LINES OF EIGENVECTORS AND SOLUTIONS TO SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT: The purpose of this paper is to describe an instructional sequence where students invent a method for locating lines of eigenvectors and corresponding solutions to systems of two first order linear ordinary differential equations with constant coefficients. The significance of this paper is two-fold. First, it represents an innovative alternative where the concept and calculation of eigenvectors precede those of eigenvalues. This “eigenvector first approach,” which seems to more closely follow students’ conceptual development, is a reversal of the typical approach where students first learn to find eigenvalues and then determine the eigenvectors. Second, we characterize the flow of this instructional sequence in terms of four types of activities: prediction, exploration, mathematization, and generalization. The approach of prediction, exploration, mathematization, and generalization represents a more general design heuristic useful for other topics in other courses.

KEYWORDS: Ordinary differential equations, student invented methods, eigenvectors, eigenvalues, instructional design.

INTRODUCTION

Advances in technology and mathematicians’ evolving interests in dynamical systems are currently prompting changes to the introductory course in
ordinary differential equations (ODEs). Traditional approaches in ODEs emphasize analytic techniques for finding closed form expressions for solutions whereas current reform efforts are emphasizing graphical and numerical approaches for analyzing and understanding the behavior of solutions. Examples of these new directions in ODEs that capitalize on geometric and numeric approaches using technology can be found in Kallacker [7] and West [12]. In addition, there is a small but growing body of research on students' understandings and difficulties in ODEs that is useful for informing teaching and instructional design (see [10] for a review of this research). In the treatment of systems of differential equations, the use of phase plane analyses for describing and predicting solutions is receiving increased attention. How students come to understand solutions in the phase plane, however, is currently not well understood. The purpose of this paper is to begin to illuminate this issue by describing an instructional sequence in which students invent an approach for locating and analytically describing solutions that, when viewed in the phase plane, fall along straight lines. In this report, we focus primarily on the flow of the instructional activities and exemplify student work and reasoning in the process. We also point to characteristics of the technology that are critical to the success of this sequence.

In addition to integrating a dynamical systems point of view and research findings on student learning, our work with students seeks to design or "engineer" learning environments where undergraduate students invent for themselves many important mathematical ideas and methods in ODEs. The design of these learning environments includes explicit attention to the patterns of interaction regarding explanation and justification in which we actively work to create a learning environment where students routinely explain their thinking, listen to and try to make sense of other students' reasoning, and respond to questions from other students and the instructor. See [11, 13] for a discussion of these social aspects of the learning environment and the proactive role of the instructor. The design of these learning environments also includes explicit attention to the use of computer and calculator tools. We have found it useful in our work to adapt the instructional design theory of Realistic Mathematics Education [5, 6], originally developed at the elementary and secondary school level, to the undergrad-

\footnote{Many current reform efforts also use computer algebra systems for efficiently determining analytic solutions. In our reform approach, we do not use a computer algebra system but instead teach students a trimmed down set of paper and pencil analytic techniques. Technology is, however, an integral part of the instructional sequence, aimed at fostering students' conceptual development and at supporting graphical and qualitative analyses.}

\footnote{The complete set of instructional activities are available from the first author.}
uate level. We find the theory of Realistic Mathematics Education useful because it provides us with a general orienting perspective that mathematics is a human activity where students should be provided opportunities to reinvent important mathematical ideas and methods. The sequence subsequently described is the result of several years of classroom based research where over the course of several semester long studies, we simultaneously created instructional materials and investigated student learning and means to support this learning. The data sources included videorecordings of each class session, videorecorded problem solving interviews with individual students, and copies of students’ in class and out of class written work (see [3] for methodological details).

**INSTRUCTIONAL SEQUENCE**

We describe the instructional sequence in which students invent an approach for first determining the slope of the line(s) of eigenvectors in the phase plane in four segments: prediction, exploration, mathematization, and generalization. In the first segment of the sequence, students predict graphical representations in the position-velocity plane for the motion of a mass attached to a spring. After developing the differential equations for this situation, students use a computer program to empirically explore changes to the vector field as the friction coefficient for the spring mass situation varies. Next, students mathematize their empirical observations by inventing an algebraic approach for determining the slope of the line(s) along which vectors point directly toward the origin. Students then figure out the equations for the pair of functions that solve the system of differential equations along this line. Finally, students generalize their approach by determining the general solution for this situation and other situations involving real and complex eigenvalues. We should point out that when this research began in 1998, the more conventional way of teaching eigenvalues first and then eigenvectors was followed. However, after listening carefully to students’ thinking, we figured out that the reverse is actually a more natural order for students⁴.

³The following colleagues have participated in one or more of these design research projects: Erna Yackel, Michelle Stephan, Karen King, Karen Whitehead, and Wei Ruan.

⁴Although further cognitive analysis is needed, the eigenvector-first approach builds on students’ ability to reason with slope and linearity, which then provides a basis for proportional reasoning that is useful for conceptualizing eigenvalues.
From an expert’s point of view, eigensolutions for a system of the form

\[
\begin{align*}
\frac{dx}{dt} &= ax + by \\
\frac{dy}{dt} &= cx + dy
\end{align*}
\]

are determined by the eigenvalues and eigenvectors for the $2 \times 2$ matrix corresponding to the system of differential equations. Typical approaches for finding the closed-form solution for eigensolutions employ techniques from linear algebra where students first find eigenvalues, then find the corresponding eigenvectors, and then form the analytic solution. Although students can often find eigenvalues and eigenvectors, the meanings of these mathematical ideas tend not to be well understood by students [4]. Our students tend to understand these ideas better because eigenvectors and eigenvalues are the result of their own constructive activity. The development of the instructional sequence we describe is, in fact, the result of student invented methods for first finding eigenvectors and then eigenvalues (a reversal of the typical order) due to their interest in locating “straight-line solutions,” namely solutions that are restricted to a one-dimensional subspace of the phase plane. In the instructional sequence, we use the term straight-line solutions because it is more aligned with student thinking at this point. In general, the instructional design is based on students’ mathematical goals (a bottom-up approach) rather than a top-down approach based on an expert’s perspective. As such, we document students’ conceptual development of straight-line solutions as they reinvent the ideas of eigenvectors and eigenvalues from their own sense-making. As stated by one of the stronger students after having completed the sequence,

I remember learning all about eigenvalues and eigenvectors in my linear algebra class last semester. While I was successful in working the assigned problems, I really didn’t have a strong conceptual understanding of what these ‘weird’ words actually represented - I didn’t have any bridge in the understanding between their algebraic and graphical implications. However, all this has changed as I developed these concepts in relation to linear systems of DEs and straight-line solutions.

Although this student had previously taken linear algebra, it is not a prerequisite, and many other students who had not taken linear algebra experienced similar feelings of accomplishment. In fact, the terms eigenvector and eigenvectors are only used near the end of the sequence and so
having had prior experience with these ideas is neither required nor necessarily helpful. Our experience has been that this sequencing is conceptually appropriate for students with or without a background in linear algebra.

**PREDICTION**

Before modeling the spring mass situation illustrated in Figure 1 with Newton’s law, students are invited to graphically predict or describe the motion of the mass in the position-velocity plane that might be possible for various values of parameters such as the stiffness of the spring, the weight of the object attached to the end of the spring, and the amount of friction along the surface that the object travels.

Typical responses on this task include closed circle-like curves if there was no friction, curves that spiral in toward the origin, as well as incorrect variations of these where the velocity is always positive. Very few students include in their predictions the overdamped case where the mass would not oscillate back and forth about the equilibrium position. The fact that many students do not include in their predictions the overdamped case is perhaps not surprising since this type of behavior is most likely not part of their prototypical image of an oscillating situation.

Rather than pointing out that students “missed” a case, we choose to move on and use Newton’s law of motion to develop the second order differential equation, which we then promptly convert to a system of two first order ODEs.\(^5\) We choose to move on for two reasons. First, we want to activate students’ sense making in which they feel free to discuss their ideas without evaluation. Second, it turns out to be pedagogically productive that students omit the overdamped case because it adds to their sense of surprise and need to figure something out as they move into the next segment of the sequence.

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\(^5\)In our course design, treatment of first order ODEs is followed by systems of ODEs rather than second order ODEs. Although this is not typical, at least one other text [2] at this level follows a similar progression.
EXPLORATION

The next part of the sequence involves vector field exploration for a specific spring-mass problem with variable friction. For convenience we choose the mass to be 1 and the spring constant to be 2 and allow the friction coefficient to vary. The resulting system of differential equations is

\[
\begin{align*}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= -2x - by
\end{align*}
\]

where \(x\) represents the position of the spring mass relative to the equilibrium point, \(y\) represents the velocity of the mass, and \(b\) is the coefficient of friction.

To support a vector field exploration of this system of differential equations, students use the software program \textit{Graphing Calculator} [8]. There are three important features of this technology that contribute to students' success. First, the program only plots vectors. It does not actually plot out graphs of solutions in the phase plane. This may strike readers as a disadvantage, but we find this to actually be pedagogically advantageous because it encourages algebraic analysis, rather than relying on empirical observations to support conjectures. More about this will be said in the section on mathematization. Second, the software allows students to interactively move a point with its associated vector about in the plane. Thus, although trajectories are not plotted out, students engage kinesthetically and imaginatively in the creation of such trajectories. Third, the program has a slider bar that allows one to dynamically change a parameter in the differential equations while simultaneously viewing corresponding changes to the vector field.

The combination of these three aspects of the software, coupled with a learning environment that supports expression and resolution of student ideas, proves to be powerful because students progress from first observing their predicted closed curve and spiral solutions to becoming invested in accounting for an apparent straight-line solution as the friction parameter \(b\) increases. Below is the open-ended task we give students.

Use the slider bar tool to vary the friction parameter to investigate how the motion of the mass, as predicted by the system of differential equations, differs when the friction parameter is 0 as compared to when it is, say 2.3, or 3, or 3.8, for example. Use graphs and descriptions of how to interpret those graphs.
to summarize the results of your investigation. Be as specific as possible. In addition, discuss changes, if any, to the type of equilibrium solution as the friction parameter changes.

Figure 2 shows two different snapshots of the vector fields students typically discuss as they explore the impact of changing the amount of friction in the system, one that spirals in (like their prediction) and one that appears to have a straight line of vectors.

![Figure 2. Spiraling and non-spiraling solutions.](image)

The open-ended nature of the task invites students to experiment with the program and discuss their findings with classmates. These activities encourage students to make important connections between the vector fields, interpretations of the physical system, other graphical representations such as $x$-$t$ and $y$-$t$ graphs, and the changing nature of the equilibrium solution$^6$. Over the many semesters of working with students, we have consistently found that students become intellectually interested in more deeply understanding the situation in Figure 2(b), often because they did not initially predict this type of behavior. As discussed next, their interest provides an opening for further mathematization.

**MATHEMATIZATION**

Following graphical exploration, algebraic formulations emerge as a way for students to express their empirical observations and test out their conjectures. Students do not a priori have terms for classifying equilibrium solutions to systems of differential equations, but because of the strong dynamical systems perspective taken when developing single differential equations, they often create terms to differentiate what they see as different types of vector fields. Conventional terminology for classifying equilibrium solutions is introduced in subsequent class periods.
tues [cf, 1]. To recap, students typically observe that for larger values of $b$, the vector field no longer spirals but rather the flow of vectors appear to get “sucked in” along a straight line. Algebraic representations then become reasoning tools for figuring out the precise slope of the conjectured straight line of vectors. Although many students have already posed this as a question for themselves, we typically follow up their graphical exploration with the following problem:

Marissa sets the friction parameter to 3 and notices that graphs of solutions in the position-velocity plane seem to get sucked into the origin via a straight line. She conjectures that along this line all vectors head directly towards the origin with a specific slope. Do you agree with Marissa’s conjecture? Figure out an algebraic way to either support or refute this conjecture. If you think the conjecture is true, (a) What is the precise slope? (b) What is the corresponding physical motion of the mass? (c) What are the corresponding $x(t)$ and $y(t)$ equations?

The typical algebraic method that students invent as a means to determine the slope of a straight line of vectors capitalizes on their knowledge that a line going through the origin is of the form $y = mx$ and that the slope of the vectors is the ratio of the two rate of change equations.

Thus, $m = \frac{y}{x}$ and $m = \frac{dy/dt}{dx/dt} = \frac{-2x - 3y}{y}$.

Equating the two expressions for the slope yields:

$$\frac{y}{x} = \frac{-2x - 3y}{y}.$$

Replacing $y$ with $mx$ and simplifying: $m = \frac{-2 - 3m}{m}$. Solving for $m$ yields: $m^2 + 3m + 2 = 0 \rightarrow m = -1, m = -2$.

Much to students’ surprise, their algebraic work yields two straight lines of vectors. While most had “seen” one straight line of vectors (see Figure 2b), the second straight line of vectors was only noticeable after they knew where to look. Using a similar algebraic method, students also figure out the smallest value of $b$ for which we get solutions that, when viewed in the position-velocity plane, lie along a straight line.

Next, students are asked to figure out the $x(t)$ and $y(t)$ equations for different initial conditions that fall along the line $y = -x$ and $y = -2x$. For example, to figure out the $x(t)$ and $y(t)$ equations for the solution with initial condition $(-2, 2)$, students typically substitute $y = -x$ into the equations.
for \( \frac{dx}{dt} \) and \( \frac{dy}{dt} \) to get \( \frac{dx}{dt} = -x \) and \( \frac{dy}{dt} = -y \), which they readily solve to yield \( x(t) = -2e^{-t} \) and \( y(t) = 2e^{-t} \). Similarly, for the initial condition \((-2,4)\), they obtain the equations \( x(t) = -2e^{-2t} \) and \( y(t) = 4e^{-2t} \). Students also quickly figure out\(^7\) that their method works to find the \( x(t) \) and \( y(t) \) equations for any initial condition along either of the straight lines, \( y = -x \) or \( y = -2x \).

Next, we describe the portion of the instructional sequence where students determine the \( x(t) \) and \( y(t) \) equations for a solution with initial condition not on a straight line of vectors. That is, students find the general solution to the system of differential equations.

**GENERALIZATION**

Now that students have figured out the \( x(t) \) and \( y(t) \) equations for any initial condition along either of the straight lines, we pose the following problem:

Suppose you were to start with an initial condition somewhere in the second quadrant between the two straight line solutions, say at \((-4,6)\). Sketch what you think the solution as viewed in the phase plane looks like.

![Figure 3. What if the initial condition is \((-4,6)\)?](image)

Note that rather than immediately ask students to figure out the \( x(t) \) and \( y(t) \) equations for initial condition \((-4,6)\), we invite students to make graphical predictions for the basic shape of the graph in the phase plane.

\(^7\)Although not necessary at this point, some students cast this finding in terms of multiplying the \( x(t) \) and \( y(t) \) equations for one initial condition along the straight line \( y = -x \) by a suitable constant to obtain the \( x(t) \) and \( y(t) \) equations for any other solution with initial condition along the line \( y = -x \). (Similarly for solutions along the line \( y = -2x \)).
Most students tend to think that the graph will head into the origin, curving closer to the line $y = -2x$. Their tentative reasoning appears to be based on the fact that this line is steeper and/or the fact that $(-4, 6)$ is closer to the line $y = -2x$ than it is to the line $y = -x$. Inviting students to make such predictions has three benefits: it maintains the spirit of the entire sequence, it tends to activate their sense making, and it motivates them to figure out whether they are correct or not. Of course the next problem we ask is,

Figure out what the $x(t)$ and $y(t)$ equations are for the solution with initial condition $(-4, 6)$. According to these equations, what should the solution in the phase plane look like?

Motivated by the fact that this initial condition is the sum of two earlier initial conditions (i.e., $(-2, 2)$ and $(-2, 4)$), students conjecture that the $x(t)$ and $y(t)$ equations for this initial condition is the sum of these earlier equations and verify that their solution satisfies the system of differential equations. Moreover, students actually use the $x(t)$ and $y(t)$ equations for the initial condition $(-4, 6)$ to reason that the graph in the phase plane heads into the origin, but actually curving closer to the line $y = -x$!!

As students determine suitable combinations of two straight line solutions to find the $x(t)$ and $y(t)$ equations for other initial conditions, they develop increased confidence that adding two different straight line solutions in fact yields the general solution. Students are then offered a number of other systems of differential equations without a physical situation like the spring mass and are asked to figure out the general solution. This helps minimize the chance that they develop the mistaken conception that the exponent for the $x(t)$ and $y(t)$ equations is always the same as the slope of the straight line.

As the sequence progresses, students actually verify the property of superposition for linear systems of the form

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

when they begin work on cases when solutions in the phase plane spiral. Finally, grounded in a firm conceptual understanding, students are eventually introduced to the conventional terminology of eigenvectors, eigenvalues, and the more traditional method of finding eigenvalues before eigenvectors. As such, students are in a much better conceptual position to appreciate the significance of first finding eigenvalues. In the words of the same student we quoted at the beginning of this paper,
While I later came to prefer the eigenvalue method, I think that [the slope-first method] helped me gain a conceptual understanding of what a straight-line solution was. I think that if the eigenvalue method had been presented first, I would have had a more difficult time in gaining this understanding.

CONCLUDING REMARKS

Finding eigensolutions to linear systems of differential equations is often considered one of the most difficult conceptual tasks for students in an introductory course to differential equations. As reported in one study [13], students in a more traditional class could easily find eigenvalues by setting up and solving a characteristic equation, but even the best students could not explain where the characteristic equation comes from or how it is motivated. The instructional sequence discussed here and our ongoing research provide some evidence that students can develop both conceptual understanding and computational proficiency, provided that we connect with and build from students’ current understandings.

As noted in the introduction, a unique feature of this sequence is that eigenvectors develop conceptually and computationally before eigenvalues. Traditionally, one expresses a linear system of differential equations with matrix notation, finds the characteristic equation of the matrix, and solves for the eigenvalues. Starting with computations that seek out eigenvalues first, however, does not appear to connect with students’ conceptual development.

Another notable feature of this sequence is the activities of prediction, exploration, mathematization, and generalization. Central to these activities is the opportunity for students to conjecture and develop algebraic and analytic means to verify or refute their conjectures. This process of prediction, exploration, mathematization, and generalization engages students in reinventing concepts and methods associated with eigenvalues and eigenvectors. Rather than supplying or giving students ready-made algorithms for finding eigenvalues first and then conducting analysis from there, students build from their current understandings of slope and straight lines to develop increasingly sophisticated ways of reasoning about solutions to differential equations - knowledge that they come to view as their own.
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**BIOGRAPHICAL SKETCHES**

Chris Rasmussen is an Associate Professor of Mathematics Education in the department of mathematics, computer science, and statistics at Purdue University Calumet. He received an undergraduate degree in mechanical engineering, a master’s degree in mathematics, and his PhD in mathematics education at the University of Maryland. He is interested in the learning and teaching of undergraduate mathematics and how approaches that have been successful at promoting student learning in earlier grade levels can be adapted to the university setting. As part of his work in differential equations, he has developed frameworks for interpreting and understanding students’ conceptual development and he has written course material for a first course in differential equations.

Michael Keynes is an Assistant Professor of Mathematics Education in the department of mathematics, computer science, and statistics at Purdue University Calumet. He received his bachelor’s degree in mathematics from MIT, and his master’s degree and PhD in mathematics at the University of Washington studying representation theory of real reductive Lie groups. He was subsequently an NSF post-doctoral researcher in mathematics education at the University of California at Berkeley. Keynes has been involved in preservice and inservice preparation of mathematics teachers since 1996. His research interests in undergraduate mathematics education include how students develop conceptual understandings of differential equations and what content knowledge preservice and inservice teachers need to know in order to teach high school mathematics.