Differential Equations class of DR Sheila Miller at
New York City College of Technology

Colleague in Learning - DR Brian Winkel

Thursday Evening, 16 May 2013
Overly Ambitious Outline of Topics

M&M Birth and Death and Immigration model
    difference and differential equation model, equilibrium value, stability
Ant Tunnel modeling
Chemical kinetics
Sublimation of carbon dioxide
Torricelli's Law modeling and video data collecting
LSD compartment model
Spring Mass (and Damper) System – Resonance and Frequency Response
Tuned Mass Damper – they are everywhere

United States Military Academy
West Point   New York

M&M Population Simulation with Immigration

Count out 50 M&M's form your "source"

(1) Put them in the cup.
(2) Gently toss them onto the plate.
(3) Remove those M&M's with "m" facing up – they die.
(4) From your "stash" add 10 immigrants.
(5) Count remaining M&M's for that generation.
Go to (1).

DO NOT eat M&M's until so instructed!

At each iteration keep track of time (generation number) and number of M&M's.

Winkel, B. J. 2010. Population Modeling with M\&M's. *International Journal of Mathematical Education in Science and Technology.* 40(4): 554-558.

What assumptions might you make about the individual M&M's, the "action", and the population? What are relevant assumptions? What are facts?

What expectations do you have for the population as the generations go on?

Let us collect some data on the "life and times" of M&M's!

Let us build a mathematical model (based on our assumptions) of the number of living M&M's at generation n. I will help out.

Let $y(n)$ = number of living M&M's at generation n.

$y(0) =$

Could we just produce a formula for $y(n)$?

Could we produce a relationship between one generation and the next?

Answering the second question might produce an answer to the first question and this is what difference or differential equations is all about. – asking a question, a good question , and developing a strategy to get an answer.

Difference equation model:

$y(n+1) =$

OR

$y(n+1) - y(n) =$

Differential equation model:

$y'(t) =$

Difference equation model:

$y(n+1) = 0.5\, y(n) + 10\,, \qquad y(0) = 50$

OR

$y(n+1) - y(n) = -0.5\, y(n) + 10, \qquad y(0) = 50$

Differential equation model:

$y'(t) = -0.5\, y(t) + 10\,, \qquad y(0) = 50$

Solve the differential equation model:

$y'(t) = -0.5\, y(t) + 10 \qquad y(0) = 50$

Solve the differential equation model:

$y'(t) = -0.5\,y(t) + 10 \qquad y(0) = 50$

```
ysol[t_] =
 y[t] /. DSolve[{y'[t] == -.5 y[t] + 10, y[0] == 50},
     y[t], t][[1]] // Expand
```

$20. + 30.\, e^{-0.5\,t}$

```
Plot[ysol[t], {t, 0, 12}, PlotRange → {{0, 12}, {0, 50}},
 AxesLabel → {"Time - generation", "M&M population"}]
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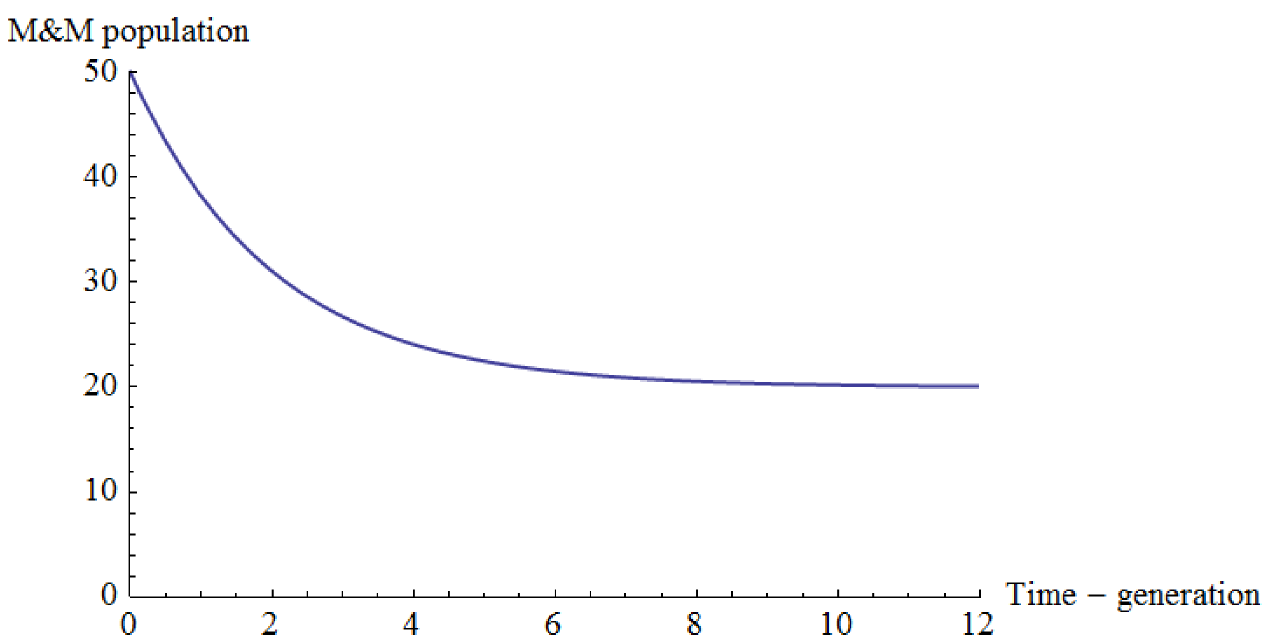
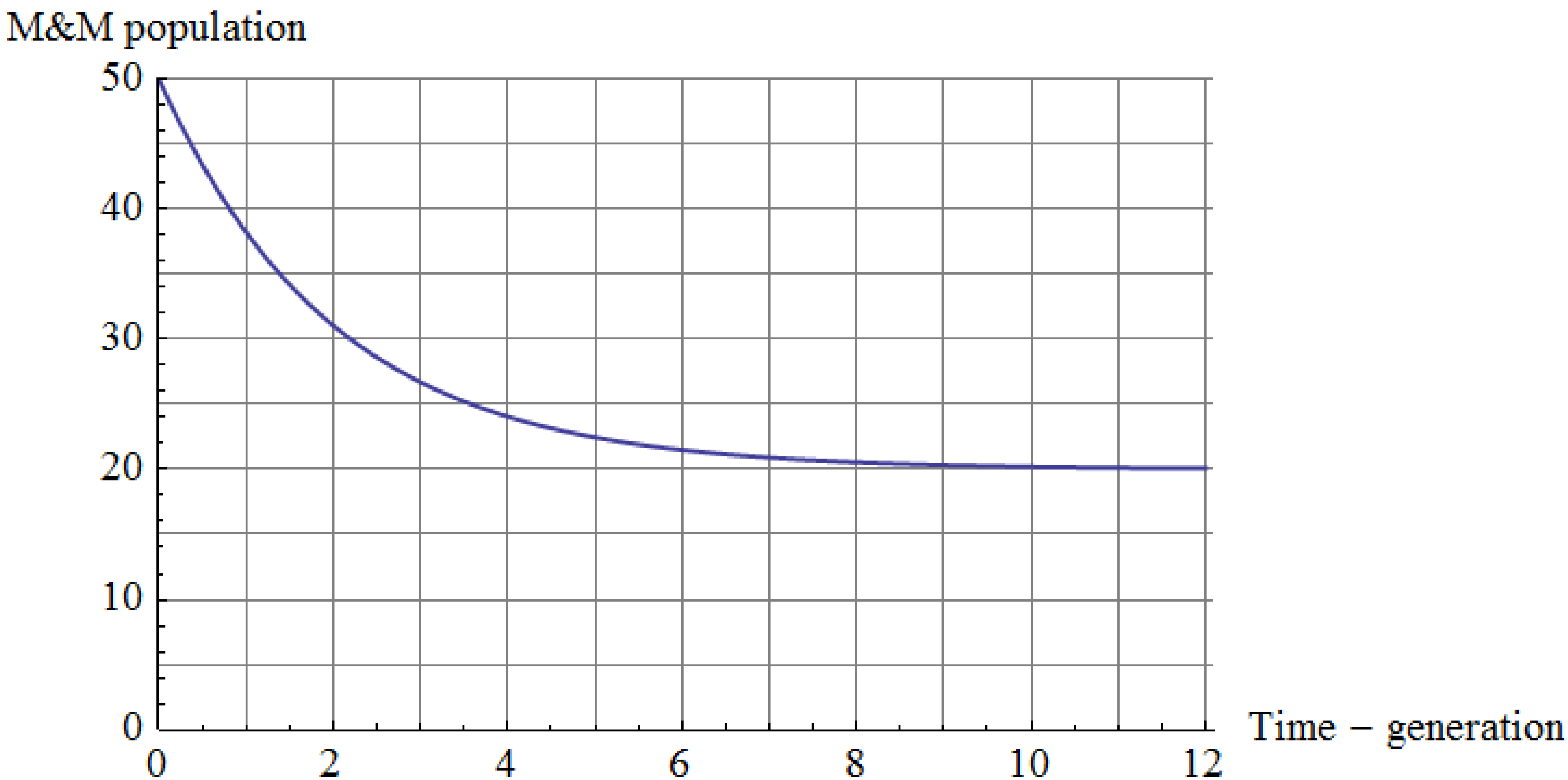


Here is a plot of our mathematical model based on assumptions:

$y(t) = 20 + 30\ e^{-0.5\,t}$.



How does our model compare with our “reality” data?

How long does it take an ant to build of tunnel of length x?

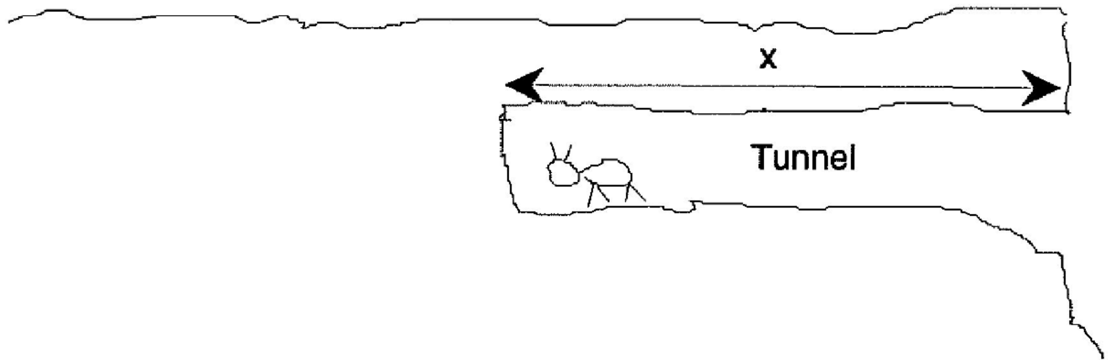


Figure 1. Crude drawing for ant tunnel building model. x is the length of the tunnel and $T(x)$ is the time it takes an ant to build a tunnel of length x.

Assumptions:

Winkel, B. J. 1994. Ants, Tunnels, and Calculus: An Exercise in Mathematical Modeling. *The Mathematics Teacher.* 87(4): 284-287.

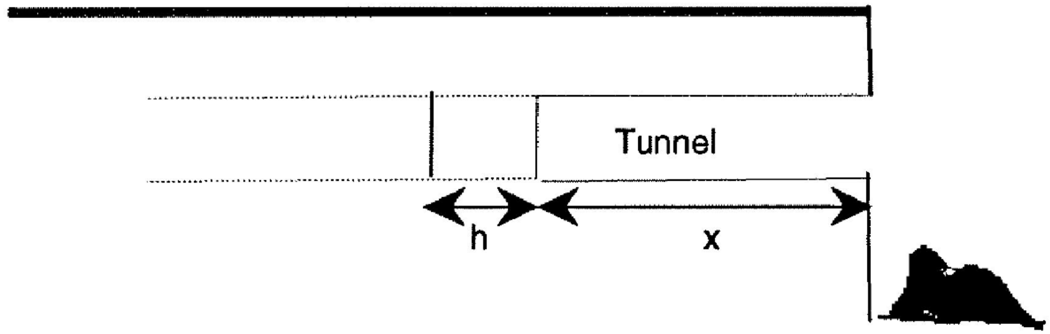


Figure 2. Useful diagram for discovering the time it takes to build a small section of the ant tunnel from distance x to $x + h$.

We define terms and build a mathematical model

Let T(x) be the time it takes an ant to build a tunnel of length x.

$T(x+h) - T(x) =$

$T(x)$ be the time it takes an ant to build a tunnel of length x.

$T'(x) = \alpha x$, $T(0) = 0$

Solve the differential equation.

What does our model tell us about the answer to t his question?

How is the time changed if we double the length of the tunnel?

Kinetics of Chemical Reactions

$y(t)$ = mass at time t. We have for $k, r > 0$ $y'(t) = -k\, y(t)^r$

$r =$ 0, 1, or 2 – zeroth, first, or second order.

Solve each differential equation:

$r = 0$

$r =$

$r = 2$

Winkel, B. J. 2010. Parameter Estimates in Differential Equation Models for Chemical Kinetics. *International Journal of Mathematical Education in Science and Technology.* 42(1): 37-51.

Kinetics of Chemical Reactions

$y(t)$ = mass at time t. We have for $k, r > 0$ $y'(t) = - k\, y(t)^r$

$r = 0, 1,$ or 2 – zeroth, first, or second order.

Solve each differential equation:

$r = 0$ $\quad y(t) = - k\, t + c$

$r = 1$ $\quad \ln(y(t)) = - k\, t + c \quad \Longrightarrow \quad y(t) = y(0)\, e^{-k\, t}$

$r = 2$ $\quad 1/y(t) = k\, t - c \quad \Longrightarrow \quad y(t) = y(0)/(1 + y(0)\, k\, t).$

Chemists like the intermediate step as they can plot a function of their mass against time and see if it is linear.

KineticsSOL [Compatibility Mode]

From Zumdahl, Steven S. Chemical Principles. 1992. Lexington MA: D.C. Heath.

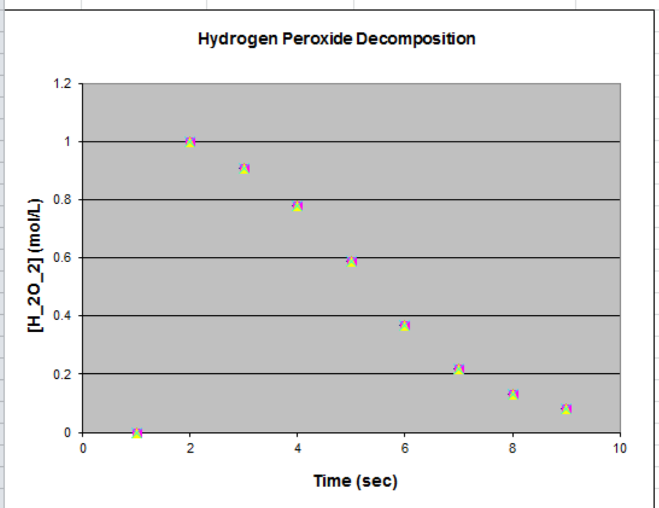
Exercise 15 on p. 682. Determine the order of the reaction 0, 1, or 2 and confirm your model.

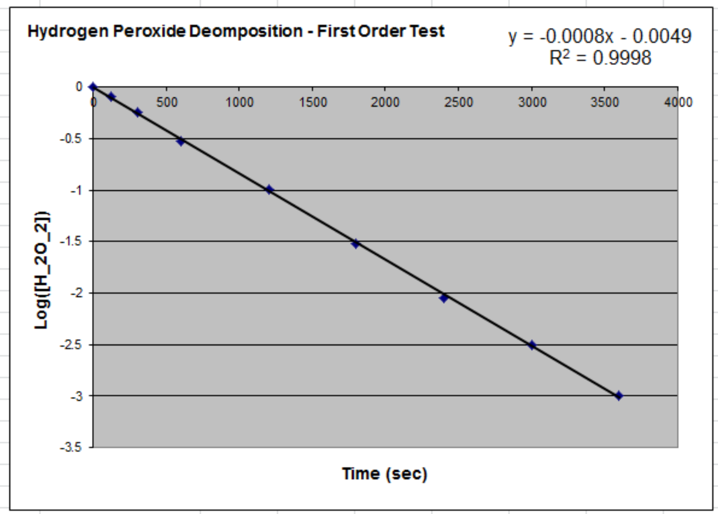
Decomposition of hydrogen peroxide - H_2O_2.

		Test for 1st Order	Test for 2nd Order		
Time (sec)	[H_2O_2] (mol/L))	LN ([H_2O_2])	1/[H_2O_2	First Order Model	
0	1	0	1	0.995111985	0.997902
120	0.91	-0.094310679	1.098901099	0.904023431	0.833518
300	0.78	-0.248461359	1.282051282	0.782782813	0.636291
600	0.59	-0.527632742	1.694915254	0.61575877	0.405717
1200	0.37	-0.994252273	2.702702703	0.3810213	0.164952
1800	0.22	-1.514127733	4.545454545	0.235769652	0.067065
2400	0.13	-2.040220829	7.692307692	0.145890345	0.027266
3000	0.082	-2.501036032	12.19512195	0.090274523	0.011086
3600	0.05	-2.995732274	20	0.055860375	0.004507

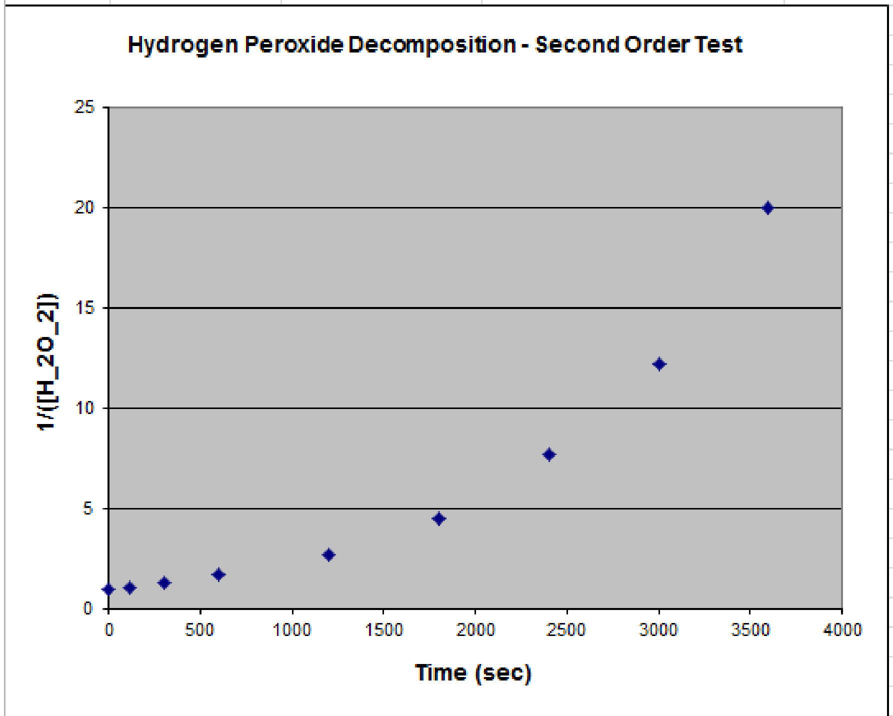
Once we have ascertained the reaction is first order we determine the rate constant k.
Here k = 0.0008 from LN ([H_2O_2]) = -k*t - c
LN ([H_2O_2]) = -.0008*t -0.0049

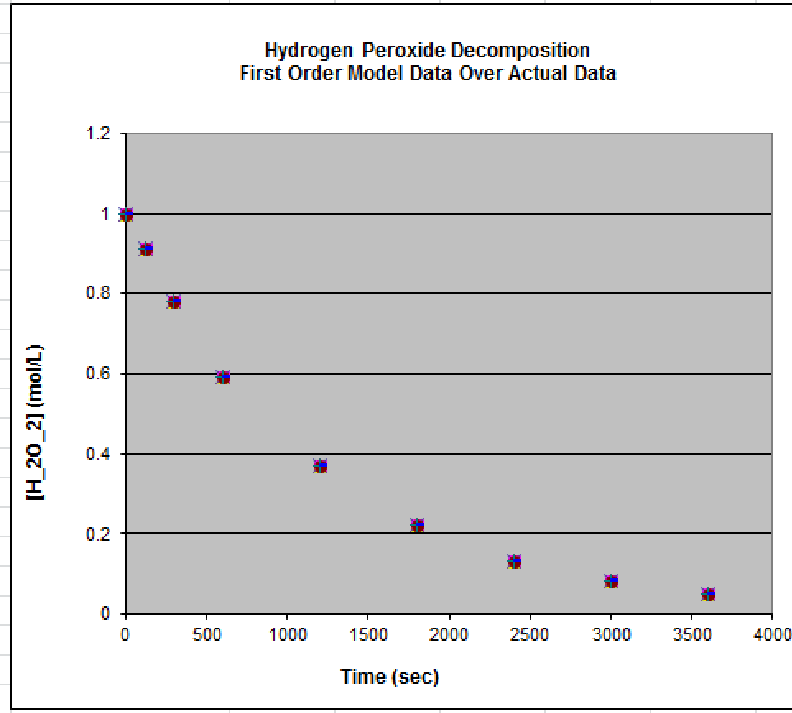
[H_2O_2] = EXP(-.0008*t -0.0049) =0.995112 EXP(-.0008*t) ~ 1.0*EXP(-.0008*t)





From the plot of 1/[H_2O_2] vs. t we see that the reaction is NOT second order.





Sublimation of carbon dioxide

How long does it take for a “block” of dry ice or solid CO_2 to sublimate?

Let us produce a mathematical model for the rate of sublimation, i.e.

$m'(t) =$

where $m(t)$ is the mass of the piece of dry ice.

Assumptions?

Winkel, B. J. 2012. Modelling sublimation of carbon dioxide. *International Journal of Mathematical Education in Science and Technology.* 43(8) 1077-1085.

Apparatus and Oops!

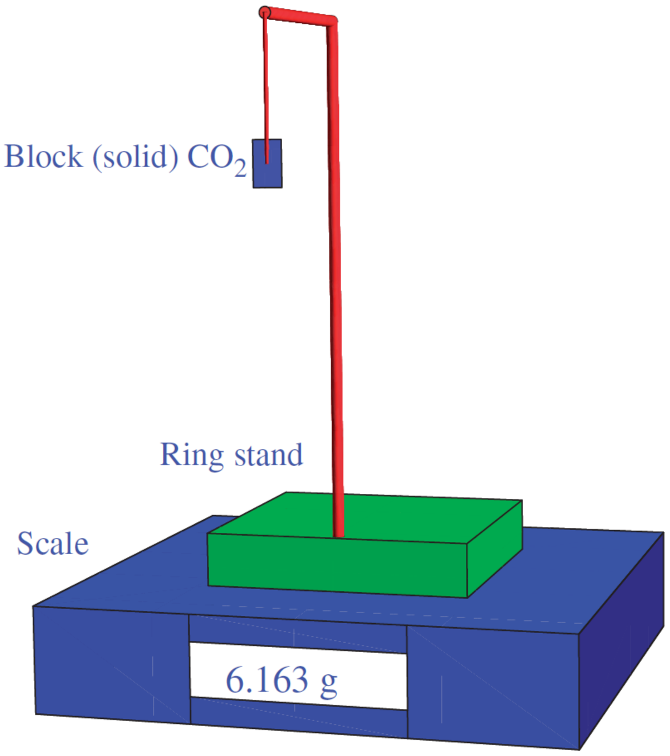


Table 1. Data collected on successful run for mass of dry ice (g) as a function of time (s).

Time (s)	Mass (g)	Time (s)	Mass (g)
0	7.570	480	5.871
30	7.457	510	5.776
60	7.338	540	5.682
90	7.220	570	5.589
120	7.110	600	5.497
150	6.995	630	5.405
180	6.885	660	5.313
210	6.778	690	5.224
240	6.673	720	5.136
270	6.571	750	5.048
300	6.464	780	4.960
330	6.363	810	4.878
360	6.265	840	4.790
390	6.163	870	4.707
420	6.067	900	4.625
450	5.969		

$$m'(t) = -km(t)^r, \quad m(0) = 7.57$$

$$m(t) = \left((r-1)\left(kt - \frac{7.57}{7.57^r(1-r)}\right)\right)^{\frac{1}{1-r}}$$

What might we expect r to be?

What might we try for r?

What is reasonable?

Table 3. Summary of data analyses for various orders r with best estimate of parameter k and respective minimum sum of square errors. We see that none of $r = 0, 1, 2$ or $r = \frac{2}{3}$ are best. Through our optimization analysis of values of r rather than presuming set values of r we obtain the best order model when $r = 0.751209$. We show the plot of the model over the data in Figure 6.

Order (r)	Parameter k	SSE
0	0.00340764	0.12126
1	0.000536791	0.0119927
2	0.0000841212	0.251683
$\frac{2}{3}$	0.00184154	0.0361216
0.751209	0.0008506	0.000529786

We seek the value of k and r in

$$m(t) = \left((r-1)\left(kt - \frac{7.57}{7.57^r(1-r)}\right)\right)^{\frac{1}{1-r}}$$

Which will minimize the sum of square errors between our model prediction m(t) = $m_1(t)$ at time t = t_i and our observation data, m_i.

$$SSE_1(k) = \sum_{i=1}^{i=n}(m_1(t) - m_i)^2$$

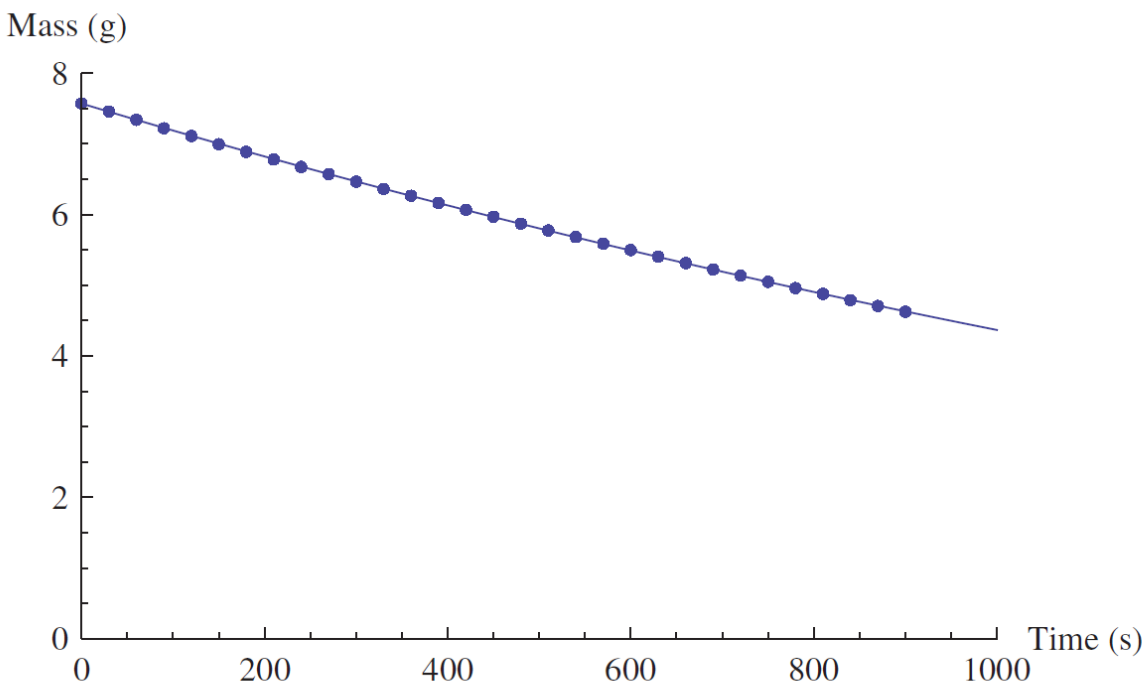
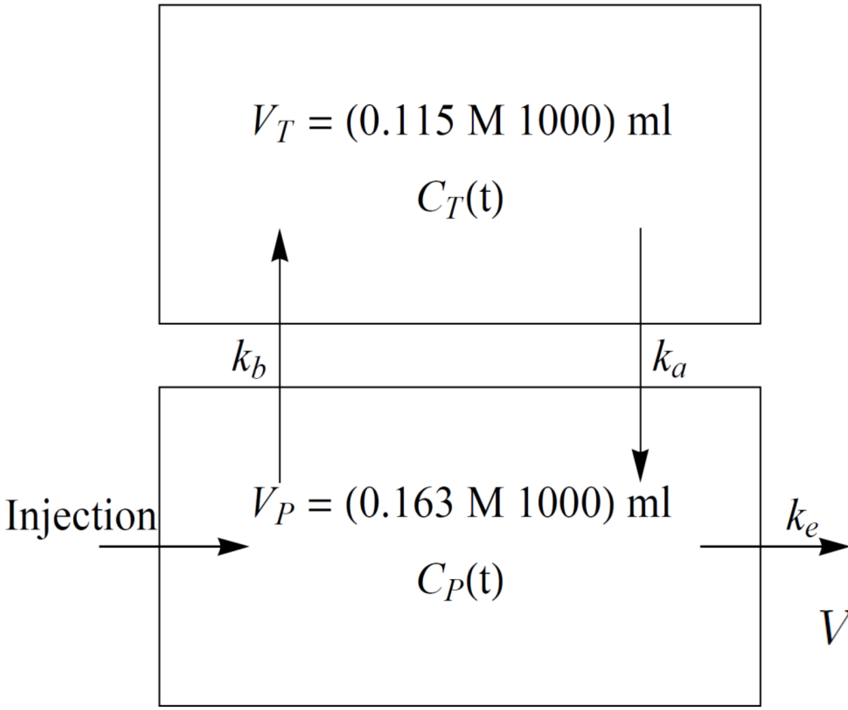


Figure 6. Plot of optimal order fit to data over the data in Table 1.
Note: Here $r = 0.751209$ and $k = 0.0008506$. This is a very convincing plot of our solution for (5) with these parameters.

Modeling Lysergic acid diethylamide (LSD) in the human body



C_P (0) = 12.2699, for originally LSD was injected at a concentration of 2,000 ng per kg of body mass for each subject and so we have an initial concentration in the plasma of12.2699 = 2000M/.163M1000 ng of LSD/kg of body mass. C_T(0) = 0.

$$V_P = \underbrace{(0.163M)}_{\text{kg of plasma}} \cdot \underbrace{(1)}_{\text{liter/kg}} \cdot \underbrace{(1000)}_{\text{ml/liter}} \cdot = \underbrace{(163M)}_{\text{ml of plasma}}$$

$$\begin{aligned} V_P C_P'(t) &= k_a V_T C_T(t) - k_b V_P C_P(t) - k_e V_P C_P(t) \\ V_T C_T'(t) &= k_b V_P C_P(t) - k_a V_T C_T(t)\,. \end{aligned}$$

Reference: Winkel, B. J. 2013. Browsing Your Way to Better Teaching. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*. 23(3): 274-290. Original references in this paper.

	Time (hr)	0.833	0.25	0.5	1.0	2.0	4.0	8.0
Subject 1	Plasma Conc (ng/ml)	11.1	7.4	6.3	6.9	5.	3.1	0.8
	Perform Score (%)	73	60	35	50	48	73	97
Subject 2	Plasma Conc (ng/ml)	10.6	7.6	7.	4.8	2.8	2.5	2.
	Perform Score (%)	72	55	74	81	79	89	106
Subject 3	Plasma Conc (ng/ml)	8.7	6.7	5.9	4.3	4.4	—-	0.3
	Perform Score (%)	60	23	6	0	27	69	81
Subject 4	Plasma Conc (ng/ml)	10.9	8.2	7.9	6.6	5.3	3.8	1.2
	Perform Score (%)	60	20	3	5	3	20	62
Subject 5	Plasma Conc (ng/ml)	6.4	6.3	5.1	4.3	3.4	1.9	0.7
	Perform Score (%)	78	65	27	30	35	43	51

Table 2. Summary of data collected [1, 14] on 5 male volunteers who were administered LSD and then tested on performance (Perform Score (%)) on simple addition questions. Both performance Score and Plasma Concentrations of LSD were recorded at 5, 15, 30, 60, 120, 40, and 480 minutes after the initial infusion of LSD.

This is the formula for $C_T(t)$, the concentration of LSD in the tissues as a function of time t.

$$\left(-17.3913\, e^{0.5\left(-1.\,ka-1.\,kb-1.\,ke-1.\,\sqrt{-4.\,kb\,ke+(1.\,ka+1.\,kb+1.\,ke)^2}\right)t}\,kb+17.3913\, e^{0.5\left(-1.\,ka-1.\,kb-1.\,ke+\sqrt{-4.\,kb\,ke+(1.\,ka+1.\,kb+1.\,ke)^2}\right)t}\,kb\right)\Big/\left(\sqrt{-4.\,kb\,ke+(1.\,ka+1.\,kb+1.\,ke)^2}\right)$$

$$SSE(k_a, k_b, k_e) = \sum_{i=1}^{\imath}(C_P(t_i) - O_i)^2 \tag{3}$$

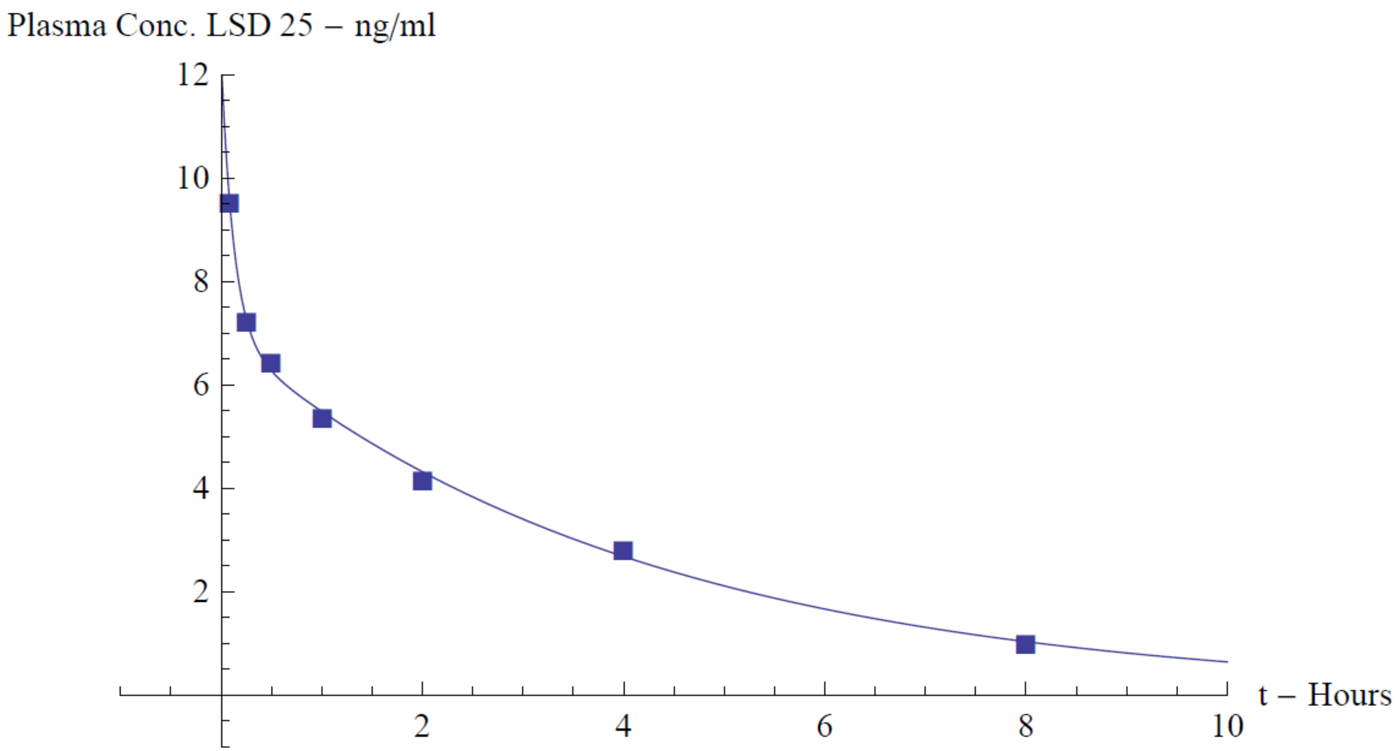


Figure 2. Plot of the observed values of the average concentration of LSD (ng/ml) (squares) and the model built from parameter (k_a, k_b, and k_e) estimates using the solution of the system of differential equations (2) and minimization of the sum of square error function (3).

$$C_T(t) = 0.128905\left(55.419e^{-0.238492t} - 55.419e^{-7.99617t}\right) . \qquad (5)$$

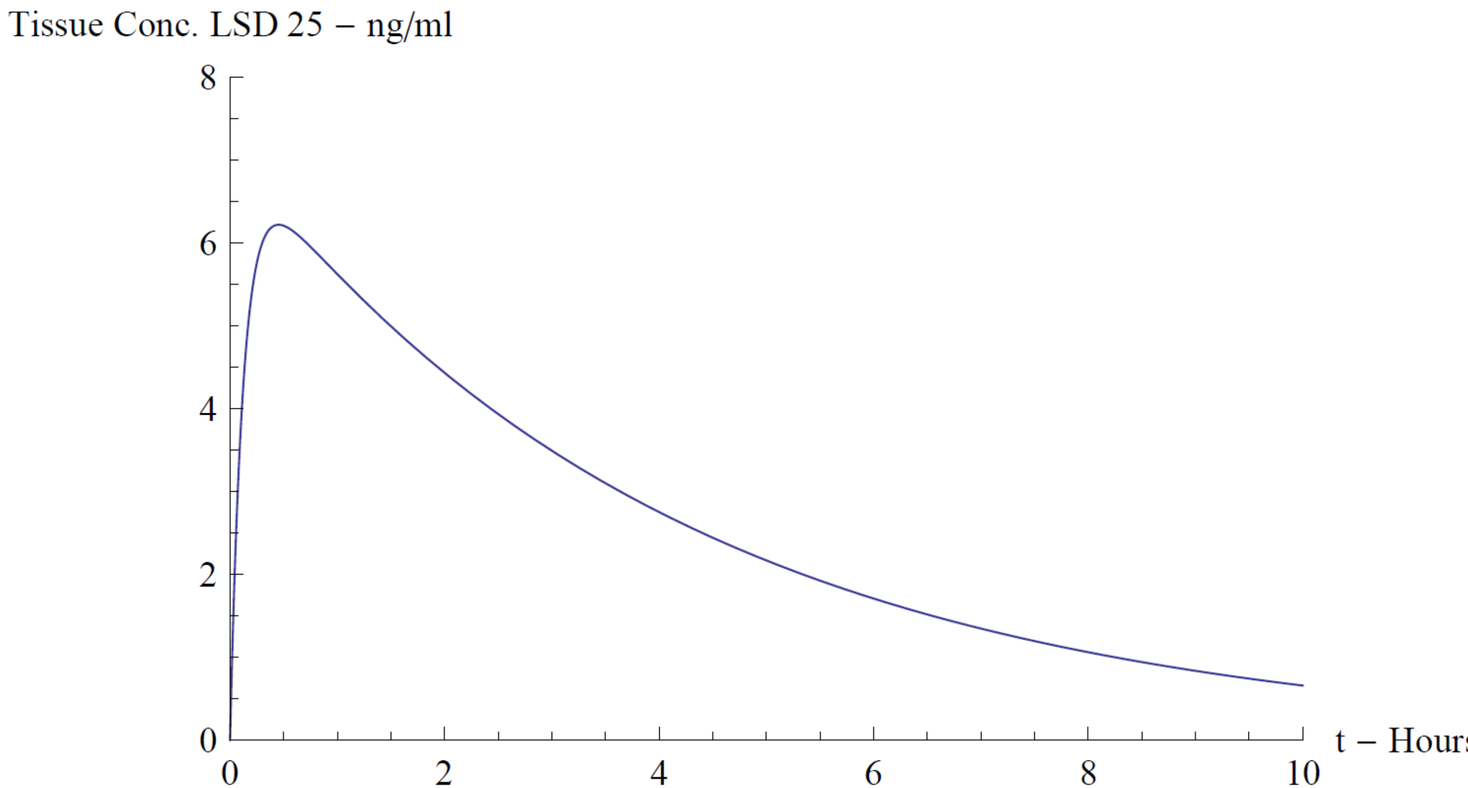


Figure 3. Plot of the model of tissue concentration of LSD in ng/ml. This model is built from parameter (k_a, k_b, and k_e) estimates using the solution of the system of differential equations (2) and minimization of the sum of square error function (3).

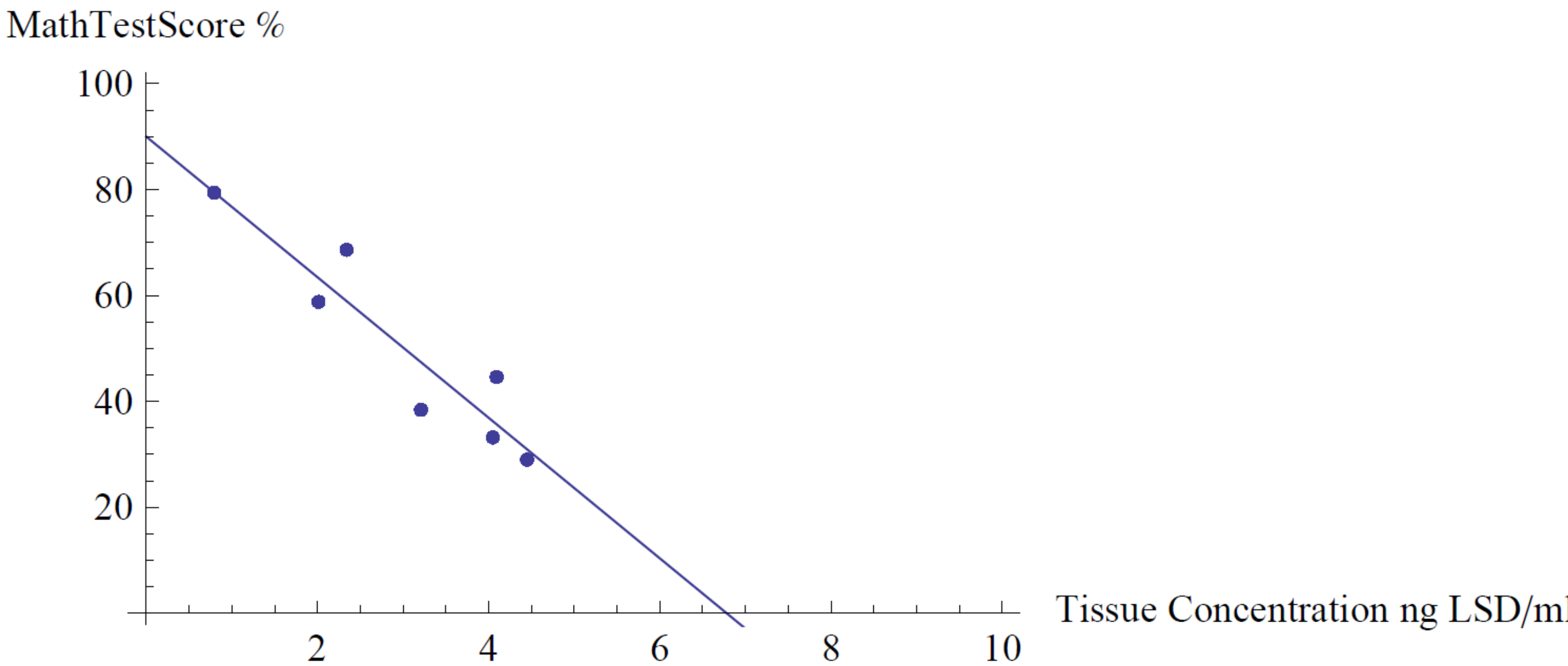


Figure 4. Plot of Performance Score (%) (PS) on the simple arithmetic problems vs. the model prediction of the concentration of LSD in ng/ml in the tissue compartment (CT). Superimposed is the best line of regression whose equation is $PS = 89.1729 - 9.32941CT$. This means for every ng/ml increase in LSD in the tissue compartment the score drops a little over 9 points.

At what rate does a column of water fall with a small bore hole for water to exit at the bottom of the column?

Based on the principle of Conservation of Energy for a small particle of water, namely the sum of Kinetic Energy and Potential Energy stays constant as the Particle falls we can derive Torricelli's Law for the height, h(t), of such a column.

$$A(h(t)) * h'(t) = \alpha\, a\sqrt{2gh(t)}\,, \quad h(0) = h_0$$

Here A(h(t)) is the cross sectional area of the cylinder at height h(t), g is the acceleration due to gravity, a is the area of the small bore hole, and α is called the *discharge or contraction coefficient*. Empirically a is about 0.70, namely the effective Discharge rate is about 70% of what it could be maximally.

For a constant cross sectional cylinder of area A (h(t)) = π r^2
Torricelli's Law

$$A(h(t)) * h'(t) = \alpha\, a \sqrt{2gh(t)}\,, \quad h(0) = h_0$$

may be realized as a simply differential equation:

$$h'(t) = -k\sqrt{h(t)}\,.$$

Of course, without recourse to physics one might conjecture something Simpler, such as a linear "law" which would lead to exponential decay In the height, h(t).

$$h'(t) = -k\, h(t)\,.$$

We compare the models after the parameters have been determined by minimizing the sum of square errors.

Clock Time t (min)	Relative Time t (s)	Linear h (cm)	Torricelli h (cm)	Observed h (cm)
1:40.419 = 100.419	0	12.5	12.5	12.5
1:45.598 = 105.598	5.179	11.3935	11.1028	11.1
1:47.568 = 107.788	7.369	10.9256	10.5369	10.5
1:49.623 = 109.523	9.104	10.5549	10.0991	10.1
1:50.985 = 110.985	10.566	10.2426	9.7374	9.8
1:56.388 = 116,388	15.969	9.08821	8.45786	8.5
2:02.057 = 122.057	21.638	7.87702	7.21218	7.2
2:08.032 = 128.032	27.613	6.60046	6.00662	6.0
2:16.386 = 136.386	35.967	4.81562	4.50576	4.5
2:25.568 = 145.568	45.149	2.85388	3.10461	3.1
2:34.116 = 154.286	53.867	0.991273	2.01507	2.0

Table 4. Computations with the linear model and Torricelli model compared to observed data.

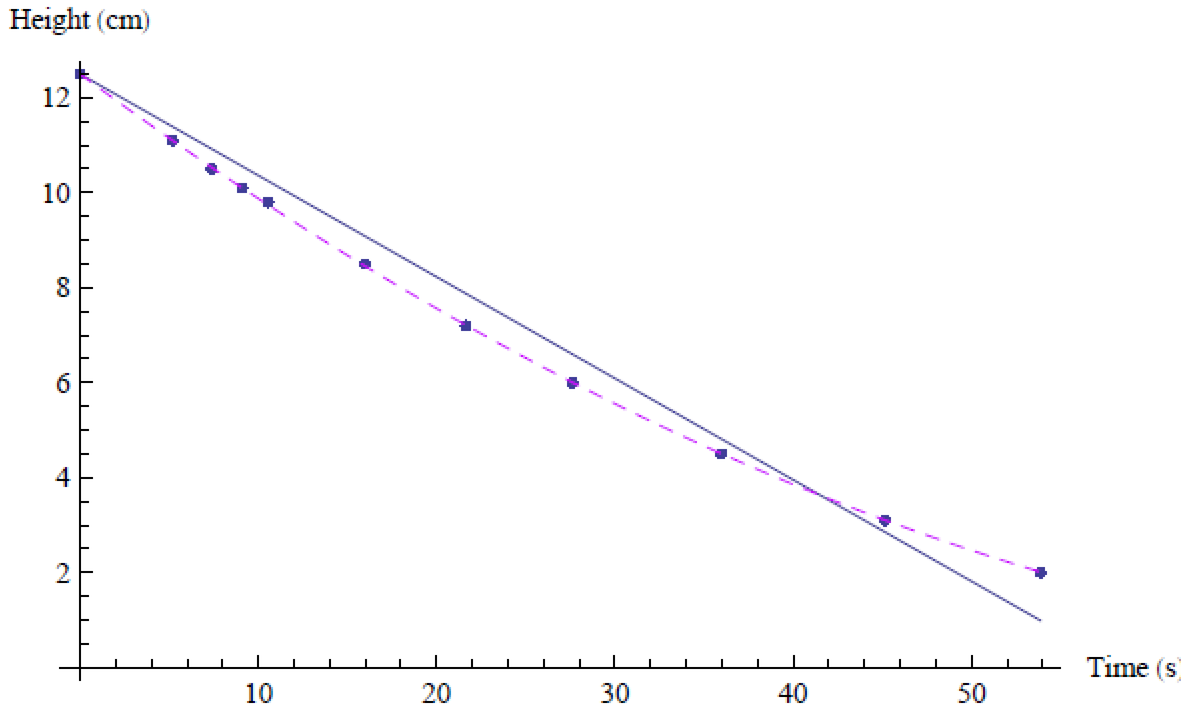


Figure 14. Plot of the observed data, the linear model (solid line) and the Torricelli model (dashed). We see the linear plat as very inadequate and the Torricelli plot as right on!

Tuned Mass Dampers

A Tuned Mass Damper (TMD) is a passive mechanical counterweight for a structure consisting of a moving mass (roughly 1–2% of the structure's mass) which is usually placed in the upper portion of the structure. The purpose of the TMD is to reduce the effects of motion caused by wind or earthquake.

The first uses of TMDs in the United States for large structures was in the John Hancock Building in Boston in 1977 and City Corp Center in New York in 1978.

TMD's are used in many, many structures and devices, including buildings and bridges, electric razors, rotating tools, surgery table platforms, etc.

So what is a TMD? Well it is just a system of two second order differential equations
And a physical device to implement the results.

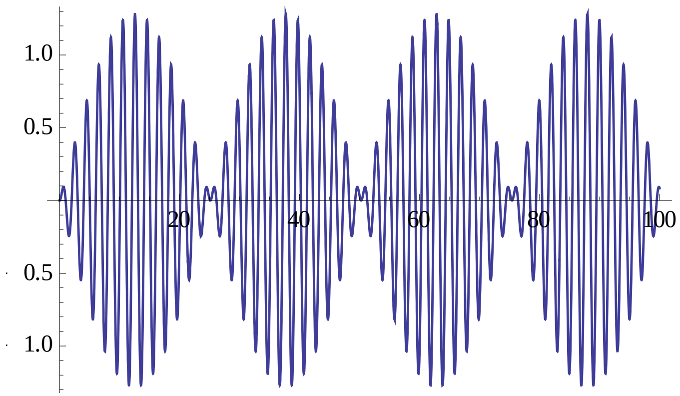
TMD's can be in the form of sliding slabs (HUGE) of concrete and steel, sloshing tanks of water, pendula, and more.

Winkel, B. J. and K. Landry. 2008. Peak Frequency Responses and Tuned Mass Dampers--Exciting Applications of Systems of Differential Equations. *The UMAP Journal*. 29(1): 15-30.

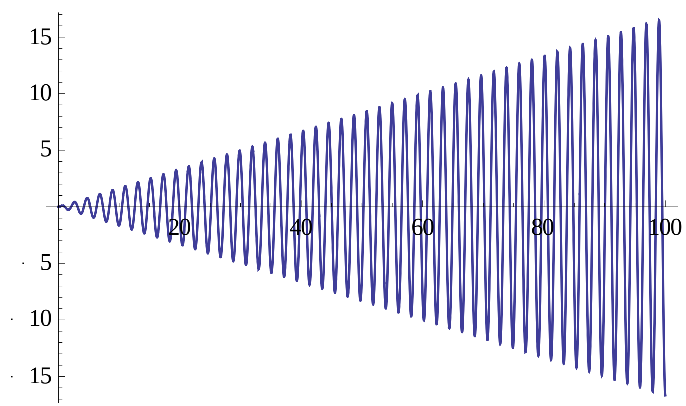
Harmonic Oscillator, e.g., building or bridge OR oboe reed.

$1\, y''(t) + \omega_0{}^2 y(t) = \cos(\omega\, t), \quad y(0) = y_0, \quad y'(0) = v_0$

When ω_0 and ω are close we get beats,
e.g. $\omega_0 = 3.0$ and $\omega = 3.25$



When ω_0 and ω are equal we get resonance,
e.g. $\omega_0 = 3.0$ and $\omega = 3.0$



Thus if in 1 $y''(t) + \omega_0{}^2 y(t) = \cos(\omega\, t), \quad y(0) = y_0, \;\; y'(0) = v_0$
We have ω_0 = ω, i. e. the driver frequency is the same as the natural frequency of the system/structure, then we can have resonance and the system./structure can have giant and dangerous oscillations. For a building this can come from wind or earthquakes. What can we do to mitigate this danger?

Differential equations comes to the rescue!

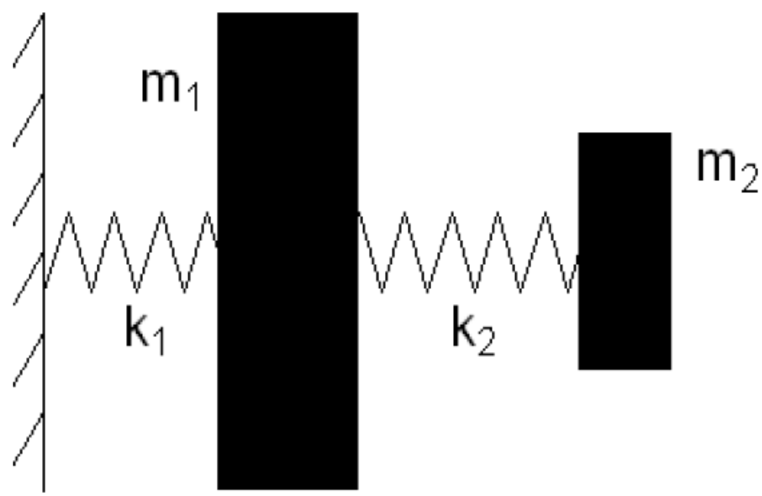
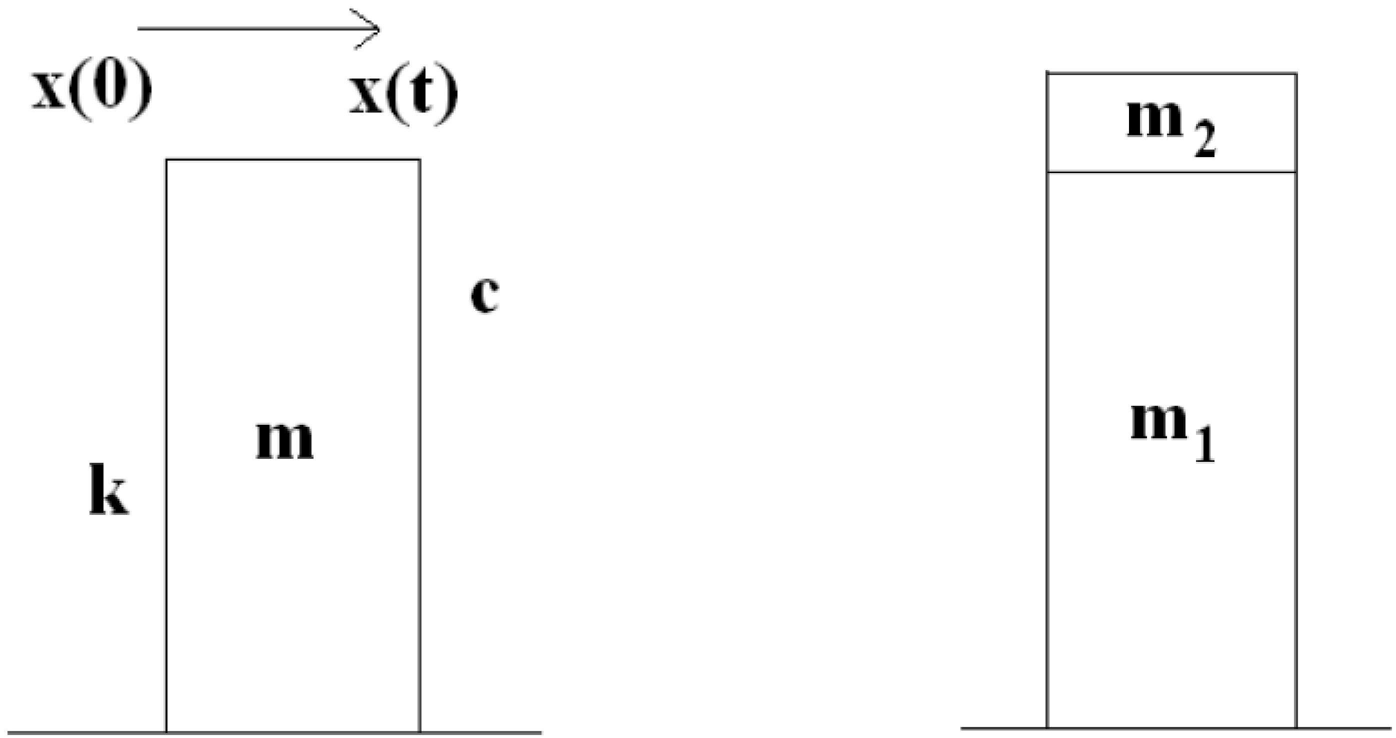


Figure 6: Horizontal depiction of two mass spring system (no damping on either mass) with smaller mass m_2 serving as Tuned Mass Damper.



Depiction of two masses for a structure – smaller mass on top.
K (spring constant) is called stiffness and c is still called damping coefficient.

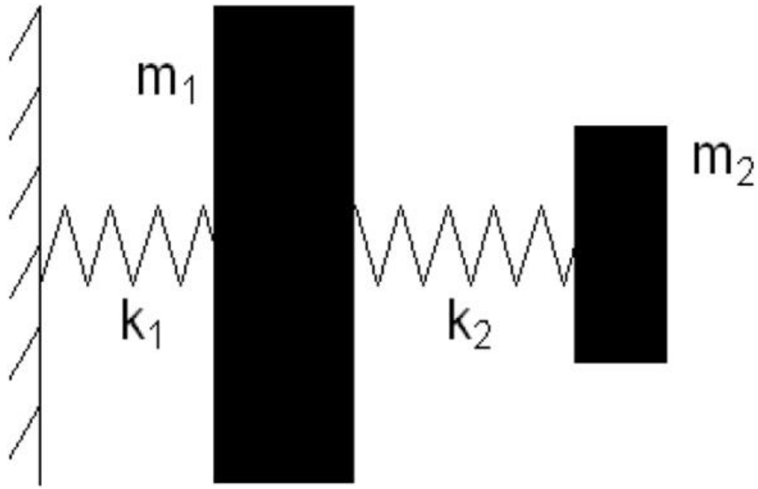


Figure 6: Horizontal depiction of two mass spring system (no damping on either mass) with smaller mass m_2 serving as Tuned Mass Damper.

$$\begin{aligned} m_1 x_1''(t) + (k_1 + k_2)x_1 - k_2 x_2 &= F(t) \\ m_2 x_2''(t) + -k_2 x_1 + k_2 x_2 &= 0\,. \end{aligned}$$

Kin the simplified case where there is no damping (i.e. c = 0) then we might seek to control for resonance by adding a second “tuned” mass.

We now "drive" the larger mass with a force. . wind, earthquake

$$m_1 x_1''(t) + k_1 x_1(t) + k_2 x_1(t) - k_2 x_2(t) = F_0 \cos(\omega t)$$

$$m_2 x_2''(t) - k_2 x_1(t) + k_2 x_2(t) = 0\,.$$

If there is no second mass then we could have resonance. So we add a second smaller mass and we "tune" it so that its system has the same frequency as that of the larger mass.

The natural frequency of our structure is $\omega = \sqrt{\frac{k_1}{m_1}}$ and of the added mass the frequency is $\sqrt{\frac{k_2}{m_2}}$. Consider k_1= 90 and $m_1 = 10$ with k_2= 0.90 and $m_1 = 0.10$.

Then both spring systems have the same frequency, namely $\omega = \sqrt{\frac{90}{10}} = \sqrt{\frac{0.9}{0.1}} = 3$.

Now, by examining the solution of our system

$$\text{mass } m_1: \quad -\cos(t\omega) + \frac{909}{10}(a\cos(t\omega) + b\sin(t\omega)) - \frac{9}{10}(c\cos(t\omega) + d\sin(t\omega))$$
$$+10\left(-a\cos(t\omega)\omega^2 - b\sin(t\omega)\omega^2\right) = 0$$
$$\text{mass } m_2: \quad -\frac{9}{10}(a\cos(t\omega) + b\sin(t\omega)) + \frac{9}{10}(c\cos(t\omega) + d\sin(t\omega))$$
$$+\frac{1}{10}\left(-c\cos(t\omega)\omega^2 - d\sin(t\omega)\omega^2\right) = 0\,.$$

we can combine terms, simplify, and determine that the amplitude of the resulting oscillation of mass is

$$\text{amp}(\omega) = 10\left|\frac{\omega^2 - 9}{100\omega^4 - 1809\omega^2 + 8100}\right|$$

A picture is worth . . . . . How many words . . . . or equations?

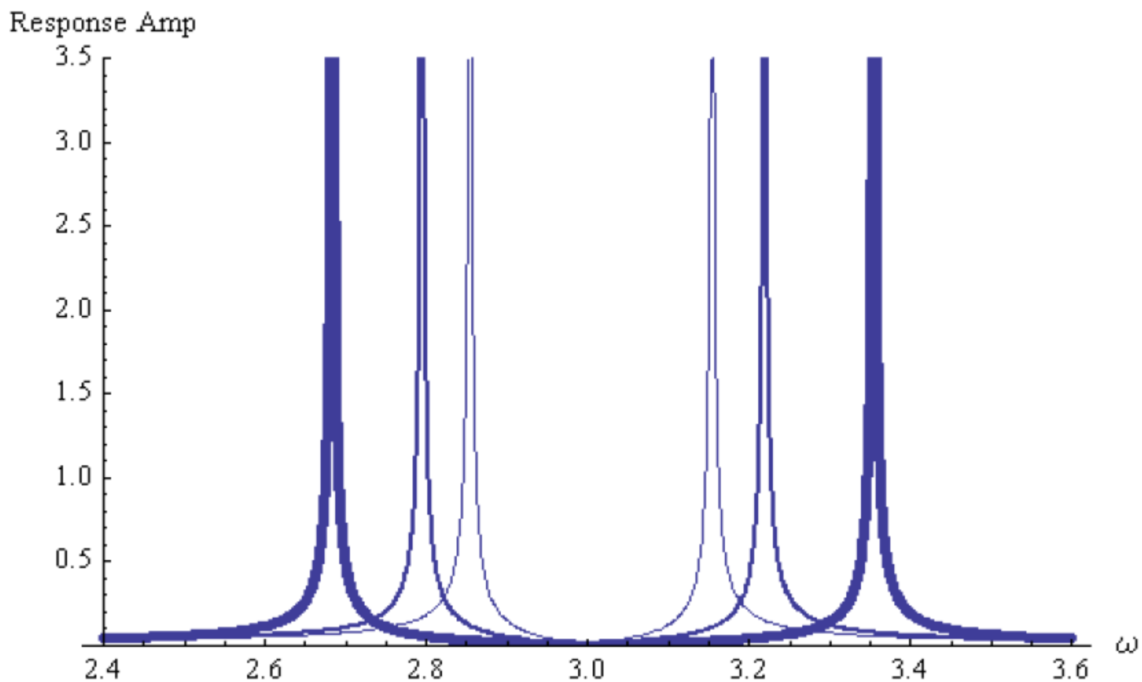


Figure 9: In case neither mass m_1 nor m_2 has damping, i.e. $c_1 = c_2 = 0$, here are plots of the response amplitudes of the primary system mass, m_1, as a function of the driver frequency ω. We see in the three plots that as the ratio $\frac{m_2}{m_1}$ of added secondary mass m_2 as a percentage of the primary mass, m_1, goes from 1%, to 2%, to 5%, (corresponding to the thin, thick, and thicker plots, respectively) the frequency region of low responses to the driver frequency surrounding the natural frequency of the structure expands.

So if we are willing to add more and more mass we get a wider "safe" region.

Incidentally, if we go back to just our one mass system

$$m \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = f(t), \quad y(0) = y_0, \quad y'(0) = v_0$$

with a driving force $f(t) = F_0 \cos(\omega t)$

Here is the steady state (non-homogeneous) solution for Equation (4) with $c \neq 0$:

$$\mathrm{ss}(t) = -\frac{F_0 m \cos(\omega t)\omega^2 + F_0 k \cos(\omega t)}{m^2\omega^4 + c^2\omega^2 - 2km\omega^2 + k^2} + \frac{cF_0 \sin(\omega t)\omega}{m^2\omega^4 + c^2\omega^2 - 2km\omega^2 + k^2}.$$

The amplitude of the steady state solution as a function of input frequency, ω, is given in Equation (5).

$$\mathrm{amp}(\omega) = F_0 \sqrt{\frac{1}{m^2\omega^4 + c^2\omega^2 - 2km\omega^2 + k^2}} = F_0 \sqrt{\frac{1}{c^2\omega^2 + (m\omega^2 - k)^2}}. \qquad (5)$$

How big a response (i.e. how big can amp(ω) really be?

We seek the maximum frequency response of amp(ω). Again, a picture . . . .

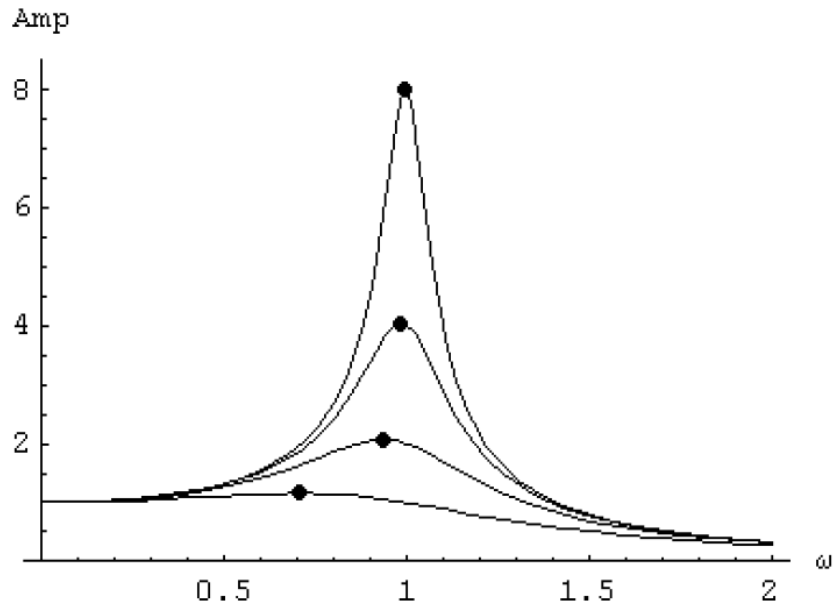


Figure 3: Plot of peak frequency response, both input frequency, ω, and amplitude, of steady state solution for $m = 1$, $k = 1$, $\omega_0 = 1$, $F_0 = 1$, for $c = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ where peak is highest for lowest value of c, i.e. $\omega_{\max} \to \omega_0$ as c decreases. These values appear in Table 1.

c value	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
Peak response frequency, $\omega_{\max}$	0.707107	0.935414	0.984251	0.996086
Maximum response amplitude	1.1547	2.06559	4.03162	8.01567

Table 1. Peak frequency response for Equation (4) with $m = 1$, $k = 1$, $F_0 = 1$, for $c = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$.