The Logistic Differential Equation

Goals

- MATH: To analyze the behavior of solutions of an ordinary differential equation geometrically .

- MATH: To analyze stability behavior of equilibria of an ordinary differential equation geometrically and symbolically.

- MATH: To introduce a basic numerical technique for approximation solutions to differential equations.

- MATH: To draw and interpret bifurcation diagrams.

- BIO: To explore the logistic model, and variations by introducing harvesting.

- COMP: To use MAPLE to analyze an ordinary differential equation

Computational Tools: MAPLE

Let $N = N(t)$ denote the size of a population at time t. The logistic differential equation is given by

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right),$$

where r and K are positive constants. The parameter $r > 0$ is a growth parameter, and the parameter $K > 0$ denotes the carrying capacity for the population. This population model is an example of a continuous, density dependent population model. The model is density dependent, because the number of offspring per individual is dependent on current population size. In this activity, we will explore geometric and numerical techniques for studying solutions of this differential equation.

1. Open a blank Maple worksheet. Since we will be using differential equation commands, we need to load the differential equation tools package *DEtools*. To do this, type the command:

    with(DEtools):

2. We will next enter the logistic equation in terms of the parameters r and K by typing the command:

    Logistic := (r, K) → diff(N(t), t) = r*N(t)*(1-N(t)/K);

    We can verify that this gives us what we desire by observing that the command

    Logistic(0.1,100);

    returns the differential equation

    $$\frac{d}{dt}N(t) = 0.1N(t)\left(1 - \frac{1}{100}N(t)\right).$$

3. Our first task is to create a direction field (slope field) plot for this differential equation with $r = 0.1$ and $K = 100$. A direction field plot enables us to observe the behavior of a family of solutions at once. To accomplish this, enter the command:

    DEplot(Logistic(0.1, 100), N(t), t = 0 .. 40, N = 0 .. 120);

    We can observe solutions on this direction field plot by plotting several solutions on the direction field. To plot solutions with initial conditions $N(0) = 0$, $N(0) = 10$, $N(0) = 60$, $N(0) = 100$, $N(0) = 120$, we modify the previous command to

    DEplot(Logistic(0.1, 100), N(t), t = 0 .. 40, [[N(0) =0],[N(0)=10],[N(0) = 60],[N(0) = 100],[N(0)=120]],N = 0 .. 120);

4. Print out the direction field plot containing the five particular solution curves from the preceding step (you would do this at the end of your lab event, or export to a bitmap or jpeg, and print those). Label each of these solution curves with its initial condition. In Maple, insert a paragraph (Insert Paragraph...) and write some text to describe the behavior of each of these solutions (in terms of monotonicity, concavity, and long-term behavior). Which of these solutions appear to be equilibrium solutions? What role does the carrying capacity play in these solutions?

5. Modify the previous command to consider other possible initial conditions. It seems that the initial value $N(0)$ and its relationship to the carrying capacity K have some relevance to the shape of the solution curve. Conjecture what you think the relationship of $N(0)$ to the solution curve's behavior might be. In particular, describe what happens in the five cases with $N(0) = 0$, $0 < N(0) < K/2$, $K/2 < N(0) < K$, $N(0) = K$, and $N(0) > K$. Use a text paragraph (Insert Paragraph...) to add this information to your Maple worksheet.

6. We now determine the equilibria of the logistic differential equation. The equilibria are determined by setting the right hand side of the differential equation equal to zero and then solving for N. The following two commands define dN/dt as a function of N and then solves for the equilibria:

$DerivPlot := (r, K) \rightarrow r * N * (1 - N/K);$
$solve(DerivPlot(r, K) = 0, N);$

7. These equilibria can be observed graphically as well by considering a plot of dN/dt versus N. The equilibrium values of N are the values of N for which the plot of dN/dt intersects the N-axis. We create this plot with the command:

$plot(DerivPlot(.1, 100), N = -10..120, labels = [N, dN/dt]);$

Print out this plot (you can export to a desktop file, then print). Place large dots on the equilibrium values of N on the N-axis of this plot. From this plot, we can determine the values of N at which solution curves are increasing by observing where $dN/dt > 0$. For each interval on which solution curves are increasing, mark the interval on the N-axis with an arrow pointing to the right. Similarly, solutions are decreasing when $dN/dt < 0$. Mark each interval for which solutions are decreasing with an arrow pointing to the left. The N-axis along with the labeled equilibria and arrows indicating increasing and decreasing behavior is called the *phase line* for the differential equation. As labeled, the phase line encodes the key dynamic behavior of the differential equation.

8. By looking at the plot of dN/dt versus N, we can also observe the concavity of solution curves by noting where dN/dt is increasing and decreasing. We will use Maple to calculate the second derivative of N directly. The command

$Deriv := (t) \rightarrow Logistic(r, K);$

defines the right hand side of the differential equation as function of t. The command

diff(Deriv(t), t);

computes the second derivative of N. We will now manipulate the right hand side of the resulting equation to easily determine points of inflection and intervals where solutions are concave up or concave down. Highlight the right hand side of this equation and copy it to a new command line. Before typing Enter, replace each occurrence of $\frac{d}{dt}N(t)$ with $rN\left(1 - \frac{N}{K}\right)$. Add a semicolon at the end of this expression and type Enter. The result is an expression for $\frac{d^2N}{dt^2}$ in terms of $N(t)$, r, and K. We can factor this expression with the command:

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factor(%);

(The "%" symbol recalls the expression on the preceding line in the Maple Worksheet.) Now using this expression, precisely determine values of N for which points of inflection occur as well as the concavity behavior of the solutions. Compare your results with the plot of dN/dt versus N.

9. The phase line also encodes the stability of each equilibrium value. If the arrows on both sides of an equilibrium point toward the equilibrium value, the equilibrium is *locally stable*. If the arrows on both sides of an equilibrium point away from the equilibrium value, the equilibrium is *unstable*. It is possible that the arrows on either side of an equilibrium point in the same direction. This indicates that solutions approach the equilibrium from one side and diverge from the equilibrium on the other side. In this case, the equilibrium is called *semistable*. Using the phase line, determine the stability of each equilibrium value. Notice that your answers do not depend on the particular values of r and K! To formulate your answer in general (for arbitrary values of $r > 0$ and $K > 0$), consider the original differential equation.

10. There is an analytic criterion to determine local stability of an equilibrium of an autonomous ordinary differential equation. The criterion can be stated as follows. Let $dN/dt = f(N)$ be a differential equation with f a differentiable function of N, and let $\hat{N}$ be an equilibrium value. Then if $f'(\hat{N}) < 0$, $\hat{N}$ is locally stable, and if $f'(\hat{N}) > 0$, the equilibrium $\hat{N}$ is unstable. Use this criterion to determine the stability of each of the equilibrium values of the logistic differential equation. Compare your results with your phase line analysis. You can have Maple perform the differentiation by modifying your function and differentiation commands to use N as independent variable. (Your analysis should result in the same conclusions!)

11. The solution curves plotted in a previous step we calculated using a numerical approximation technique for solving differential equations. We will briefly explore a simple numerical technique called *Euler's Method*, which depends on the basic calculus concept of a tangent line approximation. Euler's Method uses the idea of stringing together tangent line segments to create an approximate solution. To review the concept of tangent line approximation, we will plot the tangent line to the solution of this differential equation (again with $r = 0.1$ and $K = 100$) with initial condition $N(0) = 10$. Once you have determined the equation of this tangent line, define a function, Y, to represent this function in Maple. We use the following sequence of commands to plot the tangent line on the direction field plot:

with(plots);
plot1:=DEplot(Logistic(0.1, 100), N(t), t = 0 .. 40, N = 0 .. 120):
plot2:=plot(Y(t),t=0..40,N=0..120):
plots[display]({plot1,plot2});

(*Euler's Method*) Euler's Method is usually computed using a small stepsize for the independent variable. This means that the tangent line segments that we string together are short. The shorter these segments are, the better the approximate solution is. To illustrate how Euler's Method appears after several steps on our direction field plot, we will use a large stepsize to illustrate the process. Enter the following two commands:

DEplot(Logistic(.1, 100), N(t), t = 0 .. 40, [[N(0) =0],[N(0)=10],[N(0) = 60],[N(0) = 100],[N(0)=120]], N = 0 .. 120, method = classical[foreuler], stepsize = 5);

DEplot(Logistic(.1, 100), N(t), t = 0 .. 40, [[N(0) =0],[N(0)=10],[N(0) = 60],[N(0) = 100],[N(0)=120]], N = 0 .. 120, method = classical[foreuler], stepsize = 1);

Notice that the approximate solutions obtained with a stepsize of 1 are closer to the solution curves obtained above than the approximate solutions obtained with a stepsize of 5. Euler's Method is not used much in practice, but it does illustrate the basic idea behind numerically approximating solutions

to differential equations.

12. *(Conclusion)* To formulate your conclusion for this activity, describe the long term behavior of a population that behaves according to the logistic differential equation for arbitrary initial populations, $N(0)$. When are population trends increasing? When are these populations increasing most rapidly? When are population trends increasing? What role does the carrying capacity play in this long term behavior?

Part TWO: Harvesting

1. We will now consider the effects of different assumptions of harvesting. You might want to begin a new Maple worksheet. Modify the differential equation so that is now of the form:

$$\frac{dN}{dt} \quad = \quad rN\left(1 - \frac{N}{K}\right) - H. \tag{1}$$

The parameter H is included to account for harvesting. What harvesting assumption is being made here? When might the harvesting behavior presented in the model actually occur? What restriction must we place on H?

2. In general, determine the equilibria of the new differential equation in terms of H, r, and K. (You must solve a quadratic equation algebraically.) To do this in Maple, type the following command:

$solve(r * N * (1 - N/K) - H = 0, N)$;

3. For what values of H are there two equilibrium values N? For what value(s) of H is there one equilibrium? For what values of H are there no equilibria? Insert a paragraph answering these questions.

4. We will now create a phase portrait so that we can perform phase line analysis of the differential equation (1) as the parameter H varies. We will set $r = 0.1$ and $K = 100$. In your Maple worksheet do the following:

    (a) Click on the Components tab and then select the Plot component (don't hit ENTER!!). Now select the Slider component (still don't hit ENTER!!). Right-click on the plot window and select Component Properties to verify that it is named "Plot0" (and OK). Right-click on the slider and check Component Properties to verify that the slider is named "Slider0" (lots more to do here, in items (b), (c), and (d)).

    (b) Update the Slider values as follows:

Value at Lowest Position	0
Value at Highest Position	600
Current Position	0
Spacing of Major Tick Marks	100
Spacing of Minor Tick Marks	50

    (c) Make sure check marks are placed in EACH of the Options.

    (d) Click on the Edit button. Delete all of the content preceded by "#" characters. (These are components.) In between the remaining two lines of code type the following:

    $H0$:=GetProperty('Slider0','value'); $r := 0.1$; $K := 100$; SetProperty('Plot0','value',plot($r * N * (1 - N/K) - H0/100, N = 0..120, y = -3..3$));

    Click OK on the Edit window and on the Slider Properties window. You have now created a slider that varies from 0 to 600. The value of this slider is stored in the variable $H0$. We are interested

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in varying the parameter H in Equation (1) from 0 to 6. The parameter H is equal to $H0/100$. When we vary the slider from 0 to 600, we are really varying H from 0 to 6 as desired. The slider code we typed will plot the various curves.

5. Now plot the phase portrait and vary H (just drag the slider bar and see the graphs). For each of the three cases in the second item above, print out a phase portrait and determine the stability of any equilibria. Draw the phase line on the $N-$axis of each of the three printouts. For each of the three cases, describe the difference between the equilibria of the original model to the harvesting model and describe the biological significance. Is there a value of N below which the population will die out? Describe what is happening in this model when $H > 0$ and N is very small (and close to zero). Is there a value of H above which the population will die out regardless of the initial population? Such a critical value of H is known as a **bifurcation value**, where the behavior of the system changes. Explain the biological significance of such changes if any occur.

6. We will now plot the bifurcation diagram for the differential equation with the parameter H. To make sure that other parameters are set correctly and that H is treated as a variable by Maple, type:

$r := 0.1; K := 100; H :=' H'$

With the parameter values of r and K fixed, enter the following command again:

$solve(r * N * (1 - N/K) - H = 0, N)$

Each of the solutions is a called a *branch curve*. Now use a *plot* command (right click on the solution and select the Plot Builder) to plot these two solutions on the same grid in the HN-plane where H varies from -3 to 3. Print out this plot, and on the printout, label each branch curve as (locally) stable or unstable. Label all bifurcation points and label these points as (locally stable), unstable, or semistable.

7. Now plot several solution curves for this differential equation with a particular H (say $H = 2$) and $r = 0.1$ and $K = 100$ again, and various $N(0)$. Recall you need to load the DEtools package, define your curve, then use DEplot. (Copy and Paste and edit is really nice!) Try with another value of H.

8. Describe what is happening in this model when $H > 0$ and N is very small (and close to zero). Does this behavior make sense, or does it expose any limitations of this model? Explain.

9. We will now make another harvesting assumption. To begin, you need to reset r and K to be variables rather than set values. Type in $r :=' r'$; and $K :=' K'$;
Next, replace the H by $H * N$ so that the differential equation becomes

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - HN. \tag{2}$$

You may need to refer to your earlier work (perhaps in a previous worksheet), and just copy and paste.

10. Describe in words what this new harvesting assumption means.

11. In general, determine the equilibria of the new differential equation in terms of H, r, and K. (You must solve a quadratic equation algebraically.) For what values of H are there two equilibria? For what value(s) of H is there one equilibrium? For what values of H are there no equilibria? How does this model compare with the previous model?

12. We will now create a phase portrait plot with a slider for the values of H. As before, reset $r = 0.1$ and $K = 100$. This time, H should vary between 0 and 0.12 (I used $H1$ from 0 to 12 and divided by 100 for H). Good style would include changing the names to Plot1, Slider1, $H1$, etc. You can copy and paste and modify your slider window code from previous work. The tick marks should be much smaller, say minor of 1 and major of 5.

13. Now plot the phase portrait and vary H (just click on the slider bar and go). For each of the valid cases in the previous question, print out a phase portrait and determine the stability of any equilibria. Draw the phase line on the $N-$axis on your printout. For each of the valid cases, describe the difference between the equilibria of the original model to the harvesting model and describe the biological significance. (Is there a value of N below which the population will die out? Is there a value of H above which the population will die out regardless of the initial population? Explain the significance of such changes, if they occur.)

14. Now make a bifurcation diagram. Print out this plot, and on the printout, label each branch curve as (locally) stable or unstable. Label all bifurcation points and label these points as (locally stable), unstable, or semistable.

15. For each of the valid cases, plot several solution curves.

Part THREE: Further Explorations

1. You can repeat the above exercises with a third harvesting assumption. Replace the HN by $\frac{2HN}{N+10}$ so that the differential equation becomes

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - \frac{2HN}{N+10}.$$

   (This assumption could also be used to model predation rather than harvesting.) Describe in words what this new harvesting assumption means. How does it compare to the first harvesting assumption. (Does it resolve any issues about that model? How does this model behave when N is very small and when N is very large?) When you create a phase portrait plot with a slider for the values of H,make sure $r = 0.1$ and $K = 100$; and vary H between 0 and 6 and in increments of hundredths. Answer the same questions and perform the same explorations and find bifurcation diagrams and plot sample solution curves.

2. Consider a differential equation of the form

$$\frac{dN}{dt} = Ng(N),$$

   where g is a differentiable function of N. We call the function $g(N)$ the *intrinsic growth rate* of N. The *Allee effect* (Allee [1]) is characterized by a population that has a maximal intrinsic growth rate at a nonzero intermediate density (that is, at a population below its carrying capacity). Moreover, the Allee effect is observed when a population dies off if it drops below a threshold level. (Such a threshold could occur due to the lack of suitable mates.) We will look at a typical model that exhibits the Allee effect. First, we write down the differential equation for the Allee effect in terms of the carrying capacity K and a growth parameter r. Consider the differential equation

$$\frac{dN}{dt} = f(N) = Ng(N), \quad \text{for } t \geq 0, \tag{3}$$

   where

$$g(N) = r(N - a)\left(1 - \frac{N}{K}\right), \quad r > 0 \text{ and } 0 < a < K. \tag{4}$$

   In particular, let $r = 0.25$, $a = 5$, and $K = 50$. In a new Maple worksheet, plot the phase portrait (including the phase line) for this particular differential equation in the window $[-5, 55] \times [-20, 80]$. Print out this graph and place arrows on the $N-$axis to indicate the increasing and decreasing behavior of N. Identify and determine the stability of any equilibria when for $N \geq 0$.

3. For what value of N is $g(N)$ maximal? What is this maximal value of g?

4. Where is $f(N) > 0$ and $f(N) < 0$ for $N \geq 0$?

5. Explain the biological significance of the population dynamics for each significant interval of values of N (for $N \geq 0$). Discuss both the behavior of the instrinsic growth rate $g(N)$ as well as $f(N)$. This time, make sure you refer to K in your analysis.

6. Use the stability criterion (see item 10 of part 1) to classify the equilibria of equation (3). (You may perform your calcuulations in Maple, but you must explain your calculation process and the conclusion that you draw from the calculation.)

7. Plot each equilibrium solution and each typical solution (for appropriate values of N_0) on a single grid.

8. Go back to the general model given by equation (3) in terms of parameters r, a, and K. In general, determine the stability of each equilibrium.

References

[1] W.C. Allee, *Animal Aggregations, A Study in General Sociology*, University of Chicago Press, 1931.

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[4] C. Neuhauser, *Calculus for Biology and Medicine, 2e*, Pearson Prentice Hall, 2004.