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STUDENT VERSION

Humans vs. Zombies: A Phase Plane Analysis Activity

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STATEMENT

Introductory Material

A zombie is a deceased person whose body has been revived but who now feeds on living human flesh. Often the zombie is controlled by its own insatiable desire to feed or an evil sorcerer [5]. Because of the zombie's appetite, the imagined conflict between humans and zombies has lead to the development of fanciful and engaging stories and activities. The 1968 movie *Night of the Living Dead* [8] spawned a set of movies about zombies. In 2007, Will Smith entered the human versus zombie genre with his movie *I am Legend* [7]. In 2005, a group of Goucher College students invented the game Humans versus Zombies. Versions of their game have spread to campuses all over the United States [2]. In 2010, Frank Darabont capitalized on the popularity of the human and zombie conflict and created the television show *The Walking Dead*, a show which has been received with cult-like fervor [3].

Problem Set Up

We will model the interaction of the population of humans and zombies using the SIR model originally developed by Kermack and McKendrick [4]. SIR is an abbreviation for the words Susceptible, Infected and Recovered. This model has been used to study the spread of disease in a population, such as the spread of the flu through a British boarding school [1]. Since we will consider the process of zombies turning humans into zombies as the spread of the " zombification" disease, this model is reasonable for modeling human-zombie interaction.

Consider a population of humans. One (or more) of the humans become infected with the zombie

disease. Initially, we consider two classes of individuals: Susceptible Individuals (aka humans) and Infected Individuals (aka zombies). (We will later add an R category, which we will describe as the Removed Individuals.) Zombies then catch humans and infect them through zombification. We assume that the zombie disease spreads relatively quickly so that the only way the population of each of these groups changes can be attributed to zombification.
Let t denote time in days. Let $H(t)$ denote the fraction of the population that can catch the zombie disease. Let $Z(t)$ represent the fraction of the population that have been infected by the zombie disease. For example, if the initial population is 100, with $H(0) = 0.9$ and $Z(0) = 0.1$, then there are 90 humans and 10 zombies.

To find how the populations of humans and zombies change, we consider the rate of change of the human population $\frac{dH}{dt}$ and the rate of change of the zombie population $\frac{dZ}{dt}$. Interaction between the humans and zombies can be expressed simply through the algebraic expression HZ. In addition, we need a parameter α (with $\alpha > 0$) that will indicate the likelihood of the interaction of humans and zombies inside the population as well as the likelihood of the disease spreading during an interaction. The system of differential equations that represents the interaction between the two groups can then be expressed as

$$\frac{dH}{dt} = -\alpha HZ$$
$$\frac{dZ}{dt} = \alpha HZ$$

where $\alpha > 0$ represents the "contagion" parameter.

Phase Plane Software

As part of our analysis of the interaction of humans and zombies, we will use phase plane software. One robust software for drawing phase planes is `http://math.rice.edu/~dfield/dfpp.html`. Originally developed in Matlab by John Polking of Rice University, this free software was converted to Java applets by Joel Castallanos of University of New Mexico [6]. In this activity, all software references will be to pplane.jar. If you have access to other phase plane software, feel free to use it as well.

Activities

- **Question 1** Show that $H(t) + Z(t) = 1$ for all values of time. What does this imply for the range of values for $H(t)$ and $Z(t)$? Next, explain how each of the equations models the assumptions about the changes in the human and zombie populations. Then, show that the only equilibrium point of the the system is $H(t) = Z(t) = 0$. Explain the meaning of the equilibrium point in terms of the human and zombie populations.
- **Question 2** Graph the phase plane for the system, using the value $\alpha = 0.2$. See Figure1 for

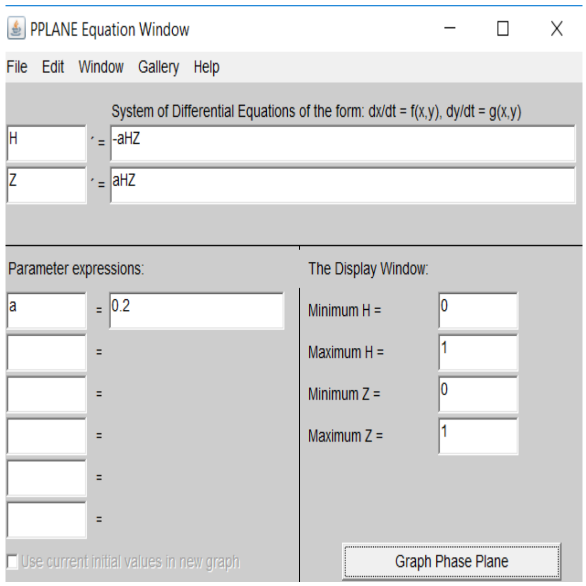

Figure 1. Entering the system in pplane.jar

hints on entering the system into pplane.jar.

For the initial condition $H(0) = 0.9$ and $Z(0) = 0.1$, graph the solution in the phase plane. See Figure 2 for hints on plotting a particular initial condition onto the phase plane. Then produce the graphs of the individual functions $H(t)$ and $Z(t)$. See Figure 3 for hints on plotting individual time functions in pplane.jar. What is the long term behavior of the human and zombie populations if $H(0) > 0$ and $Z(0) > 0$? Justify your answer with graphs and verbal explanations. If you vary the values of α, does the graph in the phase plane change? Explain why or why not.

- **Question 3** We now introduce a third group into our population, zombies who have been destroyed, either by having their heads removed or by destruction of their brain [5]. Let $R(t)$ represent the fraction of the population that are the removed zombies. These persons/zombies are now simply dead and do not re-enter the susceptible population. Assume that the entire population fits into one of the three categories H, Z or R. Then the following system provides a model for the population changes.

$$\frac{dH}{dt} = -\alpha HZ$$
$$\frac{dZ}{dt} = \alpha HZ - \beta Z$$
$$\frac{dR}{dt} = \beta Z$$

where α represents the "contagion" parameter and β represents the "removal" parameter. Explain the significance of the βZ term. Explain directly from the system of equations why

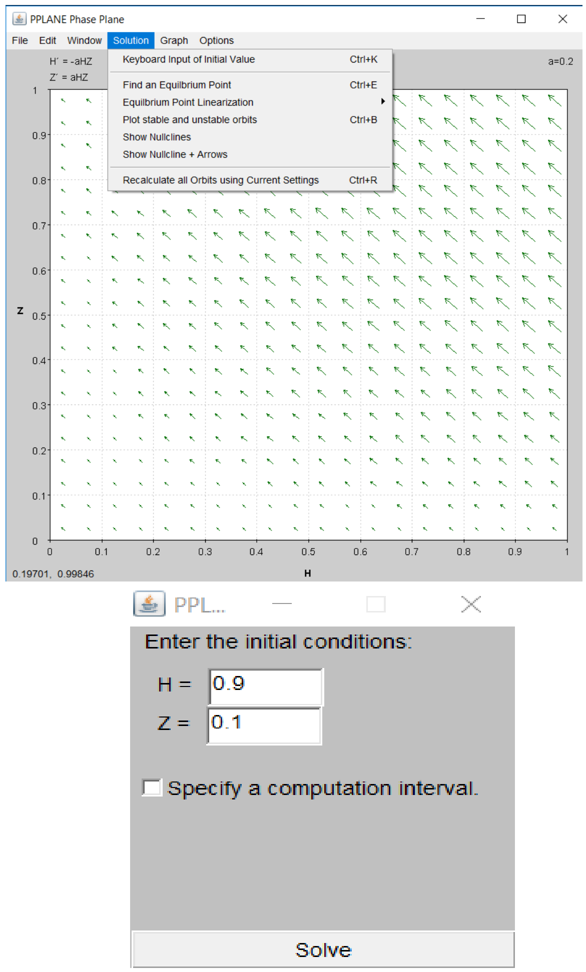

Figure 2. Entering the initial condition into pplane.jar

$H(t)+Z(t)+R(t) = 1$. Then explain why we can convert the system to a planar system in only H and Z and show the new system. Find all the equilibrium solutions to the planar system. Explain their significance in terms of the model.

- **Question 4** Graph the phase plane of the model with $\alpha = 0.2$ and $\beta = 0.1$. Then plot the three solutions corresponding to three different initial conditions: $(H(0), Z(0)) = (0.9, 0.1)$, $(H(0), Z(0)) = (0.5, 0.5)$, and $(H(0), Z(0)) = (0.3, 0.7)$. Describe the similarities and differences between the three solutions. Then, for each initial condition, plot the time graphs $H(t)$ and $Z(t)$. Describe the features of each set of time graphs, including the long term behavior of each population. Then describe the similarities and differences between the three sets of time graphs. Does varying the initial conditions result in differing long term behavior?
- **Question 5** Consider the equation describing $\frac{dZ}{dt}$ from Question 3 again. Find the value of

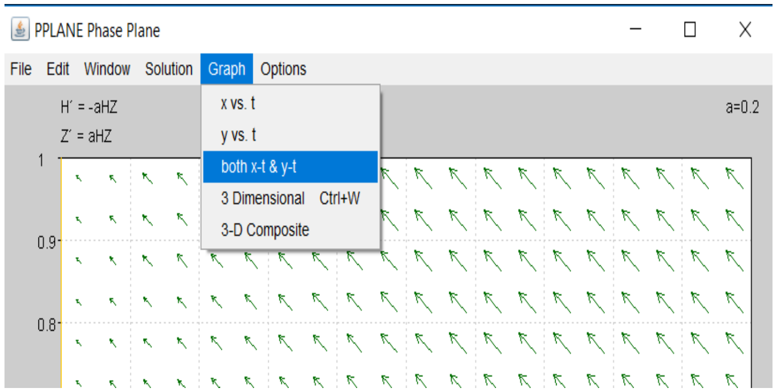

Figure 3. Plotting the time graphs

H for which $\dfrac{dZ}{dt} = 0$. We will call this value C. Describe the shape of the curve $H = C$ in the phase plane. Describe the behavior of the vectors in the phase plane along $H = C$. What happens to the solution if $H(0) < C$? What happens to the solution if $H(0) > C$? Explain why C would be called the **threshold value** for the system. (Hint: it may be helpful to calculate the threshold value for the system with $\alpha = 0.2$ and $\beta = 0.1$ and look at the three solutions plotted as part of Question 4.)

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