case with real-world problems, the models that describe their dynamical behavior are approximations to the physical reality. Therefore, there is no compelling reason to suggest that an exact or numerically accurate solution to an imperfect model would be superior to the analytical approximate solution produced by perturbation methods. Indeed, solutions obtained by perturbation can often be used to validate numerical solutions and provide further insight into the mathematical model of a physical problem.

This paper presents regular perturbation for ordinary differential equations through examples. In addition, electronic spreadsheet models have been developed to allow the reader to interactively alter parameters (the small parameter ε) and observe the effect of the changes on the solutions.

A Brief Introduction to Regular Perturbation for Ordinary Differential Equations

Consider an nth-order ordinary differential equation of the form

$$F(y^{(n)}, y^{(n-1)}, \ldots, y', y, x; \varepsilon) = 0 \tag{1}$$

where $y^{(n)} \equiv \frac{d^n y}{dx^n}$ and ε is a small parameter ($\varepsilon \ll 1$).

The idea behind perturbation is to generate an analytical approximation to the solution of a problem such as (1). The approximate solution is a power series in the small parameter ε. This series, known as *perturbation series*, has the form

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \varepsilon^3 y_3(x) + \dots \tag{2}$$

and quantifies successively decreasing corrections ($\varepsilon y_1(x)$, $\varepsilon^2 y_2(x)$, $\varepsilon^3 y_3(x)$, $\dots$) on the exact solution $y_0(x)$ to the simple *unperturbed problem* (when $\varepsilon = 0$). The functions $y_1(x)$, $y_2(x)$, $y_3(x)$, $\dots$ are the higher order terms. An approximate perturbation solution is obtained by truncating the series, usually by retaining only the first two terms, i.e. a first-order approximation $y(x) \approx y_0(x) + \varepsilon y_1(x)$.

Although this paper is concerned with regular perturbation problems, there are many other perturbation methods that are suitable to types of problems not discussed here (e.g., singular perturbation, multiple scales, boundary layers, among others). The interested reader can learn more about perturbation methods from these excellent sources [1]–[3].

The examples that follow involve differential equations for which exact or numerical solutions are obtainable. By doing so the exact or numerical solutions serve as benchmarks to assess the reasonableness of the analytical approximations produced by perturbation. It is pointed out, however, that the existence of an exact or numerical solution to a problem is not a requirement for the applicability of perturbation methods.

Suggested Activities

Activity 1. Bernoulli Differential Equation

Consider the *Bernoulli differential equation*

$$y' + y = \varepsilon y^3, \qquad y(0) = 1. \tag{3}$$

The exact solution to this equation is

$$y(x) = \frac{1}{\sqrt{(1-\varepsilon)e^{2x} + \varepsilon}}, \tag{4}$$

which can be verified either by direct substitution into (3) or solving (3) through the change of variables $u = y^{-2}$.

Now consider obtaining an approximate solution to (3) using perturbation. Follow these steps:

Step 1. Start with the perturbation series

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \varepsilon^3 y_3(x) + \dots \tag{5}$$

where $y_0(x)$, $y_1(x)$, $y_2(x)$, $\dots$ are functions to be determined.

Step 2. Substitute (5) into (3) (both the differential equation and the initial condition), i.e.,

$$\begin{aligned} &\big(y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \varepsilon^3 y_3(x) + \dots\big)' \\ &+ \big(y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \varepsilon^3 y_3(x) + \dots\big) \\ = \ &\varepsilon\big(y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \varepsilon^3 y_3(x) + \dots\big)^3 \end{aligned}$$

along with

$$y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0) + \varepsilon^3 y_3(0) + \ldots = 1$$

and expand in powers series of ε to obtain

$$\begin{aligned} & y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \varepsilon^3 y_3' + \ldots \\ & + y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon^3 y_3 + \ldots \\ = \; & \varepsilon y_0^3 + 3\varepsilon^2 y_0^2 y_1 + \varepsilon^3(3y_0^2 y_2 + 3y_0 y_1^2) + \ldots \end{aligned}$$

where the dependence of the functions on x has been dropped to simplify the writing.

Notice that the initial condition has to hold for any ε. Thus,

$$y_0(0) = 1, \quad y_1(0) = 0, \quad y_2(0) = 0, \quad y_3(0) = 0, \quad \ldots$$

Step 3. Equate powers of ε:

$$\begin{array}{lll} \varepsilon^0 : & y_0' + y_0 = 0, & y_0(0) = 1, \\ \varepsilon^1 : & y_1' + y_1 = y_0^3, & y_1(0) = 0, \\ \varepsilon^2 : & y_2' + y_2 = 3y_0^2 y_1, & y_2(0) = 0, \\ \varepsilon^3 : & y_3' + y_3 = 3y_0^2 y_2 + 3y_0 y_1^2, & y_3(0) = 0, \\ \ldots & & \end{array}$$

Observe that once y_0 is determined (the ε^0-equation), it is then used to determine y_1 (the ε^1-equation), and so on.

Step 4. Successively solve the sequence of equations from Step 3. The first equation (ε^0) yields the solution to the unperturbed problem (let $\varepsilon = 0$ in (3)), namely

$$y_0(x) = e^{-x}.$$

The second equation (ε^1) then becomes

$$y_1' + y_1 = e^{-3x}, \quad y_1(0) = 0$$

resulting in

$$y_1(x) = \frac{1}{2}e^{-x} - \frac{1}{2}e^{-3x}.$$

With y_0 and y_1 known, the third equation (ε^2) becomes

$$y_2' + y_2 = \frac{3}{2}e^{-3x} - \frac{3}{2}e^{-5x}, \quad y_2(0) = 0$$

resulting in

$$y_2(x) = \frac{3}{8}e^{-x} - \frac{3}{4}e^{-3x} + \frac{3}{8}e^{-5x}.$$

Substituting y_0, y_1, and y_2 thus obtained in the fourth equation (ε^3)

$$y_3' + y_3 = \frac{15}{8}e^{-3x} - \frac{15}{4}e^{-5x} + \frac{15}{8}e^{-7x}, \quad y_3(0) = 0$$

yields

$$y_3(x) = \frac{5}{16}e^{-x} - \frac{15}{16}e^{-3x} + \frac{15}{16}e^{-5x} - \frac{5}{16}e^{-7x}.$$

Step 5. Write the perturbation solution to the desired approximation. For instance, the perturbation solution to first order is

$$y(x) = e^{-x} + \varepsilon\Big(\frac{1}{2}e^{-x} - \frac{1}{2}e^{-3x}\Big) + O(\varepsilon^2).$$

The perturbation solution to second order is

$$y(x) = e^{-x} + \varepsilon\Big(\frac{1}{2}e^{-x} - \frac{1}{2}e^{-3x}\Big) + \varepsilon^2\Big(\frac{3}{8}e^{-x} - \frac{3}{4}e^{-3x} + \frac{3}{8}e^{-5x}\Big) + O(\varepsilon^3)$$

and to third order

$$\begin{aligned} y(x) &= e^{-x} + \varepsilon\Big(\frac{1}{2}e^{-x} - \frac{1}{2}e^{-3x}\Big) + \varepsilon^2\Big(\frac{3}{8}e^{-x} - \frac{3}{4}e^{-3x} + \frac{3}{8}e^{-5x}\Big) \\ &\quad + \varepsilon^3\Big(\frac{5}{16}e^{-x} - \frac{15}{16}e^{-3x} + \frac{15}{16}e^{-5x} - \frac{5}{16}e^{-7x}\Big) + O(\varepsilon^4). \end{aligned}$$

Activity 2. Exploring Perturbation with Electronic Spreadsheet

A Microsoft Excel macro-enabled workbook has been developed for (3) to interactively investigate the effect of the small parameter ε. Figure 1 shows a screen capture of the user interface. The scroll bar allows the user to change ε by sliding the bar; the value of ε is displayed in cell E4. The table of results is updated automatically as the user varies the parameter ε.

The workbook also exploits the graphical capabilities of Excel. As the user varies ε, a graph displaying the perturbation solution is automatically updated. For the Bernoulli differential equation (3), the graphs for two instances of the parameter ε are shown in Figure 2. The graphs include the exact solution as well as the different orders of perturbation approximations derived in Activity 1.

As can be seen in Figure 2, even for values of ε as large as 0.5 (see Figure 2(a)) the first-order perturbation approximation compares very well with the exact solution; the higher-order approximations are almost indistinguishable from the exact solution. One has to increase the value of ε to 0.8 to clearly observe the spread between the different orders of approximation and the exact solution (see Figure 2(b)).

However, it is pointed out that perturbation only provides useful information when the problem depends on extreme (very small or very large) values of the parameter ε, and the solution to the problem is readily available when $\varepsilon = 0$. For intermediate values of ε, perturbation provides no useful information. One has to bear in mind that the smallness of ε is not indicative of the convergence of a perturbation solution (ε may actually represent a physical parameter in a real application); if it is desired to obtain a better approximation for a given value of ε, one would have to include more higher-order terms in the perturbation approximation at the expense of more work. To ascertain the accuracy of a perturbation solution, one has to compare the perturbation results with some other characterization of the expected answer. The examples presented in this paper lead to exact

	A	B	C	D	E
1	**Bernoulli differential equation**				
2	$y' + y = \varepsilon y^3$, $y(0) = 1$ $(\varepsilon \ll 1)$				
3					
4				$\varepsilon =$	0.5
5					
6		**Exact**	**1st-order**	**2nd-oder**	**3rd-order**
7	x	$y(x)$	$y(x)$	$y(x)$	$y(x)$
8	**0.00**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
9	**0.05**	**0.9747**	**0.9739**	**0.9747**	**0.9747**
10	**0.10**	**0.9489**	**0.9458**	**0.9486**	**0.9488**
11	**0.15**	**0.9226**	**0.9165**	**0.9219**	**0.9225**
12	**0.20**	**0.8959**	**0.8862**	**0.8946**	**0.8957**
13	**0.25**	**0.8690**	**0.8554**	**0.8667**	**0.8686**
14	**0.30**	**0.8418**	**0.8244**	**0.8385**	**0.8412**
15	**0.35**	**0.8146**	**0.7934**	**0.8101**	**0.8136**
16	**0.40**	**0.7874**	**0.7626**	**0.7817**	**0.7860**
17	**0.45**	**0.7603**	**0.7322**	**0.7533**	**0.7585**
18	**0.50**	**0.7334**	**0.7024**	**0.7251**	**0.7311**
19	**0.55**	**0.7067**	**0.6732**	**0.6972**	**0.7039**
20	**0.60**	**0.6804**	**0.6447**	**0.6698**	**0.6771**

Figure 1. Microsoft Excel spreadsheet user interface for exploring perturbation methods. The user can slide the bar on the scroll bar to change the value of the small parameter ε.

solutions that are expressible as closed-form formulae in terms of ε. Unfortunately, most real-world problems do not lend themselves to the luxury of having exact solutions; in these cases, one has to rely on other methods of analysis or numerical solutions to ascertain whether the perturbation results are reasonably accurate.

Use the Excel spreadsheet to investigate solutions to (3) for small negative values of ε. What do you observe?

Although not related to perturbation, set the value of ε to 1. Notice the graph of the exact solution. What do you observe? Explain the nature of this solution.

Activity 3. Second-Order Differential Equation with Constant Coefficients

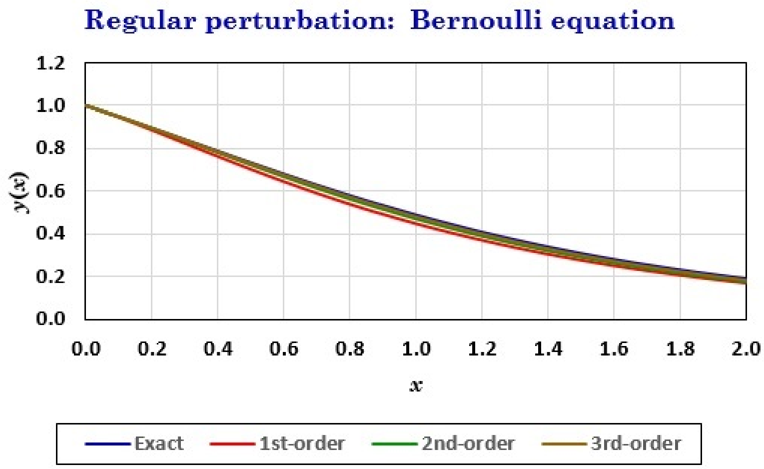
Consider the differential equation

$$y'' + y' = -\varepsilon, \qquad y(0) = 0, \quad y'(0) = 1 \tag{6}$$

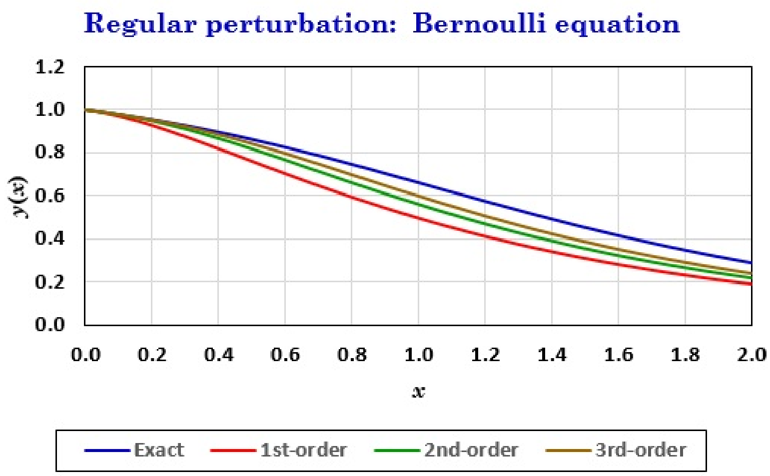
where $0 < \varepsilon \ll 1$.

Verify that the exact solution to (6) is given by

$$y(x) = (1 + \varepsilon)(1 - e^{-x}) - \varepsilon x. \tag{7}$$



(a) $\varepsilon = 0.5$: Even for this value of ε, the first-order perturbation solution approximates the exact solution very well; the higher-order perturbation approximations are almost indistinguishable from the exact solution.



(b) $\varepsilon = 0.8$: For this value of ε, the spread between the exact solution and the different orders of approximation is noticeable. It is worth mentioning that perturbation only gives useful information for very small (or very large) values of ε; for intermediate values of ε it is better to use other methods of analysis.

Figure 2. Two instances of perturbation approximation to the solution of the Bernoulli differential equation (3): $\varepsilon = 0.5$ (top) and $\varepsilon = 0.8$ (bottom).

Use the regular perturbation technique outlined in Activity 1 to obtain an analytical approximation to (7). You will find the perturbation solution surprising. Do you think the results are valid for negative values of ε?

Figure 3 shows a graph of the exact (and perturbation) solution to (6) for $\varepsilon = 0.2$. This solution is compared with the numerical solution obtained by the Runge-Kutta method (which is implemented in the accompanying Excel workbook).

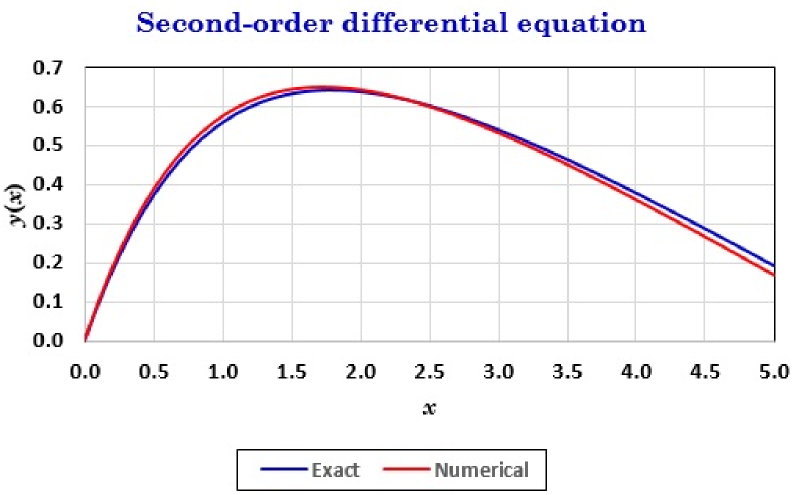


Figure 3. Graphs of the exact and numerical solution to the differential equation (6) for $\varepsilon = 0.2$; both solutions show good agreement.

Activity 4. Two-Point Boundary Value Problem

Consider the differential equation

$$y'' + 2\varepsilon y' + (1+\varepsilon^2)y = 1, \qquad y(0) = 0, \quad y(\pi/2) = 0. \tag{8}$$

For $0 < \varepsilon \ll 1$, verify that the exact solution to this problem is

$$y(x) = -\frac{e^{-\varepsilon x}}{1+\varepsilon^2}(\cos x + e^{\varepsilon\pi/2}\sin x) + \frac{1}{1+\varepsilon^2}. \tag{9}$$

Use regular perturbation and obtain the following result (make sure to impose the boundary conditions at each order)

$$\begin{aligned} y(x) &= -\cos x - \sin x + 1 + \varepsilon\Big(-\frac{\pi}{2}\sin x + x(\cos x + \sin x)\Big) \\ &\quad + \varepsilon^2\Big[\cos x + \Big(1 - \frac{\pi^2}{8}\Big)\sin x + \frac{\pi}{2}x\sin x - \frac{x^2}{2}(\cos x + \sin x) - 1\Big] + O(\varepsilon^3). \end{aligned} \tag{10}$$

Figure 4 compares the graphs of the exact solution (9) and the perturbation approximation (10) for $\varepsilon = 0.5$; it can be seen that the perturbation solution approximates the exact solution very well.

CONCLUDING REMARKS

This paper introduced the method of regular perturbation for ordinary differential equations. The examples showed good agreement between the exact solutions and their corresponding analytical

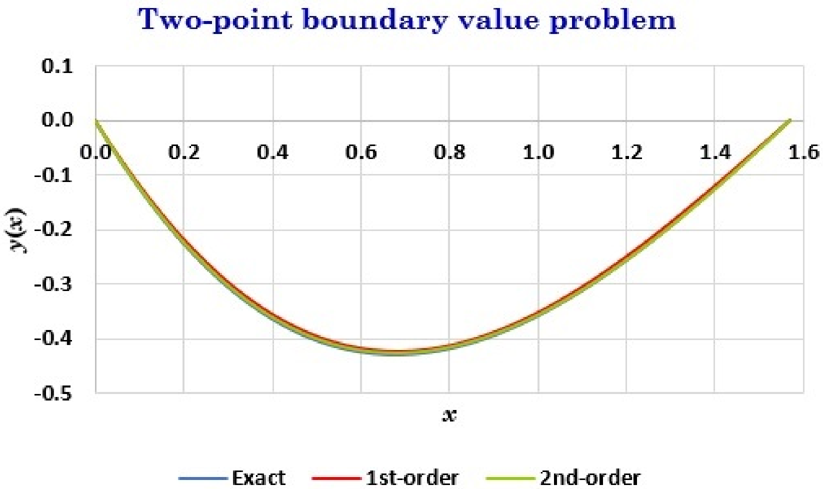


Figure 4. Graphs of the perturbation approximation and the exact solution (for $\varepsilon = 0.5$) to the two-point boundary value problem (8).

approximations obtained by perturbation. The accompanying Excel spreadsheet features a number of dynamic controls that permit the user to change the value of the small parameter ε that appears in the differential equations; the interactivity feature enables the user to gain further insight into the mathematical modeling of dynamical systems.

REFERENCES

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[3] Kevorkian, J. and J. D. Cole. 2010. *Perturbation Methods in Applied Mathematics*. New York NY: Springer-Verlag.