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SIMIODE A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

STUDENT VERSION

CHEBYSHEV POYNOMIAL SOLUTION

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STATEMENT

Introduction: Differential equations are considered *classical* if they have been studied for many years and their solutions have a variety of applications. These equations appeal to the pure mathematician because of their aesthetic beauty and to the scientist and engineer for their applicability.

One such example is the ***Chebyshev*** differential equation, which is given by:

$$\boxed{(1-x^2)y''-xy'+k^2y=0,\ \ k=0,1,2,3,\ldots}$$

This equation possesses **finite** polynomial solutions (as opposed to infinite series solutions), and we call these solutions the *Chebyshev Polynomials.*

The series solution technique: way to obtain finite polynomial solutions to the aforementioned differential equation is by using the *series method* studied in class. That is, since $x=0$ is an *ordinary point* for this differential equation, i.e. the lead polynomial, $P(x) = 1 - x^2$, is not 0 when $x=0$, we can assume a solution of the form $y=\sum_{n=0}^{\infty} a_n x^n$ and, after differentiating term-by-term, we substitute y, y' and y'' into the differential equation. After completing the appropriate manipulations, we collect like terms, compare coefficients of powers of *x*, and eventually obtain a *recursion formula*. We then "split off" the two linearly independent solutions which are guaranteed by existence theorems. We recall that both a_0 and a_1 are arbitrary.

The method verified: Let $k=2$ in the Chebyshev differential equation. Obtain two solutions to the differential equation via series techniques.

- One finite polynomial solution exists. What is the degree of this polynomial?

- What observations can be made about the other solution?

- What values of a_0 and a_1 yield only a polynomial solution and no "infinite solution"?

- What happens if $k = 3, 4, 5, 6$ and 7? Make a plausible conjecture as to the degrees of the various finite polynomial solutions and the values imposed on a_0 and a_1 to yield only polynomial solutions.

Historical background: Investigate the history of this equation. Who was Chebyshev? What properties do the polynomial solutions enjoy? What are some of the concepts associated with this differential equation? What are some of the applications which use this equation and the associated polynomial solutions?

A similar differential equation with two polynomial solutions: Consider the differential equation below, where A and B are integers which can regarded as parameters:

$$\boxed{(1-x^2)y'' + Axy' + By = 0}$$

Note that this equation is similar to the Chebyshev differential equation.

- Can we find conditions on the parameters *A* and *B* such that the differential equation will possess *two* polynomial solutions?

- Furthermore, can we determine the degrees of the polynomial solutions from the values of *A* and *B*? Can we answer these questions by merely *inspecting* the differential equation?

Formalize the results into a **Theorem**, supported by a **Proof**.