Using Real Data to Study the Heat Equation

Kim Spayd[1] and James Puckett[2]

[1]Department of Mathematics, Gettysburg College
[2]Department of Physics, Gettysburg College

Joint Mathematics Meetings
January 19, 2019
Using Modeling to Motivate the Study of Differential Equations

PRIMUS 26(10), 2016

Outline

- Background
- Experimental Setup
- Model Building and Analysis
- Student Project: Analysis, Numerics, Errors
- Reflections

Gettysburg College

- Liberal arts college
- Located in Gettysburg, Pennsylvania
- Approximately 2600 students
- Average class size: 18
- 9.6:1 student-faculty ratio
- Approximately 15-20 math majors per year

Glatfelter Hall, photo by Shawna Sherrell

Course: Partial Differential Equations

- Prereqs: Multivariable Calculus, Ordinary Differential Equations
- 15 students, sophomore through senior
- Areas of study: math, physics, economics, biology, chemistry, music, environmental science, computer science
- Specified interest in application to real-world problems
- Textbook: Haberman's *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*

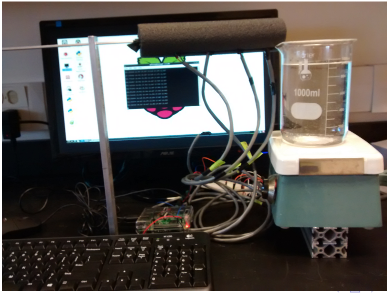
One-Dimensional Heat Equation

$$\boxed{u_t = k u_{xx}, \quad x \in (0, L), \quad t > 0}$$

- Standard material in introductory PDE course
- First topic covered in textbook: Chapters 1 and 2

Experimental Setup

- Lab experiment during second class meeting: heat transfer through a thin rod
- Groups of 3-4 students, each with different type of metal: brass, copper, aluminum, steel
- Four temperature sensors attached to metal rod of length $L = 300$ mm
- Boundary conditions: hot water bath at one end, room temperature water bath at other end



Model Derivation

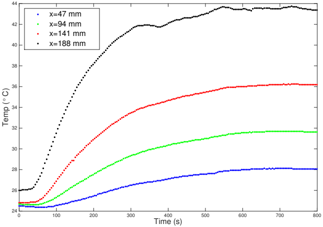
- Arguable physical assumptions
    - perfectly insulated rod
    - single, uniform material
    - no internal heat generation
    - rod is heated or cooled only at the ends

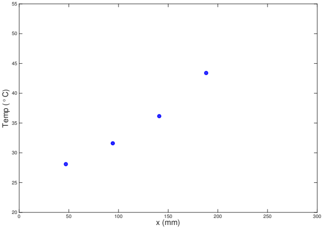
Model Derivation

- Arguable physical assumptions
  - perfectly insulated rod
  - single, uniform material
  - no internal heat generation
  - rod is heated or cooled only at the ends
- Observed physical principles
  - Conservation of energy:
    $$\frac{d}{dt}\int_a^b u(x,t)A dx = A\phi(a,t) - A\phi(b,t) + \int_a^b Q(x,t)A dx$$
  - Fourier's law of heat conduction:
    $$\phi(x,t) = -k\frac{\partial}{\partial x}u(x,t)$$

Equilibrium/Steady-State Solution

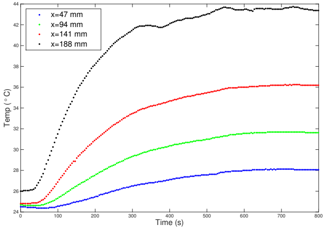
- Observed temperature changes level off
- Heat equation reduces to $u_{xx} = 0$
- Analysis gives linear steady-state solution $u(x) = \frac{T_L - T_0}{L} x + T_0$
- Linear pattern in data supports equilibrium solution

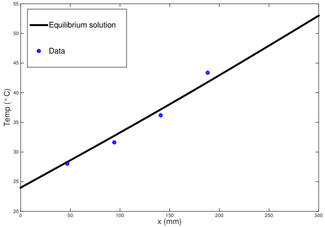




Equilibrium/Steady-State Solution

- Observed temperature changes level off
- Heat equation reduces to $u_{xx} = 0$
- Analysis gives linear steady-state solution $u(x) = \frac{T_L - T_0}{L} x + T_0$
- Linear pattern in data supports equilibrium solution





Student Project: Analysis (Separation of Variables)

1. Write initial boundary value problem observed in the lab
   Nonhomogeneous boundary conditions differ from examples done in class

Student Project: Analysis (Separation of Variables)

1. Write initial boundary value problem observed in the lab
   Nonhomogeneous boundary conditions differ from examples done in class
2. Determine steady-state solution, $u_s(x)$
   Straight forward use of linear equilibrium formula

Student Project: Analysis (Separation of Variables)

1. Write initial boundary value problem observed in the lab
   Nonhomogeneous boundary conditions differ from examples done in class
2. Determine steady-state solution, $u_s(x)$
   Straight forward use of linear equilibrium formula
3. Why does separation of variables technique fail when trying to solve for $u(x, t)$?
   Nonhomogeneous boundary conditions require alternate solution technique

Student Project: Analysis (Separation of Variables)

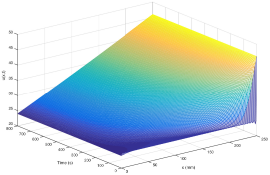
1. Write initial boundary value problem observed in the lab
   Nonhomogeneous boundary conditions differ from examples done in class
2. Determine steady-state solution, $u_s(x)$
   Straight forward use of linear equilibrium formula
3. Why does separation of variables technique fail when trying to solve for $u(x, t)$?
   Nonhomogeneous boundary conditions require alternate solution technique
4. New approach: let $v(x, t) = u(x, t) - u_s(x)$, write initial boundary value problem for v
   Homogeneous boundary conditions for v

Student Project: Analysis (Separation of Variables)

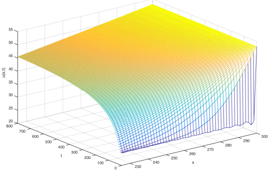
1. Write initial boundary value problem observed in the lab
   Nonhomogeneous boundary conditions differ from examples done in class
2. Determine steady-state solution, $u_s(x)$
   Straight forward use of linear equilibrium formula
3. Why does separation of variables technique fail when trying to solve for $u(x,t)$?
   Nonhomogeneous boundary conditions require alternate solution technique
4. New approach: let $v(x,t) = u(x,t) - u_s(x)$, write initial boundary value problem for v
   Homogeneous boundary conditions for v
5. Solve for $v(x,t)$ using separation of variables; then determine $u(x,t)$
   Solution is sum of infinite series (v) and steady-state solution (u_s)

Student Project: Numerics

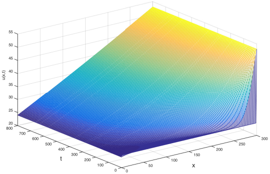
1. Graph the (analytical) series solution for $u(x, t)$
2. Use Matlab's PDE solver **pdepe** to simulate solution



(a) Truncated Analytical Solution



(b) Closer look at oscillation in (a)

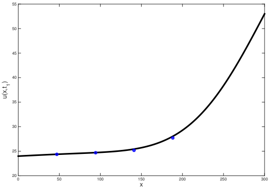


(c) Numerical Simulation

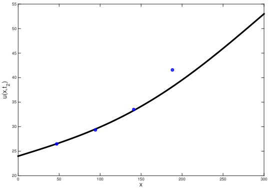
Figure: Plots used in comparing the analytic solution and numerical simulation

Student Project: Numerics (cont.)

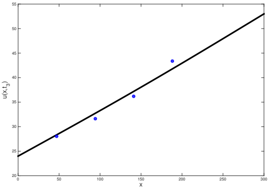
3. Overlay the four temperature values on the simulations, at various times; compare



(a) $t_1 = 60.3379$ s



(b) $t_2 = 302.1575$ s



(c) $t_3 = 799.3076$ s

Figure: Collected temperature data (dots) and numerical simulation (curve)

Student Project: Errors and Fine-Tuning

1. Identify sources of error between experiment and model

   Answers included measurement error from thermocouples, imperfect insulation, variable boundary conditions

Student Project: Errors and Fine-Tuning

1. Identify sources of error between experiment and model

   Answers included measurement error from thermocouples, imperfect insulation, variable boundary conditions

2. Identify tuning parameters; try different values to see if model can match data any better

   Popular answer was thermal diffusivity constant (k), others included experimentation with time-dependent boundary conditions and space-dependent thermal diffusivity function

Reflections: Student Evaluations

Statement	Agreement (1-5)
The demonstration added to my understanding of the physical context for the heat equation.	4.47
The demonstration added to my understanding of the derivation of the heat equation model in class.	3.93
The demonstration added to my understanding of the various boundary conditions discussed in class.	4.27
The project added to my appreciation of applied mathematics and mathematical modeling.	4.27
The demonstration, resulting data and analysis enhanced my educational experience in this class.	4.67

Table: Statements and mean response from student evaluations

Further Reflections: Our thoughts

- Lab experience was fun for the students
- Increased enthusiasm for material
- More engagement and discussion in class
- Students experienced mathematics as applied and interdisciplinary field
- Data, modeling and numerics work together to enhance education experience