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STUDENT VERSION
Differential Equations and Resonance
Can a human singing voice shatter a wine glass?

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STATEMENT

Can a human singing voice shatter a wine glass?

A) *Sure. I could do it.*

B) *A highly trained opera singer might be able to do it.*

C) *No. Humans can't possibly sing loudly enough or precisely enough at the right frequency. This is just an urban legend.*

Any structure is susceptible to damage by the force of resonance. In this project, you will study the phenomena of mechanical resonance and understand its role using second-order linear ordinary differential equations.

Shattering wine glass with human voice

In 2005 rock singer Jamie Vendera had the right frequency (556 Hz) and, after 20 attempts, worked up the volume (105 decibels - that is 12 times as loud as a vacuum cleaner) needed to shatter a wine glass. This was confirmed by the MythBusters and captured on video for the first time [3].

Watch "wine glass resonance in slow motion" [8] (highly recommended). This video demonstrates the vibration and shattering of a wine glass in front of a loudspeaker.

Every object has a resonant frequency – the natural frequency at which it vibrates. If you rub the rim of a wine glass with a damp finger, you might hear a faint hum – that is the resonant frequency of the glass. Or you can simply tap the glass and hear the same frequency. How do the tones sound at different frequencies? Have a listen at the online tone generator [4]!

Marching soldiers cause suspension bridge collapse

Built in 1826, the Broughton Suspension Bridge was an iron chain suspension bridge in England and one of Europe's first suspension bridges. It collapsed five years later due to mechanical resonance induced by troops marching over the bridge in step, throwing 74 soldiers into the river [1]. As a result of the incident, the British Army issued an order that troops should "break step" when crossing a bridge.

Crash of aircrafts

On September 29, 1959, a commercial Braniff airline, a Lockheed L-188 Electra, crashed after 23 minutes into the flight. The officials determined that the left wing had failed, but they could not determine the cause of the catastrophe. Six months later a second Electra aircraft crashed on March 17, 1960, with its right wing found 5 miles from the crash site [5]. NASA, Boeing, and Lockheed engineers determined that violent flutter had torn the wings off the two planes. They were able to duplicate the flutter in wind tunnel simulations [6]. The weakened engine mounts would develop the propeller-whirl flutter. The fatal resonance could build up and tear the plane apart in 30 seconds.

Figure 1. MythBusters: Breaking Glass (Discovery Channel) (L); Broughton suspension bridge (rebuilt in 1883) (C); NASA: a powered model of the Lockheed Electra mounted in the transonic dynamics tunnel. (R)

How did these happen? And why?

When the frequency of a periodic external force matches the natural frequency of the mechanical system, resonance may occur which builds up the oscillation to such tremendous magnitudes that the system may fall apart. Because of potential damages, mechanical resonance is in general to be avoided, especially in designing infrastructures and vibrating systems.

To illustrate the effect of resonance, let us model the system with an oscillator driven by a periodic external force,

$$m\ddot{x} + b\dot{x} + kx = F(t). \tag{1}$$

For simplicity, take $F(t) = F_0 \sin(\omega t)$ or $F_0 \cos(\omega t)$, where ω is the angular frequency.

In the vibration of a wine glass, $x(t)$ is the amplitude of vibration at time t. In the torsional dynamics of a suspension bridge, $x(t)$ represents the angle deflection from the horizontal position (φ in Figure 2). In the motion of the aircraft wing, $x(t)$ is the vertical position of the center of mass of the wing.

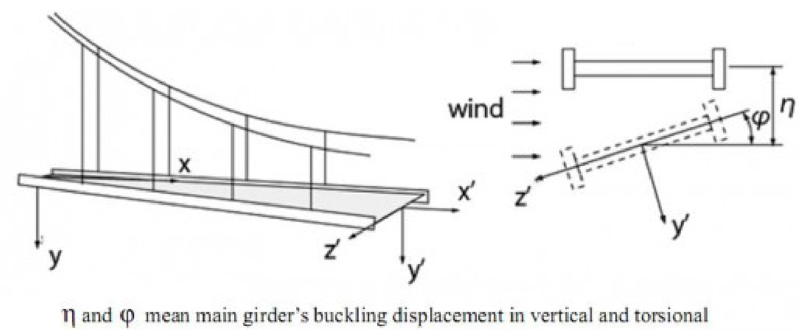

Figure 2. Theoretical model of suspension bridge [2].

Work through the following activities to understand resonance.

A suspension bridge is supported by cables (springs), illustrated in Figure 2. Viewed from the cross-section, an inflexible rod of the bridge is suspended by cables at both ends and is free to move and rotate. Suppose that the rod has a mass of 1,000 kg and the spring constant of the cable is 16,000 newtons/meter. Further, suppose that the wind or marching soldiers create an external force to the bridge, $F(t) = 3,000\sin(\omega t)$ newtons.

1. Let's first examine **pure resonance** by neglecting damping. Equation(1) becomes

$$1000\ddot{x} + 16000x = 3000\sin(\omega t),$$

$$\ddot{x} + 16x = 3\sin(\omega t). \tag{2}$$

   Solve (2) using method of undetermined coefficients for $\omega = 8, x(0) = \dot{x}(0) = 0$. Plot $x(t)$ for t in $[0, 20]$ and describe the motion.

2. Solve and plot $x(t)$ for $\omega = 5$. What difference do you observe?
3. Solve and plot $x(t)$ for $\omega = 4$. Describe the motion. What phenomenon do you observe and why does this occur?
4. In the physical world, **damping** is always present. We modify (2) and add damping.

$$\ddot{x} + 2\dot{x} + 16x = 3\sin(\omega t). \tag{3}$$

   Solve (3) in Mathematica or Matlab, and plot the solutions for the following conditions. Plot (a) and (b) on the same graph, (b), (c) and (d) on the same graph, with different colors and line styles. Label solutions.

(a) $\omega = 8$, $x(0) = \dot{x}(0) = 0$.

(b) $\omega = 4$, $x(0) = \dot{x}(0) = 0$.

(c) $\omega = 4$, $x(0) = 0$, $\dot{x}(0) = 5$.

(d) $\omega = 4$, $x(0) = 1$, $\dot{x}(0) = 0$.

5. Explain the behaviors of the solutions and your analyses. What happens as $t \to +\infty$?
6. **Practical resonance** occurs when the external frequency ω leads to the largest amplitude in steady-state. This frequency and largest amplitude are given below [7].

$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}, \qquad A = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (b\omega)^2}}.$$

The terminology **steady-state** refers to the remaining part of solution $x(t)$ with the transient portion removed, i.e. when all terms containing negative exponentials are dropped since they approach zero as t gets large. Compute this ω value and largest amplitude for (3).

7. What other situations of destructive resonance can you think of?

(You may use Mathematica's "NDSolve" to find the numerical solutions for Problem 4, and "Plot" to graph the solutions.

`http://reference.wolfram.com/language/ref/NDSolve.html`

`http://reference.wolfram.com/language/ref/Plot.html`)

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