Modeling and Applications

MathFest Summer 2013, 0830-0840,
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Modeling Opportunities with Differential Equations in the Classroom

PROF Emeritus, Mathematical Sciences,
United States Military Academy
West Point NY 10996 USA
BrianWinkel@hvc.rr.com

Director SIMIODE
www.simiode.org

We present an approach we have used for years in our own teaching and are developing as a web-based NSF developed HUB community of teachers and learners called

SIMIODE

Systemic **I**nitiative for **M**odeling **I**nvestigations and **O**pportunities with **D**ifferential **E**quations

www.simiode.org

We present and discuss examples of activities to enable students to develop mathematical modeling skills using differential equations while using technology throughout their learning and doing mathematics.

Sources for modeling opportunities with data are all about us.

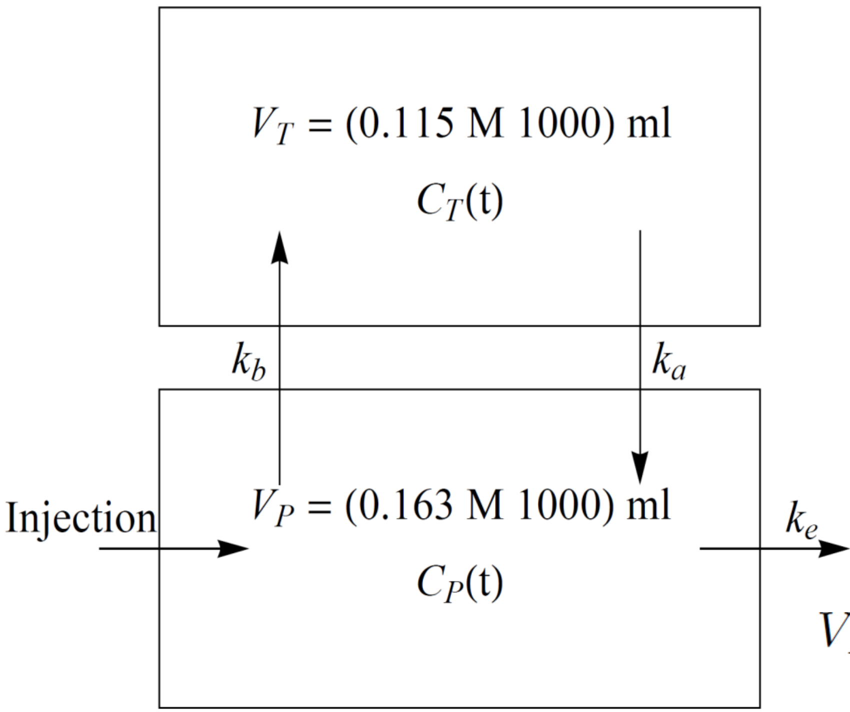
We need to look around

(1) do the Internet thing,
(2) browse library journals – current and the stacks in fields not known to you,
(3) talk to colleagues OUTside mathematics,
(4) do your own, e.g., spread of slime or cookie dough!

Take risks, learn yourself, jump into the modeling process with your students, and have fun learning with them!

Reference: Winkel, B. J. 2013. Browsing Your Way to Better Teaching. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*. 23(3): 274-290. Original references in this paper.

Modeling Lysergic acid diethylamide (LSD) in the human body



$C_P(0) = 12.2699$, for originally LSD was injected at a concentration of 2,000 ng per kg of body mass for each subject and so we have an initial concentration in the plasma of12.2699 = 2000M/.163M1000 ng of LSD/kg of body mass. $C_T(0) = 0$.

$$V_P = \underbrace{(0.163M)}_{\text{kg of plasma}} \cdot \underbrace{(1)}_{\text{liter/kg}} \cdot \underbrace{(1000)}_{\text{ml/liter}} \cdot = \underbrace{(163M)}_{\text{ml of plasma}}$$

$$\begin{aligned} V_P C_P'(t) &= k_a V_T C_T(t) - k_b V_P C_P(t) - k_e V_P C_P(t) \\ V_T C_T'(t) &= k_b V_P C_P(t) - k_a V_T C_T(t)\,. \end{aligned}$$

Reference: Winkel, B. J. 2013. Browsing Your Way to Better Teaching. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*. 23(3): 274-290. Original references in this paper.

	Time (hr)	0.833	0.25	0.5	1.0	2.0	4.0	8.0
Subject 1	Plasma Conc (ng/ml)	11.1	7.4	6.3	6.9	5.	3.1	0.8
	Perform Score (%)	73	60	35	50	48	73	97
Subject 2	Plasma Conc (ng/ml)	10.6	7.6	7.	4.8	2.8	2.5	2.
	Perform Score (%)	72	55	74	81	79	89	106
Subject 3	Plasma Conc (ng/ml)	8.7	6.7	5.9	4.3	4.4	—-	0.3
	Perform Score (%)	60	23	6	0	27	69	81
Subject 4	Plasma Conc (ng/ml)	10.9	8.2	7.9	6.6	5.3	3.8	1.2
	Perform Score (%)	60	20	3	5	3	20	62
Subject 5	Plasma Conc (ng/ml)	6.4	6.3	5.1	4.3	3.4	1.9	0.7
	Perform Score (%)	78	65	27	30	35	43	51

Table 2. Summary of data collected [1, 14] on 5 male volunteers who were administered LSD and then tested on performance (Perform Score (%)) on simple addition questions. Both performance Score and Plasma Concentrations of LSD were recorded at 5, 15, 30, 60, 120, 40, and 480 minutes after the initial infusion of LSD.

This is the formula for $C_T(t)$, the concentration of LSD in the tissues as a function of time t.

$$\left(-17.3913\, e^{0.5\left(-1.\,ka-1.\,kb-1.\,ke-1.\sqrt{-4.\,kb\,ke+(1.\,ka+1.\,kb+1.\,ke)^2}\right)t}\,kb+17.3913\, e^{0.5\left(-1.\,ka-1.\,kb-1.\,ke+\sqrt{-4.\,kb\,ke+(1.\,ka+1.\,kb+1.\,ke)^2}\right)t}\,kb\right)\Big/\left(\sqrt{-4.\,kb\,ke+(1.\,ka+1.\,kb+1.\,ke)^2}\right)$$

$$SSE(k_a, k_b, k_e) = \sum_{i=1}^{l}(C_P(t_i) - O_i)^2 \qquad (3)$$

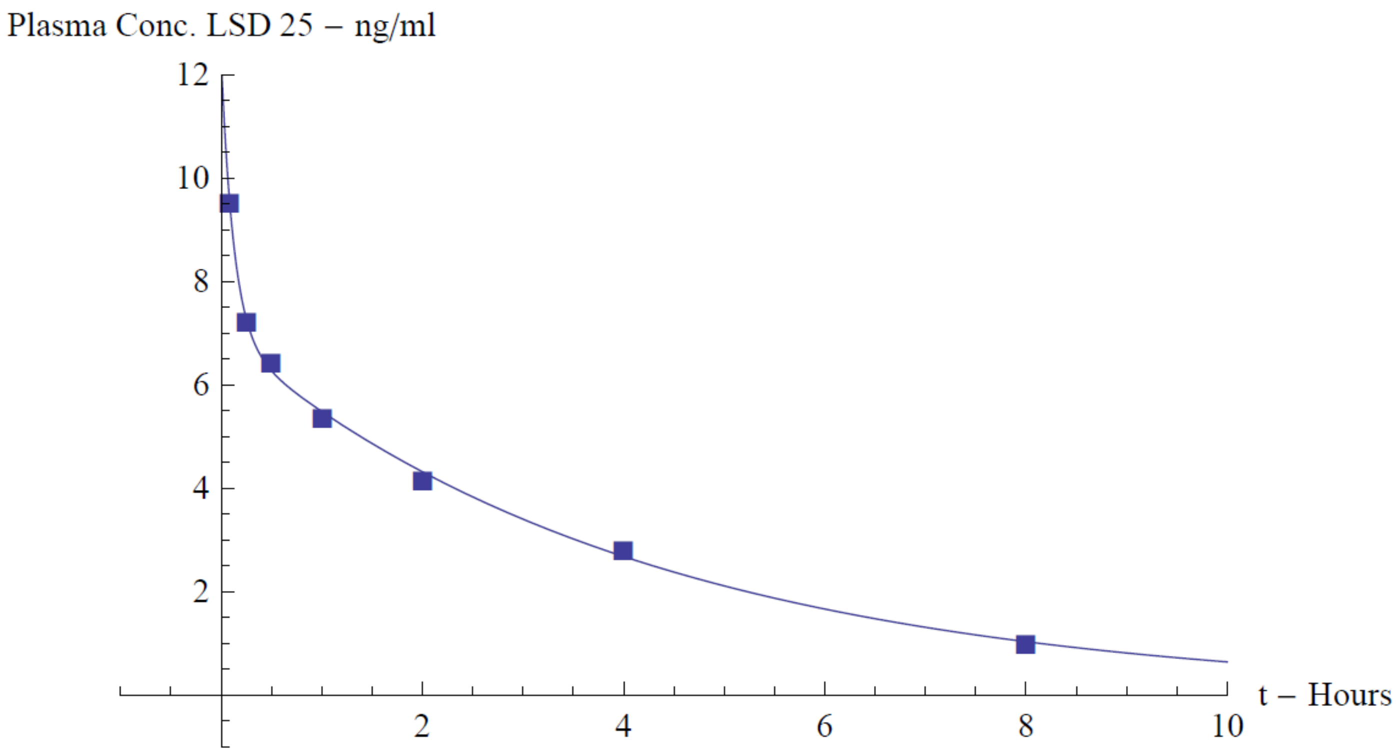


Figure 2. Plot of the observed values of the average concentration of LSD (ng/ml) (squares) and the model built from parameter (k_a, k_b, and k_e) estimates using the solution of the system of differential equations (2) and minimization of the sum of square error function (3).

$$C_T(t) = 0.128905\left(55.419e^{-0.238492t} - 55.419e^{-7.99617t}\right) \,. \tag{5}$$

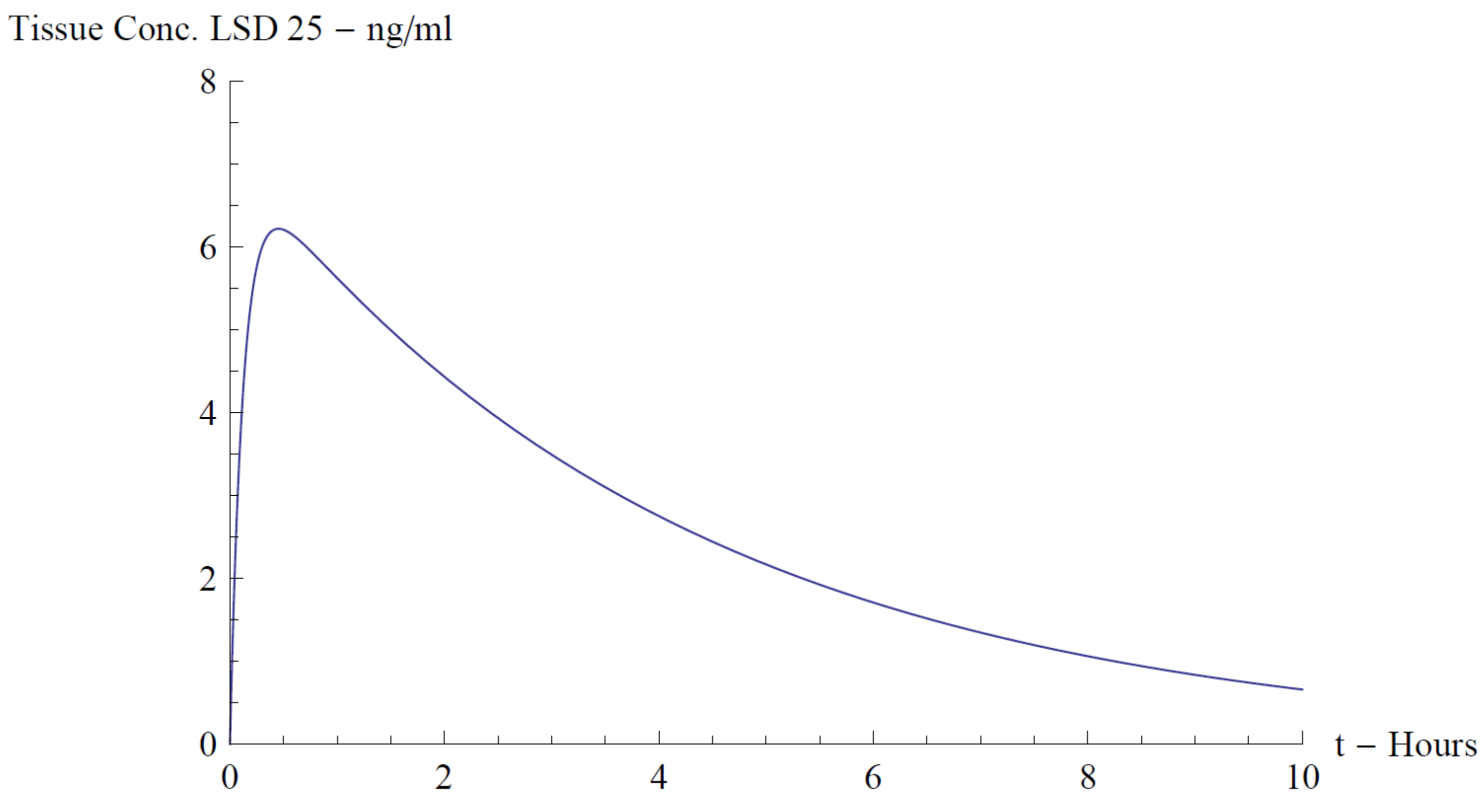


Figure 3. Plot of the model of tissue concentration of LSD in ng/ml. This model is built from parameter (k_a, k_b, and k_e) estimates using the solution of the system of differential equations (2) and minimization of the sum of square error function (3).

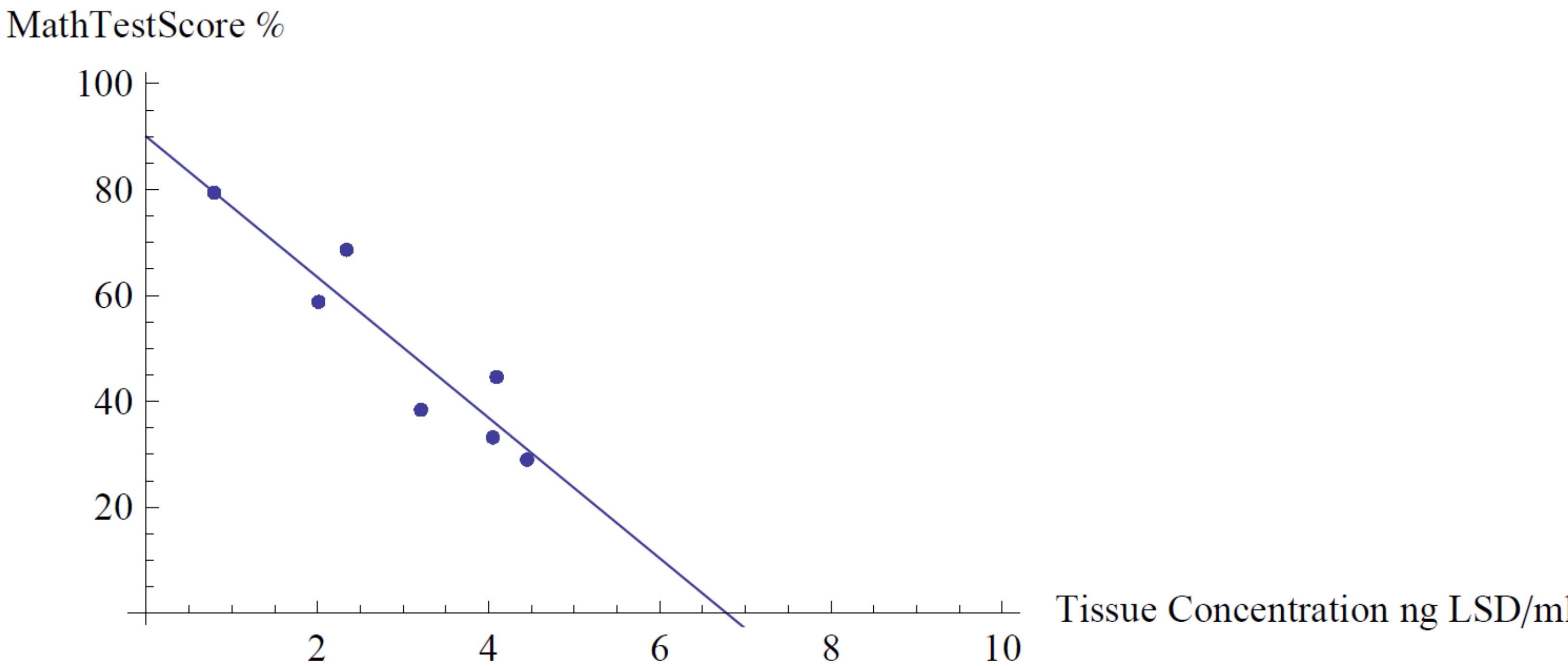


Figure 4. Plot of Performance Score (%) (PS) on the simple arithmetic problems vs. the model prediction of the concentration of LSD in ng/ml in the tissue compartment (CT). Superimposed is the best line of regression whose equation is $PS = 89.1729 - 9.32941CT$. This means for every ng/ml increase in LSD in the tissue compartment the score drops a little over 9 points.

Tuned Mass Dampers

A Tuned Mass Damper (TMD) is a passive mechanical counterweight for a structure consisting of a moving mass (roughly 1–2% of the structure's mass) which is usually placed in the upper portion of the structure. The purpose of the TMD is to reduce the effects of motion caused by wind or earthquake.

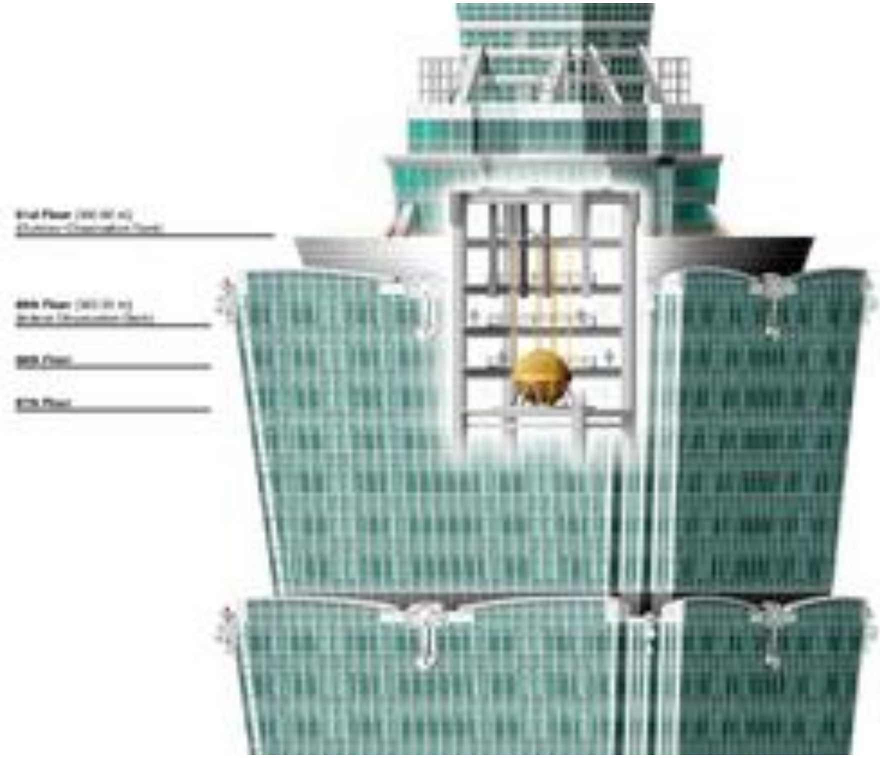
The first uses of TMDs in the United States for large structures was in the John Hancock Building in Boston in 1977 and City Corp Center in New York in 1978.

TMD's are used in many, many structures and devices, including buildings and bridges, electric razors, rotating tools, surgery table platforms, etc.

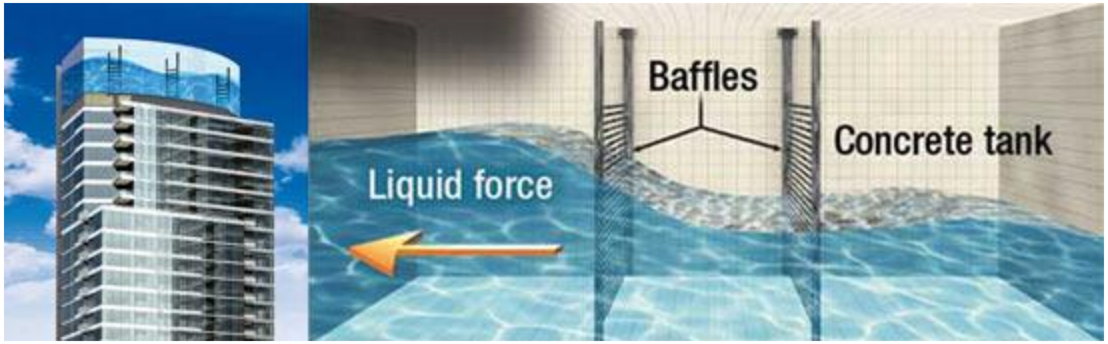
So what is a TMD? Well it is just a system of two second order differential equations And a physical device to implement the results.

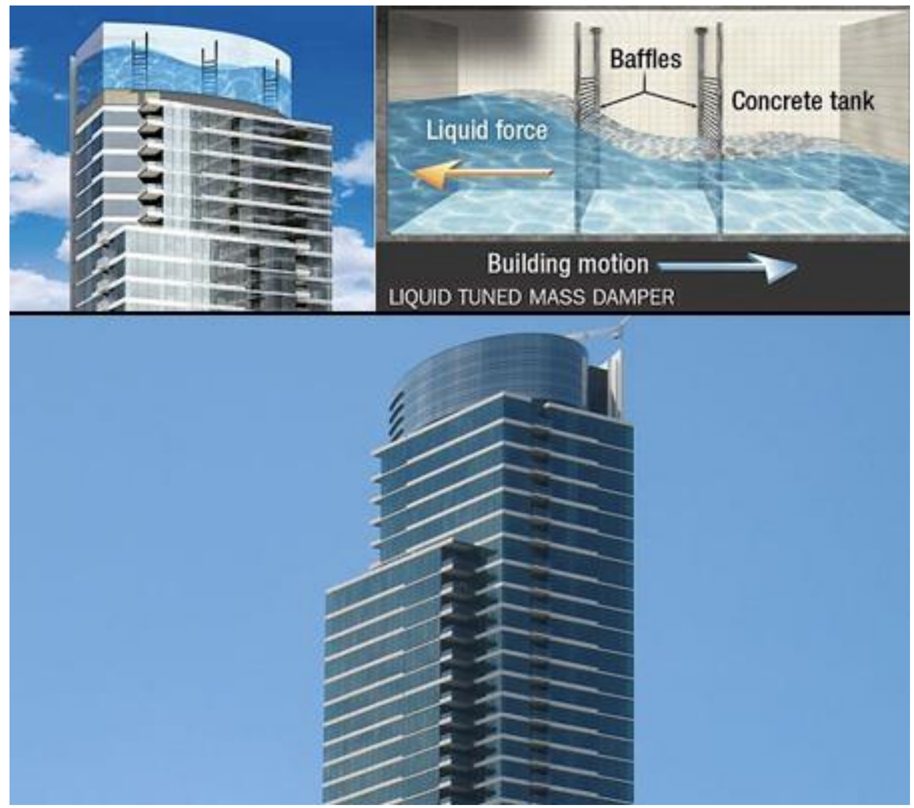
TMD's can be in the form of sliding slabs (HUGE) of concrete and steel, sloshing tanks of water, pendula, and more.

Winkel, B. J. and K. Landry. 2008. Peak Frequency Responses and Tuned Mass Dampers--Exciting Applications of Systems of Differential Equations. *The UMAP Journal*. 29(1): 15-30.




TAIPEI 101
5.5 m in diameter
660 metric tonnes
The largest tuned mass damper in the world

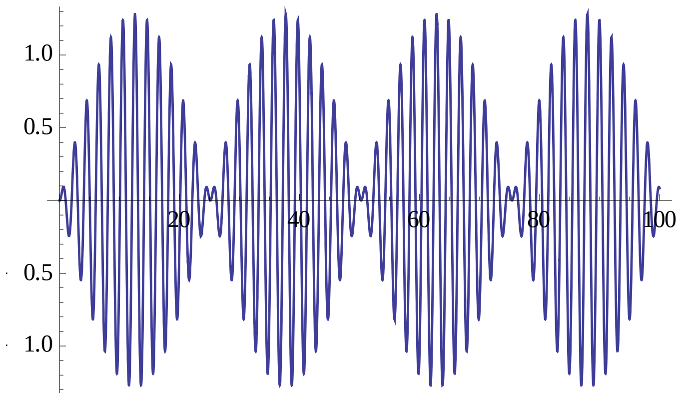

Baffles
Concrete tank
Liquid force


Baffles
Concrete tank
Liquid force
Building motion
LIQUID TUNED MASS DAMPER

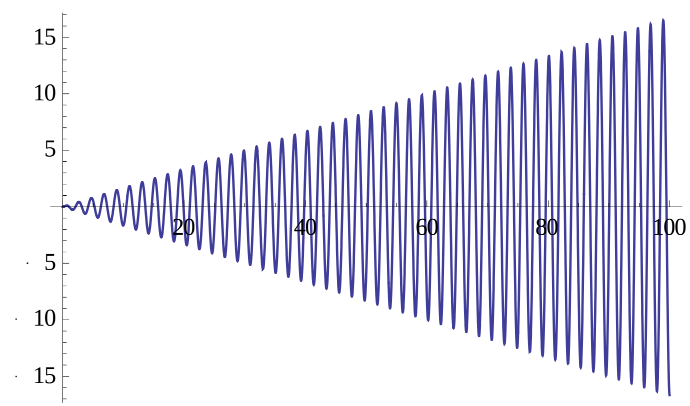
Harmonic Oscillator, e.g., building or bridge OR oboe reed.

$1\,y''(t) + \omega_0{}^2 y(t) = \cos(\omega\, t), \quad y(0) = y_0, \;\; y'(0) = v_0$

When ω_0 and ω are close we get beats,
e.g. $\omega_0 = 3.0$ and $\omega = 3.25$



When ω_0 and ω are equal we get resonance,
e.g. $\omega_0 = 3.0$ and $\omega = 3.0$



Thus if in 1 $y''(t) + \ \omega_0{}^2 y(t) = \cos(\omega\, t), \quad y(0) = y_0, \ \ y'(0) = \ v_0$
We have $\omega_0 = \omega$, i. e. the driver frequency is the same as the natural frequency of the system/structure, then we can have resonance and the system./structure can have giant and dangerous oscillations. For a building this can come from wind or earthquakes. What can we do to mitigate this danger?

Differential equations come to the rescue!

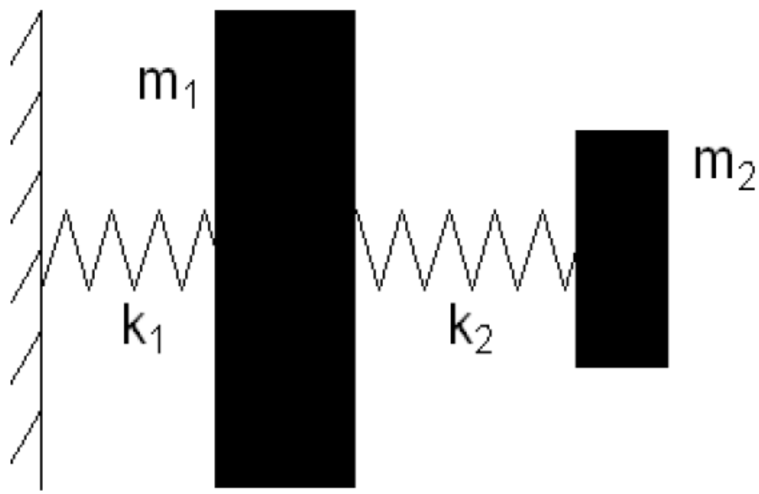
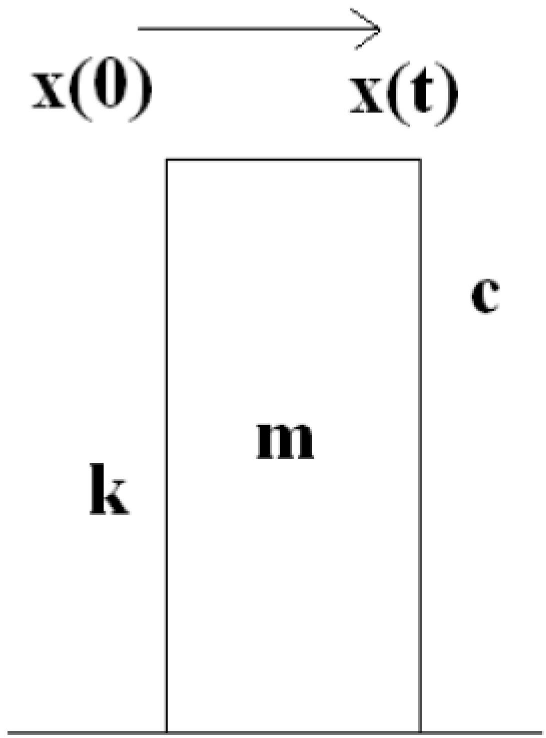
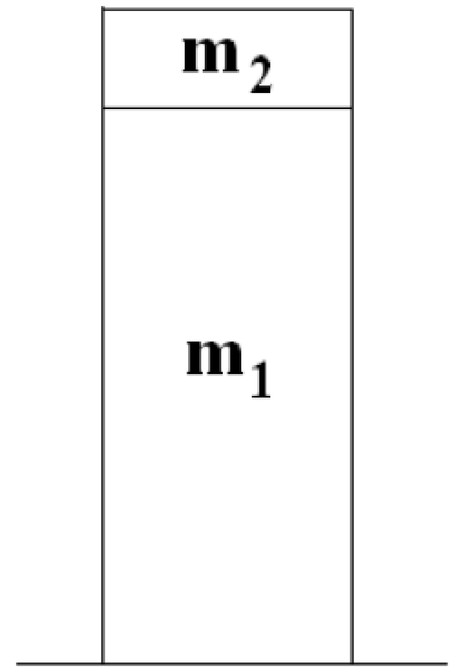


Figure 6: Horizontal depiction of two mass spring system (no damping on either mass) with smaller mass m_2 serving as Tuned Mass Damper.





Depiction of two masses for a structure – smaller mass on top.
K (spring constant) is called stiffness and c is still called damping coefficient.

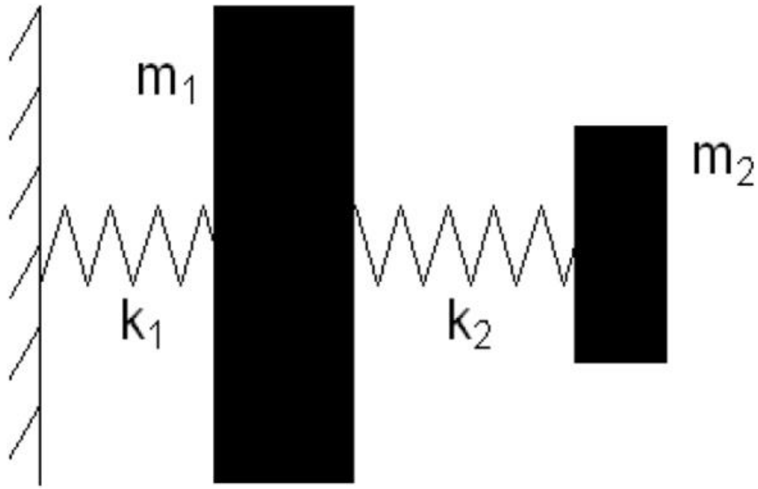


Figure 6: Horizontal depiction of two mass spring system (no damping on either mass) with smaller mass m_2 serving as Tuned Mass Damper.

$$\begin{aligned} m_1 x_1''(t) + (k_1 + k_2)x_1 - k_2 x_2 &= F(t) \\ m_2 x_2''(t) + -k_2 x_1 + k_2 x_2 &= 0\,. \end{aligned}$$

In the simplified case where there is no damping (i.e. c = 0) then we might seek to control for resonance by adding a second "tuned" mass.

We now "drive" the larger mass with a force. . wind, earthquake

$$m_1 x_1''(t) + k_1 x_1(t) + k_2 x_1(t) - k_2 x_2(t) = F_0 \cos(\omega t)$$

$$m_2 x_2''(t) - k_2 x_1(t) + k_2 x_2(t) = 0\,.$$

If no second mass then we could have resonance (c=0) or high amplitude.

So we add a second smaller mass and we "tune" it so that its system has the same frequency as that of the larger mass.

The natural frequency of our structure is $\omega = \sqrt{\frac{k_1}{m_1}}$ and of the added mass the frequency is $\sqrt{\frac{k_2}{m_2}}$. Consider k_1= 90 and $m_1 = 10$ with k_2= 0.90 and $m_1 = 0.10$.

Then both spring systems have the same frequency, namely $\omega = \sqrt{\frac{90}{10}}$ = $\sqrt{\frac{0.9}{0.1}}$ = 3.

Now, by examining the solution of our system

$$\text{mass } m_1: \quad -\cos(t\omega) + \frac{909}{10}(a\cos(t\omega) + b\sin(t\omega)) - \frac{9}{10}(c\cos(t\omega) + d\sin(t\omega)) + 10\left(-a\cos(t\omega)\omega^2 - b\sin(t\omega)\omega^2\right) = 0$$

$$\text{mass } m_2: \quad -\frac{9}{10}(a\cos(t\omega) + b\sin(t\omega)) + \frac{9}{10}(c\cos(t\omega) + d\sin(t\omega)) + \frac{1}{10}\left(-c\cos(t\omega)\omega^2 - d\sin(t\omega)\omega^2\right) = 0\,.$$

we can combine terms, simplify, and determine that the amplitude of the resulting oscillation of mass   is

$$\text{amp}(\omega) = 10\left|\frac{\omega^2 - 9}{100\omega^4 - 1809\omega^2 + 8100}\right|$$

A picture is worth . . . . . How many words . . . . or equations?

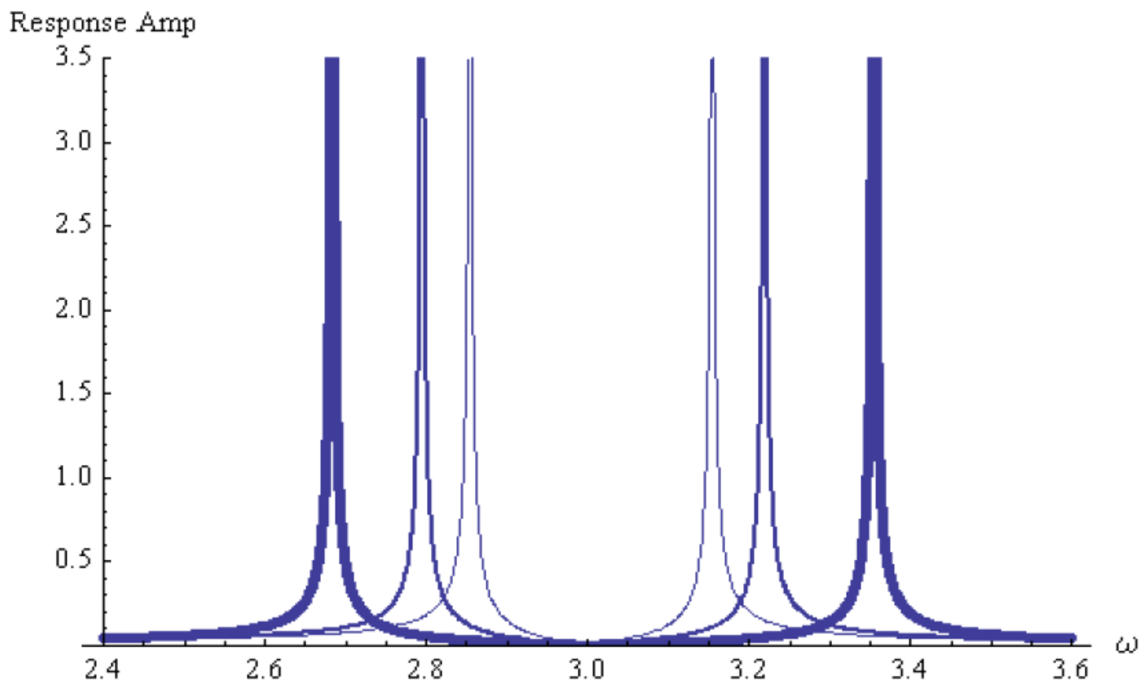


Figure 9: In case neither mass m_1 nor m_2 has damping, i.e. $c_1 = c_2 = 0$, here are plots of the response amplitudes of the primary system mass, m_1, as a function of the driver frequency ω. We see in the three plots that as the ratio $\frac{m_2}{m_1}$ of added secondary mass m_2 as a percentage of the primary mass, m_1, goes from 1%, to 2%, to 5%, (corresponding to the thin, thick, and thicker plots, respectively) the frequency region of low responses to the driver frequency surrounding the natural frequency of the structure expands.

So if we are willing to add more and more mass we get a wider “safe” region.

Incidentally, if we go back to just our one mass system

$$m \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = f(t), \quad y(0) = y_0, \quad y'(0) = v_0$$

with a driving force $$f(t) = F_0 \cos(\omega t)$$

Here is the steady state (non-homogeneous) solution for Equation (4) with $c \neq 0$:

$$\mathrm{ss}(t) = -\frac{F_0 m \cos(\omega t)\omega^2 + F_0 k \cos(\omega t)}{m^2\omega^4 + c^2\omega^2 - 2km\omega^2 + k^2} + \frac{cF_0 \sin(\omega t)\omega}{m^2\omega^4 + c^2\omega^2 - 2km\omega^2 + k^2}.$$

The amplitude of the steady state solution as a function of input frequency, ω, is given in Equation (5).

$$\mathrm{amp}(\omega) = F_0\sqrt{\frac{1}{m^2\omega^4 + c^2\omega^2 - 2km\omega^2 + k^2}} = F_0\sqrt{\frac{1}{c^2\omega^2 + (m\omega^2 - k)^2}}. \tag{5}$$

How big a response (i.e. how big can amp(ω) really be?

We seek the maximum frequency response of amp(ω). Again, a picture . . . .

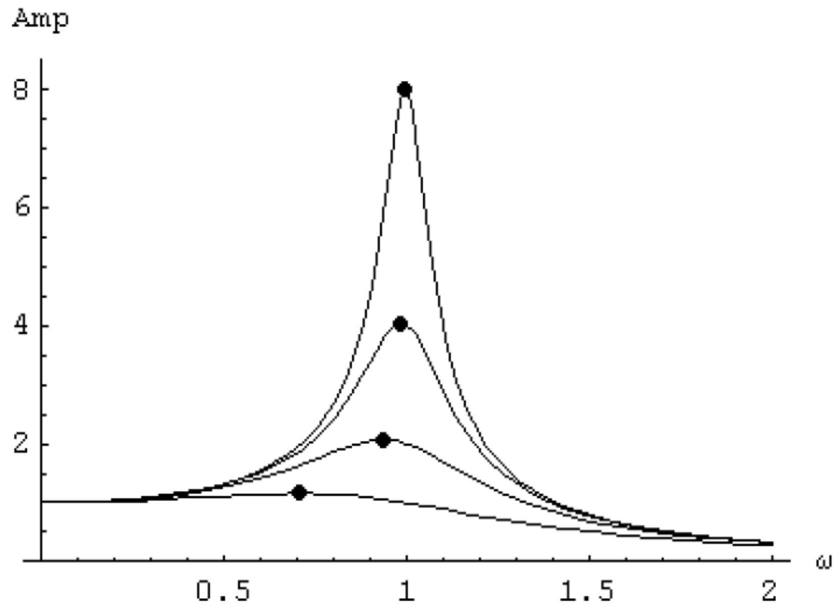
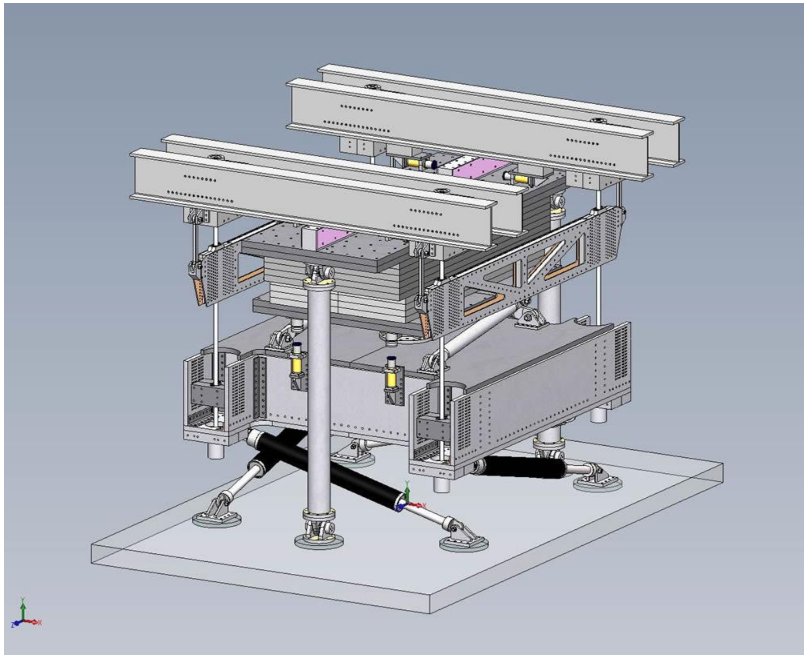
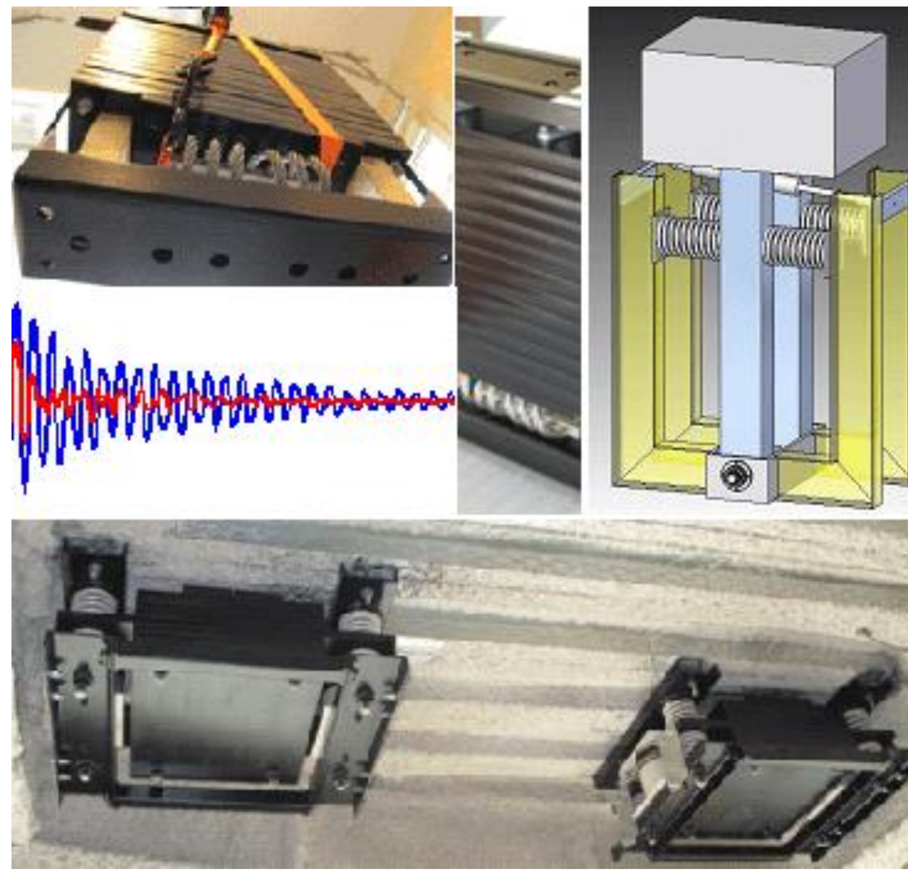


Figure 3: Plot of peak frequency response, both input frequency, ω, and amplitude, of steady state solution for $m = 1$, $k = 1$, $\omega_0 = 1$, $F_0 = 1$, for $c = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ where peak is highest for lowest value of c, i.e. $\omega_{\max} \to \omega_0$ as c decreases. These values appear in Table 1.

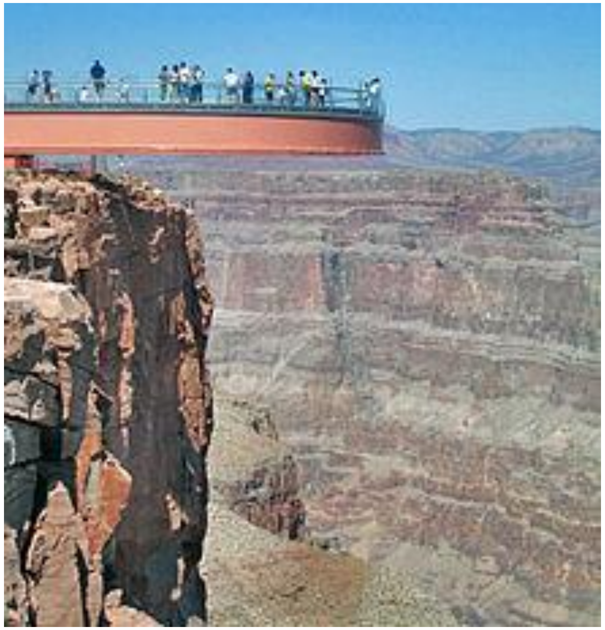
c value	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
Peak response frequency, $\omega_{\max}$	0.707107	0.935414	0.984251	0.996086
Maximum response amplitude	1.1547	2.06559	4.03162	8.01567

Table 1. Peak frequency response for Equation (4) with $m = 1$, $k = 1$, $F_0 = 1$, for $c = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$.





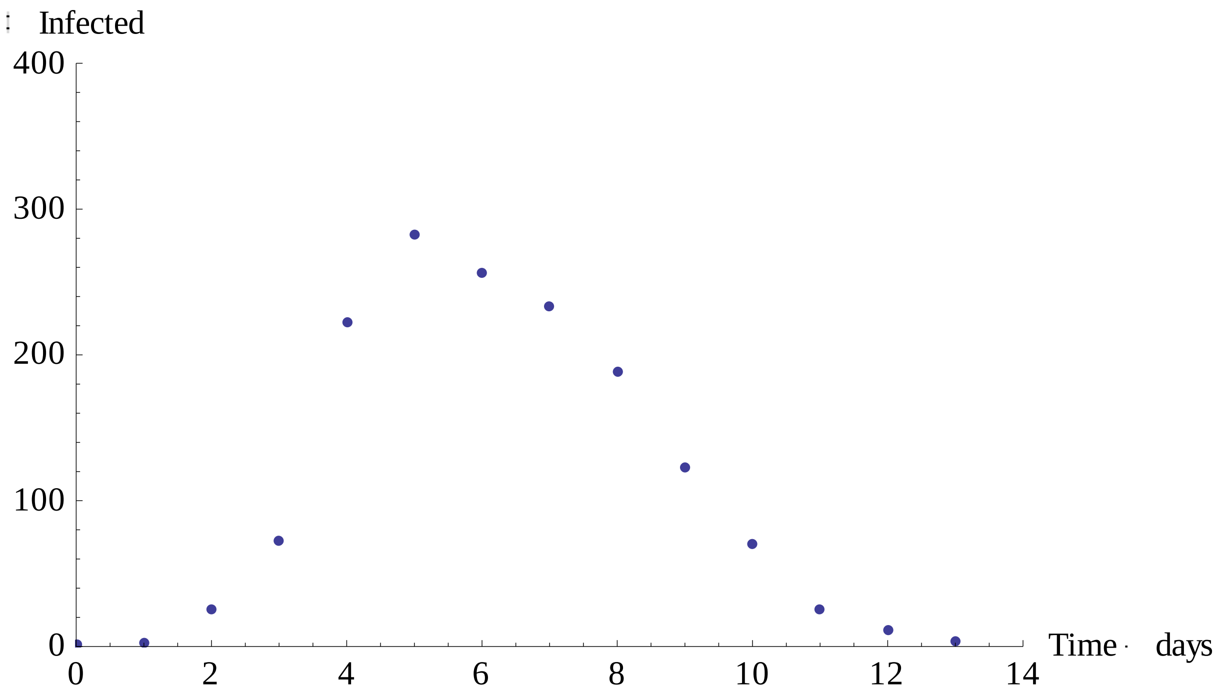

CSD




THE HANCOCK AT 30 BEHIND THE LOOKING GLASS
The Boston Globe boston.com
The tuned mass dampers

Time-days	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Infectives	1	3	25	72	222	282	256	233	189	123	70	25	11	4

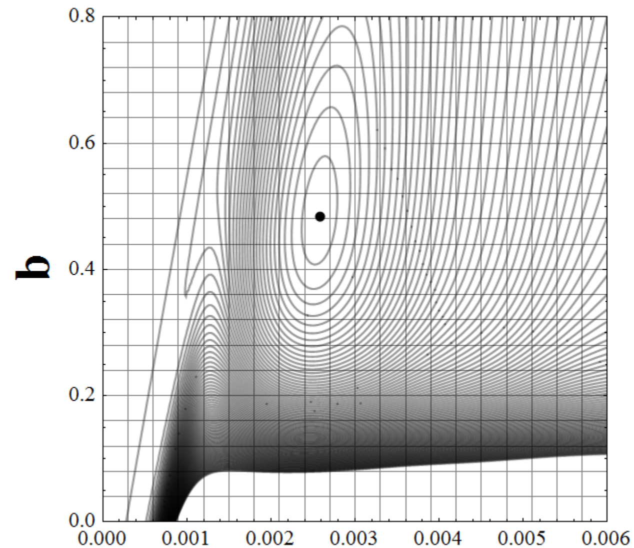
Table 1. Total Number of Bedridden Boys on Day t.



Source: Communicable Disease Surveillance Center. 1978. News and Notes: Influenza in a Boarding School. *British Medical Journal*. 1(6112). http://www.pubmedcentral.nih.gov /picrender.fcgi?artid=1603269&blobtype.pdf. Accessed 5 September 2008.

Continuous Model

$$S'(t) = -aS(t)I(t)$$
$$R'(t) = bI(t)$$
$$I'(t) = aS(t)I(t) - bI(t)$$

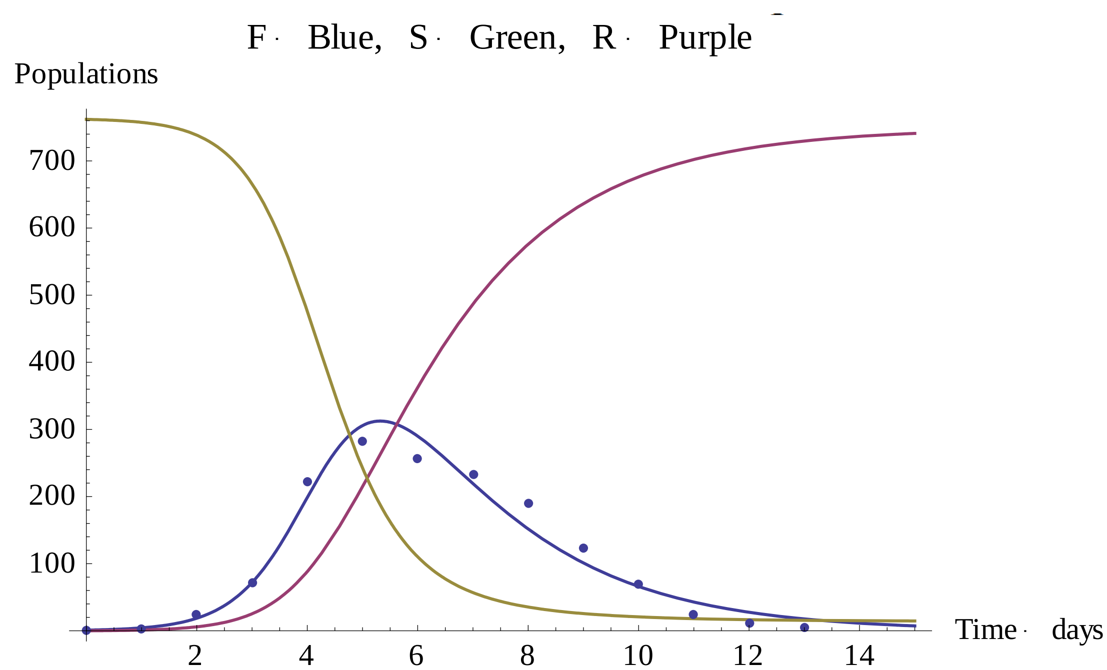


Using grid in parameter space to minimize

n

$$\text{SSE} = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

with S(0) = 763-1, I(0) = 1, and R(0) = 0 we obtain a = 0.00218 and b = 0.441.



Discrete Model

$$
\begin{aligned}
S(n+1) &= S(n) - aS(n)I(n) \\
R(n+1) &= I(n) + bI(n) \\
I(n+1) &= aI(n) + S(n)I(n) - bI(n)
\end{aligned}
$$

Using EXCEL's Solver to minimize

$$\mathrm{SSE} = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

with S(0) = 763-1, I(0) = 1, and R(0) = 0 we obtain a = 0.00291 and b = 0.72.

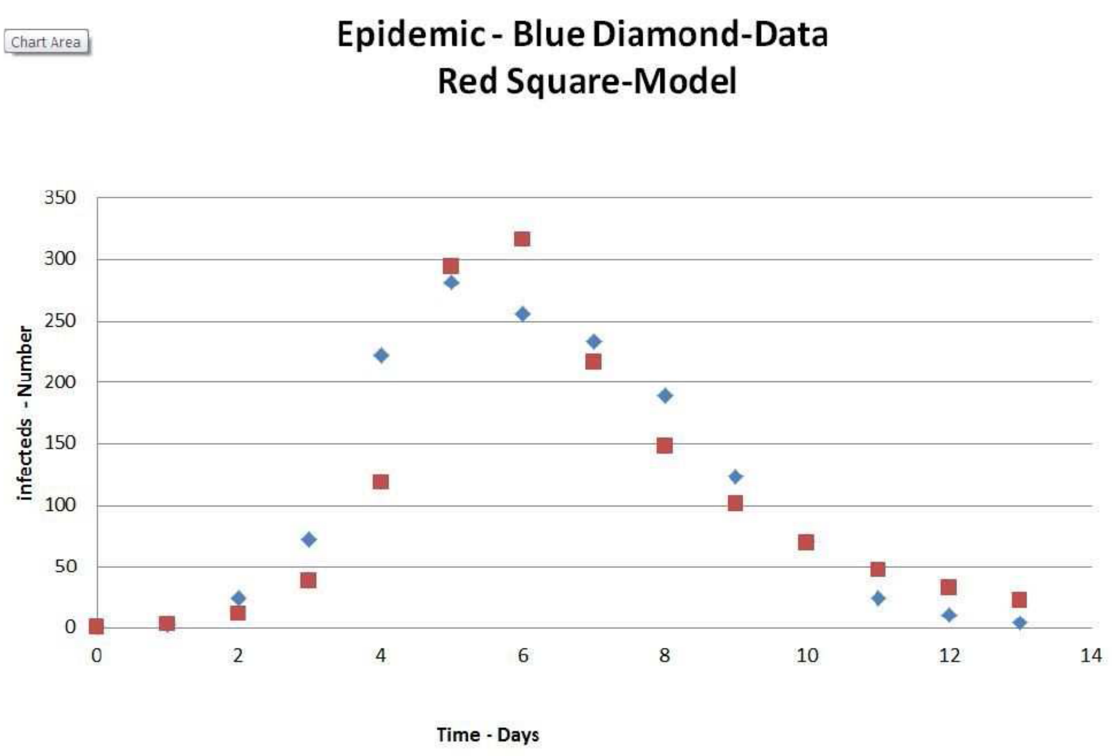
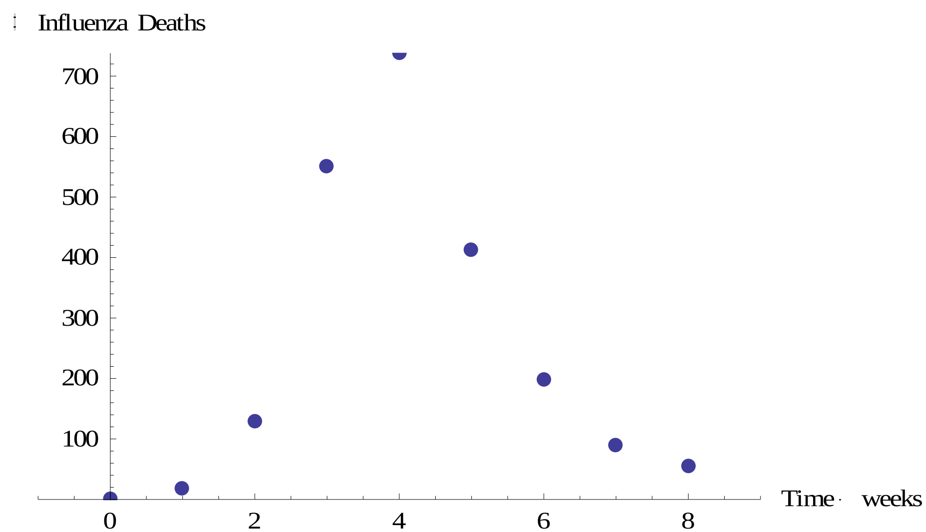


TABLE 1A
WEEKLY REPORTS FROM SAN FRANCISCO TO THE CALIFORNIA STATE DEPARTMENT OF PUBLIC HEALTH

For the week ending		*Cases*		*Deaths* *Influenza*	*Pneumonia*	*Total*
Oct.	5	36				
	12	531				
	19	4,233		88	42	130
	26	8,682		494	58	552
Nov.	2	7,164		679	59	738
	9	2,229		376	38	414
	16	600		174	24	198
	23	164		72	18	90
	30	57		42	14	56
			Total	1,925	322	2,257
Dec.	7	722				50
	14	1,517				71
	21	1,828				137
	28	1,539				178
Jan.	4	2,416				194
	11	3,148				290
	18	3,465				310
	25	1,440				149



Source: Hrenoff, A. K. 1941. The Influenza Epidemic of 1918-1919 in San Francisco. *The Military Surgeon*. November: 805-811.

Source: Miller, S, and J. Helms. 2010. The Disease that Infected Half the World: Mathematics, Biology, and History. ***PRIMUS*.** 20(3): 245-260.

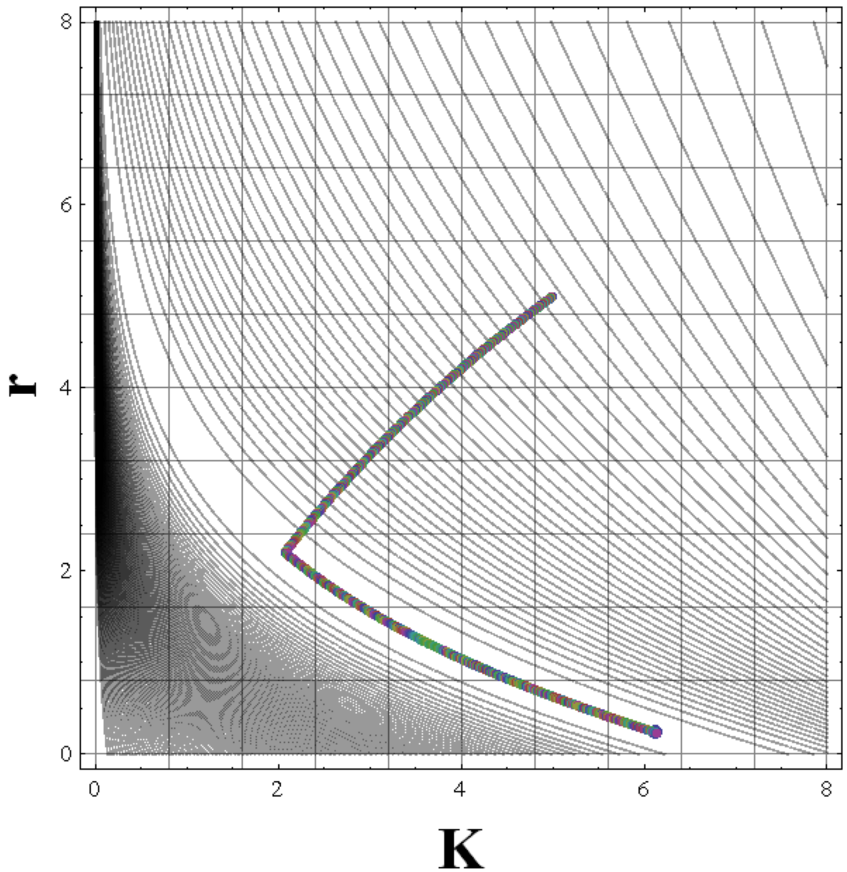
Collected data on falling coffee filters, (time in s, position in m). Using Vernier GoMotion detector.

t	pos(t)
0.	0.
0.0313	0.042
0.0626	0.0873
0.0938	0.1353
0.1252	0.188
0.1565	0.2404
0.1878	0.2961
0.2191	0.3537
0.2504	0.4155
0.28177	0.4791
0.313	0.5445
0.3443	0.6095
0.3756	0.6844
0.4069	0.761
0.4382	0.8361

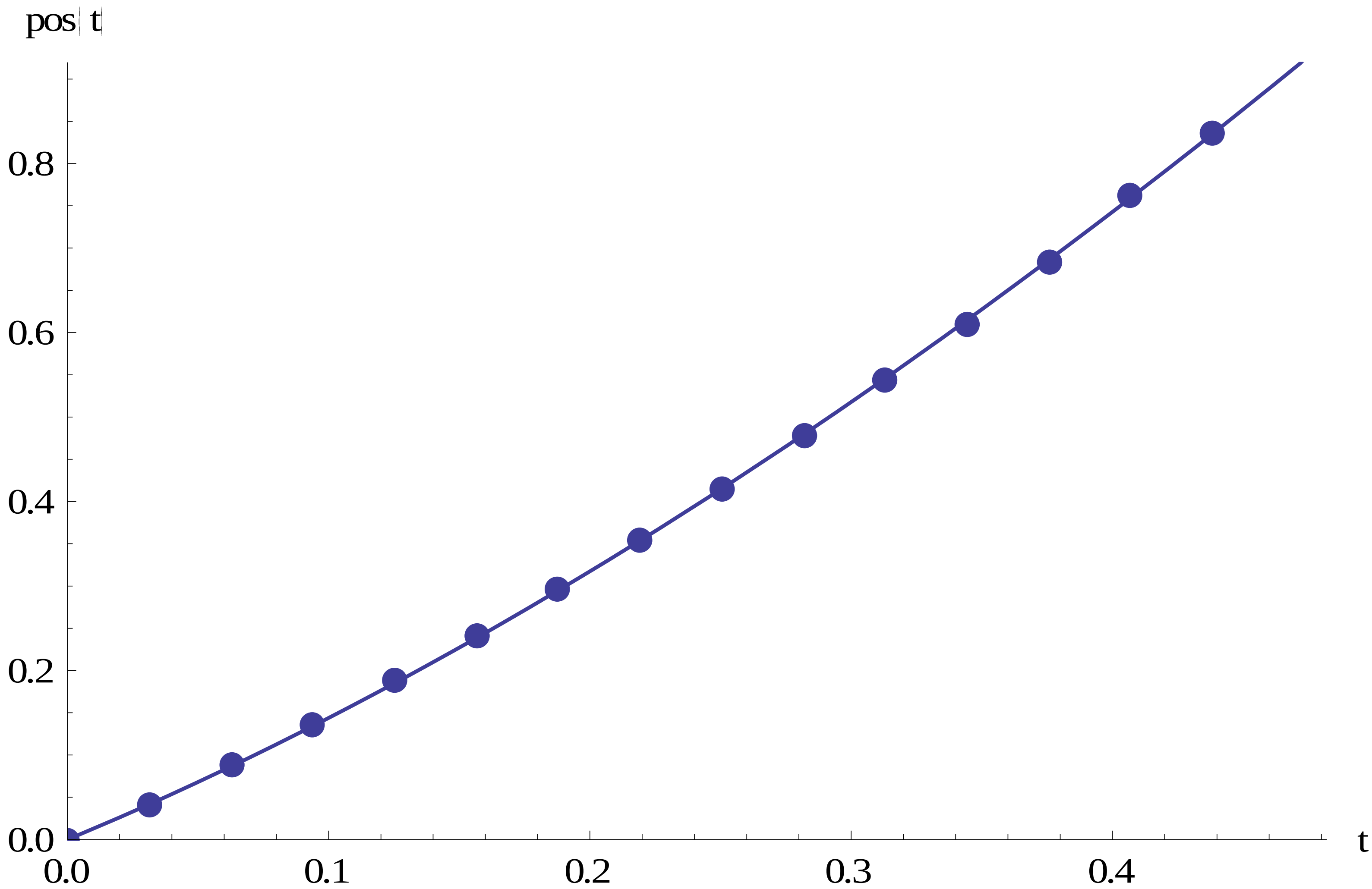
Model built from Newton's Second Law

m y''(t) = m*g - k y' (t)^r , y(0) = 0, y'(0) = v0

Usual cases r = 1 or r = 2. However, from data k = 6.12 and r = 0.24 with SSE = 0.0000511



Grid in parameter space (K, r, SSE(K,r)) and result of gradient search.



The fit is . . . .

Spread of Disease Process

Experiment conducted at Cornwall NY USA by
DR Brian Winkel, PROF Emeritus US Military Academy,
West Point NY

Prepare a grid (print out M&MGridToUseForSimulations pdf file), mark 8 randomly selected cells as infected and write numbers 1 through 8 in each cell, respectively. Make a corral surrounding the grid so when we toss the M&M's they stay on the grid.

Source: Winkel, B. J. 2012. Sourcing for Parameter Estimation and Study of Logistic Differential Equation. *International Journal of Mathematical Education in Science and Technology*. 43(1): 67-83.

Place 54 M&M’s in a cup and

(a) gently toss them out onto the grid,

(b) if an M&M contacts an infected cell/M&M mark the cell with next number of infected, first 9, then 10, then 11, etc. and remove infected M&M’s from population,

(c) when all M&M’s are marked from this toss or generation collect M&M’s and toss again until all M&M’s are infected.

If you cannot perform this simulation then we have done this for you and enclose 12 generations of our simulation from which you are to gather data on the number of infected at each generation or time.

Equipment:

1 small bag regular M&M’s
1 small cup
1 level surface

Early actions:
Prepare grid with 8 randomly selected infected cells. Number them as seen here.
Open bag and count out 54 M&M’ s.
Place M&M’s in cup.
Gently toss M&M’s onto surface.
If M&M contacts an infected cell/M&M mark the cell with next number of infected, first 9, then 10, then 11, etc.
Remove infected M&M’s from population.
When all M&M’s are marked from this toss Or generation. Collect M&M’s and toss again.

Later action:

Yummy!!!

Collect data for each generation on $y(t)$, the number of M&M cells which are infected.

Plot the number of infecteds as a function of the generation (time).

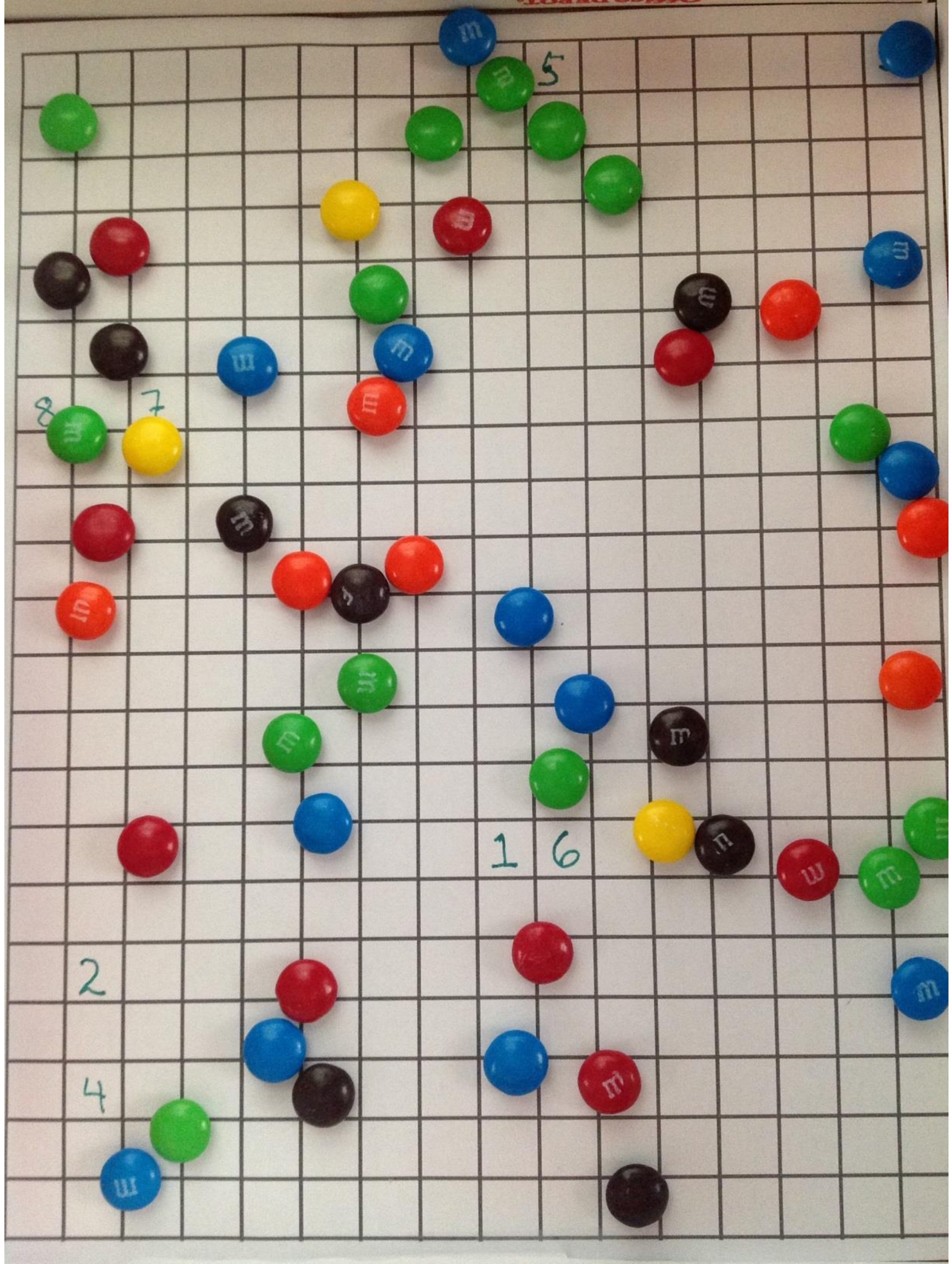
Propose a mathematical model, using a differential equation for the rate of spread of the disease.

Estimate the parameters using a sum of square errors method.

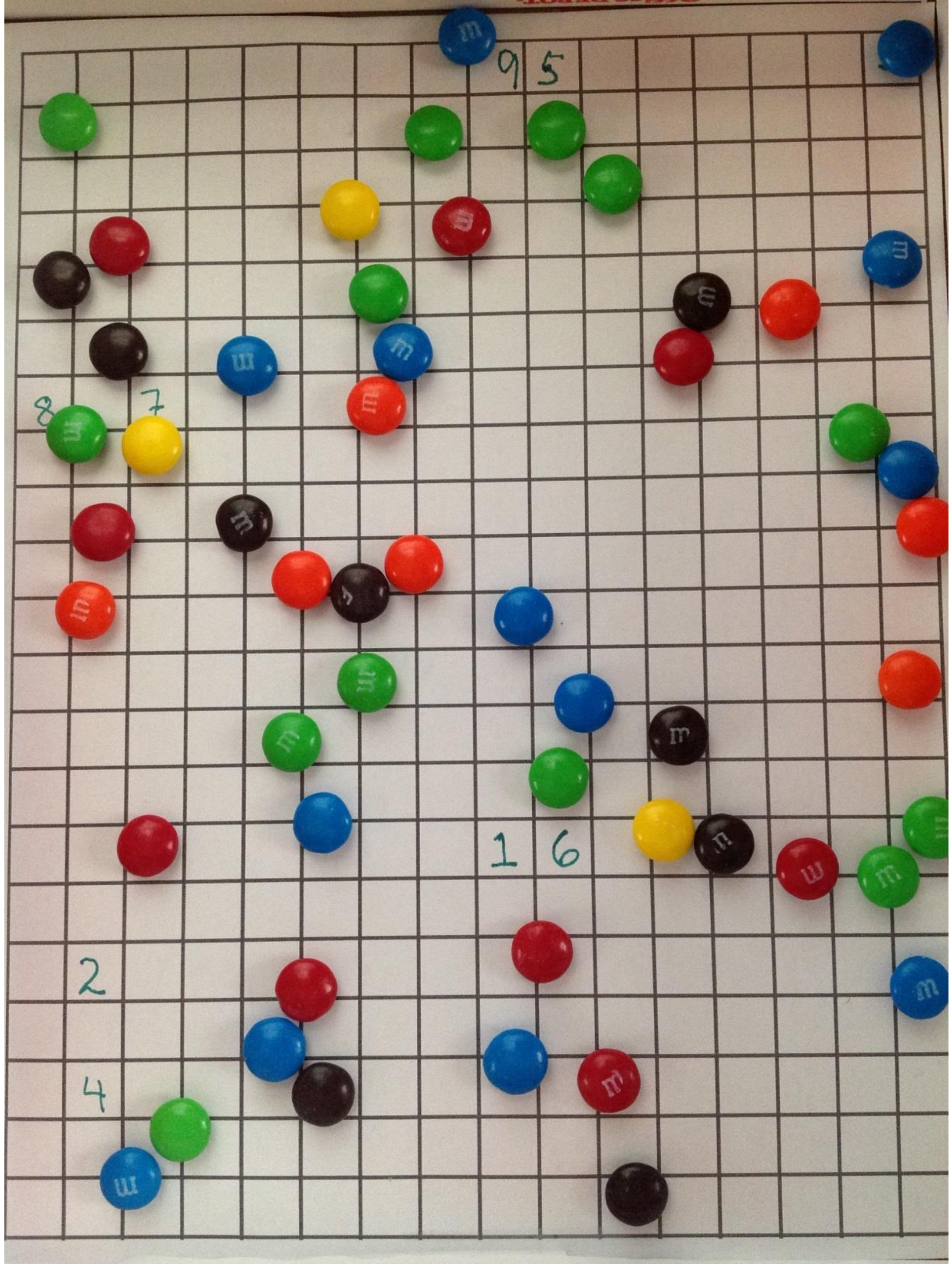
Compare your model to your data, numerically and graphically.

Does your model do a reasonable and reasoned job of modeling this data and hence this phenomenon?

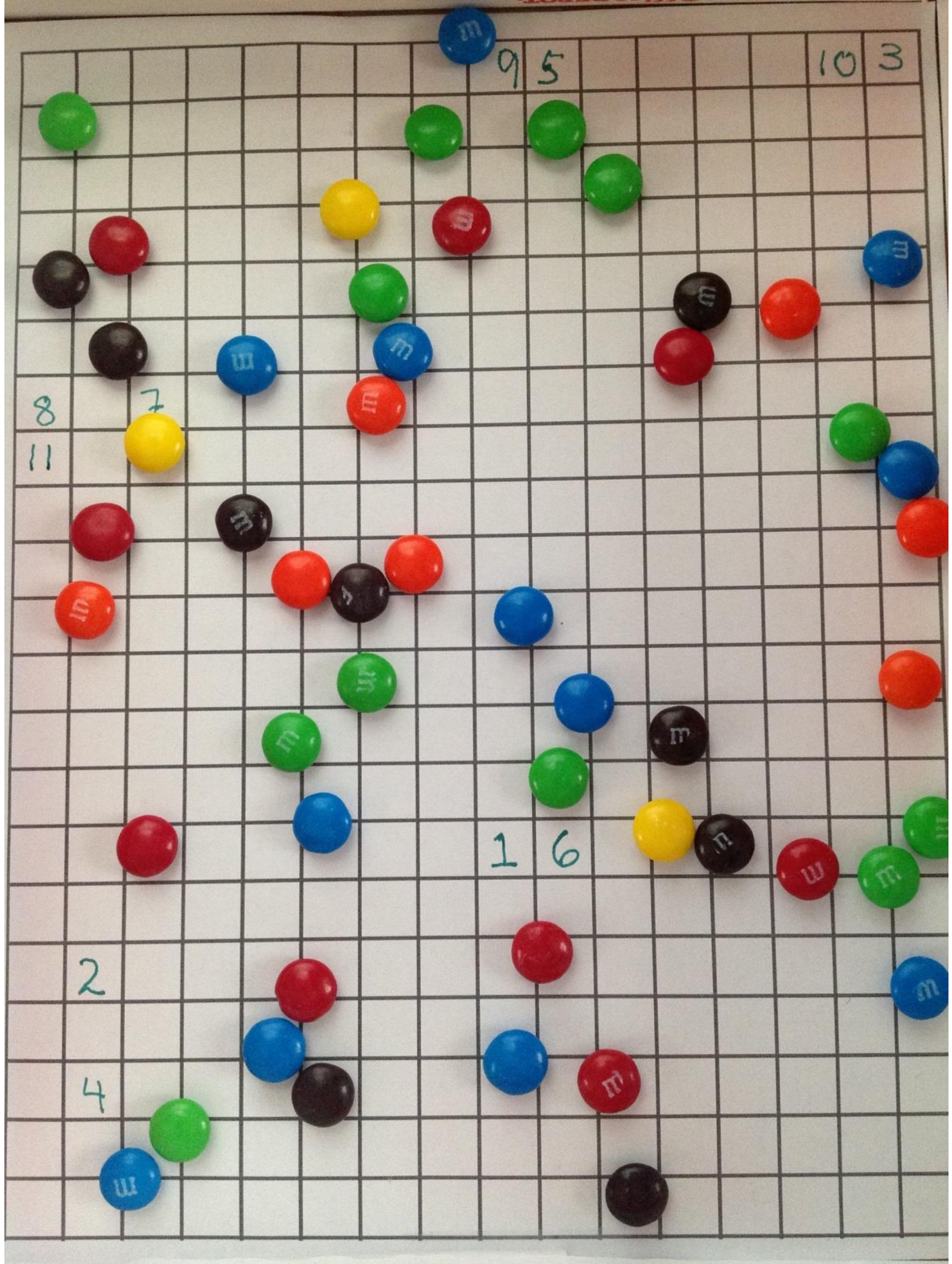
Generation 1



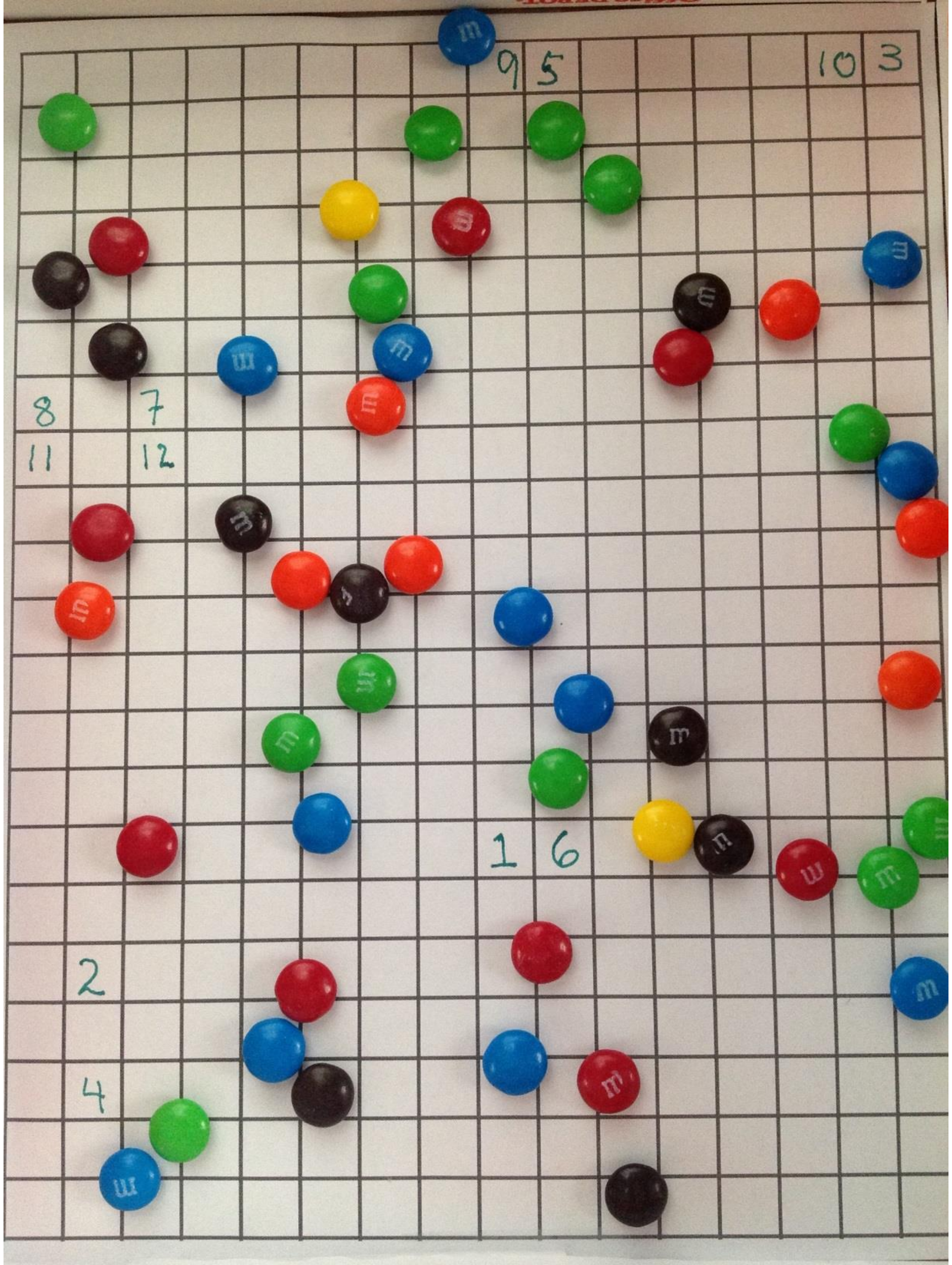
Generation 1



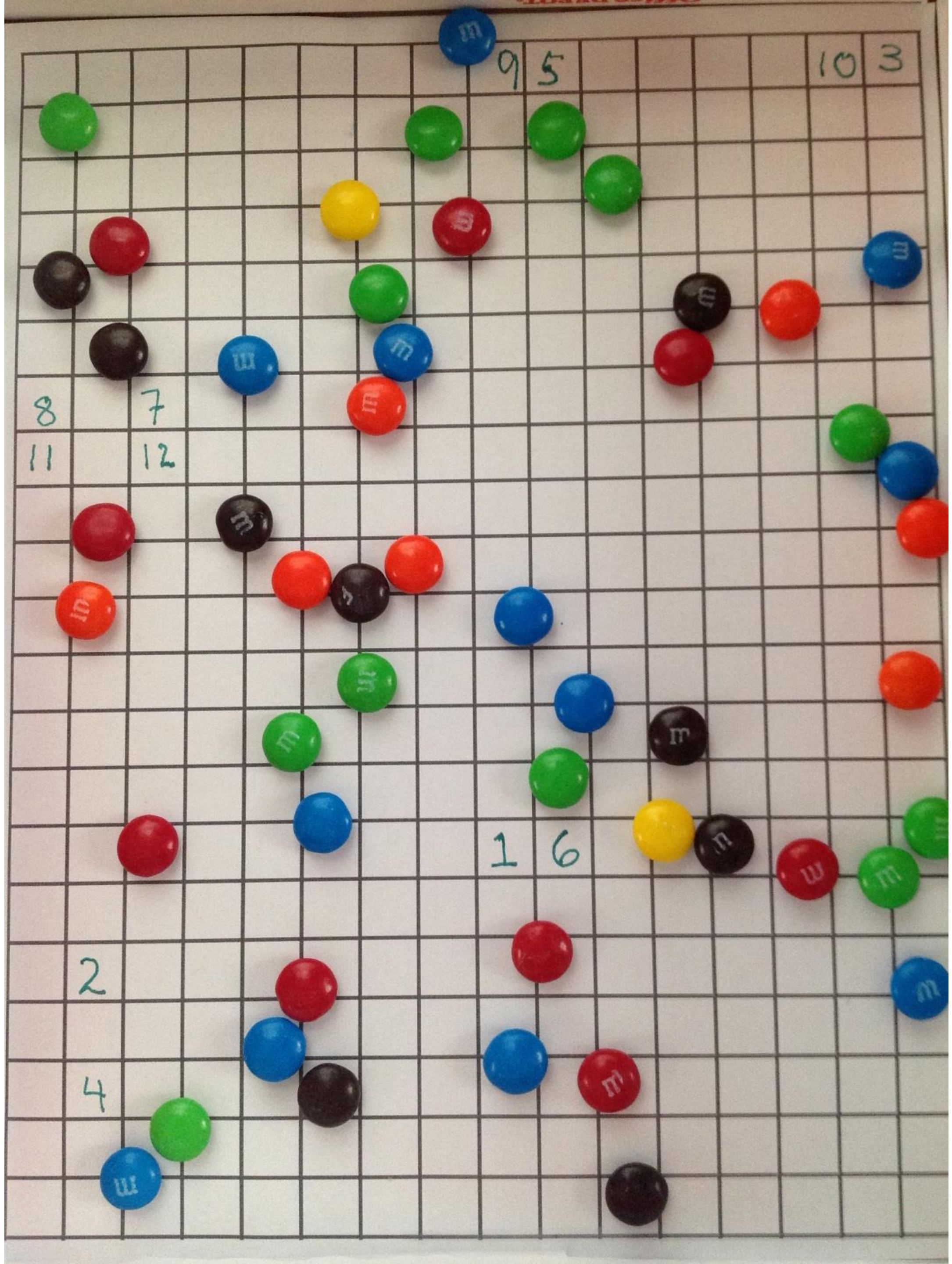
Generation 1



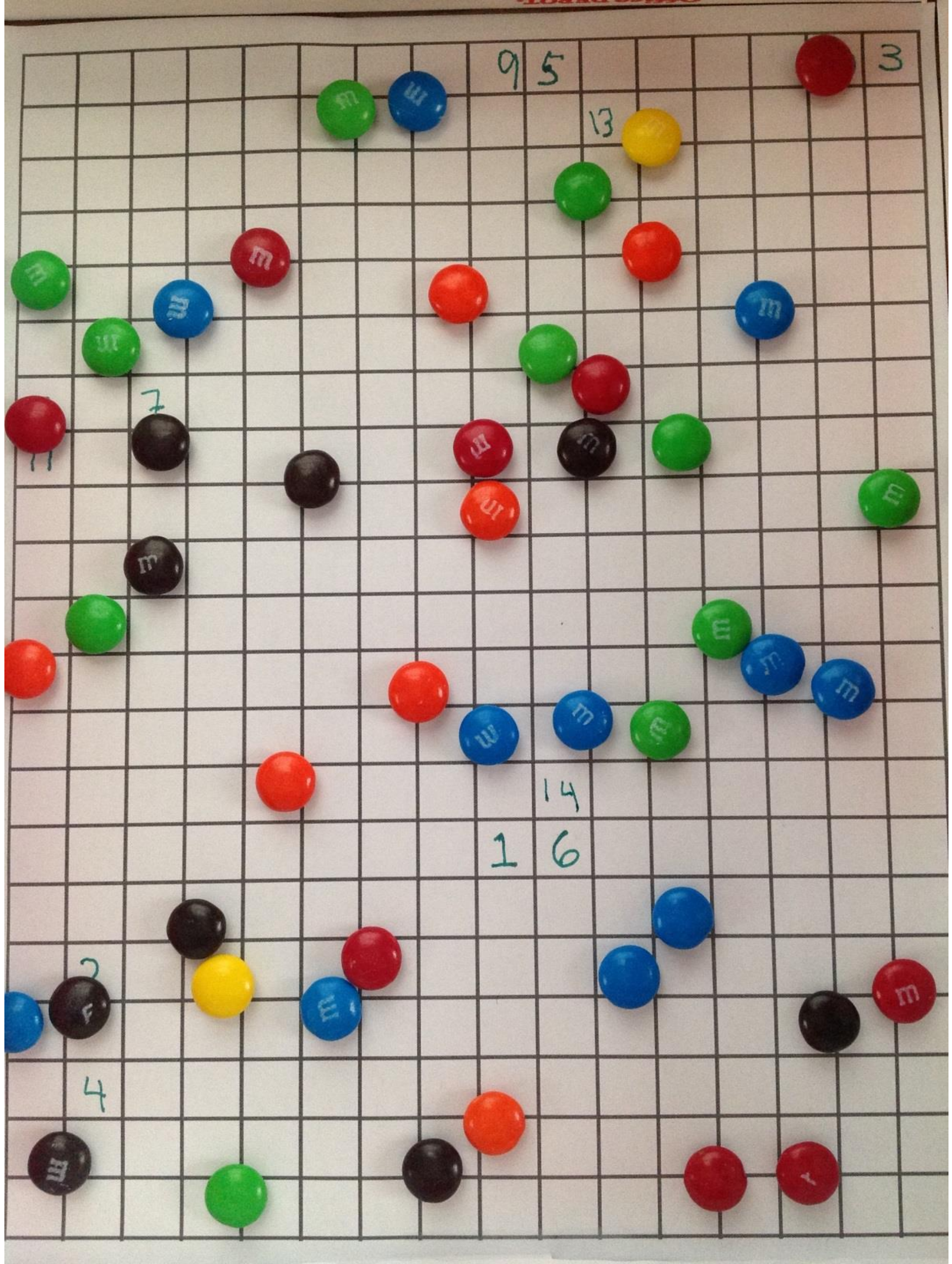
Generation 1



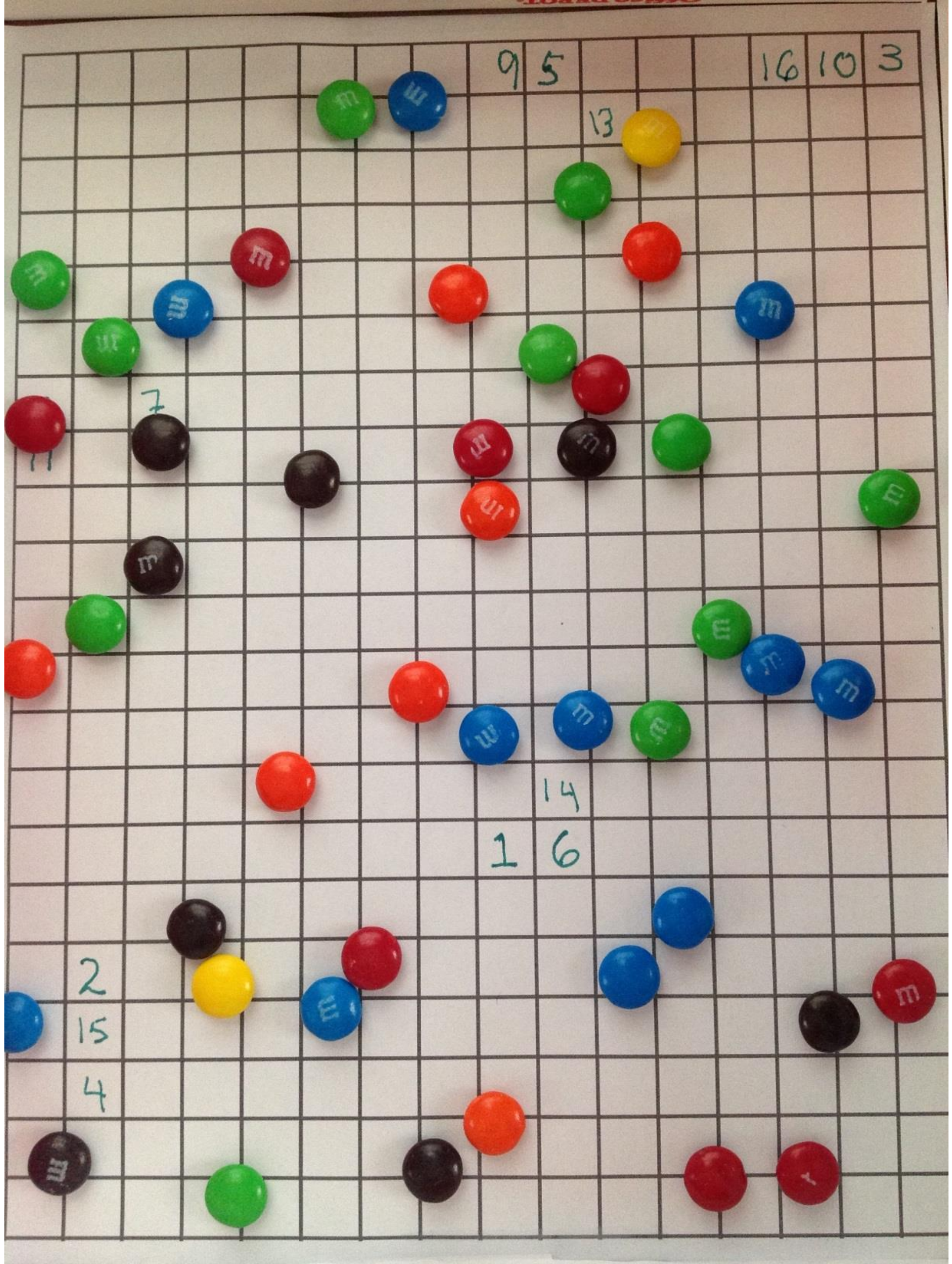
Generation 1



Generation 2



Generation 2



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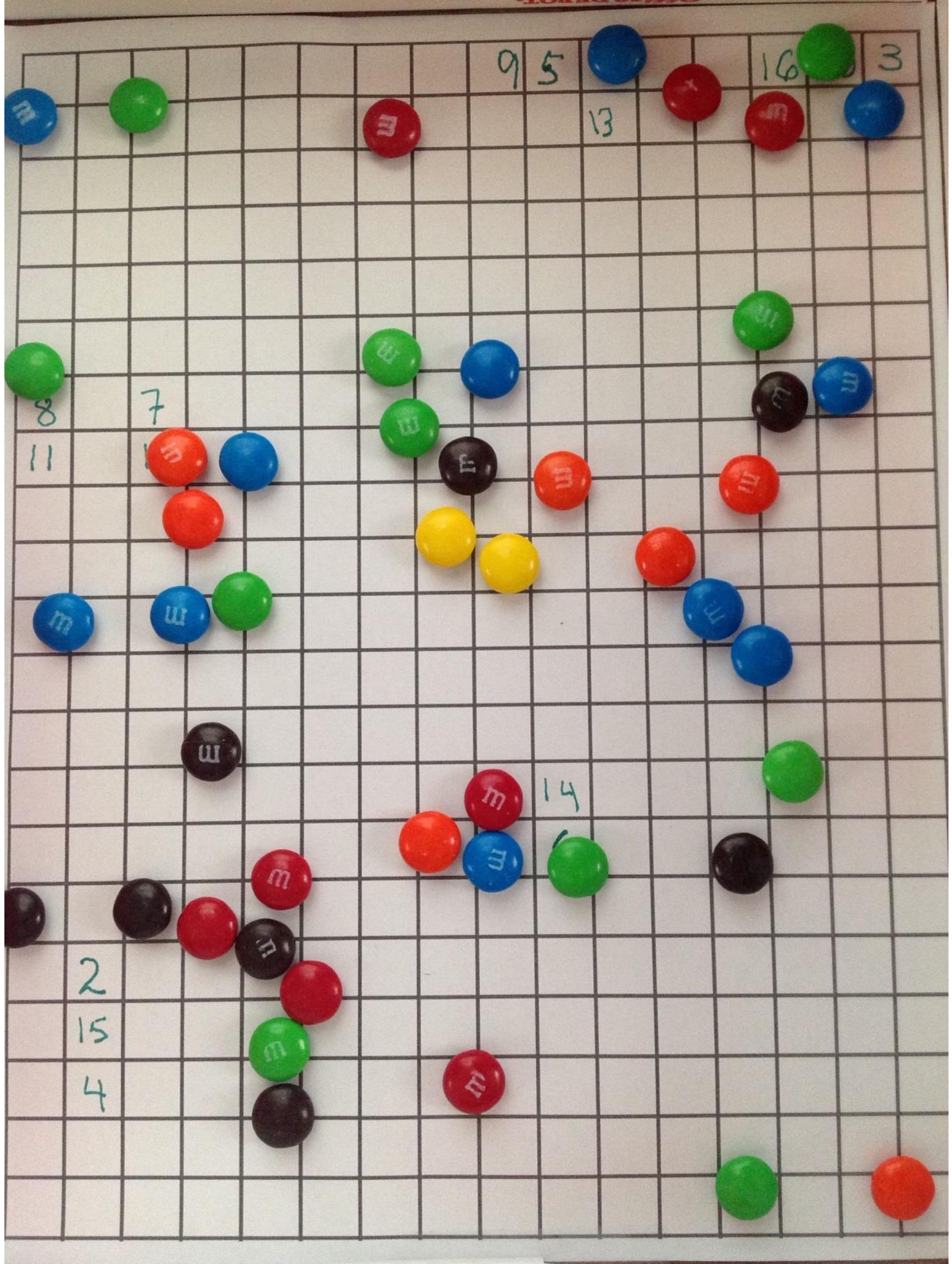

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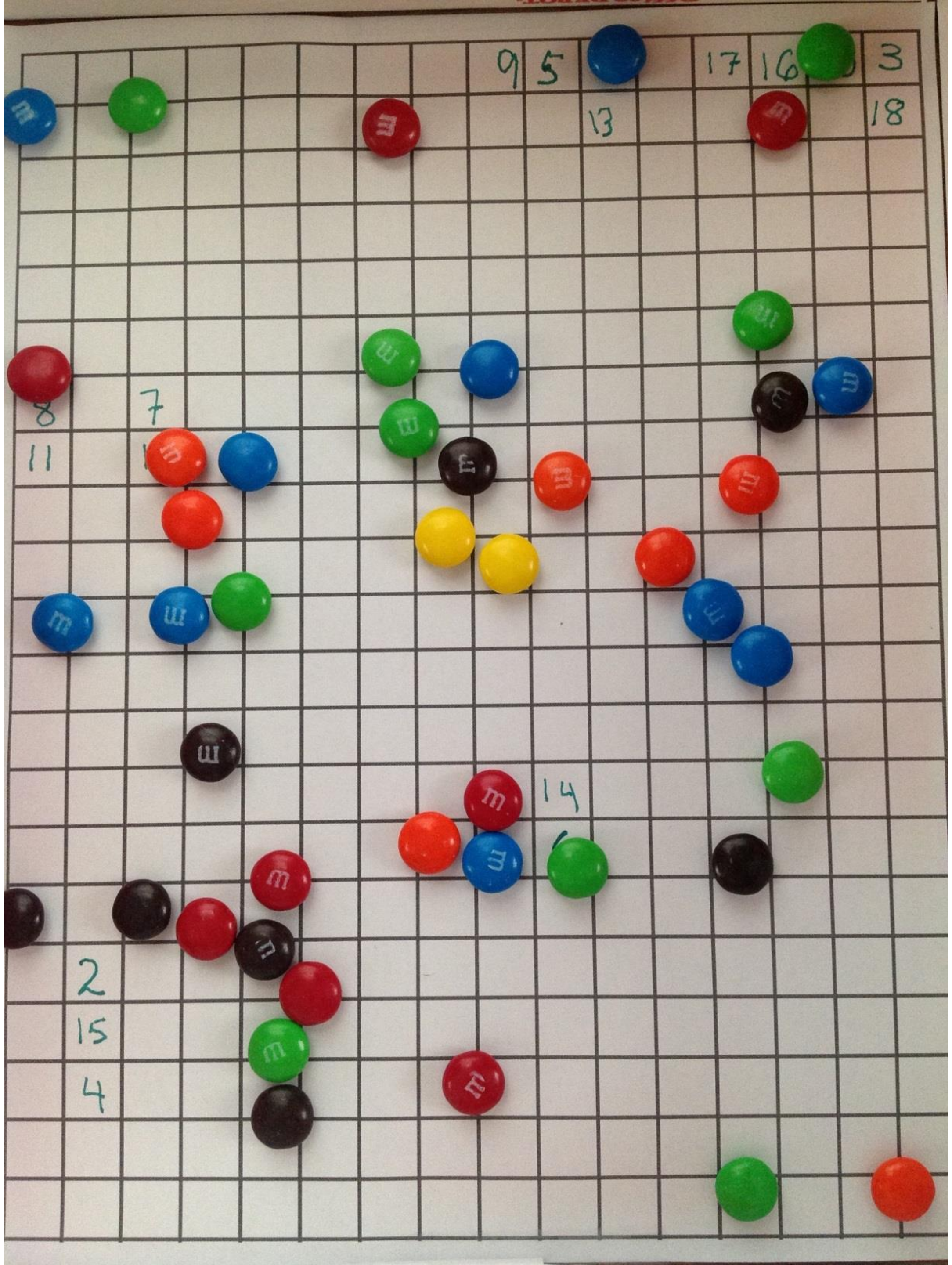
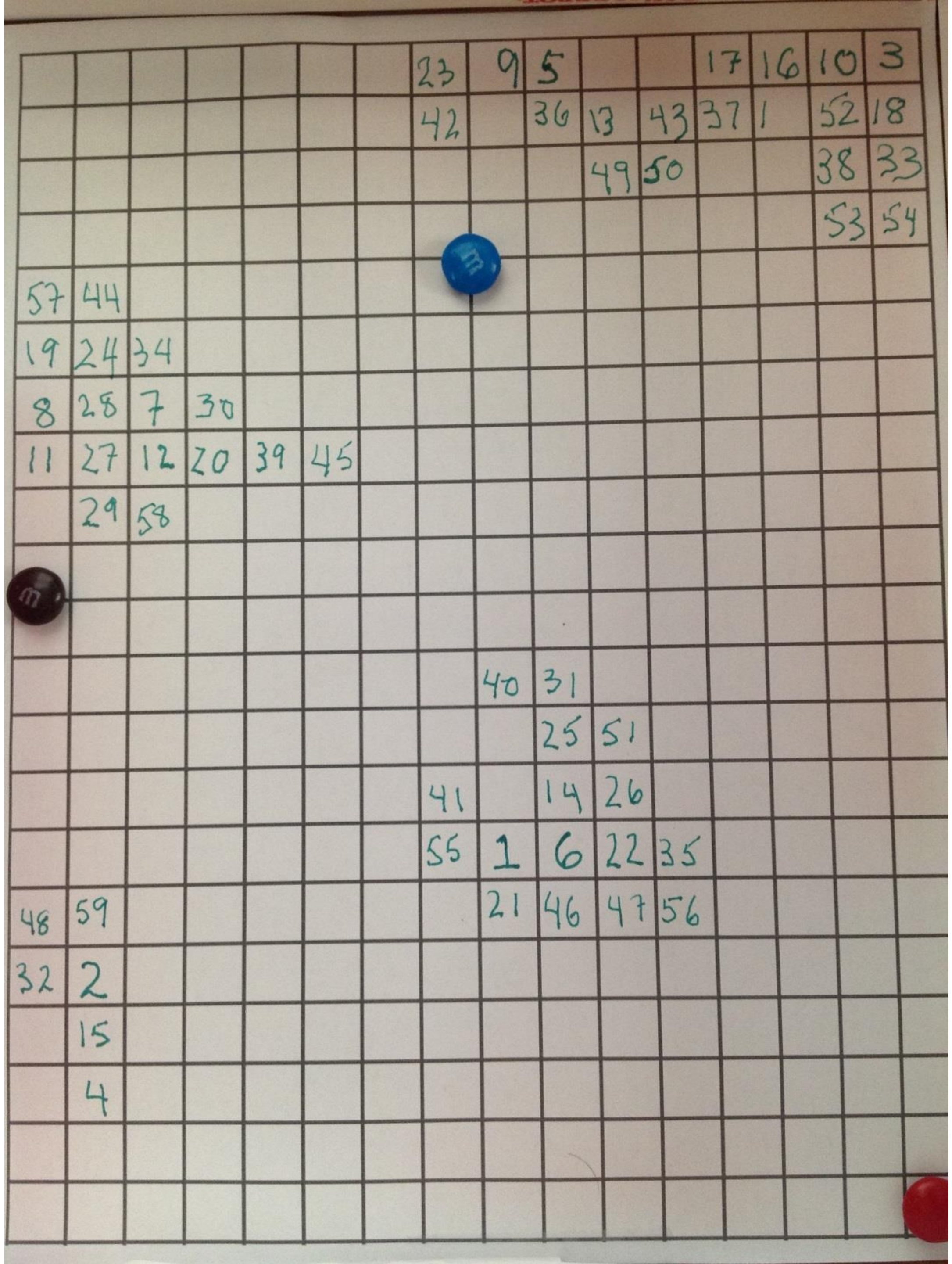
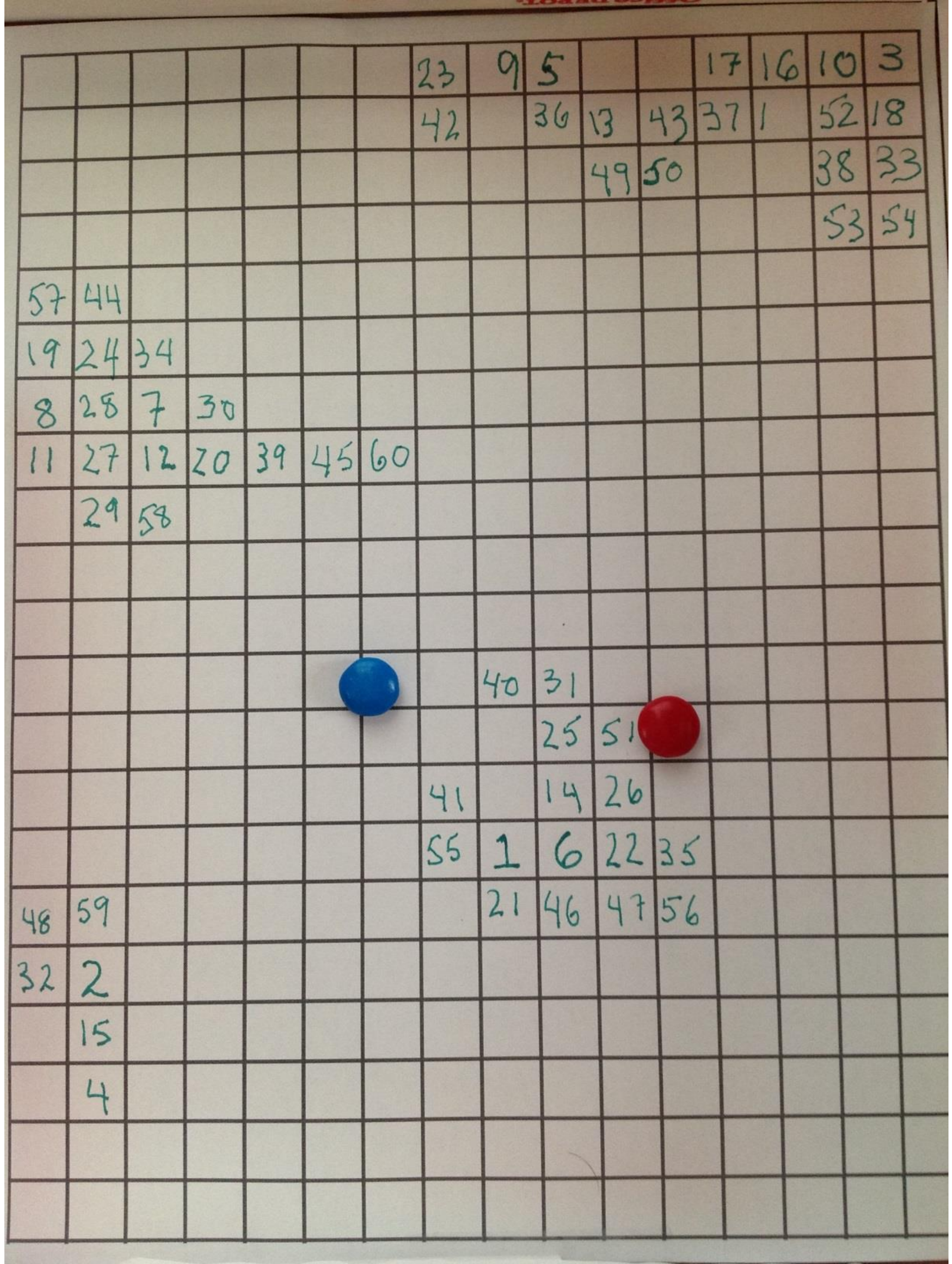


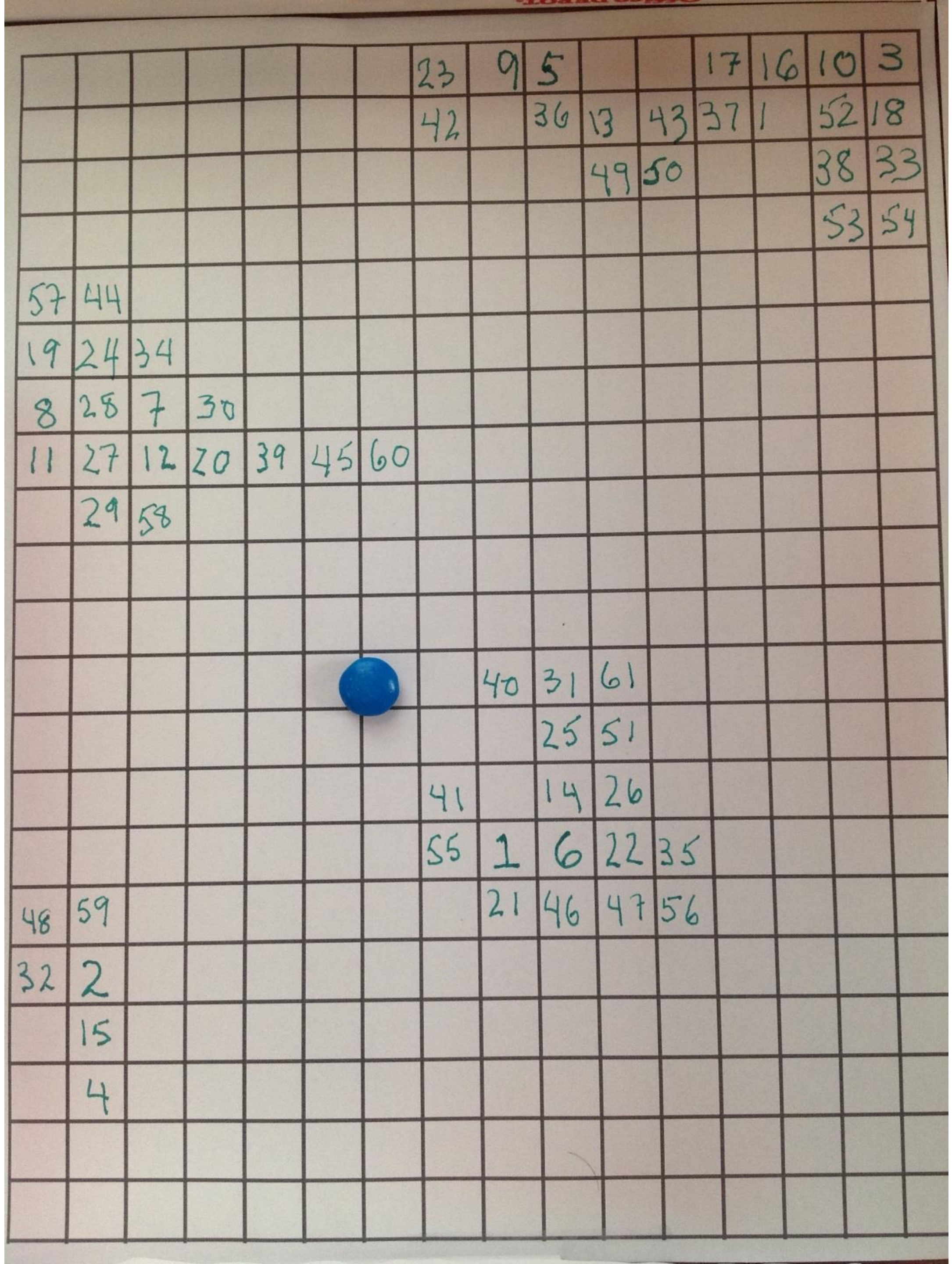
Generation 9



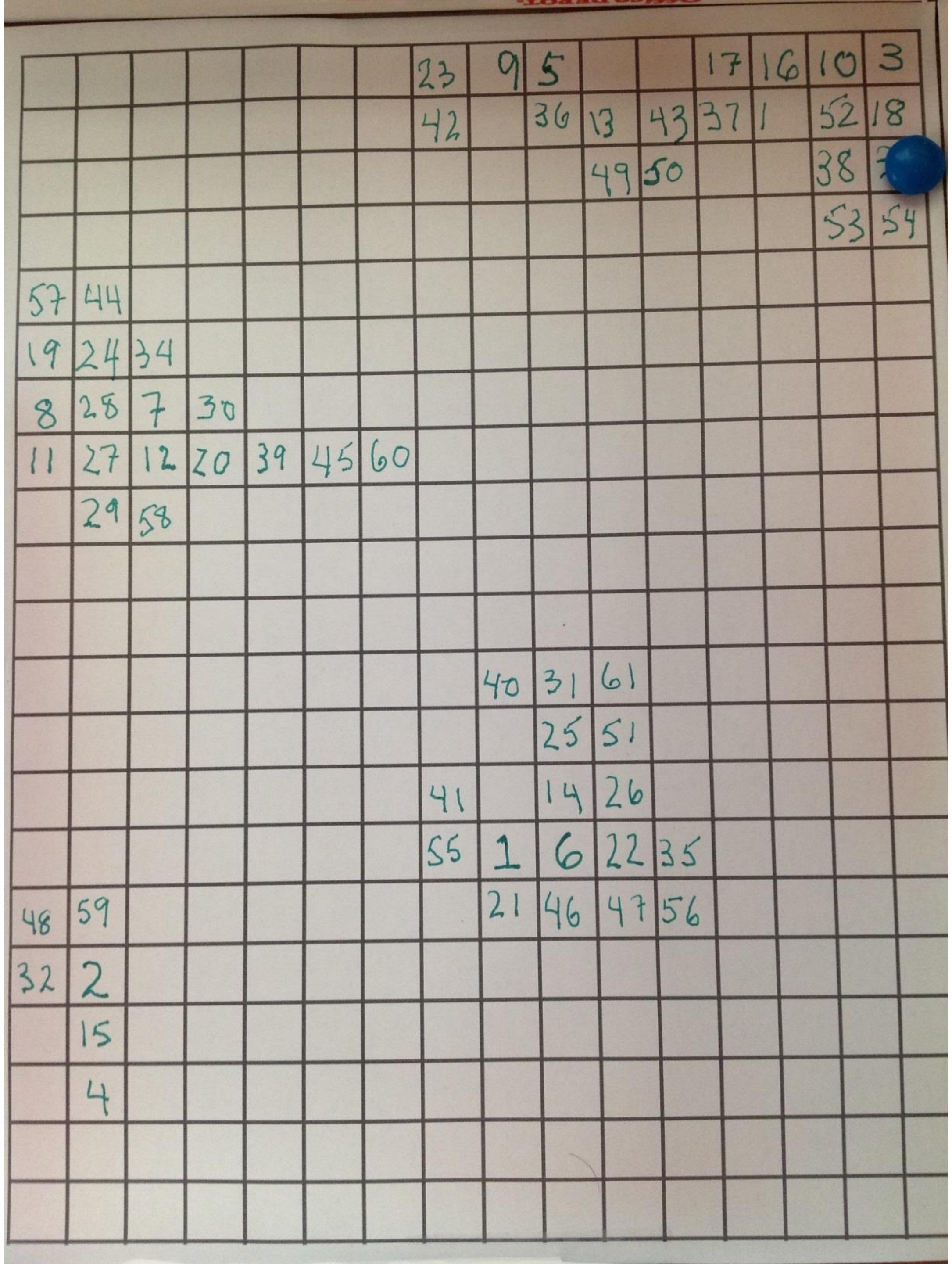
Generation 11



Generation 11

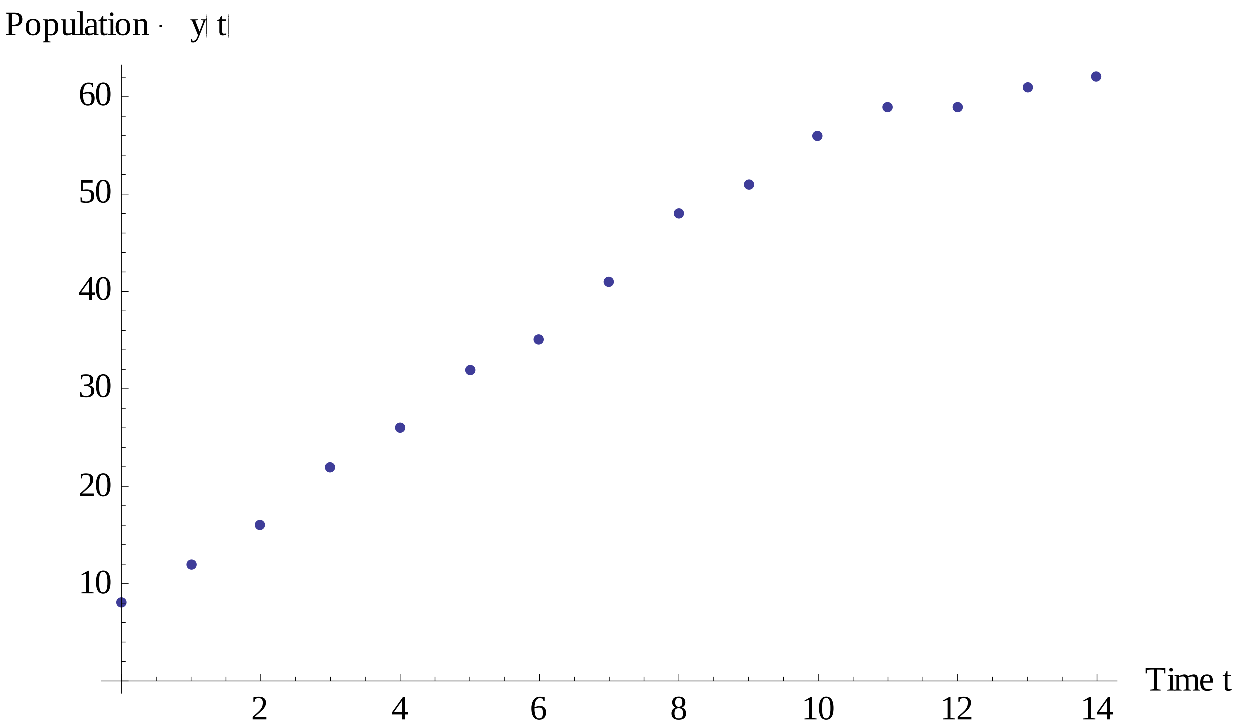


Generation 12



Generation 12

							23	9	5			17	16	10	3
							42		36	13	43	37	1	52	18
										49	50			38	33
														53	54
57	44														62
19	24	34													
8	28	7	30												
11	27	12	20	39	45	60									
	29	58													
								40	31	61					
									25	51					
							41		14	26					
							55	1	6	22	35				
48	59							21	46	47	56				
32	2														
	15														
	4														



Plot of data from M&M Generation Efforts.

Logistic mathematical model of spread of disease.

$$y'(t) = ry(t)\frac{K - y(t)}{K}, \qquad y(0) = y_0$$

```
ys[t_] = y[t] /. DSolve[{y'[t] == r y[t] (K - y[t]) / K, y[0] == 8}, y[t], t][[1]]
```

$$\frac{8\, e^{r t} K}{-8 + 8\, e^{r t} + K}$$

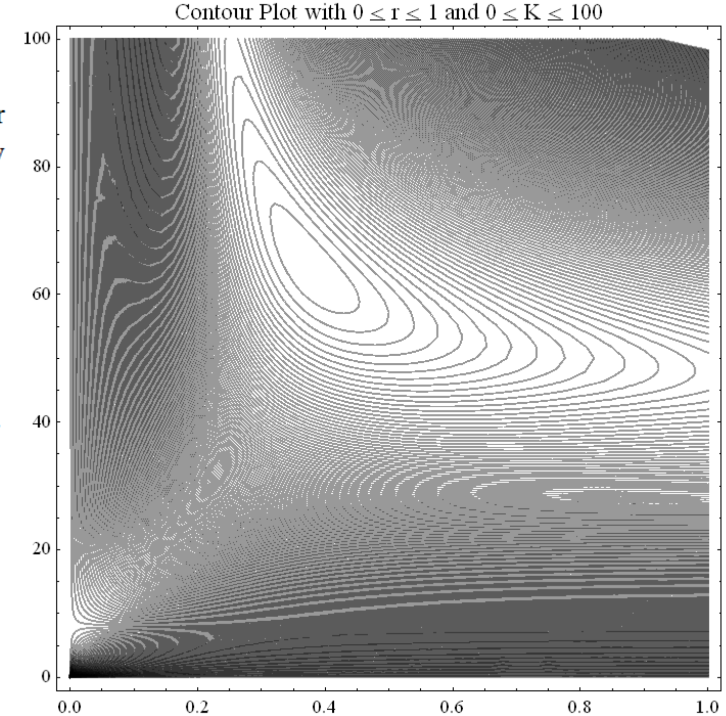
Solve DE and form SSE between Data and model.

```
SSE[r_, K_] = Sum[(data[[i, 2]] - ys[data[[i, 1]]])^2, {i, 1, Length[data]}]
```

$$\left(12 - \frac{8 e^{r} K}{-8 + 8 e^{r} + K}\right)^2 + \left(16 - \frac{8 e^{2 r} K}{-8 + 8 e^{2 r} + K}\right)^2 + \left(22 - \frac{8 e^{3 r} K}{-8 + 8 e^{3 r} + K}\right)^2 + \left(26 - \frac{8 e^{4 r} K}{-8 + 8 e^{4 r} + K}\right)^2 +$$
$$\left(32 - \frac{8 e^{5 r} K}{-8 + 8 e^{5 r} + K}\right)^2 + \left(35 - \frac{8 e^{6 r} K}{-8 + 8 e^{6 r} + K}\right)^2 + \left(41 - \frac{8 e^{7 r} K}{-8 + 8 e^{7 r} + K}\right)^2 + \left(48 - \frac{8 e^{8 r} K}{-8 + 8 e^{8 r} + K}\right)^2 + \left(51 - \frac{8 e^{9 r} K}{-8 + 8 e^{9 r} + K}\right)^2 +$$
$$\left(56 - \frac{8 e^{10 r} K}{-8 + 8 e^{10 r} + K}\right)^2 + \left(59 - \frac{8 e^{11 r} K}{-8 + 8 e^{11 r} + K}\right)^2 + \left(59 - \frac{8 e^{12 r} K}{-8 + 8 e^{12 r} + K}\right)^2 + \left(61 - \frac{8 e^{13 r} K}{-8 + 8 e^{13 r} + K}\right)^2 + \left(62 - \frac{8 e^{14 r} K}{-8 + 8 e^{14 r} + K}\right)^2$$

Plot SSE = SSE(r, K) as contour plot.

Contour Plot with $0 \le r \le 1$ and $0 \le K \le 100$



We see roughly the values of r and K which could minimize our sum of square errors and hence give us our best fitting, in the sense of least square errors at around r = 0.35 and K = 65. This latter estimate is no surprise for our maximum number of M&M's in our population was 8 + 54 = 62, for the 8 originally infected and the 54 uninfected who eventually become infected in our simulation by touching squares/M&M's who are infected.

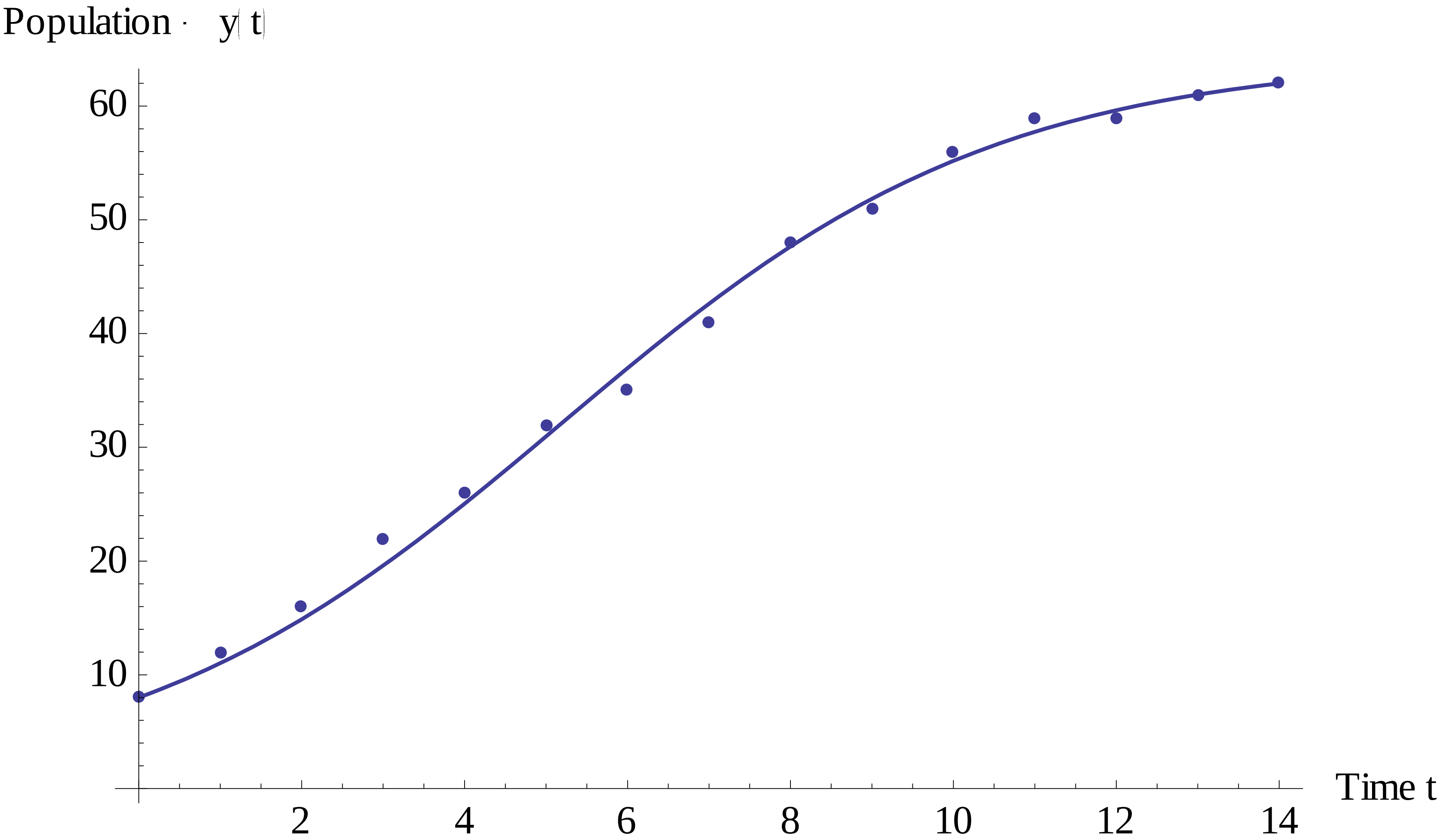
```
solrK = FindMinimum[SSE[r, K], {r, .35}, {K, 65}]
{19.9619, {r → 0.37555, K → 64.2538}}
```

We substitute our r and K values which make our sum of square errors between model and data into the model and plot this model on its own and then over the data to see how good our model is.

```
ysol[t_] = ys[t] /. solrK[[2]]
```

$$\frac{514.031\, e^{0.37555 t}}{56.2538 + 8\, e^{0.37555 t}}$$

```
ysolPlot = Plot[ysol[t], {t, 0, 14},
```



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Thank you.