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STUDENT VERSION

MODELING WITH SIGMOID CURVES

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Abstract:

The assignment considers two well-known models of population growth, Verhulst-Pearl and Gompertz models, for which qualitative and quantitative analyses are provided. The graphs of the corresponding functions have a sigmoidal or S-shape. The assignment contains two opposite, but related tasks: solving a differential equation and constructing a differential equation for a given function as its solution.

The Verhulst-Pearl and Gompertz models have been widely used, not only to describe dynamics of biological population for which they were initially suggested, but also to provide a mathematical formulation of engineering, economic, chemical, and environmental processes. The models have become a foundation for many other models.

STATEMENT

The assignment aims to analyze two functions described by an 'S'-shaped curve. It offers two opposite, but related tasks: solving a differential equation and constructing a differential equation for a given function as its solution. The assignment contains short theoretical comments, brief historic essay, and questions at the end.

To follow the material easier, a letter that refers to a formula is either the first letter of the model for a differential equation, like (L) for the Logistic model, or F is added for the corresponding function, like (LF) for the logistic function.

A sigmoid curve

A sigmoid curve has the shape of "S" and, therefore, it is often called an "S-shaped" curve or an S-curve. A mathematical formulation of a sigmoid curve varies but all functions possess common features, which include real-valued domains, two horizontal asymptotes, differentiability, a bell-shaped first derivative, acceleration from slow beginning through fast growth to saturation at large values, and many other features.

Mathematical sciences, biological sciences, demography, economics, chemistry, sociology, and political science extensively employ a sigmoid curve, for it almost perfectly describes a wide range of real life phenomena and physical processes which occur in these disciplines. Applications include, but are not limited to mathematical descriptions of growth in biological populations with limited resources, artificial neural networks, complex learning curves, probability distributions, the error function, etc.

Special cases of sigmoid functions include the logistic function and a Gompertz function.

A logistic function

A logistic function is of the form

$$x(t) = \frac{x_0 e^{rt}}{1 + x_0 (e^{rt} - 1)/K}, \qquad \text{(LF)}$$

where

- t might be considered as time,
- $x(t)$ is as a size of a population,
- x_o is the size of the population at the initial moment,
- K is a positive constant that can be interpreted as the capacity of the population,
- r is the coefficient of a population change that can be proportional to the population density and amount of available resources among others; if $r>0$ the population growths and decays otherwise.

If values of t range from $-\infty$ to $+\infty$, the curve described by (LF) is of an S-curve shape.

The model (LF) was suggested in 1838 by the Belgian mathematician Pierre-Francois Verhulst (1804-1849). To prevent the geometrical growth of a population as in the Malthus demographic model, his collaborator advised the growth rate to be proportional to the square of the speed of a population growth. Inspired by this idea, Verhulst published *a Note on the law of population growth* in 1838 where he proposed the model to describe the self-limiting growth of a biological population. He compared analytical outcomes of the model with the data on populations in France, Belgium, and Russia. There he found a good correspondence between the model and actual data, and used the term “the logistic equation” while referring to (LF) in his papers of 1845. This model was re-discovered independently several times by Robertson in 1908, McKendrick in 1911, Pearl in 1920, Reed in 1920, and others. Fortunately, Pearl noticed the work of Verhulst in 1922 and returned the term “the logistic” to the model. Therefore, the logistic model is often referred as the Verhulst-Pearl Model. Alfred J. Lotka (1880-1949) derived the equation (LF) again in 1925, calling it the “law of population growth.”

The Gompertz function

The Gompertz function satisfies the following differential equation

$$x'(t) = rx(t) \cdot \ln \frac{K}{x(t)}, \quad x(0)=x_o, \qquad \text{(G)}$$

where the description of all variables and paraments coincides with the ones offered the model (L).

If values of t are in the range from $-\infty$ to $+\infty$, then the curve described by (G) also has an S-shape.

The solution to (G) is named after the British mathematician and actuary Benjamin Gompertz (1779-1865). Self-educated Benjamin Gompertz became a Fellow of the Royal Society. Working on new tables of mortality for the Royal Society he suggested in 1825 the "law of human mortality," that rests on an assumption that a person's resistance to death decreases as their years increase.

The Gompertz model was employed by insurance companies to calculate the cost of life and is currently used in many applications, for instance, to model the growth of tumors or mobile phone uptake. Indeed, in the latter application, the costs of a mobile phone are initially high (so uptake is slow), then there is a period of rapid growth, and, finally, uptake slows down as saturation is reached.

Questions

I. Master understanding of differential equations.

   a. Solve the differential equation (G) and denote its solution as (GF).

   b. Derive a differential equation (L) for the logistic function (LF) as its solution.

II. Qualitative analysis of solutions.

   a. Provide a qualitative analysis of the logistic function (LF) and the Gompertz function (GF). Are they one-to-one functions? Are they symmetric? Do they have vertical / horizontal asymptotes? Find intervals of increase and decrease as well as points of inflection.

   b. Describe similarities and differences between the logistic and Gompertz functions.

   c. Explain why the parameter x_o is the population size at the starting point and the parameter K is called a *carrying capacity*.

   d. Provide a qualitative analysis of behavior of the functions under different parameters x_0, K, r.

   e. Summarize your observations.

III. Quantitative analysis.

   a. Plot the functions with some numerical values for parameters of your choice. Do both functions have a sigmoid shape?

   b. Describe similarities and differences between the logistic and Gompertz curves.

   c. Graph the functions for different parameters x_0, K, r.

d. Summarize your observations.

IV. Other questions.

a. Compare analytic and computational results.

b. Compare the rates of grows of the logistic and Gompertz functions.

c. Give examples of other sigmoid functions.

d. One of sigmoid functions is $x(t)=\arctan(t)$. Modify it to a new function that has the carrying capacity K and satisfies $x(0)=x_o$.

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