**How would you model the length of time it takes an ant to build a horizontal tunnel of length x into a cliff of moist sand?**

# Inquiry-Based Learning Best Practices

**MathFest Summer 2013, 1050-1100,**
**3 August 2013 CCC Rm 16,Hartford CT**

**SIMIODE – Modeling with Differential Equations in Inquiry Mode**

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We present an approach we have used for years in our own teaching and are creating as a web-based NSF developed HUB community of teachers and learners called

**SIMIODE** - **S**ystemic **I**nitiative for **M**odeling **I**nvestigations and **O**pportunities with **D**ifferential **E**quations.

www.simiode.org

We present and discuss examples of activities to enable students to develop mathematical modeling skills using differential equations while using technology throughout their learning and doing mathematics.

We are about Learning-Based Inquiry . . . .

We are about Learning-Based Inquiry in which
Inquiry about a scenario drives student Learning.

From student's perspective it is Inquiry-Based Learning.

From the teacher's perspective it is Learning-Based Inquiry, for the teacher wants the student to discover a need for and a sense as to where the mathematics under study has its roots. Thus the Inquiry of modeling and effective use of technology drives Learning.

How long does it take an ant to build a tunnel of length x?

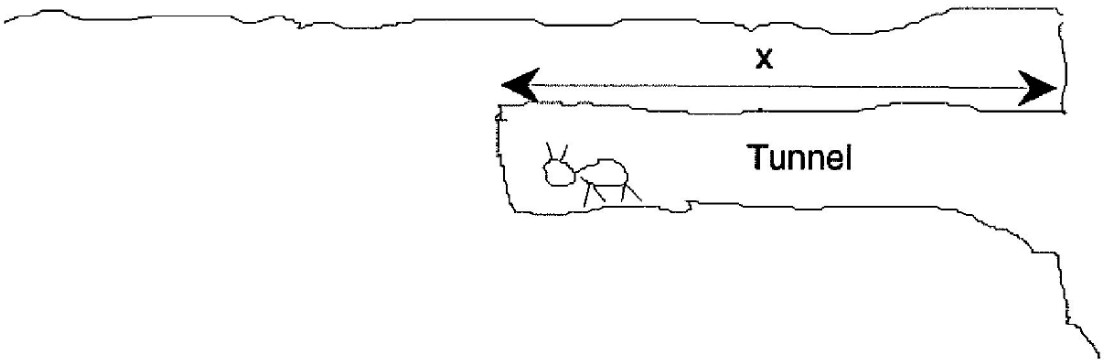


**Figure 1.** Crude drawing for ant tunnel building model. $x$ is the length of the tunnel and $T(x)$ is the time it takes an ant to build a tunnel of length $x$.

Assumptions and Rationale:

Winkel, B. J. 1994. Ants, Tunnels, and Calculus: An Exercise in Mathematical Modeling. *The Mathematics Teacher.* 87(4): 284-287.

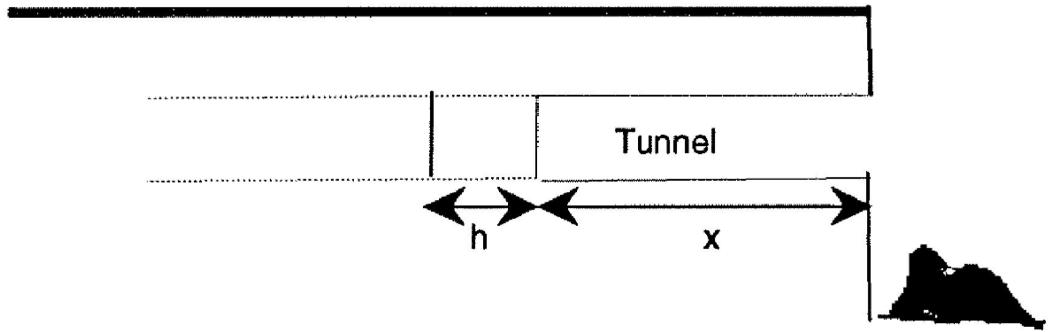


**Figure 2.** Useful diagram for discovering the time it takes to build a small section of the ant tunnel from distance $x$ to $x + h$.

We define terms and build a mathematical model

Let T(x) be the time it takes an ant to build a tunnel of length x.

$T(x+h) - T(x) =$

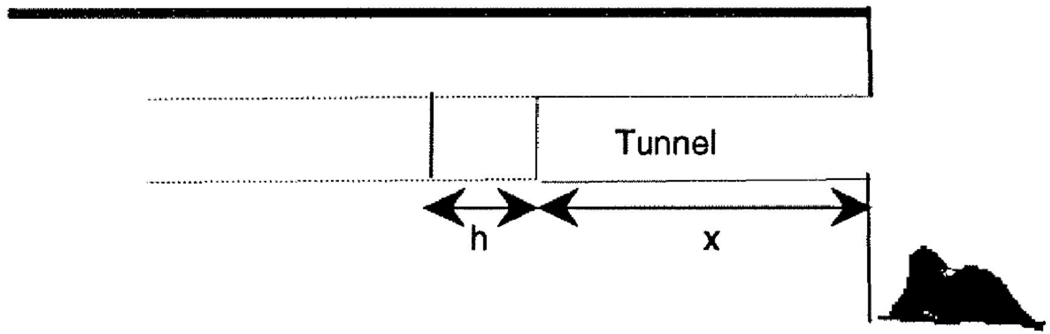


**Figure 2.** Useful diagram for discovering the time it takes to build a small section of the ant tunnel from distance $x$ to $x + h$.

We define terms and build a mathematical model

Let T(x) be the time it takes an ant to build a tunnel of length x.

$T(x+h) - T(x) =? \alpha x + h$ (why or why not? If h = 0 . . . )

$=? \alpha x - h$ (why or why not? If h = 0 . . .)

$=? \alpha x^h$ (why or why not? If h = 0 . . .)

$=? \alpha x h$ (why or why not? Well, why not?)

T(x) is the time it takes an ant to build a tunnel of length x.

From $T(x+h) – T(x) = \alpha\, x\, h$ we obtain a differential equation

$T'(x) = \alpha\, x$ , with sensible initial condition, $T(0) = 0$

Solve the differential equation.

What does our model tell us about the answer to t his question?

How is the time changed if we double the length of the tunnel?

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Solve the differential equation.

$T(x) = \alpha \frac{x^2}{2} + c$ where $0 = T(0) = c$, so $T(x) = \alpha \frac{x^2}{2}$ .

What does our model tell us about the answer to t his question?

How is the time changed if we double the length of the tunnel?

"How long does it take an ant to build a tunnel?"
Through reasoned assumptions on the rate of change in the time it takes to build a tunnel of length x, i.e., the function $T(x)$, we get the students to arrive at $T(x+h) - T(x) = a\,x\,h$ .

From there it is on to a differential equation when we divide by h and then a solution from which we ask reality check kinds of questions.

Incidentally, along the way we may get $T(x+h) - T(x) = a\,x + h$. Here we make lots of points as we question our model all the way through the process with issues like,

"Does the model work if $x = 0$ or if $h = ?$" This is all by hand and has not any data, although with an ant colony and pheromones one could imagine collecting it.

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The **Inquiry**,

"How long does it take an ant to build a tunnel of length x?"

has led to **Learning**

(a) what differential equation is,
(b) where it might come from,
(c) how to solve a simple differential equation – calculus, and
(d) how to interpret the solution.

The **Inquiry** was set before the students SO THAT the **Learning** of (a) – (d) might occur, and perhaps even more, e.g., testing assumptions and what an initial value offers.

## M&M Population Simulation with Immigration

Count out 50 M&M’s form your “source”

(1) Put them in the cup.
(2) Gently toss them onto the plate.
(3) Remove those M&M’s with “m” facing up – they die.
(4) (Optional) From your “stash” add 10 immigrants.
(5) Count remaining M&M’s for that generation.

Go to (1).

**DO NOT eat M&M’s until so instructed!**

**At each iteration keep track of time (generation number) and number of M&M’s.**

Winkel, B. J. 2010. Population Modeling with M\&M's. *International Journal of Mathematical Education in Science and Technology.* 40(4): 554-558.

What assumptions might you make about the individual M&M's, the "action", and the population? What are relevant assumptions? What are facts?

What expectations do you have for the population as the generations go on?

Let us collect some data on the "life and times" of M&M's!

Let us build a mathematical model (based on our assumptions) of the number of living M&M’s at generation n.  I will help out.

Let $y(n)$ = number of living M&M’s at generation n.

$y(0) =$

Could we just produce a formula for $y(n)$?

Could we produce a relationship between one generation and the next?

Answering the second question might produce an answer to the first question and this is what difference or differential equations is all about. – asking a question, a good question , and developing a strategy to get an answer.

Difference equation model:

$y(n+1) =$

OR

$y(n+1) - y(n) =$

Differential equation model:

$y'(t) =$

Difference equation model:

$y(n+1) = 0.5\, y(n) + 10\,, \qquad y(0) = 50$

OR

$y(n+1) - y(n) = -0.5\, y(n) + 10, \qquad y(0) = 50$

Differential equation model:

$y'(t) = -0.5\, y(t) + 10\,, \qquad y(0) = 50$

Solve the differential equation model:

$y'(t) = -0.5\ y(t) + 10 \qquad y(0) = 50$

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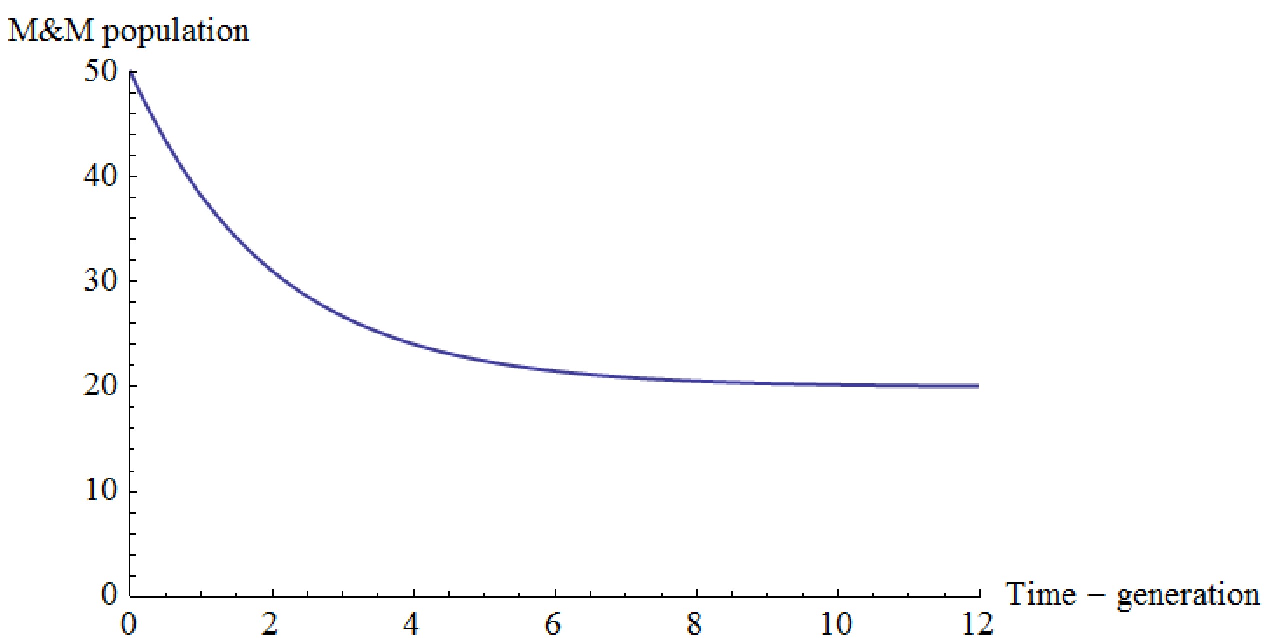
```
ysol[t_] =
 y[t] /. DSolve[{y'[t] == -.5 y[t] + 10, y[0] == 50},
     y[t], t][[1]] // Expand
```

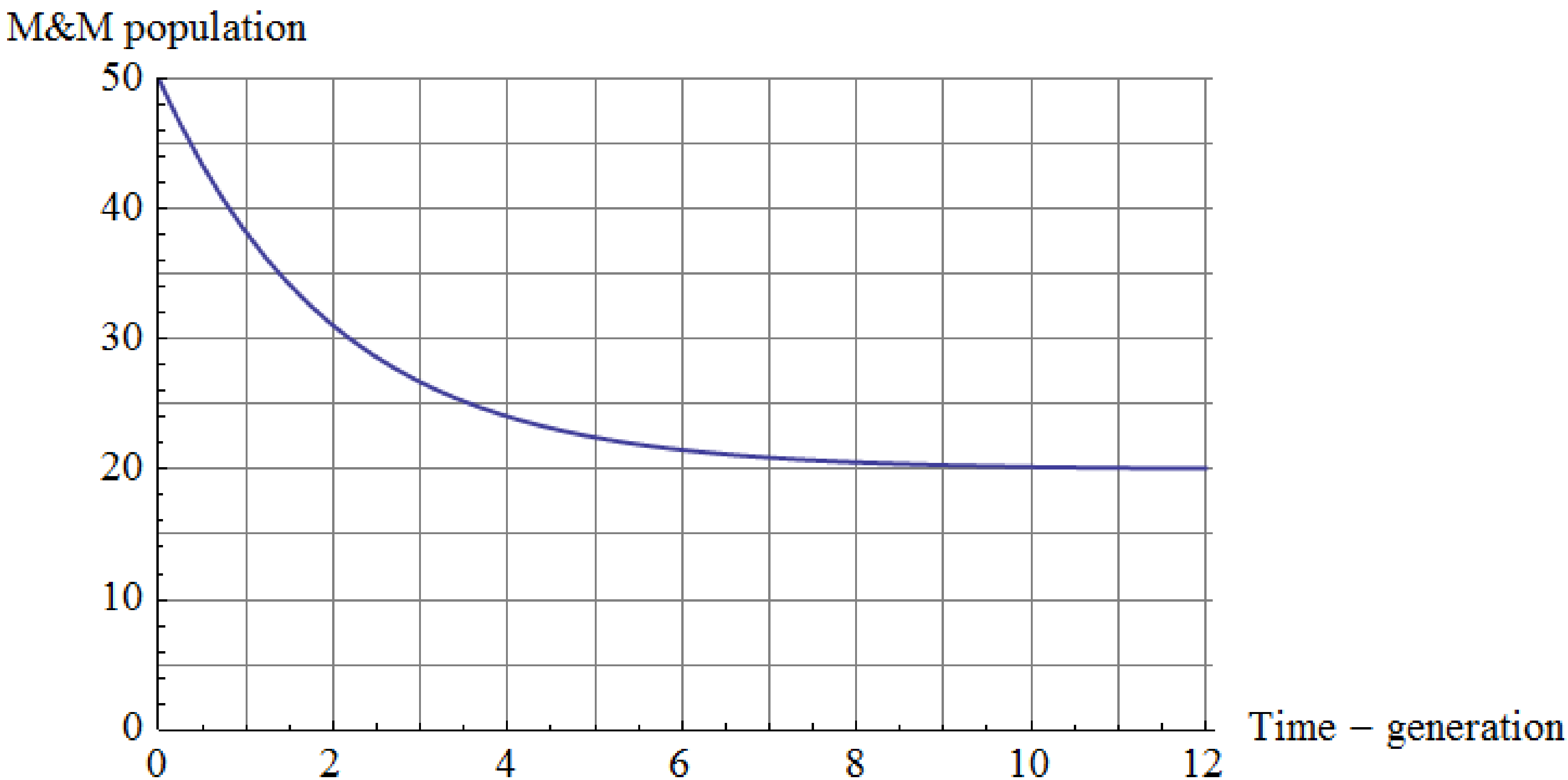
$20. + 30.\, e^{-0.5\, t}$

```
Plot[ysol[t], {t, 0, 12}, PlotRange → {{0, 12}, {0, 50}},
 AxesLabel → {"Time - generation", "M&M population"}]
```



Here is a plot of our mathematical model based on assumptions:

$y(t) = 20 + 30\ e^{-0.5\,t}$.




How does our model compare with our “reality” data?

## Sublimation of carbon dioxide

How long does it take for a “block” of dry ice or solid $CO_2$ to sublimate?

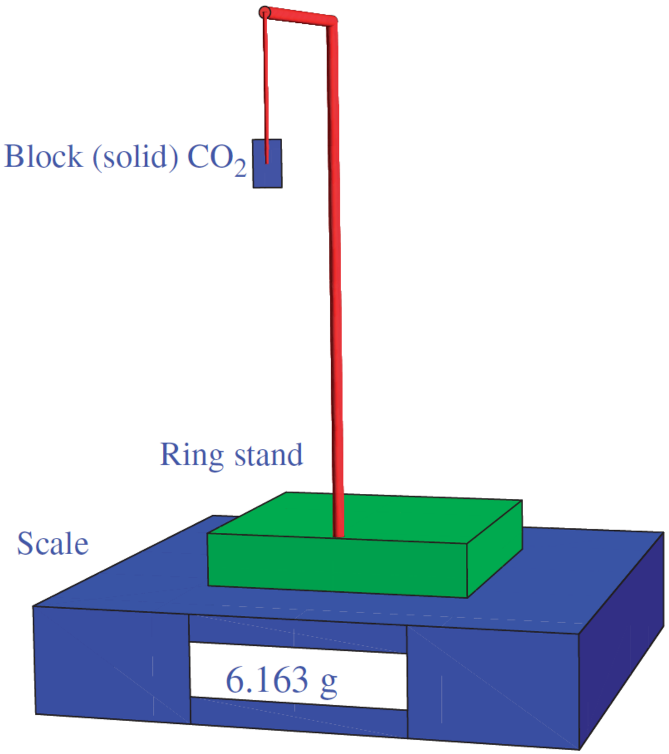
Let us produce a mathematical model for the rate of sublimation, i.e.

$m'(t) =$

where $m(t)$ is the mass of the piece of dry ice.

Assumptions?

Winkel, B. J. 2012. Modelling sublimation of carbon dioxide. *International Journal of Mathematical Education in Science and Technology.* 43(8) 1077-1085.




# Apparatus and Oops!

Table 1. Data collected on successful run for mass of dry ice (g) as a function of time (s).

| Time (s) | Mass (g) | Time (s) | Mass (g) |
|---|---|---|---|
| 0 | 7.570 | 480 | 5.871 |
| 30 | 7.457 | 510 | 5.776 |
| 60 | 7.338 | 540 | 5.682 |
| 90 | 7.220 | 570 | 5.589 |
| 120 | 7.110 | 600 | 5.497 |
| 150 | 6.995 | 630 | 5.405 |
| 180 | 6.885 | 660 | 5.313 |
| 210 | 6.778 | 690 | 5.224 |
| 240 | 6.673 | 720 | 5.136 |
| 270 | 6.571 | 750 | 5.048 |
| 300 | 6.464 | 780 | 4.960 |
| 330 | 6.363 | 810 | 4.878 |
| 360 | 6.265 | 840 | 4.790 |
| 390 | 6.163 | 870 | 4.707 |
| 420 | 6.067 | 900 | 4.625 |
| 450 | 5.969 | | |

$$m'(t) = -km(t)^r, \quad m(0) = 7.57$$

$$m(t) = \left((r-1)\left(kt - \frac{7.57}{7.57^r(1-r)}\right)\right)^{\frac{1}{1-r}}$$

What might we expect r to be?

What might we try for r?

What is reasonable?

We seek the value of k and r in

$$m(t) = \left((r-1)\left(kt - \frac{7.57}{7.57^r(1-r)}\right)\right)^{\frac{1}{1-r}}$$

which will minimize the sum of square errors between our model prediction m(t) = $m_1$(t) at time t = $t_i$ and our observation data, $m_i$.

$$SSE_1(k) = \sum_{i=1}^{i=n} (m_1(t) - m_i)^2$$

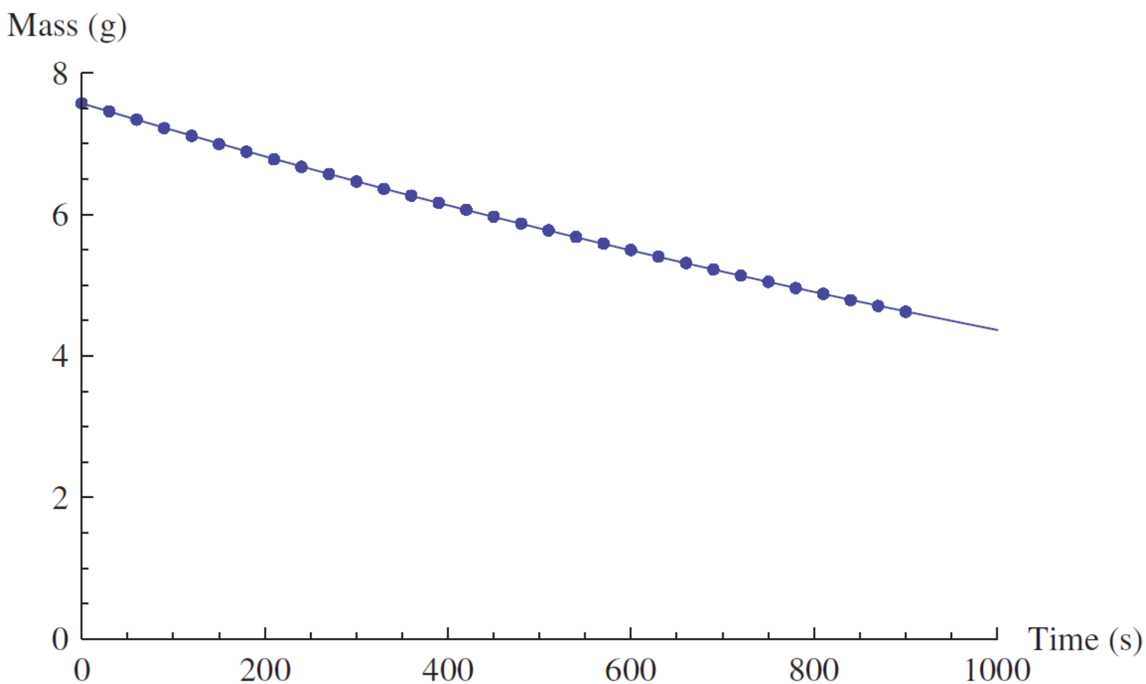



Figure 6. Plot of optimal order fit to data over the data in Table 1.
Note: Here $r = 0.751209$ and $k = 0.0008506$. This is a very convincing plot of our solution for (5) with these parameters.

Table 3. Summary of data analyses for various orders $r$ with best estimate of parameter $k$ and respective minimum sum of square errors. We see that none of $r = 0, 1, 2$ or $r = \frac{2}{3}$ are best. Through our optimization analysis of values of $r$ rather than presuming set values of $r$ we obtain the best order model when $r = 0.751209$. We show the plot of the model over the data in Figure 6.

| Order ($r$) | Parameter $k$ | SSE |
|---|---|---|
| 0 | 0.00340764 | 0.12126 |
| 1 | 0.000536791 | 0.0119927 |
| 2 | 0.0000841212 | 0.251683 |
| $\frac{2}{3}$ | 0.00184154 | 0.0361216 |
| 0.751209 | 0.0008506 | 0.000529786 |

We collected data on the sublimation of a chunk of carbon dioxide and work with the students  to model the phenomenon with a differential equation for the mass at time t, m(t), as

$$m'(t) = - k\, m(t)^r$$

where the interest is in what r might be and mean.

The discussion about what r might be is a good one, usually.

We use *Mathematica* to fit the general solution to the data, thus estimating the parameters k and r.

We then talk about why things “ain’t” perfect – although they are quite good.

# At what rate does a column of water fall with a small bore hole for water to exit at the bottom of the column?

Based on the principle of Conservation of Energy for a small particle of water, namely the sum of Kinetic Energy and Potential Energy stays constant as the Particle falls we can derive Torricelli's Law for the height, h(t), of such a column.

$$A(h(t)) * h'(t) = \alpha\, a\sqrt{2gh(t)}\,, \quad h(0) = h_0$$

Here A(h(t)) is the cross sectional area of the cylinder at height h(t), g is the acceleration due to gravity, a is the area of the small bore hole, and $\alpha$ is called the *discharge or contraction coefficient*. Empirically a is about 0.70, namely the effective Discharge rate is about 70% of what it could be maximally.

For a constant cross sectional cylinder of area A (h(t)) = $\pi$ r$^2$
Torricelli's Law (using Conservation of Energy).

$$A(h(t)) * h'(t) = \alpha\, a \sqrt{2gh(t)}\,, \quad h(0) = h_0$$

may be realized as a simple differential equation:

$$h'(t) = -k\sqrt{h(t)}\,.$$

Of course, without recourse to physics one might conjecture something simpler, such as a linear "law" which would lead to exponential decay In the height, h(t).

$$h'(t) = -k\, h(t)\,.$$

We compare the models after the parameters have been determined by minimizing the sum of square errors.

| Clock Time $t$ (min) | Relative Time $t$ (s) | Linear $h$ (cm) | Torricelli $h$ (cm) | Observed $h$ (cm) |
|---|---|---|---|---|
| 1:40.419 = 100.419 | 0 | 12.5 | 12.5 | 12.5 |
| 1:45.598 = 105.598 | 5.179 | 11.3935 | 11.1028 | 11.1 |
| 1:47.568 = 107.788 | 7.369 | 10.9256 | 10.5369 | 10.5 |
| 1:49.623 = 109.523 | 9.104 | 10.5549 | 10.0991 | 10.1 |
| 1:50.985 = 110.985 | 10.566 | 10.2426 | 9.7374 | 9.8 |
| 1:56.388 = 116,388 | 15.969 | 9.08821 | 8.45786 | 8.5 |
| 2:02.057 = 122.057 | 21.638 | 7.87702 | 7.21218 | 7.2 |
| 2:08.032 = 128.032 | 27.613 | 6.60046 | 6.00662 | 6.0 |
| 2:16.386 = 136.386 | 35.967 | 4.81562 | 4.50576 | 4.5 |
| 2:25.568 = 145.568 | 45.149 | 2.85388 | 3.10461 | 3.1 |
| 2:34.116 = 154.286 | 53.867 | 0.991273 | 2.01507 | 2.0 |

**Table 4.** Computations with the linear model and Torricelli model compared to observed data.

Data can be collected at video of cylinder of water exiting small bore on YouTube!
[www.simiode.org](http://www.simiode.org)

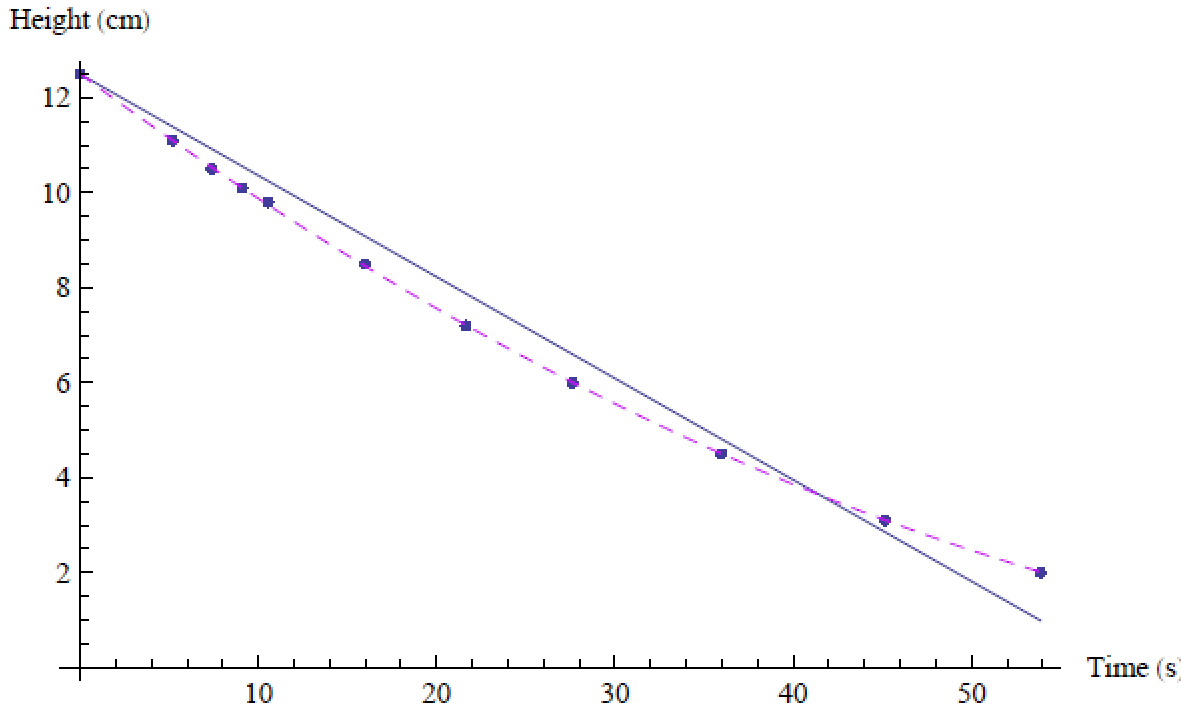



**Figure 14.** Plot of the observed data, the linear model (solid line) and the Torricelli model (dashed). We see the linear plat as very inadequate and the Torricelli plot as right on!

OR we could just play the

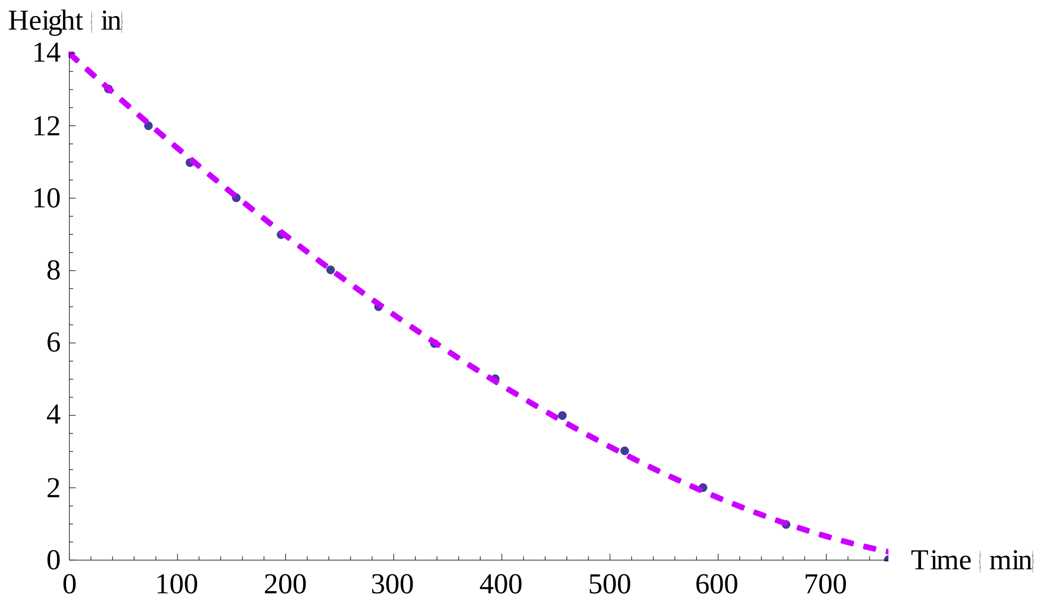
"I don't know. Who's on first. What's on second."

approach and conjecture just

$$h'(t) = -k\, h(t)^r \quad \text{with} \quad h(0) = 12.5$$

and see if we can fit for k and r. Here is what we get,

$k = 0.0101845$ and $r = 0.371881$.



We are acting upon our conviction that

**Learning-Based Inquiry** occurs when students are **Learning** mathematics (and modeling and use of technology and asking “what if” and living with data, etc.) through **Inquiry** into just what is at the heart of phenomena and then building a mathematical model to describe and better understand the phenomena.

We are doing this through

**SIMIODE** - **S**ystemic **I**nitiative for **M**odeling **I**nvestigations and **O**pportunities with **D**ifferential **E**quations

Consider joining us at www.simiode.org . Thank you.