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MS Excel[®] for higher maths: Solving initial-value second order ordinary differential equations on an Excel[®] spreadsheet

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Abstract

This paper presents a friendly method of solving initial-value second order differential equations (DEs) analytically on a MS Excel[®] workbook. Upon entering the constant coefficients and boundary conditions, the type of solution is highlighted and the user can visualize the full solution accompanied by a graphical representation, on a mouse click. Graphical artefacts and data misrepresentation have been explored and remedial procedures recommended. This tool is designed to help engage students and contribute in tackling the rather familiar “maths problem” to many lecturers today. It is also intended as a help to academics wishing to generate DEs for teaching and assessment purposes; in addition to being a handy aid to simulation and development engineers. Parameters can be varied and equations with solutions saved on a separate worksheet for future use. There is room for customising this solver to adopt user’s terminology. This present version is the first and it has been restricted to equations of the type:

$$ay'' + by' + cy = R\sin(\omega t + \varphi)$$

where a, b, c, R, ω and φ are all constants and b or R can be zero.

Introduction

Mathematics education (ME) is of extreme importance to technological development; since science and engineering - the backbone of prosperity and quality of life - rely heavily on mathematics. In the United Kingdom alone, consortia of universities have been set up for this purpose and the Institute of Mathematics and its Applications (IMA) has run regular conferences on ME since 1994 [1] - four years before the First International Conference on the Teaching of Mathematics took place in Greece [2]. The European Society for Engineering Education (SEFI) shares the same interests and concerns across that continent. SEFI and IMA jointly hosted a conference on MEE, 6 – 9 April, 2008 [3]. The Higher Education Academy’s Subject Centre on Mathematics, Statistics and Operational Research, has a full catalogue of efforts and publications addressing the same “maths problem”.

Higher mathematics underpins almost all applications and spans all science and engineering disciplines. The extent of grasp of mathematical concepts often differentiates between peer scientists and engineers. The study of differential equations (DEs) is one of the topics common to all engineering and most science syllabi. Because DEs are not easily generated and solved mentally, lecturers often resort to textbooks and laborious preparation work to develop suitable sets of DEs that illustrate particular points of interest and salient features in scientific and engineering applications. In response to this challenge, a range of software has been developed

and sold commercially. Some are rather powerful (like Matlab™) and others are limited to specific tasks. However, all are additional software packages that need to be purchased, installed and users trained to manipulate them. These requirements often deter potential users from benefiting of the features of such programs.

This paper presents an easy MS Excel® workbook that solves, analytically, initial value second order DEs with constant coefficients. This first version has been restricted to equations of the type:

$$ay'' + by' + cy = R\sin(\omega t + \phi) \tag{1}$$

where a, b, c, R, ω and ϕ are all constants and b or R can be zero.

The widespread use of MS Office on nearly every personal computer, at home or at work/study, justified the choice of Excel® as a support application. This saves the inconvenience and costs associated with buying additional programs. Although the literature states that “Excel does not have the capabilities to solve systems of ordinary differential equations” [4], this paper suggests that perhaps more exploration work is needed before such a statement can be confirmed.

The choice of equation (1) is based on its extensive use in several fields of studies including electricity, vibration, dynamics, wave propagation and acoustics. In the following sections, the method of developing this program is described along with the resources and references used and validation process. Graphical representation and potential associated errors are explored before highlighting some of the features and scope for further development. Examples and illustrative screen shots are also presented. A traditional spring-mass vibration system has been chosen as a working example.

Method of investigation

Homogeneous and non-homogenous DEs have been solved symbolically through extensive transposition and algebraic manipulation of the relevant formulae. Logical functions and formatting features are used to improve the display and presentation of solutions.

Seven possible cases were envisaged, grouped in Table 1, according to the nature of the roots of the characteristic

R>0?	roots R/C?	Form of Solution	Solution Code
= 0	R	$\alpha = \beta,$ $y = (At + B)e^{\alpha t}$	1
		$\alpha \neq \beta,$ $y = Ae^{\alpha t} + Be^{\beta t}$	2
= 0	C	$y = e^{\alpha t}(A\cos\beta t + B\sin\beta t)$	3
> 0	R	$\alpha = \beta,$ $y = (At+B)e^{\alpha t} + C\cos\omega t + D\sin\omega t$	4
		$\alpha \neq \beta,$ $y = Ae^{\alpha t} + Be^{\beta t} + C\cos\omega t + D\sin\omega t$	5
> 0	C	$\beta = \omega, \alpha = 0,$ $y = (A+Ct)\cos\beta t + B\sin\beta t$	6
		$\beta \neq \omega,$ $y = e^{\alpha t}(A\cos\beta t + B\sin\beta t) + C\cos\omega t + D\sin\omega t$	7

R: real numbers, C: complex numbers, ω : the frequency in equation (1)

Table 1 – Summary of the seven possible cases of equation (2)

quadratic equation (2) and the values of R and ω in equation (1). Initial values and manipulation of the solution to the relevant particular integral lead to determination of the full solution. Several examples from textbooks [5-13] had been used to validate this program and cross-examine the solutions.

$$ar^2 + br + c = 0 \tag{2}$$

The roots of equation (2) are denoted by α and β when real, and $\alpha \pm i\beta$ when complex.

The seven cases are coded 1 – 7 as shown in Table 1. Each solution is referred to by its code. These cases are, however, grouped under 5 variations of equation (1) in the main user interface page as shown in Fig 1.

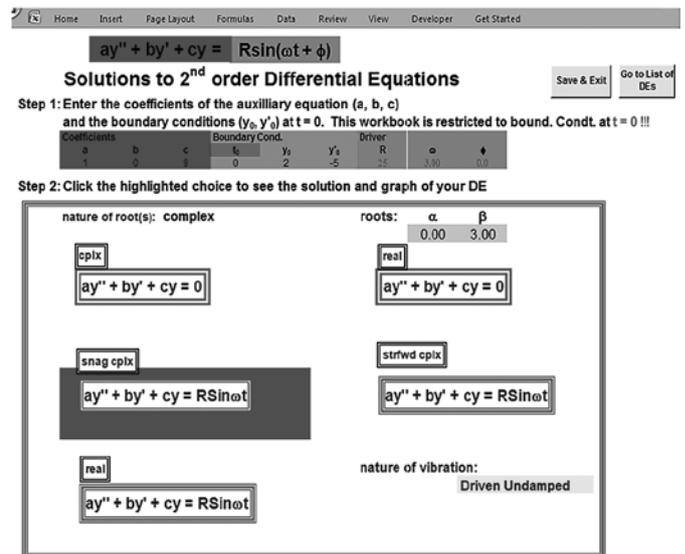


Fig 1 – A screen shot of the main interface page (switchboard) showing the 5 variations of equation (1) and the solution highlighted

Upon entering the input parameters (Fig 1), a set of macros and other Excel® routines carry out algorithms and identify the right solution code which then gets highlighted. On clicking the highlighted link, a new window is opened stating the detailed solution, giving values of all coefficients to two decimal places and displaying a 1000 data point graph of the solution over an adjustable range. Embedded buttons allows navigation between pages of the solver.

Graphical representation

Each graph represents 1000 data points within the limits

stated by user. The lower limit is set to zero by default. The upper limit, however, should be selected with care. Depending on the type of solution, this limit can influence the apparent shape of the curve.

With the exception of codes 1 and 2, all solutions exhibit some forms of periodic functions. For these, the upper range is selected as a *multiple of the primary period*. This multiple is the product of the primary period by any positive real number (the range factor). Range factor and period are highlighted with double circles in Fig.2. The primary period is taken as:

$$T = \frac{2\pi}{\omega} \tag{3}$$

when $\beta \neq \omega$, otherwise, it is:

$$T = \frac{2\pi}{\beta} \tag{4}$$

The increment in time (along the horizontal axis) is given by the formula:

$$step = \frac{(Upper\ value - Lower\ value)}{1000} \tag{5}$$

In Fig 2, the data points are plotted in scatter (no solid line) to show the drop of density of data points per cycle as the upper limit increases, as expected from this particular function.

The range factor (RF)

RF indicates how many primary periods over the 1000 data points are plotted. RF must not be equal to 1000. If it is, each cycle will be represented by a single point, thereby reducing the real graph to one side of its envelope. This is a side effect of numerical solutions where finite elements replace continuous intervals - The Sampling Theorem. An assortment of associated potential artifacts is presented in the next section.

The RF is related to the *Sampling Rate*, number of data points per cycle, by the simple formula:

$$Sampling\ Rate = \frac{1000}{RF} \tag{6}$$

Graphical artefacts/aliasing

Fig 2 shows the correct shape of the function displayed $y = (2 - 6.25t)Cos3t + 0.42Sin3t$ (7)

This function represents a driven undamped vibration, an out of control system, a system in resonance. Its main feature is a steady growth of amplitude with time. However, uncalculated modulation of the RF can lead to variation of the range that in turn can entail erroneous graphs. Fig.2 shows 10 cycles as expected from the RF 10. It is noteworthy that spacing between consecutive data points is expected to increase as the amplitudes increases without altering the sampling rate (equation (6)). This is manifested on the graph by an apparent drop in “density” of points towards the upper limit of the range as shown on Fig 2.

When the RF is set to 1000, the graph becomes that shown on Fig 3: only one “edge” of the curve’s envelope is picked. That is the data points at the amplitudes - a sampling rate of 1.

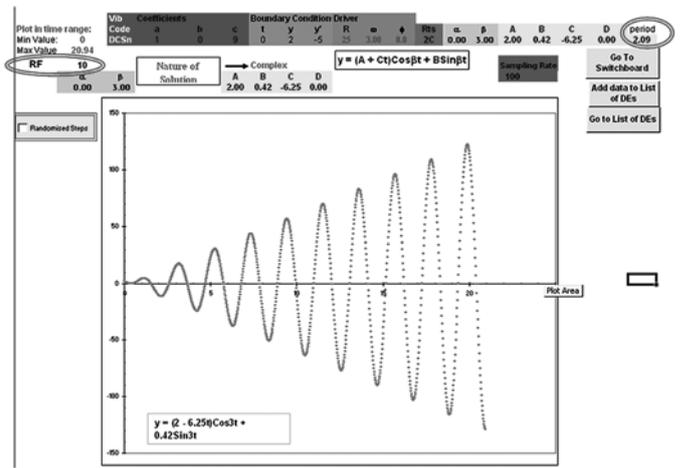


Fig 2 – A screen shot displaying the solution of a code 6 (Table 1) 2nd Order DE with a graphical representation over 10 primary periods. Notice that the spacing between adjacent data points increases rapidly while the sampling rate remains constant.

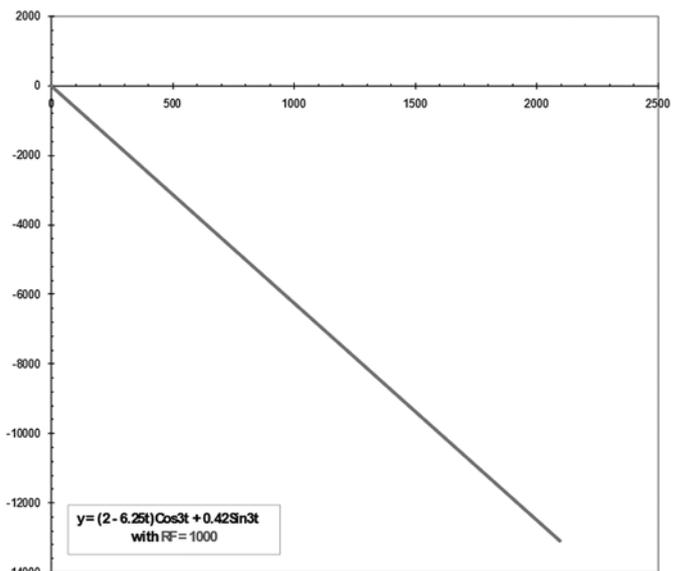


Fig 3 – Function on Fig 2 displayed over 1000 cycles. The periodicity is masked by one data point per cycle. The curve is reduced to the bottom outline of its envelope.

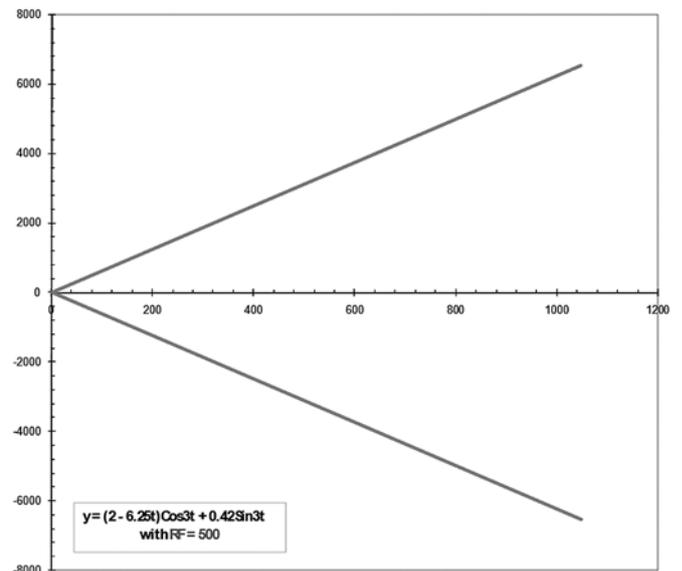


Fig 4 – Function on Fig 2 displayed over 500 cycles. Again, the periodicity is masked by the sampling rate. The envelope, however, is revealed when data is sampled at 2 points per cycle.

At RF = 500, when both amplitudes are hit, the graph turns into two symmetrical straight lines with the same absolute value gradient of 6.25, as shown in Fig 4. Now both edges of the envelope are revealed at a sampling rate of 2.

The symmetry about RF = 500 suggests that qualitatively, curves at [RF = n] and [RF = (1000 - n)] are similar in shape with difference in amplitudes and a phase of either 0 or 180°- this is known in digital signal processing (DSP) as *aliasing* [14].

Without drifting into DSP or even Fourier transforms, it is easy to visualize the correct shape of the graphs by selecting a low RF (say 5), then 995 to watch out for aliasing (180° phase change). Close to RF = 500, 250 or 750, serious aliasing can occur as illustrated in Fig 5 and Fig 6.

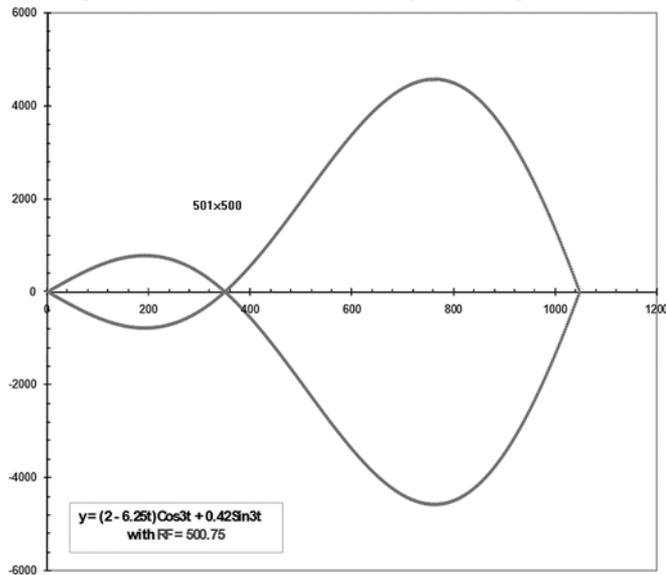


Fig 5 – Function on Fig 2 displayed over 500.75 cycles. Departure from the correct shape is visible. Slight changes engender serious aliasing.

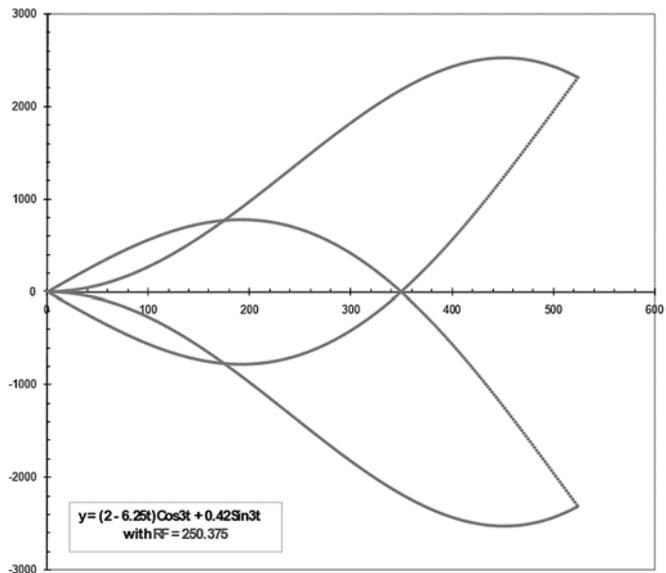


Fig 6 – When the RF of Fig 5 is halved, more “wild” shapes appear.

A RF = 500.75 has been selected to generate an initial “wild” shape and then progressively half this RF to explore restoring of the correct shape. The idea is that as RF decreases, sampling rate increases and, eventually, a

suitable sampling rate will be reached to restore the shape of the graph.

Fig 6 clearly shows that a sampling rate of about 4 is not sufficient to restore the correct shape of the graph. Fig 7, however, displays the correct shape at lower amplitudes but proves that more than 16 data points per cycle are needed to avoid aliasing at higher amplitudes. The correct shape was restored at a sampling rate close to 64 as shown in Fig 8.

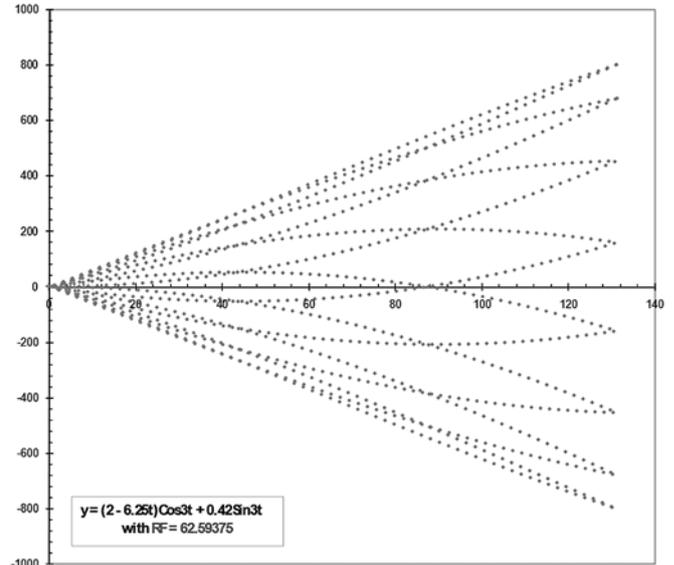


Fig 7 – At a sampling rate of about 16, only low amplitude cycles are correctly displayed.

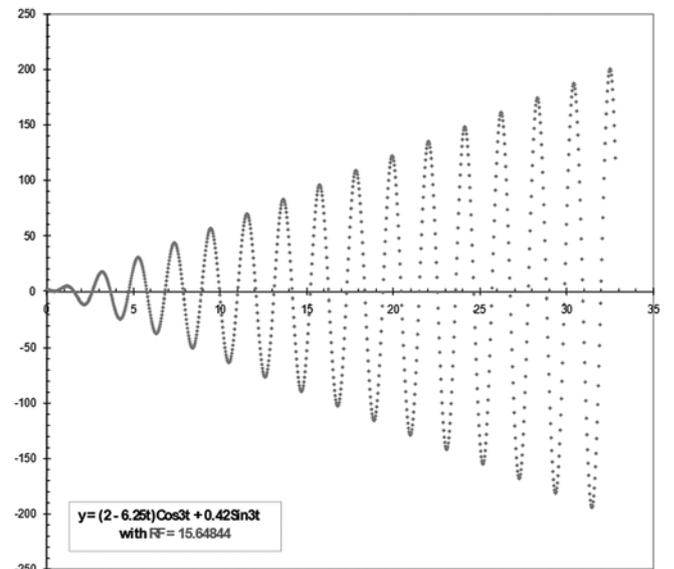


Fig 8 – A sampling rate approaching 64 is necessary to display the full range correctly.

A much easier way to guarantee the correctness of the shape of graphs is to tick the *Randomised Step* button (Fig 2). This introduces a random factor in the step formula (5) without compromising the range of interest. Each term T_n of the 1000 terms is now calculated as:

$$T_n = Rand() \times [step(n + 1) - step(n - 1)] + step(n - 1) \text{ with } 0 \leq n \leq 1000 \quad (8)$$

This random factor is independent for each step giving rise to up to up to 1000 different random factors in the steps.

Replacing equation (5) by (8) turned Fig 6 into Fig 9. The envelope of the curve has been restored and the correct shape recovered.

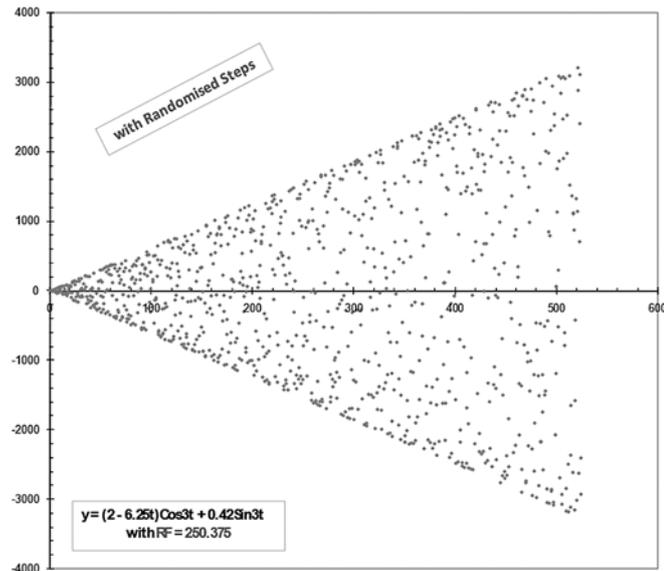


Fig 9 – Graph of Fig 6 with a randomised step increment. The genuine envelope is restored.

A good case study for students is to explore the shapes of the graphs when RF is greater than 1000.

Key features of this solver

Key features and benefits of using the solver are:

1. Helps students appreciate the importance of understanding what they do and the danger of blindly relying on machines.
2. Easy in-situ visualization of response to parameters variation. This is particularly useful when exploring responses close to critical values (e.g. natural frequency).
3. Desirable DEs can be saved on a separate sheet with full solution and possibility to attach comments to appropriate parameters.
4. The benefits of familiar Excel® environment.
5. Flexible and open for user-defined contribution.

Conclusions and further development

This work has demonstrated that MS Excel® can be useful in widening participation into mathematics education. Tackling DEs work in the familiar environment of Excel® means that lecturers and learners can engage in a more interactive learning experience, and can both have their inputs and lasting contribution. The room for manoeuvre within an Excel® spreadsheet provides opportunities for innovation and improvement of this solver. This is seen as an attraction to young learners eager to make their marks. Advanced learners, simulation and modelling engineers can find in this solver a handy aid for quick tasks as well as an opportunity to expand and develop their own skills.

It is planned to extend this solver to other DE cases, where the right hand side of equation (1) is a polynomial, an

exponential or other standard functions. The following challenge will be to write compact algorithms to solve simultaneous linear DEs. The author welcomes collaboration with interested colleagues in order to both improve and speed up the delivery of the next versions.

Despite the full range of commercial software addressing the same task (and more) we believe that this solver is not an exercise in “re-inventing the wheel”. The solver is rather seen as an inclusion tool to further the understanding of DEs, develop innovation, planning and optimizing skills of students; and involve them in creation of the solution. Students can excel and expand their working knowledge when encouraged to switch from passive operators to active designers, from mere consumers to confident producers.

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