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SIMIODE A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

STUDENT VERSION
SPRING MASS FIRST TRY

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STATEMENT

Consider a mass hanging at the end of a spring as depicted in Figure 1.

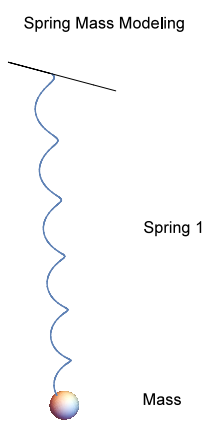

Figure 1. Simple spring mass configuration.

If we let the mass hang still, thus extending the spring from its natural length, we will see that the mass comes to rest at what is called *static equilibrium*. In attempting to model the vertical motion of this mass we would impose a coordinate system for $y(t)$, the vertical displacement from the static equilibrium. Engineers refer to such a system as a Single Degree of Freedom System

(SDOFS), as we are tracking only one variable, namely, $y(t)$, vertical displacement from the static equilibrium. For consistency, let us say $y = 0$ is the spring's vertical displacement at the spring's static equilibrium and is the distance of mass displacement from that static equilibrium and denote positive in the downward direction and negative in the upward direction. That is, if we extend the spring 3 cm downward then $y = 3$, while if we compress the spring upward 2 cm then $y = -2$.

We are going to use a Free Body Diagram (Figure 2) to depict all the vertical forces acting on the mass in this case. Figure 2 depicts the mass pulled down just a bit beyond the static equilibrium position with the only force acting on it being the restoring force of the spring which acts in an upward vertical direction.

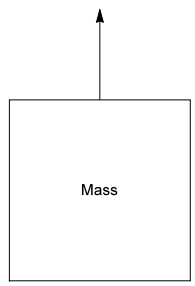

Figure 2. Template for Free Body Diagram.

Newton's Second Law of Motion says that for a given body the mass of that body times the acceleration of that mass is equal to the sum of the external forces acting on the mass. Again, we will presume the spring mass system is at rest and this means the downward force due to gravity, $m \cdot g$ (mass m times acceleration due to gravity g) cancels out the upward restoring force of the spring when the mass is extended. Thus our Free Body Diagram (Figure 2) has only one force when extended, namely the restoring force when the spring is extended beyond its static equilibrium by distance $y(t)$ cm.

In high school you may have conducted an experiment on springs to verify that when a spring is extended a distance y beyond its equilibrium then there is a force of restoration, $F = k \cdot y$, acting to restore the spring to its equilibrium and that force is proportional to the displacement only. The constant of proportionality, k, is called the *spring constant* and it can be determined experimentally by suspending different masses on the spring, each mass producing a corresponding force. Recall $F = m \cdot g$, where g is the acceleration due to gravity. If we denote the corresponding displacement y, then when plotting F vs. y the data is linear and goes through the origin, as 0 units of force displaces the spring 0 units of distance. k can thus be determined by fitting the line $F = k \cdot y$ to the data F vs. y for a given spring. This equation is called Hooke's Law and is named after 17th century British physicist Robert Hooke.

A spring is thus said to have *stiffness* which we can characterize as a force proportional to its extension from static equilibrium.

Modeling Activities

1. Identify the one force acting on the mass shown in Figure 2 and if y is positive and the spring is extended downward then in what direction is this force? If y is negative and the spring is compressed upward then in what direction is this force?
2. Complete the differential equation (1) using this force in Newton's Second Law of Motion (again, here m is the mass):

$$m \cdot y''(t) = \underline{\hspace{4cm}} \tag{1}$$

3. Write out an initial value problem using (1) if the spring's initial displacement from static equilibrium is y_0 and its initial velocity is v_0.

You now have a mathematical model of a mass bobbing up and down on a spring in which no resistance to the motion is assumed, i.e. there is no damping. Often, we jokingly say the spring was purchased from the "Ideal Spring Company" because without damping, the mass will never return to a position of static equilibrium once released.

We shall study this simple spring model in some depth before moving on to a spring mass model which experiences damping and eventually comes to rest.

4. If we believe our model without damping will just oscillate up and down forever without any decay then what kind of function(s) does that suggest for our vertical displacement of the mass, $y(t)$? Why, some kind of sine or cosine function, perhaps something like $y(t) = A\sin(\omega t)$ or $y(t) = B\cos(\omega t)$. But if we were to pick the sine function, then since it always has $y(0) = 0$ we could never model a mass with a non-zero initial displacement, so we would also need the cosine function in our solution. Similarly, if we were only to use the cosine function then since the derivative of cosine is sine we would always have $y'(0) = 0$ which would limit our model as well. So why not have a little of each, i.e. a of $\sin(\omega t)$ and b of $\cos(\omega t)$. Thus our candidate or conjectured solution for the vertical displacement from static equilibrium for our mass on a spring without resistance might look like

$$y(t) = a\sin(\omega t) + b\cos(\omega t)\,. \tag{2}$$

   One way to tell how good this conjecture is would be to just check it by substituting it into your differential equation model you built from (1). Go ahead and do that to see where it leads. Remember you are trying to figure out if (2) is a solution and then if it is a solution what might a, b, and ω have to be to fulfill the initial conditions $y(0) = y_0$ and $y'(0) = v_0$?
5. What information does k and m give you in your confirmed conjectured solution? Explain.
6. Let us take an actual spring with a mass and use your model with some initial conditions to find a complete solution. Use $m = 3$ gram, stiffness coefficient or spring constant $k = 27$ dyne/cm, $y(0) = 1$ cm and $y'(0) = 2$ cm/sec. Units are in grams, centimeters, seconds, and

dynes appropriately. Solve the differential equation in the above manner and graph its motion over a reasonable interval of time. Explain what you see.

7. If you have access to a computer algebra system then solve the differential equation with the appropriate initial conditions and compare that solution with your solution from the method above.

Supporting Theory

There is supporting theory (called existence and uniqueness theorems) in mathematics which says that there will be a solution and if we find a candidate solution for our differential equations which also satisfies the initial conditions then this solution is the unique solution and we need look no further. Thus our intuition has helped us to find the unique solution. We shall develop other means of finding "candidate" solutions in our further studies.