An Effort to Assess the Impact a Modeling First Approach has in a Traditional Differential Equations Class

JMM 2019

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MANHATTAN COLLEGE

Manhattan College Facts

Location: Bronx, N.Y. Founded: 1853

Full-time Enrollment: 3,664

Schools:

Business

Education and Health

Engineering,

Liberal Arts

Science

Continuing and Professional Studies

Background:

The differential equations course is required of every engineering student.

There is a **common cumulative final** that typically includes traditional methods of solving DE **by hand**.

Spring 2018:

- It was agreed that only 80% of the final was common.

SIMIODE provides a comprehensive and cohesive community approach to studying differential equations.

www.simiode.org

Spring, 2018:

- There were 9 differential equations classes and 1 of them was taught with the modeling first approach advocated by SIMIODE.

- 222 students took the cumulative common final including 28 who were in the modeling first classroom.

What we decided to do:

- Adopt and adapt a modeling first approach in our differential equations classes. Since we are **bound** to cover a specific syllabus we decided to modify some of the scenarios found on SIMIODE to suit our needs.

- We made extensive use the computer algebra system Maple. We have found that the modeling first approach works best when students actually solve problems and get **answers** that make sense. Any computer algebra system will work.

- Hold one of three 50 minute classes each week in a computer lab where we do modeling first. So we committed a full **1/3** of our in class time modeling first activities.

- Supplement lecture material with **videos**.

- Ask questions on the **tests** that covered the ideas from lab.

Every new topic was motivated by a modeling first activity.

First Order DE
- M&M Death and Immigration
- Ant Tunnel Building
- Mixing It Up

Second Order DE
- Models Motivating Second Order
- Spring Mass Data Analysis

Systems of DE
- Whales and Krill
- Salt Compartments

Each scenario was adopted but adapted for our classes.

First lab day: M&M Death and Immigration—Students use M&Ms to simulate death and immigration.

Students collect data, build a mathematical model, estimate parameters, and discuss long term behavior.

Students had to believe that what they did in lab was **important.** They needed it to "count."

A test #1 question:

Suppose that you conduct the following experiment. You start with 60 pennies in a cup. You toss them onto a desk and remove any penny that is heads up. Then you add 10 more pennies. You put the pennies back into the cup. You perform this same experiment repeatedly.

Let P(n) = the number of pennies on the desk after you perform this experiment n times.

a. Find a formula for P(n+1) that involves P(n).

b. What is the equilibrium value E? That is, how many pennies do you expect to have on the table in the long run? Show algebraically how you get this value.

Ant Tunnel Building

How long does it take an ant to build a tunnel?

To answer the question we might need some narrowing of scope, some simplification, and certainly some identification of terms and variables before we can get a nice answer. Let us identify some variables and then together make some assumptions which will lead to a mathematical model.

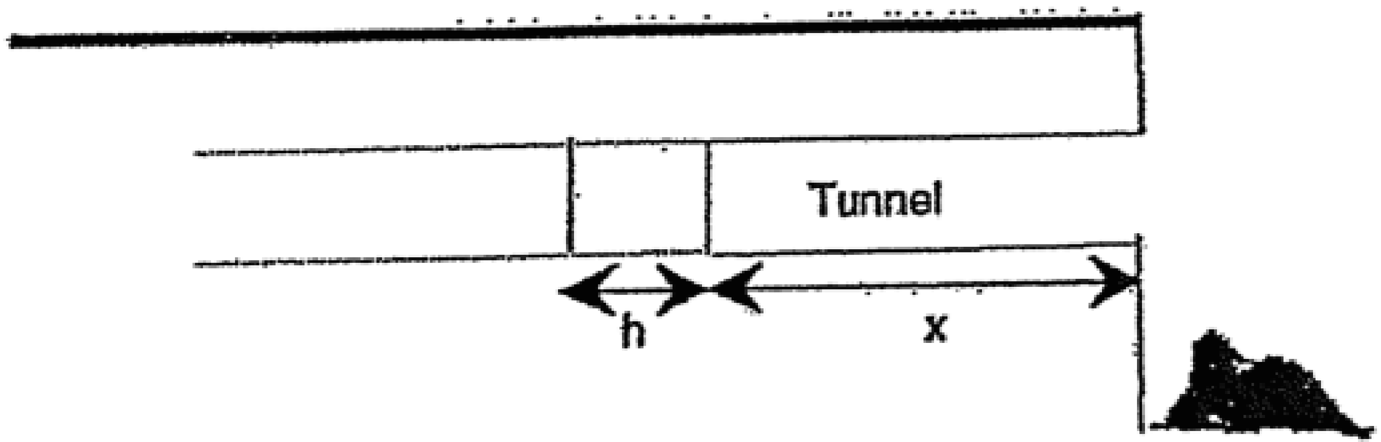


Figure 2. Useful diagram for discovering the time it takes to build a small section of the ant tunnel from distance x to $x+h$.

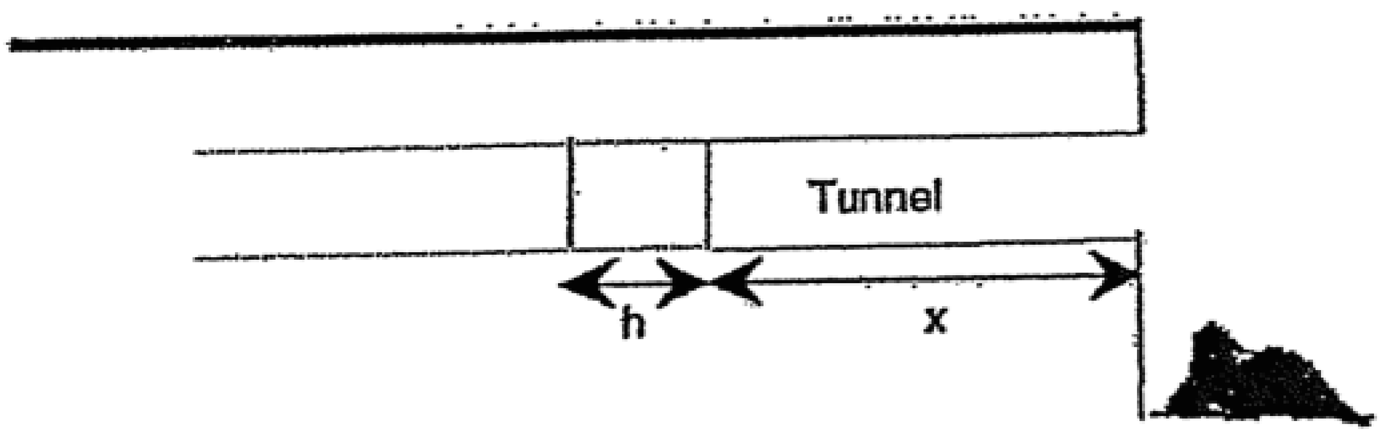


Figure 2. Useful diagram for discovering the time it takes to build a small section of the ant tunnel from distance x to $x + h$.

Let T(x) be the time in hours it takes the ant to build the tunnel of length x starting from the beginning of the tunnel.

The time it takes to build the small section of tunnel:

$$T(x + h) - T(x)$$

$T(x + h) - T(x)$ is proportional to both x and h.

$T(x + h) - T(x) = k\,x\,h$ so $T'(x) = k\,x$.

It starts snowing in the morning and continues steadily throughout the day. A snowplow that removes snow at a constant rate starts plowing at noon. It plows 2 miles the first hour, and 1 miles in the second hour. Assume that the rate the snowplow travels is inversely proportional to the height of the snow.
What time did it start snowing?

Something remarkable happened.

The students decided that to answer the question they needed some narrowing of scope, some simplification, and certainly some identification of terms and variables.

They identified some variables and made some assumptions which will led to a mathematical model.

Let t be the time measured in hours after noon.

Let $h(t)$ be the height of the snow at time t.

Let $b > 0$ be the (unknown) number of hours before noon
that it started snowing.

Let $x(t)$ be the distance the snowplow has traveled.

Let k be the constant rate at which snow falls.

The snow is falling at a constant rate so $h(t) = k(b+t)$.

Now the rate at which the snowplow travels is inversely proportional to the height of the snow.

$$eq1 := x'(t) = \frac{c}{k \cdot t + k \cdot b}$$

$$eq1 := \mathrm{D}(x)(t) = \frac{c}{k\,b + k\,t}$$

Also we have an initial condition, so we will solve an IVP.

$$dsolve([eq1, x(0) = 0], x(t))$$

$$x(t) = \frac{c\ln(b+t)}{k} - \frac{c\ln(b)}{k}$$

Since it plows 2 miles the first hour and 1 mile in the second hour.

$evalf(solve(\{x(1) = 2, x(2) = 3\}))$

$$\{b = 0.6180339888, c = c, k = 0.4812118252\, c\}$$

Remember that b is the number of hours before noon that it started snowing.

So it started snowing at about 0.618 hours before noon.

It started to snow at approximately 37 minutes before noon.

That is, it started to snow at approximately 11:23 AM.

Prior to the Spring Mass Data Analysis we did Models Motivating Second Order.

The students are given a diagram of a mass at the end of a spring. The mass is in equilibrium when the force due to gravity equals the restorative force of the spring.

$y(t)$ = the distance the mass is from equilibrium.
$m > 0$ is the mass
$c > 0$ is the resistance coefficient
$k > 0$ is the spring constant

They use Hooke's Law and Newton's Second Law of Motion to find their first second order initial value problem

$$m\,y''(t) + c\,y'(t) + k\,y(t) = 0, \quad y(0) = y_{0,} \quad y'(0) = y_1$$

3-001-T-Spring Mass Data Analysis

This scenario considers a spring mass system. A spring is attached to a horizontal rod on a ring stand and a Vernier Motion Detector Is placed under the mass to capture its displacement from the motion detector at time intervals of 0.02 seconds for about 30 seconds. To make it a bit more complicated, 4 index cards are taped to the base of a 200 g mass to offer more resistance.

In the supporting documents for this scenario there is an excel spreadsheet that records the data captured. We used this to initiate the scenario.

	A	B	C	D	E	F	G	H
1	**Orig. Data**			**First Form Data**			**Final Form Data**	
2								
3	**Time (s)**	**Dist (m)**		**Time (s)**	**Dist (m)**		**Time (s)**	**Dist (m)**
4								
5	0.02	0.288187						
6	0.04	0.170934						
7	0.06	0.299940						
8	0.08	0.315024						
9	0.10	0.330263						
10	0.12	0.345623						
11	0.14	0.352940						
12	0.16	0.359895						
13	0.18	0.365608						
14	0.20	0.370630		0.00	0.038524		0.00	0.041907
15	0.22	0.374013		0.02	0.041907		0.02	0.044444
16	0.24	0.376550		0.04	0.044444		0.04	0.045220

	A	B	C	D	E	F	G	H
1491	29.74	0.325068		29.54	-0.007038		29.54	-0.010283
1492	29.76	0.321823		29.56	-0.010283		29.56	-0.012820
1493	29.78	0.319286		29.58	-0.012820		29.58	-0.015098
1494	29.80	0.317008		29.60	-0.015098		29.60	-0.016927
1495	29.82	0.315179		29.62	-0.016927		29.62	-0.018446
1496	29.84	0.313660		29.64	-0.018446		29.64	-0.019412
1497	29.86	0.312694		29.66	-0.019412		29.66	-0.019326
1498	29.88	0.312780		29.68	-0.019326		29.68	-0.018998
1499	29.90	0.313108		29.70	-0.018998		29.70	-0.017790
1500	29.92	0.314316		29.72	-0.017790		29.72	-0.016375
1501	29.94	0.315731		29.74	-0.016375		29.74	-0.014010
1502	29.96	0.318096		29.76	-0.014010		29.76	-0.011680
1503	29.98	0.320426		29.78	-0.011680		29.78	-0.008574
1504	30.00	0.323532		29.80	-0.008574		29.80	0.000000

How can we manipulate the data so that
$y(t)$ = the distance the mass is from equilibrium?

Again, we will use the convention that $y(t) > 0$ when the mass is below equilibrium, that is when the spring is extended, and $y(t) < 0$ when the mass is above equilibrium, that is when the spring is compressed.

Students were asked to explain how the data was “cleaned up.”

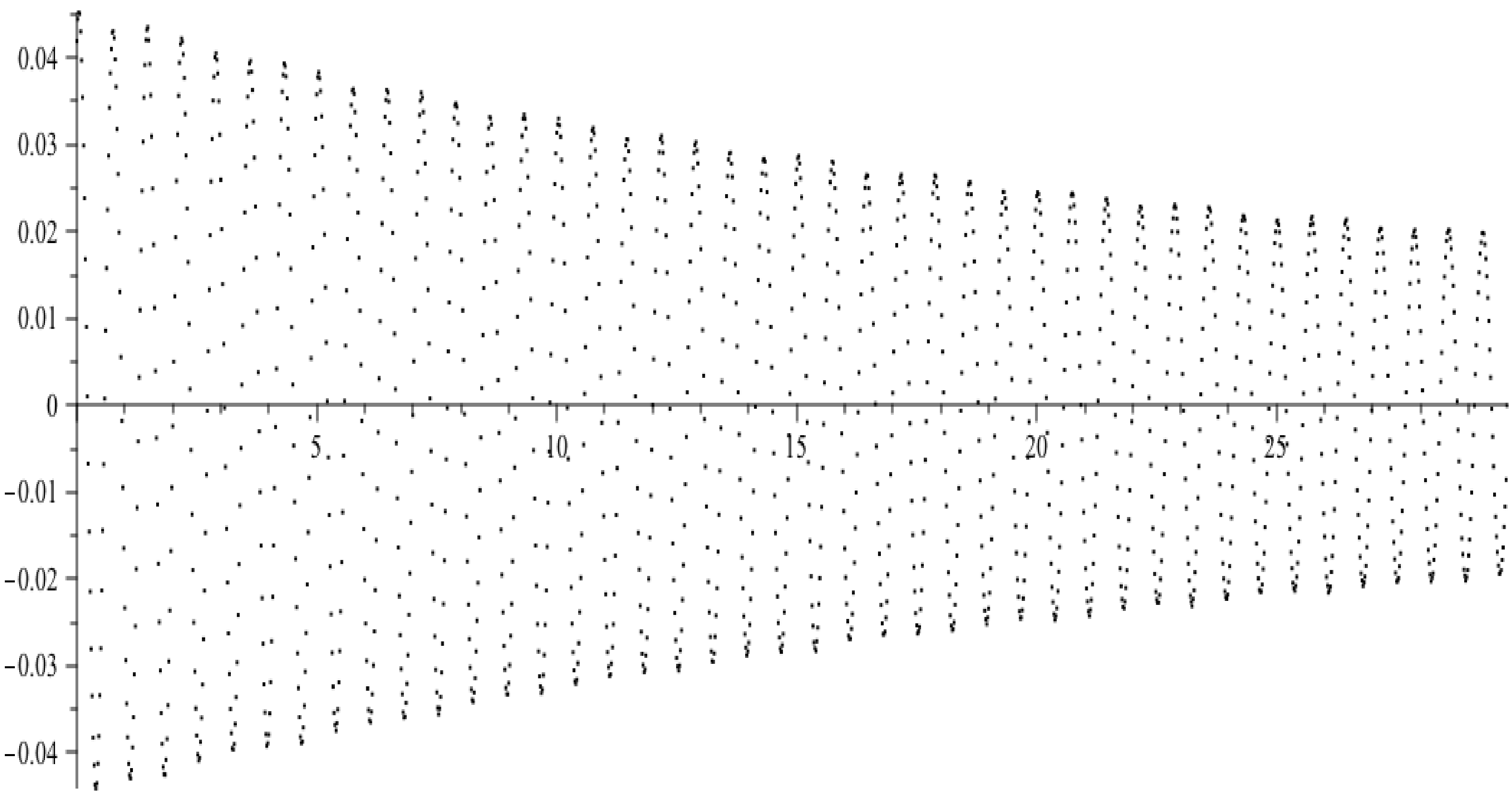
	A	B	C	D	E	F	G	H	I
1	**Orig. Data**			**First Form Data**			**Final Data**		
2									
3	**Time (s)**	**Dist (m)**		**Time (s)**	**Dist (m)**		**Time (s)**	**Dist (m)**	Aver Distance =
4									0.332106
5	~~0.02~~	~~0.288187~~							initial velocity
6	~~0.04~~	~~0.170934~~							0.148
7	~~0.06~~	~~0.299940~~							
8	~~0.08~~	~~0.315024~~							Aver Distance =
9	~~0.10~~	~~0.330263~~							=AVERAGE(B5:B1504)
10	~~0.12~~	~~0.345623~~							initial velocity
11	~~0.14~~	~~0.352940~~							=((E16-E15)/(0.02)+(E15-E14)/(0.02))/2
12	~~0.16~~	~~0.359895~~							
13	~~0.18~~	~~0.365608~~		=A14-A14	=B14-I4		=D14	=E15	
14	**0.20**	**0.370630**		**0.00**	**0.038524**		**0.00**	**0.041907**	
15	[illegible]	[illegible]		[illegible]	[illegible]		[illegible]	[illegible]	

Part 2—Using Maple—our adaption

The students were then instructed to import the cleaned up data from the Excel spreadsheet to Maple using a simple copy and paste.

With a simple Maple procedure that the students were given and asked to explain, Maple plotted time vs. displacement for all 1491 data points.

```
for i to 1491 do
P[i] := pointplot([B[i, 1], B[i, 2]], symbol = point):
L := [seq(P[i], i = 1 .. 1491)] end do; display(L)
```



The students were asked to answer the following:

- Suppose that the mass with the index cards attached is 0.20626 kg and the spring constant, as advertised when it was purchased is 17.206 N/m.

Write an initial value problem that this spring mass system must satisfy. Since we do not know the resistance constant, call it c.

SOLUTION: The IVP is $0.20626\, y'' + c\, y' + 17.206\, y = 0$;
So from the data the students found:

$$y(0) = \text{initial displacement} = 0.041907 \text{ m}$$
$$y'(0) = \text{initial velocity} = 0.148 \text{ m/sec}$$

- In the experimental data, you see that the data points are bouncing --sometimes above equilibrium--and sometimes below equilibrium. Look at the DE you wrote above. What inequality involving c must be satisfied given the experimental data you just graphed?

SOLUTION: First we agreed that $c > 0$. Given the graph of the data, the solution will definitely have a sine and/or cosine in the solution. This will only happen if

$$c^2 - 4\,m\,k < 0$$

or $$c^2 - 4\,(0.20626)\,(17.206) < 0$$

thus $$0 < c < 3.767709946$$

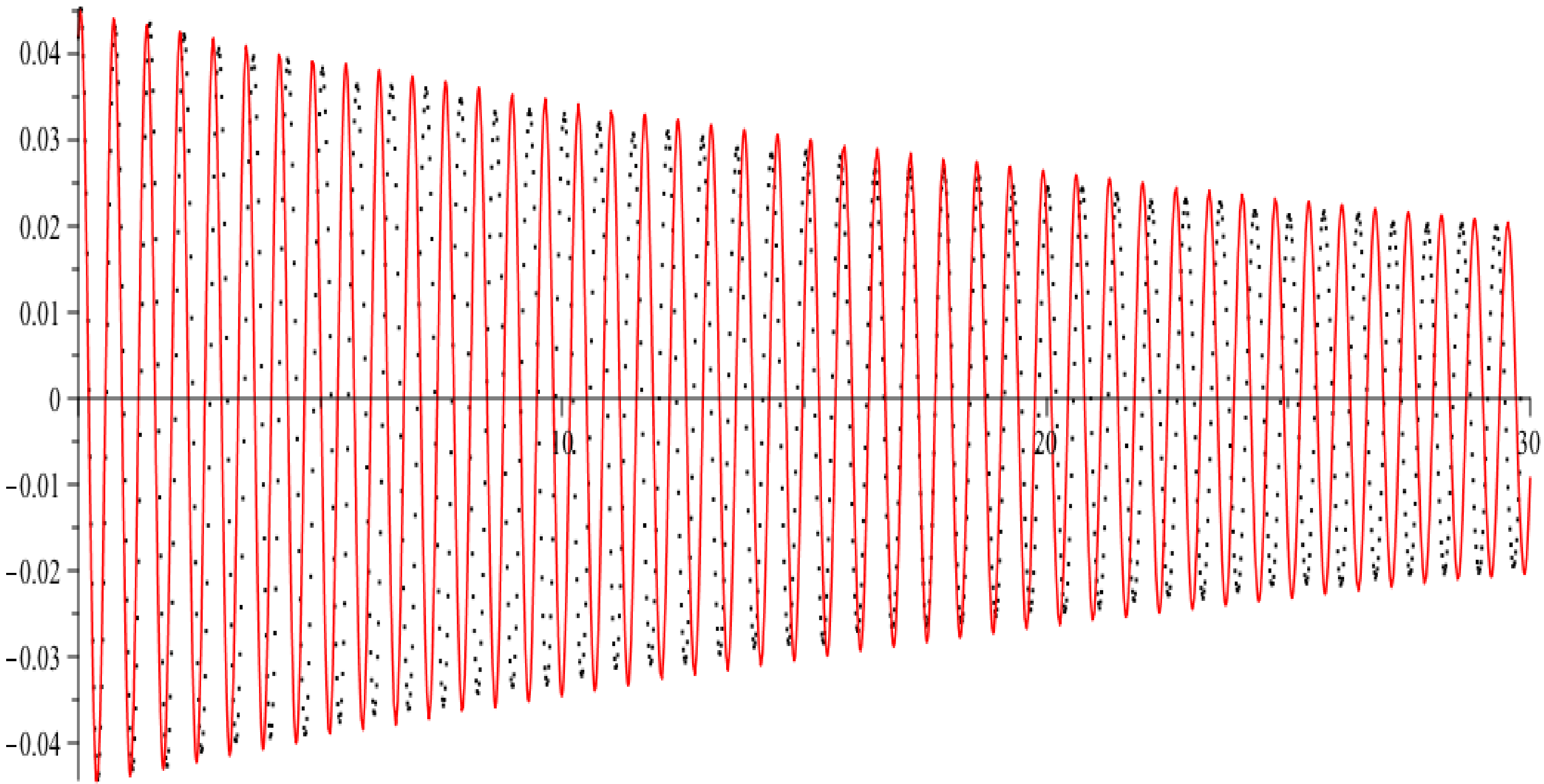
Students had already solved initial value problems using Maple and had plotted the solutions. They were able to graph their solution to the IVP and plot all the data points on the same set of axes. They were then given the following exercise:

- By experimentation with values of c and staying within the interval you found in the previous question, find the value of c that best fits the data. Plot the data and your solution together.

Students then experimented with the value of the coefficient of y'(t) to get the best fit for their data.

```
eq := 0.20626· y"(t) + 0.011· y'(t) + 17.306· y(t) = 0 :

dsolve([eq, y(0) = 0.041907, y'(0) = 0.148], y(t)) :

y1 := (unapply(rhs(%), t)) :

P := plot(y1, 0..30, color = black) :

display(L, P)
```

After much experimentation with different values of c, most students chose c to be approximately equal to 0.011.



Predator-Prey: Blue Whales and Krill:

Abstract:

Students will use Excel to observe qualitative behavior in a simulation of a predator-prey model, with blue whales and krill as the predator and prey populations, respectively.

Students are asked to explain terms in the system of differential equations, compute population values using iteration without a spreadsheet. Then they will implement the simulation in Excel and create charts to observe qualitative behavior.

Lydia Kennedy (2016), "6-25-T-WhalesAndKrill,” https://www.simiode.org/resources/1498.

“One of the favorite foods of the blue whale is krill. Blue whales are baleen whales and feed almost exclusively on krill. These tiny shrimp-like creatures are devoured in massive amounts to provide the principal food source for the huge whales. In the absence of predators, in uncrowded conditions, the krill population density grows at a rate of 25% per year. The presence of 500 tons/acre of krill increases the blue whale population growth rate by 2% per year, and the presence of 150,000 blue whales decreases krill growth rate by 10% per year. The population of blue whales decreases at a rate of 5% per year in the absence of krill. “

These assumptions yield a pair of differential equations

(a Lotka-Volterra model) that describe the population of the

blue whales (b) and the krill population density (k)

over time given by:

$$b'(t) = -0.05 \cdot b(t) + \frac{0.02}{500} \cdot b(t) \cdot k(t)$$

$$k'(t) = 0.25 \cdot k(t) - \frac{.10}{150000} \cdot b(t) \cdot \mathrm{k}(\mathrm{t})$$

Time in years	Blue Whale Population	Krill population in tons per acre
0	75000	150.0000000
1	71700	180.0000000
2	68631.24	216.3960000
3	65793.73903	260.5939828
4	63189.87018	314.3121768
5	60824.8305	379.6493239
6	58707.2732	459.1669177
7	56850.16705	555.9876887
8	55271.97842	673.9126156
9	53998.31884	817.5584471
10	53064.27416	992.5168711

Blue Whale Population	Krill population in tons per acre
75000	150
=B2-0.05*B2+(0.02/500)*B2*C2	=C2+0.25*C2-(0.1/150000)*B2*C2
=B3-0.05*B3+(0.02/500)*B3*C3	=C3+0.25*C3-(0.1/150000)*B3*C3
=B4-0.05*B4+(0.02/500)*B4*C4	=C4+0.25*C4-(0.1/150000)*B4*C4
=B5-0.05*B5+(0.02/500)*B5*C5	=C5+0.25*C5-(0.1/150000)*B5*C5
=B6-0.05*B6+(0.02/500)*B6*C6	=C6+0.25*C6-(0.1/150000)*B6*C6

Time in years	Blue Whale Population	Krill population in tons per acre
29	1460198.315	3878.7851122
30	1613740.218	1072.6177342
31	1602290.263	186.8212500
32	1534149.424	33.9653160
33	1459526.268	7.7180650
34	1387000.543	2.1377689
35	1317769.119	0.6954867
36	1251917.323	0.2583644
37	1189334.395	0.1073216
38	1129872.781	0.0490578

Time in years	Blue Whale Population	Krill population in tons per acre
91	74538.94581	0.0060056
92	70812.01642	0.0072085
93	67271.43602	0.0086703
94	63907.88755	0.0104491
95	60712.51988	0.0126162
96	57676.92453	0.0152596
97	54793.1135	0.0184877
98	52053.49835	0.0224343
99	49450.87014	0.0272643
100	46978.38057	0.0331816

$$b'(t) = -0.05 \cdot b(t) + \frac{0.02}{500} \cdot b(t) \cdot k(t)$$

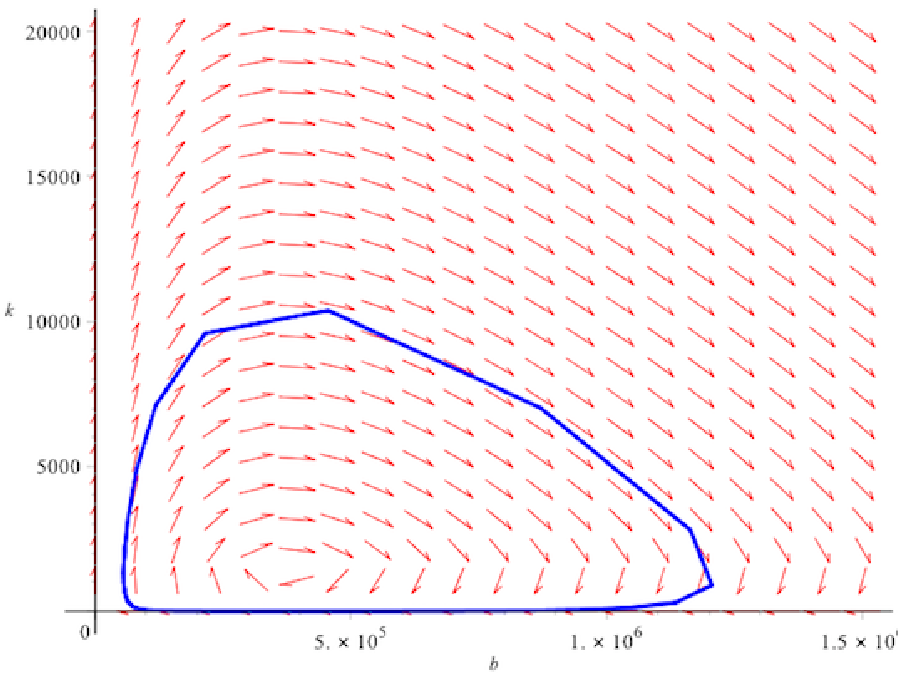
$$k'(t) = 0.25 \cdot k(t) - \frac{.10}{150000} \cdot b(t) \cdot \mathrm{k}(\mathrm{t})$$

We will not ask Maple to solve this system of differential equations.

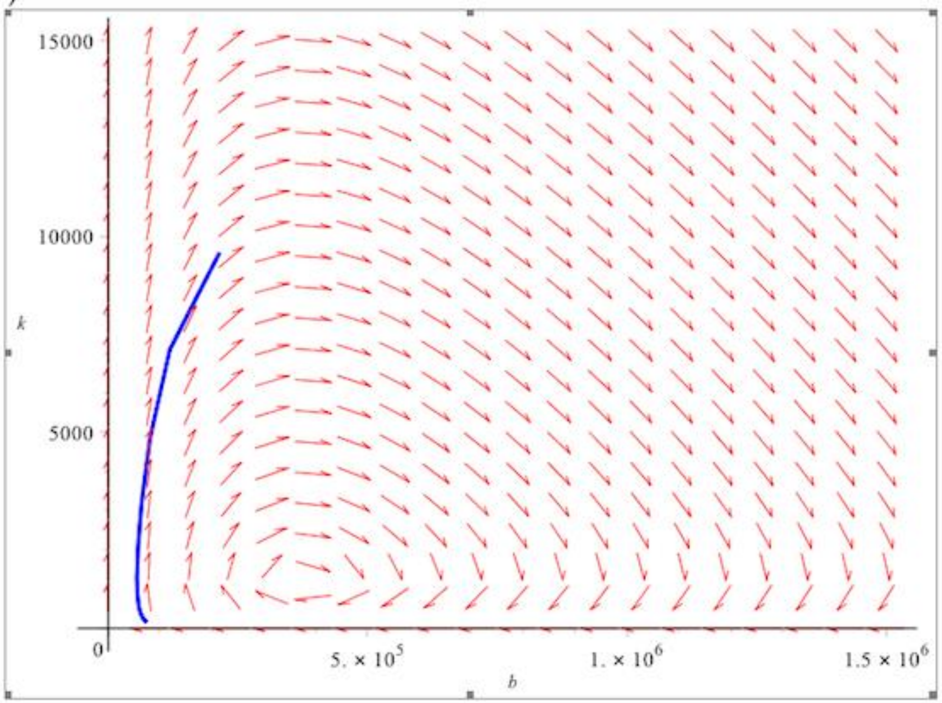
Rather, we will let Maple plot the direction field and a particular solution for certain initial conditions.

Let's again assume that the initial amount blue whale population is 75,000 and the krill population (in tons per acre) is 150.

$DEplot([eq1, eq2], [b(t), k(t)], t = 0..100, b = 0..1500000, k = 0..20000, [[b(0) = 75000, k(0) = 150]], linecolor = blue)$



$DEplot([eq1, eq2], [b(t), k(t)], t = 0..100, b = 0..1500000, k = 0..15000, [[b(0) = 75000, k(0) = 150]], linecolor = blue, animatecurves = true)$



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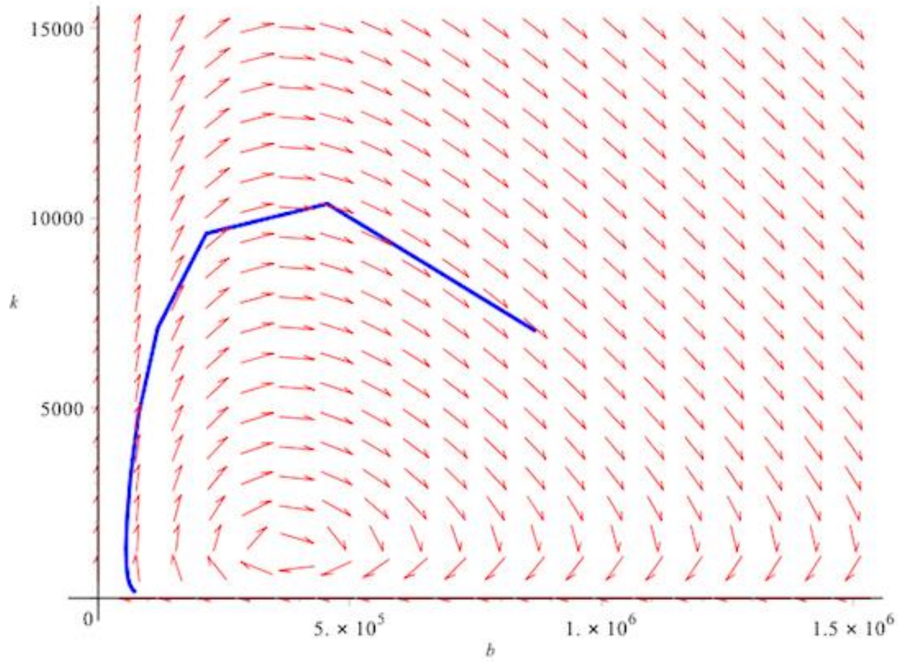
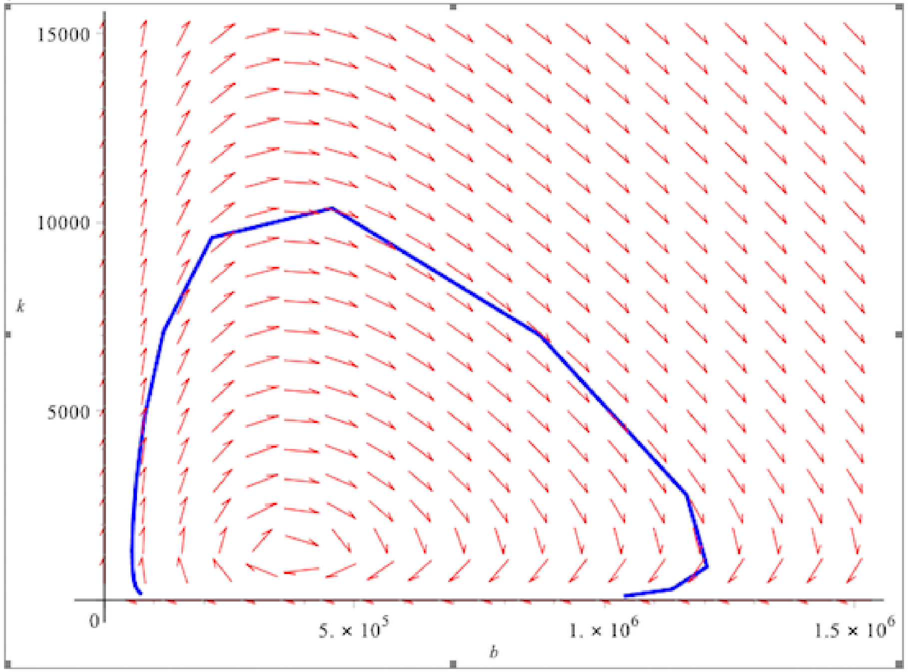
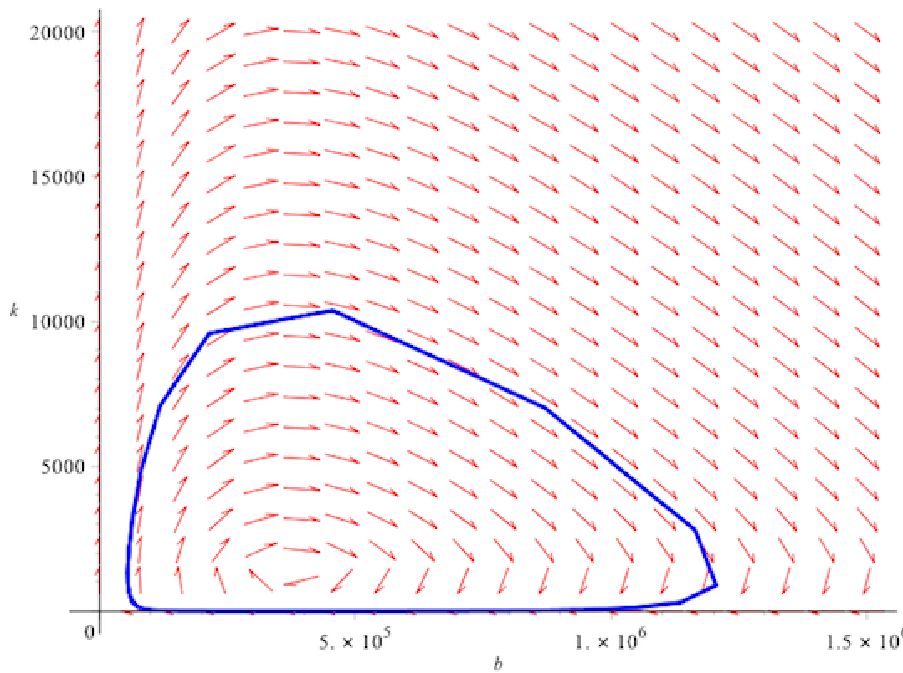


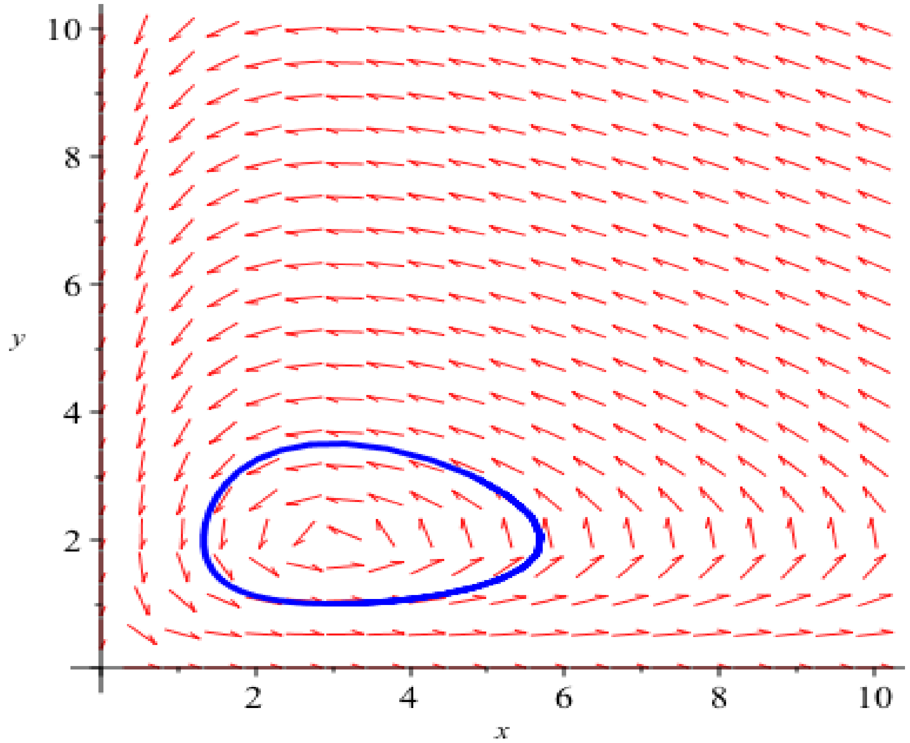
$$DEplot([eq1, eq2], [b(t), k(t)], t = 0..100, b = 0..1500000, k = 0..15000, [[b(0) = 75000, k(0) = 150]], linecolor = blue, animatecurves = true)$$



$DEplot([eq1, eq2], [b(t), k(t)], t = 0..100, b = 0..1500000, k = 0..20000, [[b(0) = 75000, k(0) = 150]], linecolor = blue)$



Test question: Consider the graph below where x(t) represents the population of the prey at time t and y(t) represents the population of the predator at time t.



Suppose that initially there are three prey and one predator. Describe fully what happens to both the predator and prey populations as functions of time.

A typical common final question:

Find all solutions of the system:

$$x_1'(t) = 2\,x_1(t) + 3\,x_2(t)$$

$$x_2'(t) = 1\,x_1(t) + 4\,x_2(t)$$

A non-common final question:

Suppose that you hang a mass of 5 kg on a spring. The spring constant is 2. You do not know the damping constant so you call it b. There is no external force.

Write a differential equation (that includes this damping constant b) that models this situation.

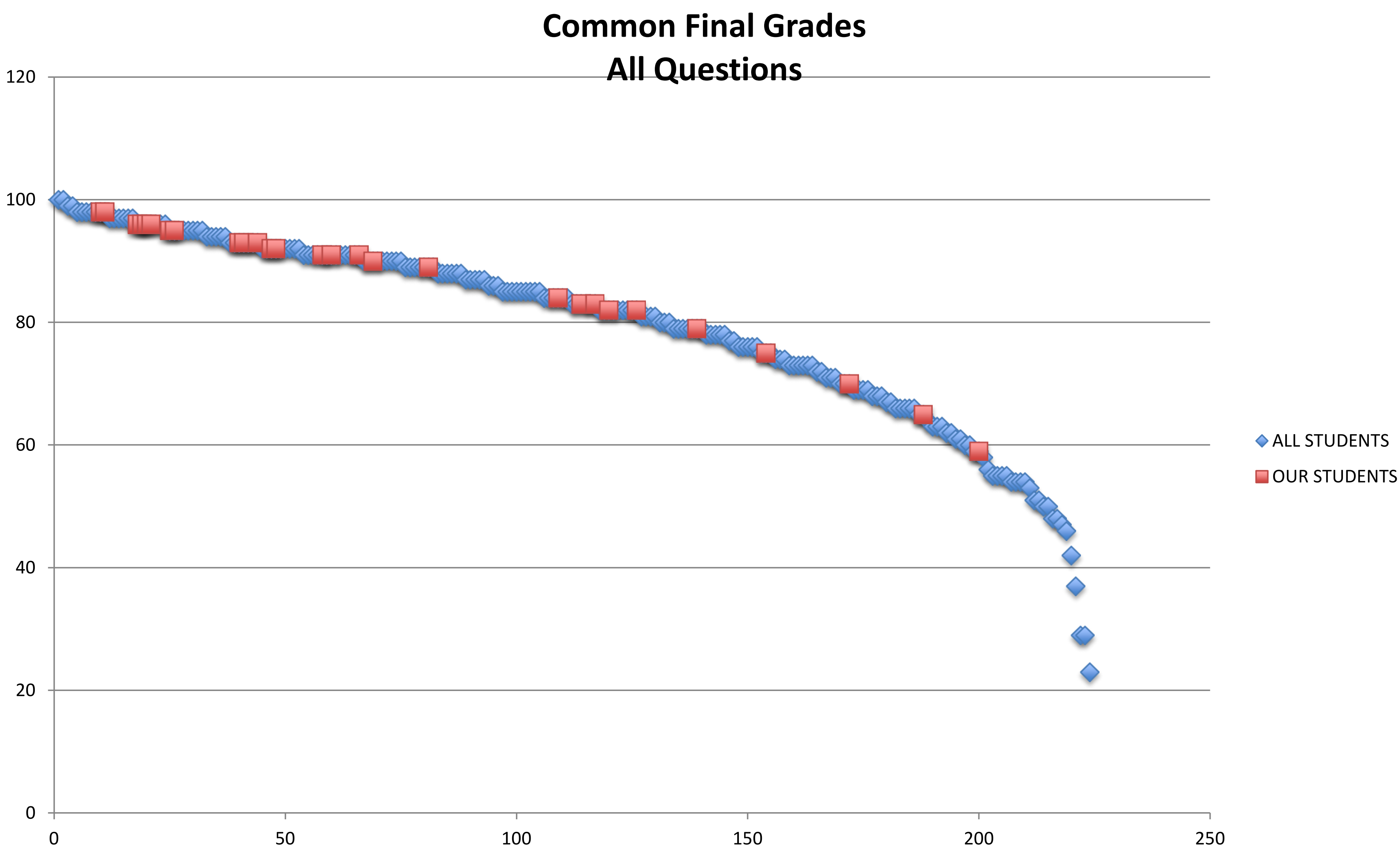
After conducting your experiment, you realize that there definitely is damping because the mass returns to equilibrium rather quickly. This is not a case of simple harmonic motion. So you know that b is not equal to 0. You plot the data of time vs. distance from equilibrium and you see a plot that does not look like a sine and/or cosine curve. There is no bouncing at all. The mass simply returns to equilibrium without bouncing up and down. Use this information and the differential equation you found in part a to find an inequality that the damping constant b satisfies. Explain fully.

Remember: Spring, 2018:

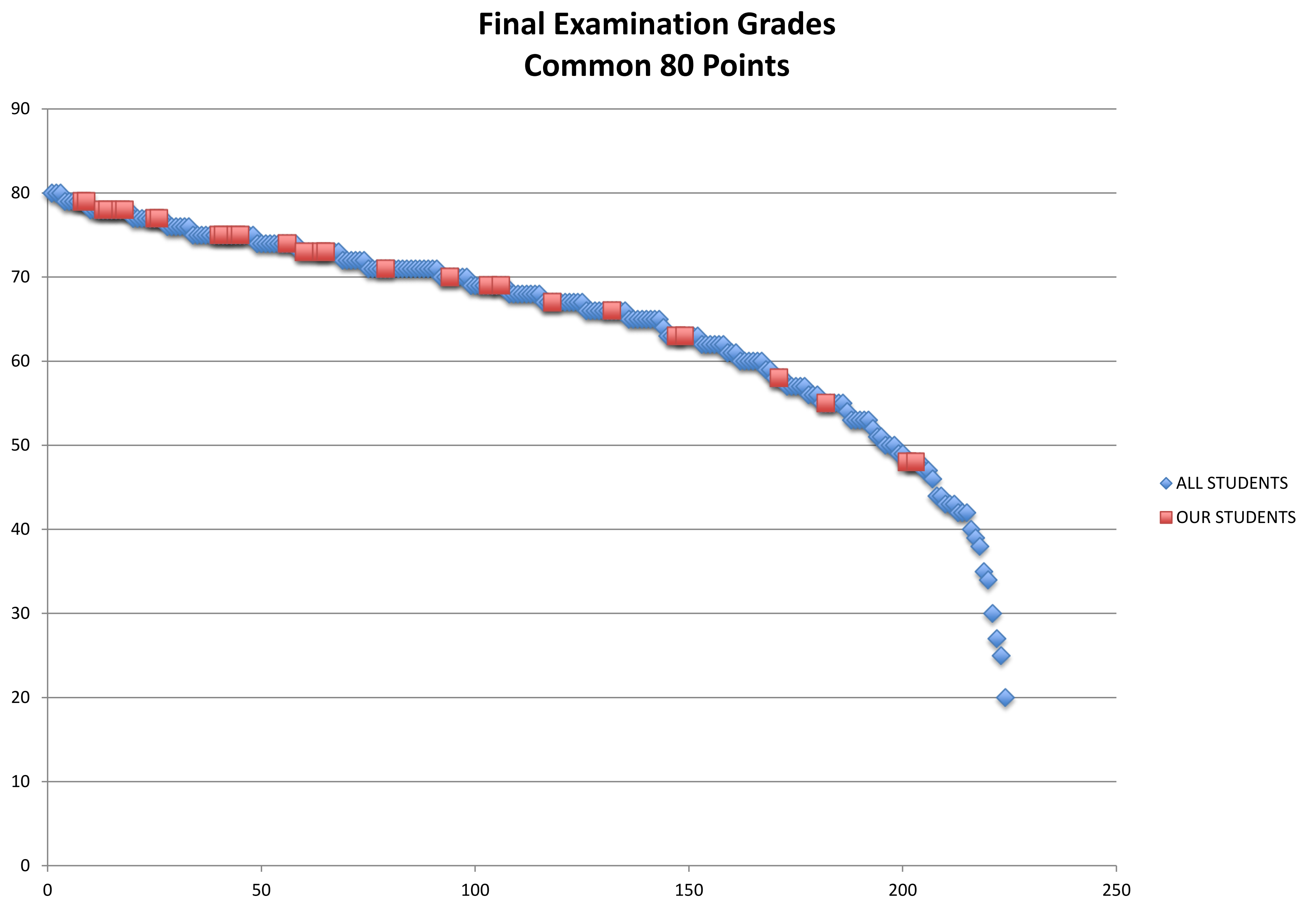
- There were 9 differential equations classes and 1 of them was taught with the modeling first approach advocated by SIMIODE.

- 222 students took the cumulative common final including 28 who were in the modeling first classroom.

- It was agreed that 80% of the final was common.



Common Final Grades
All Questions
120
100
80
60
40
20
0
0
50
100
150
200
250
ALL STUDENTS
OUR STUDENTS


Final Examination Grades
Common 80 Points
90
80
70
60
50
40
30
20
10
0
0
50
100
150
200
250
ALL STUDENTS
OUR STUDENTS

Good Evening Professor,

I had you in calculus 3. I heard that you made videos for differential equations. Would it be possible for me to have them so that I can use them as a study tool this semester?

Thanks!

Parting Words:

- Getting started with the modeling first approach is much easier with the support of the SIMIODE community.

- Check the SIMIODE site often. Scenarios are added all the time. Once you get started you might add your own scenario to the site.

- The use of technology encourages students to hypothesize, experiment, analyze their solutions, and if necessary readjust their hypothesis and try again--without frustration! And students really like answers.

Thanks!

Feel free to contact us at:

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