

## STUDENT VERSION

### Population Model Variations and MATLAB

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**Abstract:** Students will walk through a detailed derivation and review of basic population models (exponential and logistic) to create and understand variations of those models while learning some basic MATLAB functions for working with differential equations. They will also work with other utilities such as desmos.com and dfield.jar for visualizing dynamic plots and slopefields. It is assumed that students have had at least some experience with interpreting exponential and logistic differential equations. However, there are ample opportunities for review of these ideas if necessary.

**Keywords:** population ecology, population model, logistic growth, slope field, classifying equilibria

**Tags:** bifurcation, harvesting, stocking, extinction, exponential growth, autonomous differential equation, linear differential equation, non-linear differential equation, qualitative analysis, MATLAB, stability

## STATEMENT

### Instructions:

- (1) You can save all MATLAB input and output as a diary file (in case you need to come back to it later). To do so, simply type `diary pickaname` before you do anything else ... at the end of the session, be sure to enter `diary off` before closing MATLAB to close and save the diary. You can then double-click on the diary file in MATLAB's file explorer to access it as a text file from which you can copy-and-paste.
- (2) Answer all questions concisely but thoroughly. Preferably, you should type your answers and solutions as you cut and paste code and plots from MATLAB. However, some things take a long time to type, so leave some space in your document so you

can write or type your conclusions later when you no longer need access to MATLAB. Make sure you do everything that requires access to MATLAB during the lab session.

- (3) After copying MATLAB code, please remove any blank lines and error messages to make it as reader-friendly as possible.
- (4) Work through problems 1 and 2 until you obtain the expected output and are able to answer all of the questions asked – you do NOT have to include any MATLAB code or plots in your lab document, but you should include answers to the questions asked in problem 2.
- (5) For problems 3 through 9, copy all MATLAB code and any plots (MATLAB or DESMOS) you use into a Word document. Right-click on the picture in Word, select the “size” tab, and change the height to 2.25 inches (make sure the “lock aspect ratio” box is CHECKED). This seems to do the trick with being able to fit 2 plots on one line if you also set margins at 0.75in. Be sure to label each plot with the problem number to which it corresponds – using a 2x2 table can make centering and labeling a little easier than using captions.
- (6) Your final product should be a single document containing all the necessary and relevant MATLAB code, plots, and your analysis for each problem, with each problem and each part of the problem clearly labeled.

NOTE: MATLAB commands typically have extensive help available. For help on the “ezplot” command, for example, type `help ezplot` at the command prompt and press “enter.” MATLAB commands used in this lab include:

`syms dsolve clear solve ezplot diff subs simplify pretty limit`

### Exponential Model of Population Growth

- Suppose a population changes only via births and deaths (no migration or other external influences), where  $\beta(t)$  is the number of births per unit of population per unit of time at time  $t$  and  $\delta(t)$  is the number of deaths per unit of population per unit of time at time  $t$ .
- The numbers of births and deaths would be  $\beta(t) \cdot P(t) \cdot \Delta t$  and  $\delta(t) \cdot P(t) \cdot \Delta t$ , respectively, and the incremental change in population would be  $\Delta P = \{\text{births}\} - \{\text{deaths}\} \approx [\beta(t) - \delta(t)] \cdot P(t) \cdot \Delta t$ , so in the end, we would have 
$$\frac{\Delta P}{\Delta t} \approx [\beta(t) - \delta(t)] P(t).$$
- If the birth and death rates ( $\beta(t)$  and  $\delta(t)$ ) are constant, then as  $\Delta t \rightarrow 0$ , we have  $\frac{dP}{dt} = (\beta - \delta)P$ , or in other words,  $\frac{dP}{dt} = kP$ , where  $k = \beta - \delta$ .
- This leads to the familiar exponential solution  $P(t) = P(0)e^{kt}$ , which is often called the *Malthusian model* (after the English clergyman and political economist Thomas Malthus (1766-1834)). Note that if births outpace deaths, i.e. if  $\beta > \delta$ , then  $k > 0$  and we have exponential population growth, whereas if  $\beta < \delta$ , then  $k < 0$  and we have population decline (exponential decay).

**PROBLEM 1** – Use the “`dsolve`” command in MATLAB to confirm the general solution to the above differential equation. To do so, enter the following commands.

```

syms P(t) k Po
    % defines k and Po as symbolic constants and P(t) as a symbolic
    function
P(t)=dsolve(diff(P)==k*P,P(0)==Po)
    % solves the differential equation P'=kP with initial condition
    P(0)=Po and assigns the solution to the symbolic function P(t).

The expression "diff(P)" represents the derivative of P(t). (Note
the double == represents equality and NOT assignment, and note the
use of an explicit multiplication symbol *.) Also, since P(t) was
defined as a symbolic function, once MATLAB solves the differential
equation, we can evaluate the function P at specific values of t.
If the above commands worked without errors, enter a command such as
P(3) and see if you get what you would expect.

```

**PROBLEM 2** – Suppose an alligator population consisted of 12 alligators in 2008 and two dozen in 2018. Also suppose that the population grows as in the exponential model above, except that the birth rate actually increases by a factor of  $k > 0$  as the population increases (i.e. the birth rate is proportional to the population and is *not* constant). In addition, without any natural predators and a typically long lifespan, we'll consider the death rate to be zero. Hence the model we end up with in this case is

$$\frac{dP}{dt} = (\beta - \delta)P = (kP - 0)P = kP^2, \text{ with } P(0) = 12.$$

MATLAB usually assigns output to the variable “**ans**,” and each subsequent command overwrites what used to be defined as “**ans**.” So a useful practice is to name your variables with convenient names so that you can easily recall them later. With that in mind, perform the following steps in MATLAB to solve this initial value problem.

```

clear all
    % clears all variable definitions used previously - this is a good
    practice when doing multiple problems in the same session - you can
    also clear specific values by entering a command such as clear k,Po
syms P(t) k
    % defines k as a symbolic constant and P(t) as a function
DP=diff(P)
    % defines DP as the derivative of the function P(t) (or we can just
    use "diff(P)" in the dsolve command as in Problem 1)
P(t)=dsolve(DP==k*P^2,P(0)==12)
    % solves the differential equation with initial condition P(0)=12,
    and assigns the solution to the function P(t). You should get a
    result equivalent to P(t)=12/(1-12kt).
simplify(P) or simplify(P(t))
    % Used to further simplify expressions (optional)
pretty(P) or pretty(P(t))
    % Attempts to express output in a more human-readable format
    (optional)

```

Now that we have an expression for  $P(t)$ , we want to use the initial condition to solve for  $k$ . To do this, we would substitute 10 in for  $t$  and solve for  $k$ . There are a number of ways to do this in MATLAB:

```

K=solve(12/(1-12*k*10)==24,k)
    % OPTION 1 - retype the entire equation with 10 in place of t, then
    use the "solve" command to solve for k (and assign the value to K.
    Note that I used "K" instead of "k" so I can redo the problem later

```

```

    without having to retype the original "k")
K=solve(subs(P(t),t,10)==24,k)
    % OPTION 2 - use the solve AND subs commands simultaneously to have
    MATLAB do all the work. The "subs" command shown replaces all
    instances of "t" in the expression "P(t)" with 10, then solves the
    resulting equation for k and assigns the result to K.
K=solve(P(10)==24,k)
    % OPTION 3 - solve P(t)=24 with t=10 for k
P(t)=subs(P(t),k,K)
    % Redefines P(t) but replaces the k with K=1/240 found above.
pretty(simplify(P(t)))
    % See what happens

```

Regardless of the method, this should give you a value of  $\frac{1}{240}$  for  $k$ . Thus we know that

the solution to this differential equation is  $P(t) = \frac{12}{1 - \frac{1}{20}t} = \frac{240}{20 - t}$ .

- Use `solve` to determine when the alligator population will first reach (or perhaps when it did reach) four dozen (i.e. 48).
- What happens thereafter (i.e. at some point in time, our mathematical solution  $P(t)$  has an issue – when and why)? What does this imply in terms of the alligator population? Why, given the initial parameters  $\delta$  and  $\beta$ , would this make sense? You might find it useful that MATLAB will calculate limits. As an example, use the command `syms x; limit(exp(-x),x,inf)` to find the limit of  $e^{-x}$  as  $x \rightarrow \infty$  (note that we can put multiple commands on one line if we separate them by a semi-colon (suppressing output) or a comma (not suppressing output)). If the limit of your expression is approaching infinity, MATLAB will return the result “NaN,” which means “Not a Number.” Of course, we are not using  $x$  and our time variable is not quite heading to `inf`, so adjust the command accordingly to fit our alligator population context.
- MATLAB has two plotting commands, `ezplot()` and `plot()`. The `ezplot` command is used to plot symbolic functions like  $P(t)$  in this lab, while the `plot` command plots  $x$  and  $y$  vectors (like we did in the M&M lab). Type `P` and press enter to confirm that the variable `P` is still defined as `P(t)=-1/(t/240 - 1/12)`. If not, define it as such. Then type `ezplot(P)`. The `ezplot` command tries its best at coming up with a reasonable domain over which to plot, but in this case, it is not the domain we want. We can adjust the plot by including axes settings such as `ezplot(P,[xmin,xmax,ymin,ymax])` (but replace `xmin`, `xmax`, etc. with reasonable values based on your answers to parts (a) and (b)).
- Now use `dfield.jar` (you should have downloaded that from <https://math.rice.edu/~dfield/dfpp.html> in a previous lab) to plot the original differential equation (with the value of  $k$  substituted in), keeping in mind that you need to set your window settings appropriately. Finally, select “Keyboard input” from the “solutions” menu of the `dfield` Display window, and enter the initial condition (0, 12) and press “Solve” to confirm your observation from part (c) above.

**PROBLEM 3 - (Doomsday vs. Extinction)** – Consider a population of rabbits in which *both* the birth and death rates are proportional to the current population, i.e.

$\beta = k_1 P$  and  $\delta = k_2 P$ . Then the equation from the second and third points of the

“exponential model” discussion above would become

$$\frac{dP}{dt} = (k_1P - k_2P)P = (k_1 - k_2)P^2, \text{ where } P(0) = P_0.$$

- (a) For the sake of simplicity, you might set  $k_1 - k_2 = k$  since we are not really concerned with the individual values of the two constants. Then, with  $P_0 = 6$ , use MATLAB’s `dsolve` command to solve this differential equation (your solution will be in terms of  $k$ ). In addition, you might want to use the `pretty(ans)` or `simplify(ans)` commands to format your output in a way that might be a little easier to read.
- (b) If the birth rate is greater than the death rate, you should notice an “issue” with the resulting population function. (Specifically, focus on what happens to the population as time moves forward from the initial condition (0,6).) Explain what impact this has on the solution you found in part (a), and at what time (in terms of  $k$ ) does the “issue” with the population function occur? Why do you think this is called the “doomsday” scenario? Plot the solution using Desmos (accessed via <https://www.desmos.com>) and include a slider for  $k$  (you have to use  $x$  instead of  $t$  as the independent variable) and see if your analysis was correct (just type in the solution as it is reported from MATLAB’s `dsolve` command and desmos will automatically ask if you want to create a slider for  $k$ ). You can use a scroll wheel on your mouse to zoom the graph in/out, and you can zoom each axis independently if you hold the SHIFT key as you drag the appropriate axis. You can fine-tune the slider by clicking on its definition and setting the min, max, and step size – A good range for  $k$ , as a combination of constants of proportionality, would be between 0 and 2 with a small step size such as 0.005.
- (c) Now set the slider in desmos to values of  $k$  for which  $k_1 < k_2$  and describe what happens to the population as time moves forward from the initial population. Why do you think this is called the “extinction” scenario? Why, in the context of this problem and in terms of what  $k_1$  and  $k_2$  represent, does this make sense?
- (d) Do the same as in part (c) and answer the same questions for  $k_1 = k_2$ .

**Logistic Growth** In environments with limited resources, it is reasonable to assume that the birth rate actually *decreases* as the population increases. If we assume the birth rate is a decreasing linear function of the population and the death rate remains constant, we

have  $\frac{dP}{dt} = ((\beta - \beta_1P) - \delta)P$ . To make this more familiar to us, note that this is a

“logistic differential equation”  $\frac{dP}{dt} = kP(M - P)$  with  $k = \beta_1$  and  $M = \frac{\beta - \delta}{\beta_1}$ . From

previous study, we know a number of things about this particular model:

- If  $P_0 = 0$ , then  $P'(t) = 0$  and the population does not grow.
- If  $P_0 > M$ , then  $P'(t) < 0$ , and the population shrinks down towards the carrying capacity.
- If  $0 < P_0 < M$ , then  $P'(t) > 0$ , and the population grows towards the carrying capacity.
- Finally, if  $P_0 = M$ , then  $P'(t) = 0$  and the population remains at  $M$ . (Note that  $P(t) = M$  and  $P(t) = 0$  are called **equilibrium** solutions of the differential equation. Specifically,  $P(t) = M$  is called a **stable equilibrium** because solutions

- from above and below both approach  $M$  over time, while  $P(t) = 0$  is called an **unstable equilibrium** because solutions from above (and below if that made practical sense) both move away from  $P(t) = 0$ .)
- The solution curves cannot intersect (cross) the carrying capacity  $P(t) = M$  (this is “uniqueness” in theory, but in reality, populations can of course fluctuate above and below the carrying capacity).
  - $M$  is the “carrying capacity” of this particular environment, or the maximum population that the environment can support over the long run. The population is growing fastest (i.e.  $P'(t)$  reaches its maximum) when  $P(t) = M/2$ , or half of its carrying capacity.

Because the logistic model is so much more realistic than the exponential model, there are numerous variations in use. The following problems explore some of those variations.

**PROBLEM 4** Suppose we have a population of unsophisticated animals in which females rely solely on chance encounters with males for reproductive purposes. It is reasonable to assume that such encounters occur at a rate that is proportional to the product of the number of males and the number of females (if equally sized, then each would be  $0.5P$  and the product would be  $0.25P^2$ ). We might therefore assume that births occur at a rate proportional to  $P^2$  (births/unit time), which in turn means the actual birth rate (births/time/population) is proportional to  $P$  (i.e. we have divided “births/unit time” by “population,” so we lose a factor of  $P$ ). If the death rate  $\delta$  is constant, the differential equation governing this model is

$$\frac{dP}{dt} = (kP - \delta)P = kP^2 - \delta P = kP(P - M).$$

Note that  $P - M$  is the *negation* of the  $M - P$  factor present in the logistic equation and that  $\delta = Mk$ . If we use MATLAB to solve the differential equation

$\frac{dP}{dt} = 0.0004P(P - 150)$  with  $P(0) = P_0$ , we obtain a solution that can be expressed as

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{Mkt}} = \frac{M}{1 + \left(\frac{M}{P_0} - 1\right)e^{Mkt}} = \frac{150}{1 + \left(\frac{150}{P_0} - 1\right)e^{0.06t}}.$$

- (a) Notice, as in the logistic equation, the behavior of this population will depend on the following possible cases:  $P_0 = 0$ ,  $0 < P_0 < M$ ,  $M = P_0$ , or  $P_0 > M_0$ . Based on this observation and the solution above, and without really doing any labor-intensive calculations, determine what would eventually happen to the population in each of the four possible cases given for  $P_0$ . Give sufficient justification for each of your answers using (1) the differential equation and (2) the solution  $P(t)$ .
- (b) Using **dfield**, plot a slope field for  $\frac{dP}{dt} = 0.0004P(P - 150)$ . Were your analyses from part (a) correct? If not, revise them and explain any errors in your thinking.
- (c) What do you suppose the **equilibrium** solution(s) is/are in this case (if there are any)? How is it (or are they) different from the equilibrium solutions in the logistic equation (consider what happens to a solution slightly above or slightly below the equilibrium)? Would you consider them **stable**? Why or why not?

### More Logistic Variations

We might consider the differential equation  $\frac{dP}{dt} = kP(M - P) - h$  in which  $h$  represents either a “harvesting” (if  $h > 0$ ) or a “restocking” (if  $h < 0$ ) of the population. The value of  $h$  could be constant, time-dependent, or population-dependent.

**PROBLEM 5** – Assume the following logistic differential equation with harvesting holds for a population:

$$\frac{dP}{dt} = P(a - bP) - h, \quad P(0) = P_0.$$

- (a) First, identify the long-term behavior of this population if we were to ignore the effect of the harvesting term  $-h$ . Specifically, what is the carrying capacity without any harvesting?
- (b) Now, with  $a = 5$ ,  $b = 1$ , and  $h = 4$  (i.e. including harvesting), use **dfield** with the window  $[-2, 6]_x, [-2, 6]_y$  to determine the long-term behavior of solutions for which  $0 < P_0 < 1$ ,  $1 < P_0 < 4$ , and  $P_0 > 4$ . (I would recommend that you make use of the “parameters” feature of dfield so it is easier to change the values of  $a$ ,  $b$ , or  $h$  without having to retype the entire equation with each adjustment.) What impact, if any, has harvesting had on the carrying capacity?
- (c) Identify equilibrium solutions and classify each as stable, unstable, or semi-stable (can you guess what the latter means?)
- (d) With this level of harvesting, is it possible for a non-zero initial population to become extinct in finite time? Under what conditions, if any, does this occur? Keep in mind we are using small numbers to make life easier on us, but  $P = 1$  could mean that the population is one *hundred*, or one *thousand*, or one *million*, etc. Also, is it possible for the population to grow without bound?

**PROBLEM 6** – Repeat parts (b) through (d) of PROBLEM 5, but with  $h = 6.25$ . The equilibrium solutions have changed. Has their overall nature changed as well (in particular, has their stability changed)?

**PROBLEM 7** – Repeat parts (b) through (d) of PROBLEM 5, but with  $h = 7$ . Now consider the lack of an equilibrium solution. What happens to the population, regardless of  $P_0$ , in this case? How does this make sense when you consider what  $h$  represents?<sup>1</sup>

**PROBLEM 8** – Consider stocking instead of harvesting. Change the equation to

$$\frac{dP}{dt} = P(a - bP) + h, \quad P(0) = P_0.$$

- (a) Recall the equilibrium solutions without harvesting or restocking as found in part (a) of PROBLEM 5.

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<sup>1</sup> Problems 5, 6, and 7 demonstrate a concept known as **bifurcation**. In changing the value of  $h$  ever so slightly, we have fundamentally changed the nature of the solutions to the differential equation. Clearly a fisheries expert, for example, would want to avoid these *bifurcation points* in order to avoid catastrophe (extinction scenario) or unmanageable or unwanted situations (over-population and “doomsday” scenario).

- (b) Start with  $h = 0.1$  (and  $a$  and  $b$  as defined in PROBLEM 5) and use **dfield** to determine the long-term behavior of solutions as you did in PROBLEM 5. Have the equilibria changed significantly in location or stability?
- (c) Incrementally increase the value of  $h$ , using **dfield** to plot a new slope field each time, and note any changes to the equilibria in terms of location or stability. Are you able to create another *bifurcation* (where the overall behaviors of the equilibria change significantly)?
- (d) Roughly what level of restocking is necessary to achieve a stable population of  $P(t) = 10$ ?

**PROBLEM 9** – Taking the results of PROBLEMS 5 through 8 into consideration, what appears to be easier to do, overharvest, or overstock?<sup>2</sup>

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<sup>2</sup> Perhaps this can help us understand why conservation efforts can be so important!