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SIMIODE A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

STUDENT VERSION
FORCED VIBRATION AND NO DAMPING

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STATEMENT

Consider a spring mass dashpot model with mass m kg, resistance c N/(m/s), and spring constant k N/m with a forcing function $f(t)$ in units N to describe the displacement from static equilibrium, $y(t)$, (1).

$$m \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = f(t)\,, \quad y(0) = y_0 \quad \text{and} \quad y'(0) = v_0\,. \tag{1}$$

In this case we shall deal with an ideal spring and set $c = 0$. Thus we have the following differential equation as (1) becomes (2).

$$m \cdot y''(t) + k \cdot y(t) = F(t)\,, \quad y(0) = y_0 \quad \text{and} \quad y'(0) = v_0\,. \tag{2}$$

Recall that (2) without a driving function $f(t)$ produces a pure oscillating function with natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$. If now we apply a driver function and our driver is a sinusoidal function, $F(t) = F_0 \cos(\omega t)$, we seek to know what the possibilities are for solutions/outputs from (2).

For notational convenience let us divide both sides of (2) by m and noting that $\omega_0 = \sqrt{\frac{k}{m}}$ we can see that the coefficient of $y(t)$ is $\frac{k}{m} = \left(\sqrt{\frac{k}{m}}\right)^2 = \omega_0^2$. Thus (2) now looks like (3). Also let us set $y(0) = y_0 = 0$ and $y'(0) = v_0 = 0$ as the driver will move the mass.

$$y''(t) + \omega_0^2 y(t) = \frac{F_0}{m}\cos(\omega t)\,, \quad y(0) = 0 \quad \text{and} \quad y'(0) = 0\,. \tag{3}$$

We can solve (3) by hand or, say, use *Mathematica*'s `DSolve` to solve, obtaining:

$$y(t) = \frac{F_0}{m(w_0^2 - w^2)}(\cos(\omega t) - \cos(\omega_0 t))\,. \tag{4}$$

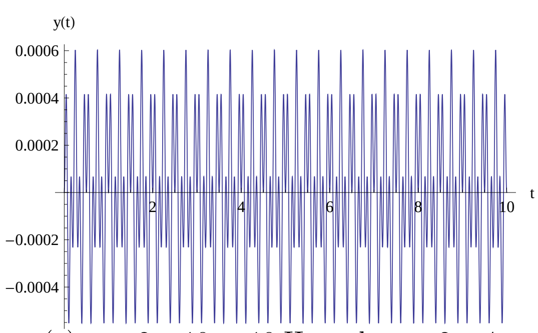
We can use the trigonometric identity in (5) to alter our solution (4) to a more learning friendly format.

$$(\cos(\omega t) - \cos(\omega_0 t)) = 2\sin\left(\frac{\omega_0+\omega}{2}t\right)\sin\left(\frac{\omega_0-\omega}{2}t\right). \tag{5}$$

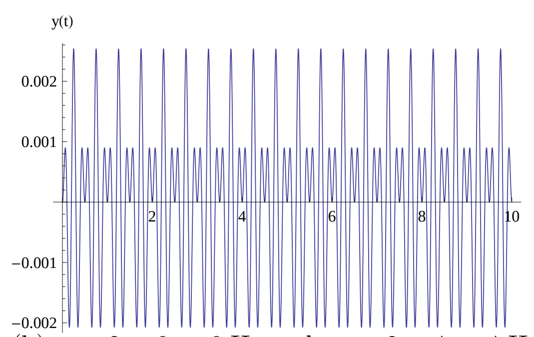
Thus our solution in (3) looks like (6),

$$y(t) = \frac{2F_0}{m(w_0^2 - w^2)}\sin\left(\frac{\omega_0+\omega}{2}t\right)\sin\left(\frac{\omega_0-\omega}{2}t\right). \tag{6}$$

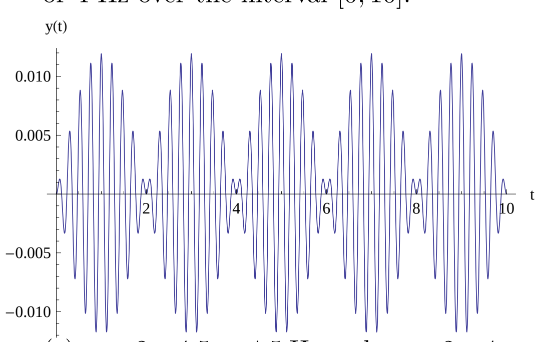
So what might this solution for (3) look like? Let us examine the two time dependent terms, $\text{Term}_1 = \sin\left(\frac{\omega_0+\omega}{2}t\right)$ and $\text{Term}_2 = \sin\left(\frac{\omega_0-\omega}{2}t\right)$, in the solution. Assuming both ω and ω_0 are positive numbers, then Term_1 has a higher frequency than Term_2. We note that Term_1 has a high frequency $\left(\frac{\omega_0+\omega}{2\cdot 2\pi}\right)$ Hz while Term_2 has a low frequency $\left(\frac{\omega_0-\omega}{2\cdot 2\pi}\right)$ Hz. We say that frequency $\left(\frac{\omega_0+\omega}{2\cdot 2\pi}\right)$ is *modulated at* $\left(\frac{\omega_0-\omega}{2\cdot 2\pi}\right)$, noting that the low frequency term forms an envelope for the higher frequency term. We demonstrate this in Figure 1(d).



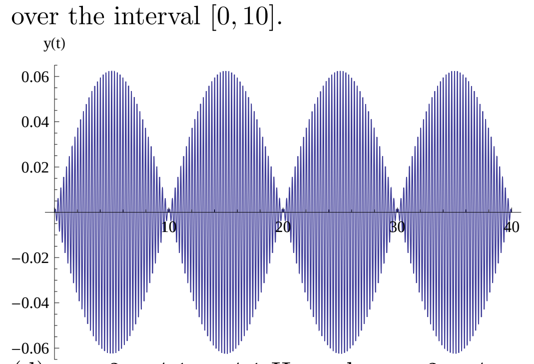
(a) $\omega = 2\pi \cdot 10$ or 10 Hz and $\omega_0 = 2\pi \cdot 4$ or 4 Hz over the interval $[0, 10]$.



(b) $\omega = 2\pi \cdot 6$ or 6 Hz and $\omega_0 = 2\pi \cdot 4$ or 4 Hz over the interval $[0, 10]$.



(c) $\omega = 2\pi \cdot 4.5$ or 4.5 Hz and $\omega_0 = 2\pi \cdot 4$ or 4 Hz over the interval $[0, 10]$.



(d) $\omega = 2\pi \cdot 4.1$ or 4.1 Hz and $\omega_0 = 2\pi \cdot 4$ or 4 Hz over the interval $[0, 40]$.

Figure 1. Plots of $y(t) = \frac{2F_0}{m(w_0^2 - w^2)}\sin\left(\frac{\omega_0+\omega}{2}t\right)\sin\left(\frac{\omega_0-\omega}{2}t\right)$, the solution for differential Equation (3), for various values of ω and ω_0.

In Figure 1 we note that as the driver frequency, ω, gets closer to the natural frequency, ω_0, that the envelope frequency of Term$_2$, i.e. $\left(\frac{\omega_0-\omega}{2\cdot 2\pi}\right)$ becomes more pronounced and sinusoidal, while the high frequency term of Term$_1$, i.e. $\left(\frac{\omega_0+\omega}{2\cdot 2\pi}\right)$ gets ever higher in frequency. When the driver frequency, ω, gets close to the natural frequency ω_0 we say the system displays *beats*, a sort of undulation in amplitude (Term$_2$) of a high frequency tone (Term$_1$).

a) We can play the sound of beats in *Mathematica*. If the frequencies in our system are in the interval $[20, 20000]$ Hz, then humans can detect the corresponding sound. Consider (3) with $\omega_0 = 2\pi \cdot 440$ (440 Hz - the tone of the A note above "middle" C note on the piano keyboard) and $\omega = 2\pi \cdot 480$ (480 Hz). Using `DSolve` in *Mathematica*, solve (3). Then, instead of plotting the obtained solution, use *Mathematica*'s `Play` command, which has the same syntax as *Plot*, to play the sound of our function solution to (3). Here the time interval offered, $\{t, t_{\text{start}}, t_{\text{stop}}\}$, refers to the playing time of the sound. Be sure your sound is on, but not too loud and not too long (say, $t_{\text{start}} = 0$ s and $t_{\text{start}} = 2$ s), lest you wake a slumbering roommate who packed it in earlier, or a classmate sitting next to you, or the librarian in the reference room where you do your homework! You should recognize this note as the ring tone [**?**] in the telephone.

b) Now you try various combinations of ω_0 and ω (close to each other) to see what beats you can produce. Certain combinations evoke the vibrato or tremolo on a "weeping" or sad organ background as the heroine is distressed in "olde timey" films. Have you ever heard the annoying signal on the radio generated by $\omega_0 = 2\pi \cdot 2000$ and $\omega = 2\pi \cdot 2200$? What is this signal used for?

c) We massage the solution (6) to (3) one more time by dividing and multiplying by $\left(\frac{\omega_0-\omega}{2}\right)$ to obtain (7):

$$y(t) = \frac{2F_0}{m(w_0+w)} \sin\left(\frac{\omega_0+\omega}{2} t\right) \frac{\sin\left(\frac{\omega_0-\omega}{2} t\right)}{\frac{\omega_0-\omega}{2}}\,. \tag{7}$$

Now we were interested in what happened as our driver frequency, ω, got close to our natural frequency, ω_0 and we saw (and heard!) beats in our solution. Well, what happens as ω gets very close, arbitrarily close, ω_0. Indeed, what happens as $\omega \to \omega_0$? In this case the term

$$\frac{\sin\left(\frac{\omega_0-\omega}{2} t\right)}{\frac{\omega_0-\omega}{2}}$$

in (7) approaches t. We can obtain this using L'Hôpital's Rule or using a general `Limit` command in *Mathematica* to see that `Limit[Sin[a t]/a, a -> 0] = t`. Thus our solution (7) to the (3) becomes in the limit (8) as $\omega \to \omega_0$.

$$y(t) = \frac{F_0}{m \cdot \omega_0} t \sin(\omega_0 t)\,. \tag{8}$$

Unlike the situation which brought about beats, namely a low frequency envelop modulating a high frequency function, now we have an oscillation with a frequency ω_0 – that of the system and the driver, since they are the same, but here the solution is multiplied by t. Thus, as time increases this solution gets bigger and bigger, wildly oscillating without end. We call this *resonance* and for the most part engineers and scientists are not happy when this happens.

d) Now you try various combinations of ω_0 and ω to see what resonance you can produce.

e) Drive (3) (use $m = 1$ and $F_0 = 1$) with a driver equal to the natural frequency, say $\omega = \omega_0 = 2\pi \cdot 440$. Solve the differential equation, plot the solution over $[0, 0.1]$ s and then $[0, 4]$ s. What do you see? Explain what is happening. Now plot/play the solution in the interval $[0, 8]$ s, BUT if you play the solution be sure you are away from those you would not want to disturb!!! Why?

f) Given (2) or its altered form (4) suppose we had a driver of the form

$$F(t) = F_0 \cos(\alpha t) + G_0 \cos(\beta t)\,, \tag{9}$$

but did not know what α or β actually were. Describe how you could change w_0 so you could "see" a solution which would tell you, first what α was and then what β was. You may assume it is easy, perhaps as easy as a radio dial, to alter ω_0. Indeed, engineers design and study a simple electrical circuit with resistors, inductors, and capacitors in which the current going through the circuit is modeled as a differential equation and changing the dial (the capacitor) picks out the signals from the ether, much as your changing ω_0 will help you determine α and β. In the latter case we call it tuning your radio dial!