Differential Equation Population Models in a Slowly Varying Environment

by

Majda Idlango

M.Math.Sc. (Alexandria University, Egypt)

B.App.Sc. (Al-Jabal Al-Gharbi University, Libya)

Dissertation submitted in fulfilment of the requirements for the degree of

Doctorate of Philosophy

School of Mathematical and Geospatial Sciences

RMIT University

Melbourne - Australia

January 2013

DECLARATION

The candidate hereby declares that the work in this thesis, presented for the award of the Doctor of Philosophy and submitted in the School of Mathematical and Geospatial Sciences, RMIT University:

- **has been done by the candidate alone and has not been submitted previously, in whole or in part, in respect of any other academic award and has not been published in any form by any other person except where due reference is given, and**
- **has been carried out under the supervision of Assoc. Prof. John J. Shepherd and Dr. John Gear**

....

Majda Idlango

Certification

This is to certify that the above statements made by the candidate are correct to the best of our knowledge.

....

Assoc. Prof. John J. Shepherd

Senior Supervisor

....

Dr. John Gear

Second Supervisor

Acknowledgements

Firstly, I would like to thank God for the opportunity to undertake this dissertation. I also wish to sincerely acknowledge those who have supported and helped me throughout my candidature. I express deep thanks for both the senior and second supervisors, Associate Professor John J. Shepherd and Dr John Gear respectively, for their patient and encouragement throughout last three years. Without their support, advice and effort in both the analysis and writing, this research would not have been as successful and rewarding.

I also wish to thank all my family and friends for their support. A special thanks goes to my husband, Ismail, who always assisted and stood by my side and to my a lovely daughter, Shayma, for her patience. I am also very grateful to my parents, relatives and friends for their prayers and belief that I could achieve this dissertation.

Contents

List of Figures

Summary

Many investigations have been carried out that analyse population dynamics using differential equations where the governing parameters of the models concerned are assumed to be a constant. However, in reality, this assumption of constancy is not always true, since the dynamics of such models can be influenced by regular environmental changes such as food availability, birth rates, etc. This may lead to the defining parameters of such population models varying with time, and this time variation could occur on one time scale or on a number of different time scales. In general, for such cases, an exact solution of population evolution problems involving the model cannot be obtained and numerical solution methods must be used. However, when the variation of the model parameters is slow relative to other quantities, analytic multi-scaling technique can be applied to obtain approximate solutions which can successfully represent the variation of the population over time. This technique is widely applied in the fields of physics and engineering and has begun to become more established as a technique in recent times in population dynamics.

This thesis considers single species differential equations population models in which the model parameters are functions of time that is slow compared to the intrinsic time changes of the population and uses multi-timing methods to obtain close-form (explicit or implicit) approximate expressions for the evolving populations represented by these models.

Chapter 1 introduces the background of the single species population models studied in this research. These are based on simple extensions of the basic logistic growth equation to a range of other population models. It also presents the development of theory of environment variations or fluctuations in recent studies and their effect on the defining parameters of these models.

Chapter 2 begins by reviewing previous analysis of a harvested logistic growth model where the population is harvested subject to constant environmental factors. The work of Nguyen [50] is extended to the situation where all of the growth rate, carrying capacity and harvesting rate vary on a time scale much longer than the time variation of the population itself. A general multi-scaled method is used to construct analytic approximate expansions in the three different categories of a subcritical harvested population with survival, subcritical harvested population with extinction and supercritical harvesting with extinction. These results are compared with a numerical solution of this model for a number of specific examples.

In Chapter 3, a contraction mapping approach is used to provide a rigorous proof of the validity of the various approximations constructed in Chapter 2, under suitable assumptions. This also establish the existence and the uniqueness of an exact solution of the slowly varying harvesting problem, in a suitable small neighbourhood of these approximations

In Chapter 2, the approximate expansions for the population for subcritical or supercritical harvesting were examined separately. Chapter 4 considers a case of transition, that is, the result of increasing the harvesting rate from the subcritical (small) case to supercritical (large) case. Here, the approximations of Chapter 2 fail to represent the solution at the transition point and so a transition analysis is required. A particular case is examined where the population is harvested subcritically and survives to its carrying capacity and as the harvesting becomes supercritical the population faces extinction and two uniform composite expansions are obtained.

Chapter 5 considers a single species population model involving an Allee effect, where by reduction of the population below a critical value cause extinction. It begins by analysing the model with constant coefficients and establishes the behaviour of the solution for a large time period. The same multiscaling analysis as applied in Chapter 2 is used to extend this analysis to a situation where the model parameters, the growth rate, the carrying capacity and the critical population rate are considered in a slowly varying environment. Implicit approximate representations for a surviving and for an extinguished population are obtained. The situation when the carrying capacity reduces to a value below the critical level is also analysed, using a transition analysis, and a uniform approximation to this transition situation is obtained.

In Chapter 6, a more complex form of the harvested logistic model discussed in Chapter 2 is studied, where the harvesting term is represented by a saturating Holling type II functional response. The stability and solution of the constant parameter model is analysed, and this is extended to the case where the harvesting rate is considered to be a function of slow time. Here, the analysis as used in Chapter 5 is employed to obtain implicit approximations for both survival and extinction states of the population. The accuracy of the approximate expansions are compared with the numerical solution using particular choices satisfying particular conditions.

Publications Resulting from this Dissertation are:

- Idlango, M.A. and Shepherd, J.J. and Nguyen, L. and Gear, J.A., Harvesting a logistic population in a slowly varying environment., *Applied Mathematics Letters*, 2012, **25**, 81-87.
- Idlango, M.A. and Shepherd, J.J. and Gear, J.A., Multiscaling analysis of a slowly varying single species population model displaying an Allee effect , *Mathematical Methods in the Applied Sciences* , `In press`.
- Idlango, M.A. and Gear, J.A. and Shepherd, J.J. , Transition in a slowly varying harvested logistic population model, *Applied Mathematics Letters*, `Accepted`.

Chapter 1

Introduction

In the study of the dynamics of single species populations based upon biological and/or ecological assumptions, different mathematical techniques have been used to model the evolution of these populations. Typically, these models take the form of initial value problems for ordinary differential equations where the governing model parameters are taken to be constants. The structure of these models is based on the simple form of Malthus-Verhulst logistic growth models.

The generalized autonomous single species population model takes the form

$$\frac{dP}{dT} = F(P) - G(P), \qquad P(0) = P_0, \tag{1.1}$$

where $P(T)$ is the population at times $T \geq 0$, $F(P)$ is the growth rate, $G(P)$ is the removal rate and P_0 is a positive constant that represents the initial size of the population.

The simplest model of this type of population evolution was introduced by Malthus [43] and also discussed in ([5, 6, 19, 52] among others). In this model, the population grows exponentially according to the *Malthus' law*:

$$\frac{dP(T)}{dT} = R\,P(T), \qquad P(0) = P_0, \tag{1.2}$$

where R represents the *growth rate*. Here, the per capita growth rate of the population $dP/P\ dT$, is a constant and when R is positive, there is growth without limit; while when

R is negative, there is exponential decay. In this model, $G(P) = 0$; i.e., the growth is *unconstrained.*

This model has been widely applied; Malthus' simple model is discussed in Haberman[32] where it is applied to the growth of human population. Under this model, human population is expected to grow exponentially and humanity would face starvation and possible extinction if food sources were outgrown.

In real world populations, species have limited resources for survival and cannot grow without limit, in contrast to the unconstrained growth predicted by the simpler exponential (Malthusian) growth model. Thus, Verhulst [62] reconsidered these effects and proposed that the population growth should be modelled by the logistic growth equation

$$\frac{dP}{dT} = R\,P(1 - \frac{P}{K}), \qquad P(0) = P_0, \tag{1.3}$$

where the positive constant K is termed the *carrying capacity* and provides an upper limit to the ability of the environment to sustain the given species. Banks ([5], Ch 2) and Brauer, et al [12] describe this in general details (see further [6, 7, 19, 45, 48].)

Gause [20] discussed an example of how the logistic equation can be applied to the growth of a yeast population. Pearl and Reed [51] presented an application of the logistic model to the growth of human population in the United States, and Banks ([5], Ch2) also applied it in the field of technology substitution.

In reality, the growth of any single species population can be affected by various factors, such as depletion of fishing stocks, hunting, natural disasters and immigration etc. Thus, if population is extracted from the logistic growth model (1.3) at a constant rate H, we arrive at the harvested logistic model which is represented by the initial value problem

$$\frac{dP}{dT} = R\,P(1 - \frac{P}{K}) - H, \qquad P(0) = P_0, \tag{1.4}$$

where R, K and H are as defined above. Here, $G(P) = H$, a positive constant.

This model has been investigated in many related studies when R, K and H are positive constants, notably Brauer and Sanchez [13], Banks ([5], §2.2) and others [5, 6, 12, 13, 19, 33, 48, 50, 55]. These studies have identified two particular situations that depend on the harvest rate. One is *subcritical harvesting*, where the population survives to a positive constant limiting value, or is extinguished in finite time. Whether this last occurs depends on the initial population P_0. The other is *supercritical harvesting* where for any initial population P_0, the population always reduces to zero in finite time; i.e., is extinguished. Banks [5] has illustrated some applications of (1.4) in fish harvesting and the population growth of the sandhill crane.

A key feature of the Verhulst model (1.3) is that the per capita growth rate of the population, $R\,(1 - P/K)$ is a maximum when the population is small and reduces to zero as the population converges to the limiting state K. However, in some real population models, when the per capita population growth rate is reduced to such low levels, the extinction of that population may occur.

Courchamp et al [17] described some causes of this behaviour in several categories that affect a single species, such as low possibility of finding mates, a reduction in fertilization etc. This phenomenon, termed the Allee effect (Allee, [2], Allee et al [3], Stephens et al[58]), contradicts the predictions of (1.3), and so a more comprehensive model is needed to accommodate this effect. Models describing the Allee effect take various mathematical forms, as discussed by Boukal et al [11] and Gonzalez-Olivares [22] and others [12, 23, 58]. One such form is described by the initial value problem

$$\frac{dP}{dT} = R\,P\left(\frac{P}{M} - 1\right)\left(1 - \frac{P}{K}\right), \qquad P(0) = P_0, \tag{1.5}$$

where M is a positive constant that represents a critical population threshold, with $0 < M < K$ (see [17]). This may be viewed as a variant of the logistic model (1.3), with a population dependent growth rate, $R\,(P/M - 1)$, negative if P is small. The model (1.5) has the property that when $0 < M < P_0 < K$, the population survives to the carrying capacity, K. However, when $0 < P_0 < M < K$, the population is driven to zero, i.e., is extinguished, displaying an Allee effect as described above.

A more general form of single species harvested population model takes the form of the initial value problem

$$\frac{dP}{dT} = R\,P\left(1 - \frac{P}{K}\right) - \frac{HP}{A+P}, \qquad P(0) = P_0, \tag{1.6}$$

where R, K, H and A are all positive.

Here, the harvesting term is represented by a Holling type II functional response,

$$\frac{HP}{A+P}. \tag{1.7}$$

The structure and the theory of this function is discussed by Berryman et al [10] and Bazykin et al [6], (see also [16, 44, 45]) and Holling [34] summarized how the size of harvesting (or predation) can have an effect on the overall population behaviour. In particular, as $P \to \infty$, (1.7) tends to H, a constant. In fact, for all $P > 0$,

$$0 \ < \ G(P) = \frac{HP}{A+P} \ < \ H;$$

i.e., this harvesting term displays a *saturation* effect. Ghansah et al [21] argued that "at low population the predator is limited by prey availability, so that $G(P)$ increases with increasing prey population; at very high population, $G(P)$ saturates to some constant H, determined by the predators' intake capacity or processing rate". Ghansah has also examined in detail the stability index of the the differential equation of (1.7).

There are many models for the evolution of single species of populations, taking the form of initial value problems. In many cases, the differential equations in the models may be solved in closed form, whether explicit or implicit. A catalogue of exact solutions for a range of such models may be found in Tsoularis and Wallace [61], and whether these solutions are used depends on how complex they are, and more often than not, the problem is solved numerically.

In all the above single species models the governing parameters of the models are considered to be constant. However, in many real populations, regular environmental changes such as food shortage, climate change, migrating species and overfishing etc can influence the dynamics of such models by causing these parameters to be time dependent. Many recent studies have attempted to take account of these changes, for example Rosenblat [54] and Legovic [41] have considered the effect of a periodic variation. Similarly, May [44] examined the effect of the time variation of parameters in the logistic model. Benardete, et al [8] and Cromer [18] among others([7, 54]) showed that a small periodic change in the harvesting rate in the harvested logistic equation (1.4) may result in driving an otherwise surviving population to extinction. Rizaner and Rogovchenko[53] and Graef et al [24] examined the behaviour of an Allee model with parameters which vary with time, obtained a periodic solution for the model, and proved the existence of this solution.

Thus the parameters of such models can and do change with time and this variation may have quite a dramatic effect on the population evolution. In such cases, an exact solution of the initial value problems describing the model is often impossible to obtain and an approximate solution must be constructed using numerical methods. However, these numerical solutions don't usually provide a general representation of the behaviour of the solution of such problems but can only show a limited response to particular input data (for example [46]). Thus, for a better understanding of the impact of time variation in model parameters and its effect on population behaviour, analytic approximate methods can be used that provide a valid general formula for different general cases.

In many situations, the model parameters vary on a time scale much larger than that intrinsic to the model itself. This may arise in response to relatively slowly changing environmental effects, for example. In this case the analysis of the model may be split in two on the basis of these two time scales. This procedure was used in a qualitative way in an analysis of the spruce budworm problem, by Ludwig et al [42]. However, no analytical solutions were obtained.

Following similar successful studies (among others, [26, 50, 59], we will here consider the particular situation where the model parameters are regarded as changing slowly compared with the variation of the overall population. This slow change may, in general, arise from model parameters changing on a range of time scales that are slow relative to the overall population time frame. This is too general for this study, so two time scales only are considered in the present research; the time scale on which the population evolves and the (single) time scale on which the model parameters vary. Slow variation of parameters thus means this second time scale is large relative to the first. To examine this situation, multi-scaling methods as presented in Bush [15] and Nayfeh [49] will be applied to obtain approximations to the solutions of the given problems.

The approach outlined in the above references comes from a family of related methods referred to as multi-timing or multi-scaling methods. They have been extensively researched as discussed in [15, 35, 47, 49] and are well-established in the literature of both the engineering and the physical sciences. A detailed and interesting survey of techniques and history of application of these methods may be found in Nayfeh [49], chapter 6. They rely on an ability to recognize that the problem consideration involves variation on two or more time scales that are disparate; i.e., they are all 'slower' than a fundamental (intrinsic) time scale. Very often, these time scales are easily recognized from the formulation of the problem; but not always.

To illustrate the method in a very simple case, consider the task of finding an approximation to the solution of the initial value problem

$$\frac{dx}{dt} + a(\epsilon t)\, x^2 = 0; \quad x(0, \epsilon) = 1, \tag{1.8}$$

where $a(.)$ is a suitably smooth function, and ϵ is a small positive parameter. This problem shows variation on two time scales- the 'intrinsic' time scale $t_0 = t$ and the 'slow' time scale $t_1 = \epsilon t$ governing the variation of $a(\epsilon t)$. To implement the multiscale technique for (1.8), we assume that $x(t, \epsilon)$, the (unknown) solution of (1.8) may be regarded as a function $\tilde{x}(t_0, t_1, \epsilon)$ of two variables t_0, t_1 and the parameter ϵ. We then assume that

$\tilde{x}(t_0, t_1, \epsilon)$ may be represented by the perturbation expansion

$$\tilde{x}(t_0, t_1, \epsilon) = \tilde{x}_0(t_0, t_1) + \epsilon \tilde{x}_1(t_0, t_1) + \dots \tag{1.9}$$

while the derivative term converts according to

$$\frac{d}{dt} = D_0 + \epsilon D_1 \tag{1.10}$$

where D_0, D_1 are partial derivatives taken with respect to t_0, t_1.

Substituting (1.9) into (1.8) and noting (1.10), we arrive at partial differential equations for $\tilde{x}_0$, $\tilde{x}_1$;

$$D_0 \tilde{x}_0 + a(t_1)\tilde{x}_0^2 = 0, \tag{1.11}$$

$$D_0 \tilde{x}_1 + D_1 \tilde{x}_0 + 2\, a(t_1)\tilde{x}_0\, \tilde{x}_1 = 0, \tag{1.12}$$

which may (in principle) be solved for $\tilde{x}_0$, $\tilde{x}_1$ in terms of arbitrary functions $c_0(t_1)$, $c_1(t_1)$. Various side conditions governing the expected form of the solutions of (1.11), (1.12) may then be applied to obtain equations determining $c_0(t_1)$, $c_1(t_1)$ in terms of constants, which are then evaluated from the known initial condition.

These methods have also been successfully used to generate approximate solutions to a number of related slowly varying population problems. Thus, Shepherd and Stojkov [57] and Stojkov [59] studied the logistic growth model (1.3) where only the carrying capacity K varied slowly. Their successful approach was based on two linear time scales, normal population time, t_0 and slow parameter time t_1, given by

$$t_0 = t, \qquad \text{and} \qquad t_1 = \epsilon t; \qquad \text{with} \qquad \epsilon \ll 1. \tag{1.13}$$

Grozdanovski et al [29] and Shepherd et al [56] considered the more general case where both of R and K were considered to be slowly varying functions and were led to introduce a more general (nonlinear) normal time scale t_0, and slow time t_1, given by

$$t_0 = \frac{1}{\epsilon}\, g(t_1), \qquad \text{and} \qquad t_1 = \epsilon t \tag{1.14}$$

respectively, where $g(t_1)$ is expected to be a positive valued function on all $t_1 > 0$, to be determined, with $g(0) = 0$. A similar analysis for the Gompertz model was carried out by Grozdanovski [25] and Grozdanovski and Shepherd [27] using the more general time scales (1.14). Grozdanovski [26] and Grozdanovski et al[28] considered a related application of population dependent harvesting where the harvesting term expressed was proportional to the population and again the general scale times (1.14) were employed to obtain approximate solutions. Nguyen [50] applied multiscaling analysis to (1.4) where only the harvesting rate, H, varied slowly, while R and K were positive constants. Idlango et al [40] extended the work of Nguyen to the population behaviour of (1.4) for the more general case where R, K and H are functions of slow time using time scales (1.14).

These investigations all involved obtaining approximate solutions using (1.14) in situations where the population evolved to some slowly varying limiting state, (survived) or was driven to zero in finite or infinite time (extinguished). The situation where there was a transition between one state and another via a transition point was analysed by Grozdanovski et al [30], using a mix of multitiming and matching (see Nayfeh [49]) methods. This approach was also used to analyse transitions in a generalized logistic model, in Shepherd et al [56].

In the following chapters, we will apply these multiscaling techniques to analyse the problems (1.4), (1.5) and (1.6) described above, where the model parameters vary on a single common slow time scale.

We will find that these techniques provide useful analytical approximations to the solutions of these problems that are valid for a general range of parameter values. We will compare these approximations with appropriate numerical solutions and find the agreement to be very good indeed.

Chapter 2

A Single Species Harvested Logistic Model

As mentioned in Chapter 1, the logistic model with harvesting was analysed in many studies (See for example [5]), where the parameters were considered to be constants. However, in reality, the model parameters of (1.4) may vary with time, T, and this variation may originate from changes (often periodic) in the surrounding environment such as seasonable variations, food shortage and damage to natural habitats etc. These factors cause us to reconsider the parameters R, K and H as functions of time, so that the initial value problem (1.4) is replaced by

$$\frac{dP}{dT} = R(T)P\left(1 - \frac{P}{K(T)}\right) - H(T), \qquad P(0) = P_0. \tag{2.1}$$

Now, let us assume that $R(T)$, $K(T)$ and $H(T)$ vary on the time scales T_R, T_K and T_H so that these functions may be expressed in the form

$$\begin{aligned} R(T) &= R_0\, r(T/T_R), \\ K(T) &= K_0\, k(T/T_K), \\ H(T) &= H_0\, h(T/T_H), \end{aligned}$$

where R_0, K_0 and H_0 are representative values of these functions respectively and r, k and h are dimensionless functions. When R and/or K and/or H are constant, this means that $R = R_0$ and so $r \equiv 1$ and/or $K = K_0$ and so $k \equiv 1$ and/or $H = H_0$ and so $h \equiv 1$.

By expressing the non-dimensional time scale by $t = R_0 T$ and non-dimensional population scale by $p = P/K_0$, (2.1) can be written in dimensionless form as

$$\frac{dp}{dt} = r(\frac{R_0^{-1}}{T_R}t)\, p \left(1 - \frac{p}{k(\frac{R_0^{-1}}{T_K}t)}\right) - \sigma\, h(\frac{R_0^{-1}}{T_H}t). \tag{2.2}$$

Assuming that R, K, H vary on the same time scale, that is $T_R = T_K = T_H = T^*$, this gives (2.2) as

$$\frac{dp(t,\epsilon)}{dt} = r(\epsilon t)\, p(t,\epsilon) \left(1 - \frac{p(t,\epsilon)}{k(\epsilon t)}\right) - \sigma\, h(\epsilon t), \qquad p(0,\epsilon) = \mu. \tag{2.3}$$

where $\epsilon = (T^* R_0)^{-1}$ measures the ratio of the intrinsic population variation time scale, R_0^{-1}, to T^*, that of R, K and H (see a similar discussion in Grozdanovski et al. [29]). The positive non-dimensional ratios of characteristic values σ (the non-dimensional harvesting rate) and μ (the non-dimensional initial population value) are given by

$$\sigma = \frac{H_0}{R_0\, K_0} \quad \text{and} \quad \mu = \frac{P_0}{K_0} \tag{2.4}$$

respectively.

Note that p, the solution of (2.3) depends on all the parameters ϵ, σ and μ (as well as time, t). However, in the calculations to follow, we will focus attention on its ϵ-dependence only; so that explicit dependence on σ and μ will be suppressed. When T^* is large relative to R_0^{-1}, the variation of R, K and H is small. This is characterised by the condition that ϵ be small.

We note that in (2.3), r, k and h all vary on the same slow time scale ϵt. This is physically reasonable, since we might expect slow variation in the carrying capacity k to be reflected in the growth rate, r. Such slow variation might arise from slowly varying environmental factors. Further, such environmental variation might reasonably be supposed to affect the harvest rate, h.

Thus, (2.3) might reasonably be expected to model a species with growth rate r and carrying capacity k being consumed by a predator which consumes at rate σh. Both species and predator are affected by the slowly varying background environmental factors.

2.1 The Harvesting Model with Constant Parameters

In this case where R, K and H are positive constants, we put $r(\epsilon t) \equiv k(\epsilon t) \equiv h(\epsilon t) \equiv 1$ as noted above. Then, (2.3) becomes

$$\frac{dp(t)}{dt} = p(t)\ (1 - p(t)) - \sigma, \qquad p(0) = \mu. \tag{2.5}$$

where σ and μ given by (2.4).

When solving the initial value problem (2.5) for p, we must consider two distinct cases:

[1] when $\sigma < \frac{1}{4}$ there is subcritical harvesting and the solution of (2.5) is

$$p(t) = \frac{1}{2}\left(1 + \sqrt{1 - 4\,\sigma}\tanh\left[\tfrac{1}{2}(\sqrt{1 - 4\sigma}\, t + c)\right]\right), \tag{2.6}$$

where

$$c = 2\text{arctanh}(\frac{2\,\mu - 1}{\sqrt{1 - 4\sigma}}). \tag{2.7}$$

Since the value of c depends on the value of μ, this leads us to discuss the following points:

(a) c has a real value when

$$|\frac{2\,\mu - 1}{\sqrt{1 - 4\sigma}}| < 1 \tag{2.8}$$

which requires that μ lies in the range

$$\tfrac{1}{2}(1 - \sqrt{1 - 4\sigma}) < \mu < \tfrac{1}{2}(1 + \sqrt{1 - 4\sigma}). \tag{2.9}$$

In this case, and from (2.6) we can see that as time progresses, the tanh(..) function tends to 1 and so the population will tend to a nonzero limiting state i.e.,

$$p(t) \to \tfrac{1}{2}(1 + \sqrt{1 - 4\,\sigma}), \tag{2.10}$$

and so, the population survives.

(b) c is a complex when μ lies outside the interval (2.8), and since in this case, c can be expressed as $c = c_R \pm i\,\pi$, where c_R is a real part then

$$\tanh(\tfrac{1}{2}(\sqrt{1 - 4\sigma}\,t + c_R \pm i\,\pi)) = \coth(\tfrac{1}{2}(\sqrt{1 - 4\sigma}\,t + c_R)).$$

This leads to replace the hyperbolic tangent in (2.6) with a hyperbolic cotangent. i.e., the solution of (2.5) becomes

$$p(t) = \frac{1}{2}\left(1 + \sqrt{1 - 4\,\sigma}\coth\left[\tfrac{1}{2}(\sqrt{1 - 4\sigma}\,t + c_R)\right]\right), \tag{2.11}$$

where

$$c_R = 2\mathrm{arccoth}(\frac{2\,\mu - 1}{\sqrt{1 - 4\sigma}}). \tag{2.12}$$

The behaviour of the solution (2.11) is now determined by the value of c_R, via the value of μ.

I). *When μ satisfies*

$$\mu > \tfrac{1}{2}(1 + \sqrt{1 - 4\sigma}), \tag{2.13}$$

so that $0 < c_R < \infty$, $p(t) \to \frac{1}{2}(1+\sqrt{1 - 4\,\sigma})$ as $t \to \infty$, i.e., the population $p(t)$ survives to a limiting positive finite state (2.10).

II). When μ *satisfies*

$$\mu < \tfrac{1}{2}(1 - \sqrt{1 - 4\sigma}), \tag{2.14}$$

so that $-\infty < c_R < 0$, $p(t) \to -\infty$ as $t \to -2\,c_R/\sqrt{1 - 4\sigma} > 0$, and the population $p(t)$ presented by (2.11) reduces to zero in finite time.

If t_d represents the extinction time of the population with subcritical harvesting, then

$$\frac{1}{2}\left(1+\sqrt{1-4\,\sigma}\coth\left[\tfrac{1}{2}(\sqrt{1-4\sigma}\,t_d+c_R)\right]\right)=0,$$

where c_R is given by (2.12) and this gives t_d as

$$t_d=\frac{-2}{\sqrt{1-4\sigma}}\left(\operatorname{arccoth}(\frac{2\mu-1}{\sqrt{1-4\sigma}})-\operatorname{arccoth}(\frac{-1}{\sqrt{1-4\sigma}})\right). \tag{2.15}$$

[2] when $\sigma>\frac{1}{4}$ the harvesting is supercritical, and so the solution of (2.5) becomes

$$p(t)=\frac{1}{2}\left(1-\sqrt{4\,\sigma-1}\tan\left[\tfrac{1}{2}\sqrt{4\sigma-1}t+\arctan(\frac{1-2\,\mu}{\sqrt{4\sigma-1}})\right]\right). \tag{2.16}$$

In this case, as $\left\{\frac{1}{2}\sqrt{4\sigma-1}\,t+\arctan(\frac{1-2\,\mu}{\sqrt{4\sigma-1}})\right\}\to\pi/2$, the tangent function tends towards $-\infty$, so that the population presented by (2.16) reduces to zero in finite time; i.e., the population will become extinct in finite time.

Now, if we suppose that at time t_e the population with supercritical harvesting extinguishes, that is, the population reaches zero at finite time t_e and so

$$\frac{1}{2}\left(1-\sqrt{4\,\sigma-1}\tan\left[\tfrac{1}{2}(\sqrt{4\sigma-1}\,t_e)+\arctan(\frac{1-2\,\mu}{\sqrt{4\sigma-1}})\right]\right)=0.$$

Thus

$$t_e=\frac{2}{\sqrt{4\sigma-1}}\left(\arctan(\frac{1-2\,\mu}{\sqrt{4\sigma-1}})+\arctan(\frac{1}{\sqrt{4\sigma-1}})\right) \tag{2.17}$$

The above subcases **(a)-(b)** of the solutions of (2.5) are displayed in Figures 2.1-2.4.

The Figure 2.1 illustrates the subcritical harvesting (where $\sigma=0.1<\frac{1}{4}$) for initial populations $\mu=0.35,\,0.6$, that satisfy the inequality (2.8), i.e.,

$$0.113<\mu<0.887,$$

and as we can see the population survives to the finite state (2.10) at the value of 0.89. Figure 2.2, with the initial populations $\mu=1.2,\,2$ satisfying the criterion (2.13) and

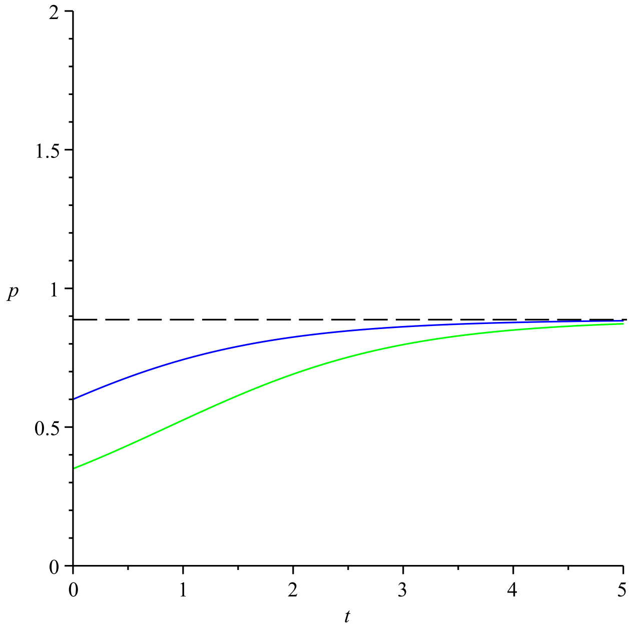

Figure 2.1: A subcritical harvesting with survival that is given by (2.6) where $\mu = 0.35, 0.6$ and $\sigma = 0.1$. The dotted line indicates the limiting state of value≈ 0.89.

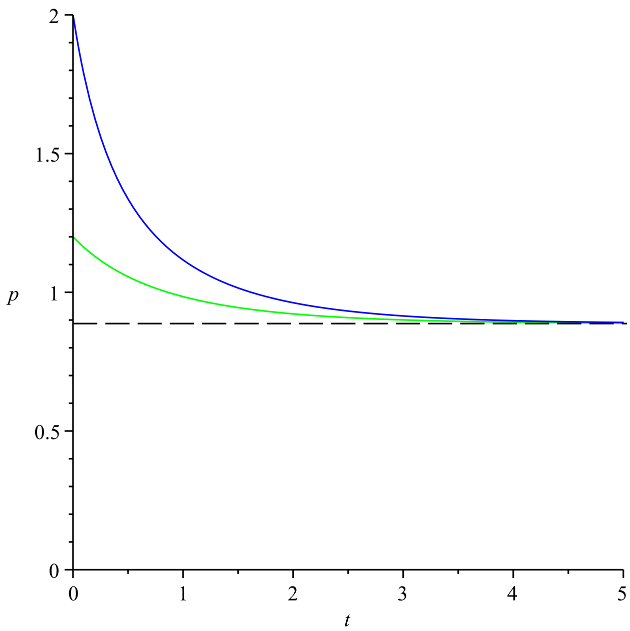

Figure 2.2: A subcritical harvesting with survival given by (2.11) with the same data as Figure 2.1 and $\mu = 1.2, 2,$.

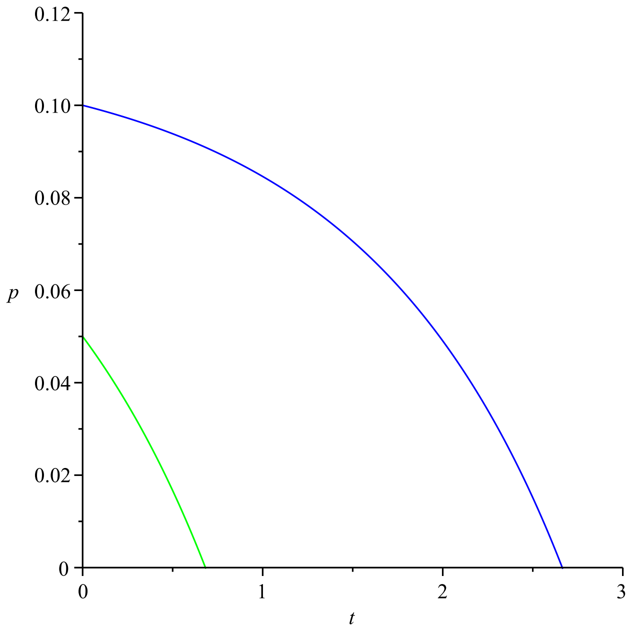

Figure 2.3: A subcritical harvesting with extinction for different μ values of 0.05, 0.1 and $\sigma = 0.1$.

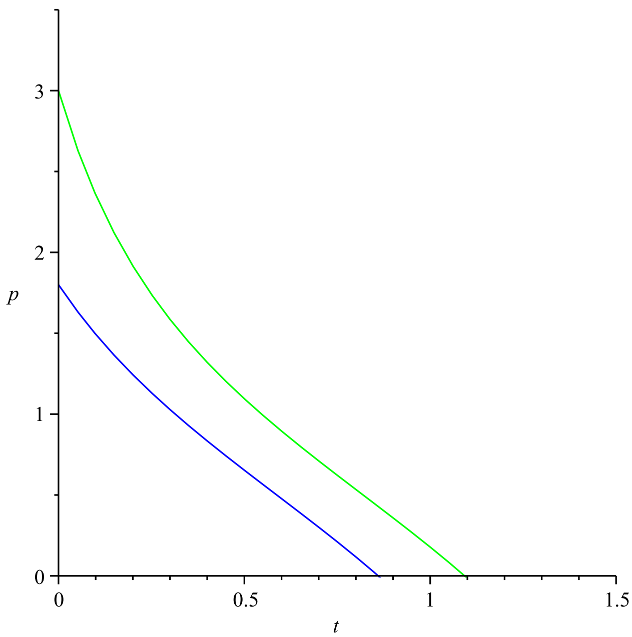

Figure 2.4: Constant supercritical harvesting with extinction (2.16) where $\mu = 1.8$, 3 and $\sigma = 2$.

higher than limiting state, shows that the population declines but to surviving state. However, in Figure 2.3, even though the harvesting is subcritical, the initial population $\mu = 0.05,\ 0.1 < 0.113$ is too low, so that it becomes extinct in finite time (2.15), where $t_d = 0.68$, 2.66 respectively.

In Figure 2.4, where $\sigma = 2 > \frac{1}{4}$, the harvesting is supercritical, satisfying (2.16). It is shown that for the values of $\mu = 1.8$, 3, populations die out in a very short period of time t_e expressed by (2.17), where $t_e = 0.86$, 1.09 respectively.

2.2 Slowly Varying Harvesting Model Parameters

We now consider the model (2.3) where r, k and h vary with time. The time variation of these parameters means that exact solution of this initial value problem is virtually impossible in general and numerical methods must be used to construct an approximate solution, with the consequent restriction of r, k, h and ϵ, σ, μ to specific functions and values respectively.

However, when ϵ is small; that is, the time scale of variation of R, K and H is large relative to that of the overall population P, the problem (2.3) may be viewed as one involving two time scales - a fast scale, t, and a slow scale, ϵt. Then, as we will see below, (2.3) may be solved approximately using a multiscaling method based on these two time scales, in the limit $\epsilon \to 0$.

2.2.1 The Multiscale Harvesting Equation

Since the problem (2.3) involves behaviour on two time scales - slow time, ϵt and normal time t, we apply a multiscaling analysis to this problem, which follows the lead of earlier investigations see [28, 26, 40] as discussed in Chapter 1. We propose the generalized normal time t_0, and a slow time, t_1 as defined in (1.14) by

$$t_0 = \frac{1}{\epsilon}\, g(t_1), \qquad \text{and} \quad t_1 = \epsilon t \tag{2.18}$$

respectively, where $g(t_1)$ is expected to be a positive valued function on all $t_1 > 0$, to be determined, with $g(0) = 0$.

Differentiating (2.18) gives

$$dt_0 = g'(t_1)\, dt$$

and we ensure one-to-one correspondence between t_0 and t by requiring that $g'(t_1) > 0$ on all $t_1 \geq 0$.

In what follows, we regard $p(t, \epsilon)$, the solution of (2.3), as a function $\tilde{p}(t_0, t_1, \epsilon)$ of the variables (2.18), i.e.,

$$p(t, \epsilon) \equiv \tilde{p}(t_0, t_1, \epsilon). \tag{2.19}$$

By applying the chain rule and substituting, we convert the ordinary differential equation in (2.3) to the multiscaled harvesting equation

$$g'(t_1) D_0\, \tilde{p} + \epsilon\, D_1\, \tilde{p} = r(t_1)\, \tilde{p}\, \left(1 - \frac{\tilde{p}}{k(t_1)}\right) - \sigma h(t_1), \tag{2.20}$$

where D_0 and D_1 denote partial derivatives taken with respect to t_0 and t_1 respectively. Note that (2.20) is a partial differential equation for the unknown function $\tilde{p}(t_0, t_1, \epsilon)$, and ϵ is displayed explicitly in (2.20) rather than implicitly as in (2.3). This will allow us to use a perturbation technique to construct an approximate solution of (2.3) that is valid for all $t \geq 0$.

2.3 Perturbation Analysis

We now express $\tilde{p}(t_0, t_1, \epsilon)$ as a Poincaré expansion in ϵ

$$\tilde{p}(t_0, t_1, \epsilon) = \tilde{p}_0(t_0, t_1) + \epsilon \tilde{p}_1(t_0, t_1) + \epsilon^2 \tilde{p}_2(t_0, t_1) + \ldots, \tag{2.21}$$

and on substituting (2.21) into (2.20), expanding in powers of ϵ and equating coefficients of like powers of ϵ, obtain a differential equation for the leading order term $\tilde{p}_0$ as

$$g'(t_1)\, D_0\tilde{p}_0 - r(t_1)\,\tilde{p}_0 \left(1 - \frac{\tilde{p}_0}{k(t_1)}\right) = \sigma h(t_1), \tag{2.22}$$

and on considering $O(\epsilon)$ terms, an analogous equation for $\tilde{p}_1$ as

$$g'(t_1)\, D_0\tilde{p}_1 - r(t_1) \left(1 - \frac{2\,\tilde{p}_0}{k(t_1)}\right) \tilde{p}_1 = -D_1\tilde{p}_0. \tag{2.23}$$

Solving the partial differential equation (2.22) for $\tilde{p}_0$ gives

$$\tilde{p}_0(t_0, t_1) = \tfrac{1}{2}k(t_1)\,(1 + \eta(t_1)\,\tanh\left[\theta(t_1)\,(t_0 + F(t_1))\right]) \tag{2.24}$$

where

$$\begin{aligned} \eta(t_1) &= \sqrt{\delta(t_1)}, \\ \delta(t_1) &= 1 - \frac{4\,\sigma h(t_1)}{r(t_1)\,k(t_1)}, \\ \theta(t_1) &= \frac{r(t_1)\,\eta(t_1)}{2\,g'(t_1)} \end{aligned} \tag{2.25}$$

and $F(t_1)$ is an arbitrary function of t_1.

Substituting (2.24) into (2.23) and solving gives a particular solution for $\tilde{p}_1$ as

$$\begin{aligned} \tilde{p}_1(t_0, t_1) = &-\frac{1}{4\,\theta(t_1)\,g'(t_1)}\,\left\{(k(t_1)\,\eta(t_1))' + k'(t_1)\tanh\left[\theta(t_1)\,(t_0 + F(t_1))\right]\right\} \\ &- \left[k'(t_1)\,t_0 + k(t_1)\,\eta(t_1)\left\{\theta'(t_1)\,\left(t_0^2 + 2F(t_1)\,t_0\right) + 2\theta(t_1)\,F'(t_1)\,t_0\right\}\right] \\ &\times\frac{1}{4\,g'(t_1)}\,\mathrm{sech}^2\left[\theta(t_1)\,(t_0 + F(t_1))\right]. \end{aligned} \tag{2.26}$$

To this point, we have placed no restriction on the parameters r, k and h. Our only condition has been that $g(t_1)$ and $g'(t_1)$ be real and positive on $t_1 \geq 0$. However, it is clear that for certain combinations of r, k and h, $\eta(t_1)$ given by (2.25) is imaginary; and this affects the nature of $\tilde{p}_0$, $\tilde{p}_1$ as functions of t_0; and consequently, that of the expansion (2.21).

Therefore, in the next sections, we divide our analysis into two parts; when

$$\delta(t_1) > 0 \qquad \text{for all} \quad t_1 \geq 0; \tag{2.27}$$

termed *subcritical harvesting*; and when

$$\delta(t_1) < 0 \qquad \text{for all} \quad t_1 \geq 0; \tag{2.28}$$

termed *supercritical harvesting.*

We now consider the behavior of $\tilde{p}_0$ and $\tilde{p}_1$, given by (2.24) and (2.26) as functions of t_0.

When the harvesting is *subcritical*, $\eta(t_1)$ and $\theta(t_1)$ are real and positive functions of t_1 on $t_1 \geq 0$. Thus, $\tilde{p}_0$ tends to a finite limit

$$k(t_1)(1+\eta(t_1))/2 \qquad \text{as} \quad t_0 \to \infty$$

and the rate of convergence to this limit is exponential, of the form of $\mathrm{e}^{-2\theta(t_1)t_0}$.

Similarly, as $t_0 \to \infty$, $\tilde{p}_1$ given by (2.26) tends to the limit

$$-\left\{k'(t_1) + (k(t_1)\,\eta(t_1))'\right\} / \left\{4\,\theta(t_1)\,g'(t_1)\right\} \qquad \text{as} \quad t_0 \to \infty.$$

However, the presence of the t_0 and t_0^2 terms in (2.26) means that this convergence is not exponential (as for $\tilde{p}_0$). This convergence to the limit may be made exponential, so that $\tilde{p}_1$ reaches its limit at the same rate as $\tilde{p}_0$ by removing these terms. To do this, we set the coefficients of t_0 and t_0^2 in (2.26) separately to zero. This leads to

$$k'(t_1) + 2\,k(t_1)\,\eta(t_1)\;(\theta(t_1)\,F(t_1))' = 0 \tag{2.29}$$

and

$$k(t_1)\,\eta(t_1)\,\theta'(t_1) = 0; \tag{2.30}$$

we have two equations for the two unknown functions $F(t_1)$ and $g(t_1)$.

When the harvesting is *supercritical*, $\eta(t_1)$ and $\theta(t_1)$ are pure imaginary. Then, $\tilde{p}_0$ is a periodic function of t_0. However, the t_0 and t_0^2 terms in (2.26) mean that $\tilde{p}_1$ is not a periodic function of t_0. If we argue that $\tilde{p}_1$ should reflect the periodic nature of $\tilde{p}_0$, we again arrive at the conclusion that the coefficients of the t_0 and t_0^2 terms in (2.26) should again be equated to zero. Thus, we again at arrive (2.29) and (2.30).

In each case, (2.30) leads to the conclusion that $\theta'(t_1) = 0$; so that $\theta(t_1)$ is a constant; while with this, (2.29) gives

$$F'(t_1) = -\tfrac{1}{2}\,\frac{k'(t_1)}{k(t_1)\,\eta(t_1)\,\theta(t_1)}. \tag{2.31}$$

We now consider the implications of this for subcritical and supercritical cases.

2.3.1 Subcritical Harvesting

Here, (2.27) applies; and by the arguments above, $\theta(t_1)$ is a real constant. Thus, choosing $\theta(t_1) = \frac{1}{2}$, we have, from (2.25)

$$g'(t_1) = r(t_1)\,\chi(t_1) \qquad \text{where} \quad \chi(t_1) = \sqrt{|\delta(t_1)|}, \tag{2.32}$$

while from (2.32) and (2.18), we obtain

$$t_0 = \frac{1}{\epsilon}\int_0^{t_1} r(s)\,\chi(s)\,\mathrm{d}s, \tag{2.33}$$

defining the *normal time* t_0.

Similarly, equation (2.31) gives

$$F(t_1) = A(t_1) + c \qquad \text{where} \quad A(t_1) = -\int_0^{t_1} \frac{k'(s)}{k(s)\,\chi(s)}\,\mathrm{d}s, \tag{2.34}$$

and c is an arbitrary constant.

Since our expansion consists of leading order terms and $O(\epsilon)$ terms, we suppose that c in (2.34) has the form $c = c_0 + \epsilon c_1 + \ldots$.

Substituting (2.24) and (2.26) back into (2.21) and considering (2.19) give the expansion for $p(t,\epsilon)$, the exact solution of (2.3) in powers of ϵ, as

$$p(t,\epsilon) = p_0(t,\epsilon) + \epsilon\,p_1(t,\epsilon) + O(\epsilon^2), \tag{2.35}$$

where

$$p_0(t,\epsilon) = \tfrac{1}{2}k(t_1)\left\{1+\chi(t_1)\,\tanh\left[\tfrac{1}{2}(t_0+A(t_1)+c_0)\right]\right\}, \tag{2.36}$$

and

$$\begin{aligned} p_1(t,\epsilon) = {} & \frac{1}{2\,r(t_1)\chi(t_1)}\left\{(k(t_1)\,\chi(t_1))' + k'(t_1)\,\tanh\left[\tfrac{1}{2}(t_0+A(t_1)+c_0)\right]\right\} \\ & + \tfrac{1}{4}\,c_1\,k(t_1)\,\chi(t_1)\,\mathrm{sech}^2\left[\tfrac{1}{2}(t_0+A(t_1)+c_0)\right], \end{aligned} \tag{2.37}$$

where t_0 and t_1 are defined as functions of t by (2.33) and $\epsilon\, t$ respectively.

Substituting the initial condition from the problem (2.3) into (2.35), using (2.36) and (2.37), expanding and equating like powers of ϵ gives equations for c_0 and c_1 which, when solved respectively, give

$$c_0 = 2\mathrm{arctanh}\left[\frac{2\mu - k(0)}{k(0)\,\chi(0)}\right], \tag{2.38}$$

$$c_1 = 2\,\frac{k'(0)(2\,\mu\, - k(0)) + k(0)\chi(0)\,\left.(k(t_1)\,\chi(t_1))'\right|_{t_1=0}}{r(0)\chi(0)\,[k^2(0)\,\chi^2(0) - (2\,\mu - k(0))^2]}. \tag{2.39}$$

From (2.39), c_1 is always real. However, as discussed in Section 2.1, c_0 given by (2.38) depends on the values of μ and contains the function arctanh(.), so there are three possibilities:

(1) When

$$\left|\frac{2\mu - k(0)}{k(0)\,\chi(0)}\right| \;<\; 1, \tag{2.40}$$

or

$$\frac{k(0)}{2}(1-\chi(0)) < \;\mu\; < \frac{k(0)}{2}(1+\chi(0)),$$

c_0 **is real valued**.

Therefore, the population (as represented by (2.35)) tends to the slowly varying limiting state

$$\tfrac{1}{2}k(t_1)\left\{1+\chi(t_1)\right\} - \epsilon\left\{\frac{(k(t_1)\,\chi(t_1))' + k'(t_1)}{2\,r(t_1)\chi(t_1)}\right\} + O(\epsilon^2). \tag{2.41}$$

This case comprises *subcritical harvesting with survival.* We note that (2.41) is corresponded to constant limiting state (2.10) where $r(t_1) \equiv k(t_1) \equiv h(t_1) \equiv 1$.

(2) When

$$\left|\frac{2\mu - k(0)}{k(0)\,\chi(0)}\right| > 1, \tag{2.42}$$

i.e., μ does not satisfy (2.40), there are two possibilities:

(a) when

$$\frac{2\mu - k(0)}{k(0)\,\chi(0)} > 1, \tag{2.43}$$

i.e.,

$$\mu > \tfrac{1}{2}k(0)\,(1 + \chi(0)), \tag{2.44}$$

then c_0 **given by** (2.38) **becomes complex**; i.e., $c_0 = c_{0R} - i\pi$, where c_{0R} and is expressed by

$$c_{0R} = 2\mathrm{arccoth}\left[\frac{2\mu - k(0)}{k(0)\,\chi(0)}\right], \tag{2.45}$$

This has the effect of replacing the hyperbolic tangent in the leading term of (2.35) with a hyperbolic cotangent, with appropriate subsequent replacement in the $O(\epsilon)$ term. i.e.,

$$\tanh[\tfrac{1}{2}(t_0 + A(t_1) + c_{0R} \pm i\pi)] = \coth[\tfrac{1}{2}(t_0 + A(t_1) + c_{0R})]$$

and so (2.35) becomes

$$p(t,\epsilon) = p_0(t,\epsilon) + \epsilon\, p_1(t,\epsilon) + O(\epsilon^2), \tag{2.46}$$

where

$$p_0(t,\epsilon) = \tfrac{1}{2}k(t_1)\left\{1 + \chi(t_1)\,\coth\left[\tfrac{1}{2}(t_0 + A(t_1) + c_{0R})\right]\right\}, \tag{2.47}$$

and

$$\begin{aligned} p_1(t,\epsilon) = {} & \frac{1}{2\,r(t_1)\chi(t_1)}\left\{(k(t_1)\,\chi(t_1))' + k'(t_1)\,\coth\left[\tfrac{1}{2}(t_0 + A(t_1) + c_{0R})\right]\right\} \\ & + \tfrac{1}{4}\,c_1\,k(t_1)\,\chi(t_1)\,\mathrm{csch}^2\left[\tfrac{1}{2}(t_0 + A(t_1) + c_{0R})\right] \end{aligned} \tag{2.48}$$

As $t_0 + A(t_1) + c_0 \to \infty$, the hyperbolic cotangent function tends downwards to 1 while the hyperbolic cosecant function tends to *zero*. Thus (2.46) tends to the limiting state (2.41) and again this case comprises *subcritical harvesting with survival.*

(b) when

$$\frac{2\mu - k(0)}{k(0)\,\chi(0)} < -1, \tag{2.49}$$

i.e.,

$$0 < \mu < \tfrac{1}{2}k(0)\,(1 - \chi(0)), \tag{2.50}$$

c_0 **is complex** such that (2.38) becomes $c_0 = c_{0R} + i\pi$ where

$$c_{0R} = 2\text{arccoth}\left[\frac{2\mu - k(0)}{k(0)\,\chi(0)}\right], \tag{2.51}$$

Again, as in a similar discussion in Section 2.1, this has the effect of replacing the hyperbolic tangent in the leading term of (2.35) with a hyperbolic cotangent, with appropriate subsequent replacement in the $O(\epsilon)$ term as (2.46). However, the hyperbolic cotangent tends to $-\infty$ at some finite $t-$value. Thus, (2.46) reaches zero in finite time, signifying an extinction of the population in finite time and this case comprises *subcritical harvesting with extinction.*

The inequality (2.50) provides a criterion by which we may determine initial populations μ for which the population is driven to extinction, even though the harvesting is subcritical.

2.3.2 Supercritical Harvesting

In this case where $\delta(t_1)$ satisfies (2.28), $\eta(t_1) = i\,\chi(t_1)$ and we choose $\theta(t_1) = \frac{i}{2}$ so that t_0 is again defined by (2.33), while

$$F(t_1) = -A(t_1) + d, \tag{2.52}$$

where d is an arbitrary constant and $A(t_1)$ is as defined in (2.34). Again, substituting $d = d_0 + \epsilon d_1 + \dots$ into (2.24) and (2.26) and considering (2.19) give the expansion for $p(t, \epsilon)$, the exact solution of (2.3) in powers of ϵ, as

$$p(t,\epsilon) = p_0(t,\epsilon) + \epsilon\, p_1(t,\epsilon) + O(\epsilon^2) \tag{2.53}$$

where

$$p_0(t,\epsilon) = \tfrac{1}{2}k(t_1)\left\{1 - \chi(t_1)\,\tan\left[\tfrac{1}{2}(t_0 - A(t_1) + d_0)\right]\right\} \tag{2.54}$$

$$\begin{aligned} p_1(t,\epsilon) = {} & \frac{1}{2r(t_1)\chi(t_1)}\left\{(k(t_1)\chi(t_1))' + k'(t_1)\,\tan\left[\tfrac{1}{2}(t_0 - A(t_1) + d_0)\right]\right\} \\ & -\tfrac{1}{4}\,d_1\,k(t_1)\,\chi(t_1)\,\sec^2\left[\tfrac{1}{2}(t_0 - A(t_1) + d_0)\right] + O(\epsilon^2). \end{aligned} \tag{2.55}$$

Substituting the initial condition (2.3) into the expansion (2.53), using (2.54) and (2.55), and equating like powers of ϵ as above, we obtain

$$d_0 = 2\arctan\left\{\frac{k(0) - 2\mu}{k(0)\,\chi(0)}\right\}, \tag{2.56}$$

$$d_1 = -2\,\frac{k'(0)(k(0) - 2\mu) + k(0)\chi(0)\;(k(t_1)\,\chi(t_1))'\big|_{t_1=0}}{r(0)\chi(0)\left[k^2(0)\,\chi^2(0) + (k(0) - 2\mu)^2\right]}. \tag{2.57}$$

where we noted that d_0 and d_1 are always real valued.

From (2.54), (2.55) as $t_0 \to \infty$, the tangent and secant functions tend to ∞, so that the population dies out in a finite time period.

2.4 Examples

In this section, we show examples of comparisons between the analytic approximations (2.35), (2.53) and numerical solution of the original initial value problem (2.3) where the solid curve represents the series approximation, while the dotted line represents the numerical solution of (2.3). In these, typical values of ϵ are $\epsilon = 0.02 - 0.05$, which could be regarded as typical of time scales relevant to weekly versus annular time variation.

Because of the complexity of the expansions for the functions r, k and h, as well as for the two term expansion (2.35), involving p_0, p_1 obtained from (2.36), (2.37), (2.47), (2.48), (2.54) and (2.55),we will not display these explicitly, but we will understand that the appropriate substitution of chosen parameters into has been made.

2.4.1 Subcritical Harvesting with Exponential Carrying Capacity

In this example, we propose that the growth rate, r, increases exponentially and the carrying capacity, k, varies exponentially with small saturation, while h is constant; that is,

$$r(\epsilon\, t) = r_0 + \Delta\, e^{\epsilon\, t}, \quad k(\epsilon\, t) = k_0 + \Omega \tanh(\epsilon\, t), \quad h(\epsilon\, t) = 1, \tag{2.58}$$

where all of r_0, k_0, Δ and Ω are positive. From (2.25) we have

$$\delta(\epsilon t) = 1 - 4\, \frac{\sigma(1)}{(r_0 + \Delta\, e^{\epsilon\, t})(k_0 + \Omega \tanh(\epsilon t))} \tag{2.59}$$

and since r and k are positive monotonically increasing functions for all $t \geq 0$, i.e., $0 < r_0 + \Delta \leq r(\epsilon\, t) < \infty$, and $0 < k_0 \leq k(\epsilon\, t) < k_0 + \Omega$, and

$$1 - 4 \frac{\sigma}{(r_0 + \Delta) k_0} < \delta(\epsilon t) < 1, \tag{2.60}$$

so that $\delta(t_1) > 0$ if

$$0 < \sigma < \frac{(r_0 + \Delta) k_0}{4}. \tag{2.61}$$

For the values of r_0, k_0, Δ, Ω, and μ used in Figure 2.5 we have

$$\frac{(r_0 + \Delta)(k_0)}{4} = \frac{(1.02)(1)}{4} = 0.255;$$

and the value $\sigma = 0.05$ satisfies (2.61). So, the harvesting is subcritical, and $0.8 < \delta(\epsilon t) < 1$ for all $t \geq 0$.

Figure 2.5 shows the growth of the population for this choice of r, k and h, when $\mu = 0.2$ (

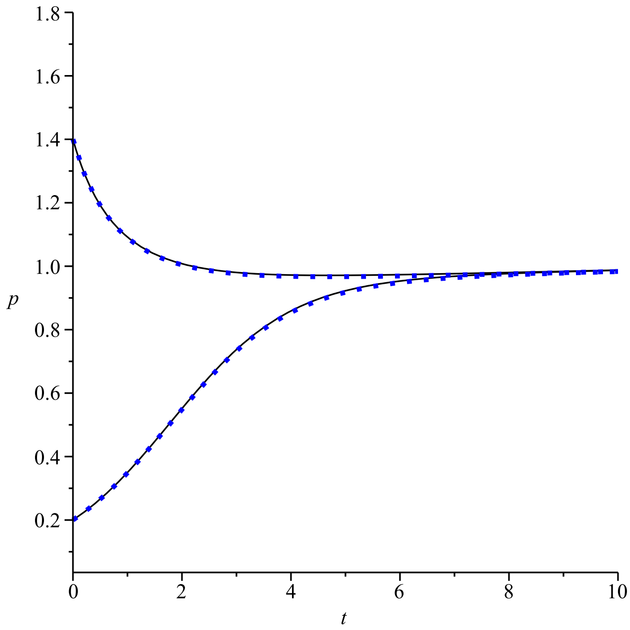

Figure 2.5: Subcritical multiscale expansion (black solid) given by (2.35) and (2.46) with numerical solution (blue dotted) using (2.58) where $r_0 = k_0 = 1$, $\Delta = 0.02$, $\Omega = 0.05$ and $\mu = 0.2, 1.4$, $\epsilon = 0.02$ and $\sigma = 0.05$.

which satisfies (2.40) since $\frac{k(0)}{2}(1-\chi(0)) = 0.05 < \mu < \frac{k(0)}{2}(1+\chi(0)) = 0.95$) gives a very good agreement between the approximate expansions (2.36), (2.37) (in terms of hyperbolic tangent) and the numerical solution of (2.3), as well as with the approximate expansions (2.47), (2.48)(in terms of hyperbolic cotangent) when $\mu = 1.4$ (that satisfies (2.44) since $\mu > \frac{k(0)}{2}(1+\chi(0)) = 0.95$). As expected, in both cases the population survives to a slowly varying limiting state given by (2.41) where $\delta(\epsilon t)$ is given by (2.60) and $\chi(\epsilon t) = \sqrt{\delta(\epsilon t)}$. However, when μ satisfies (2.50), we expect extinction to occur, Figure 2.6 shows this situation for the same choice of r, k and h as above (and when $\mu = 0.04 < \frac{k(0)}{2}(1-\chi(0)) = 0.051$; i.e., μ satisfies (2.50)). Clearly, the population is doomed to extinction. Again, the plots demonstrate good agreement between the numerical and asymptotic solutions.

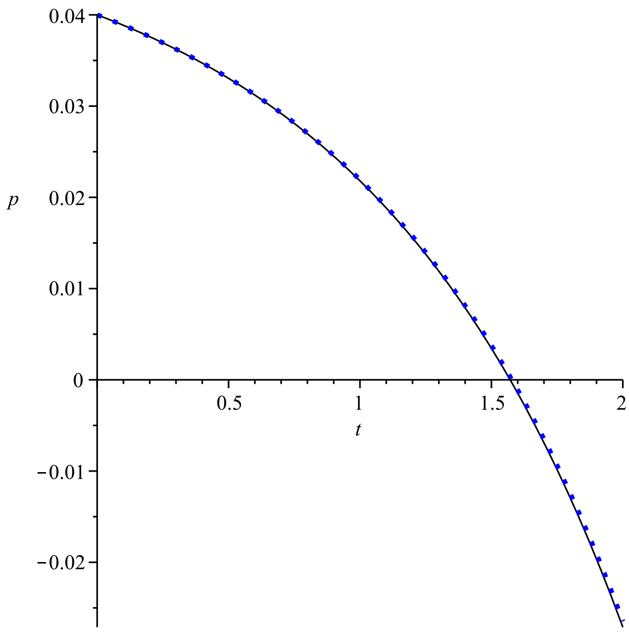

Figure 2.6: Subcritical multiscale expansion (black solid) given by (2.46) with numerical solution (dotted) using data of Figure 2.5 with $\mu = 0.04$.

2.4.2 Subcritical Harvesting with Periodic Carrying Capacity and Harvesting

For a second application, the slowly varying functions h, k are assumed to have the form

$$k(\epsilon\, t) = 1 + 0.13\, \cos(\epsilon\, t), \quad h(\epsilon\, t) = 1 + 0.2\, \tan(\epsilon\, t) \;\text{ and }\; r(\epsilon\, t) = 1, \tag{2.62}$$

From (2.25), we have

$$\delta(\epsilon\, t) = 1 - 4\frac{\sigma(1 + 0.2\, \tan(\epsilon\, t))}{1 + 0.13\, \cos(\epsilon\, t)},$$

and

$$\delta(\epsilon t) \geq 1 - \frac{4\sigma}{1.13} \;\text{ for all }\; t \geq 0;$$

so that $\delta(\epsilon t) > 0$ when $\sigma < 0.28$. Thus with the choice $\sigma = 0.11$ used in Figure 2.7, the harvesting is subcritical and $\delta(\epsilon\, t) > 0.38$ for all $t \geq 0$. .

Also, the value of initial population μ in Figure 2.7 satisfies the inequality (2.40), since

$$\frac{k(0)}{2}(1 - \chi(0)) = 0.125 < \; \mu = 0.8 \; < \frac{k(0)}{2}(1 + \chi(0)) = 1.008,$$

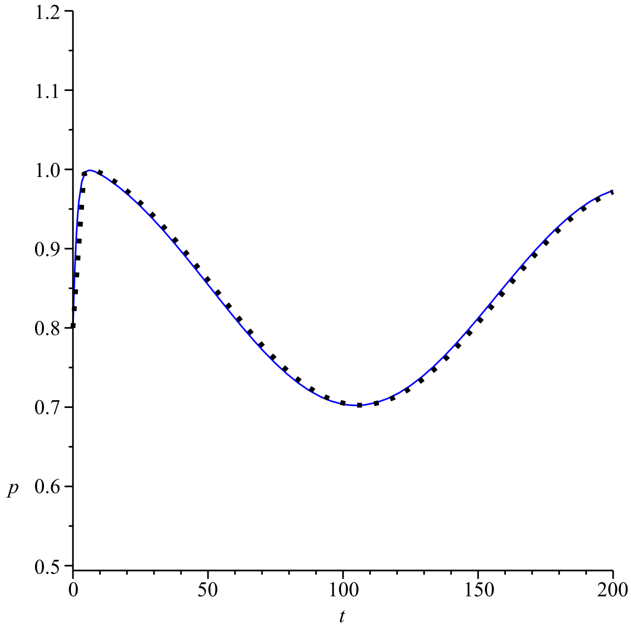

Figure 2.7: Survival population with subcritical harvesting given by (2.35) (in terms of hyperbolic tangent) involving (2.62) with $\sigma = 0.11$, $\epsilon = 0.03$ and $\mu = 0.8$.

while in Figure 2.8 $\mu = 1.2 > 1.008$ which satisfies (2.44).

Figures 2.7-2.8 show the variation of the surviving populations for k, h defined by (2.62) where $\mu = 0.8,\ 1.2$ satisfying (2.40), (2.44) respectively. The agreement between the multiscale approximation (solid curve) and numerical solution (dotted curve) is clearly very good.

However, in Figure 2.9 the population becomes extinct, since $\mu = 0.1$ satisfies (2.50), where $\mu = 0.1 < \frac{k(0)}{2}(1 - \chi(0)) = 0.125$. Again the graphs show a good agreement between analytic approximation and the numerical solution.

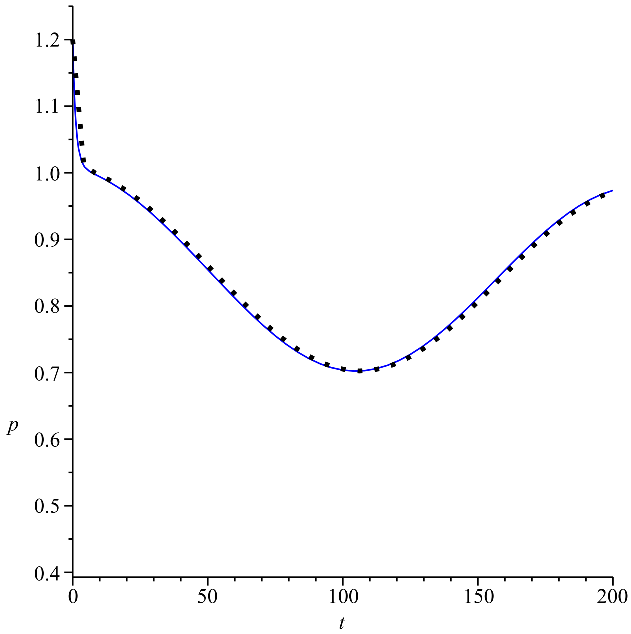

Figure 2.8: Survival with subcritical harvesting given by (2.46) (in terms of hyperbolic cotangent) with the same data as Figure 2.7, but $\mu = 1.2$.

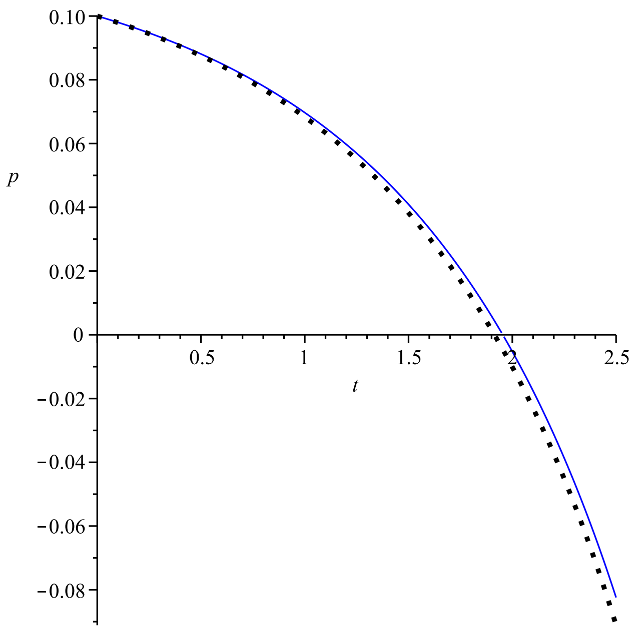

Figure 2.9: Subcritical harvesting with extinction given by (2.46) (in terms of hyperbolic cotangent) using same data as Figure 2.7, but $\mu = 0.1$.

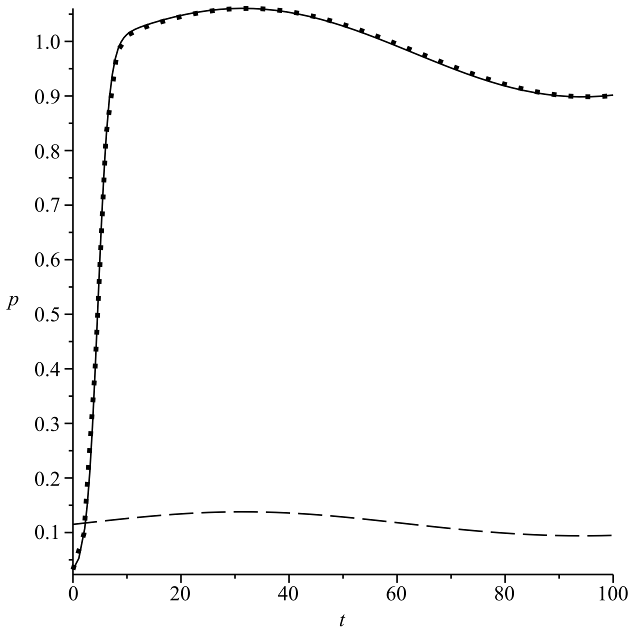

Figure 2.10: Evolution of the population subject to subcritical harvesting with survival to a slowly varying state given by (2.35) using (2.63) where $\sigma = 0.02$, $\mu = 0.032$ and $\epsilon = 0.05$.

2.4.3 Subcritical Harvesting with Slowly Varying Periodic Growth Rate, Carrying Capacity and Harvesting

Here, we consider r, k and h to be periodically slowly varying as follow

$$r(\epsilon t) = 1 + 0.1\sin(\epsilon t), \quad k(\epsilon t) = 1 + 0.08\sin(\epsilon t), \quad h(\epsilon t) = 1 + 0.05\sin(\epsilon t). \tag{2.63}$$

Here the harvesting is subcritical since

$$\begin{aligned}\delta(\epsilon t) &= 1 - 4\frac{\sigma(1 + 0.05\sin(\epsilon t))}{(1 + 0.1\sin(\epsilon t))(1 + 0.08\sin(\epsilon t))} \\ &> 1 - 4\,\sigma \quad \text{for all} \quad t \geq 0\end{aligned}$$

and with the choice $\sigma = 0.02$ used in Figure 2.10, thus $\delta(\epsilon t) > 0.92$.

Figure 2.10 shows population evolution under subcritical harvesting from an initial $\mu = 0.032$ for these r, k and h. Here, $\mu = 0.032 > \frac{1}{2}k(0)\,(1 - \chi(0)) \approx 0.02$, , satisfying the criterion (2.44).

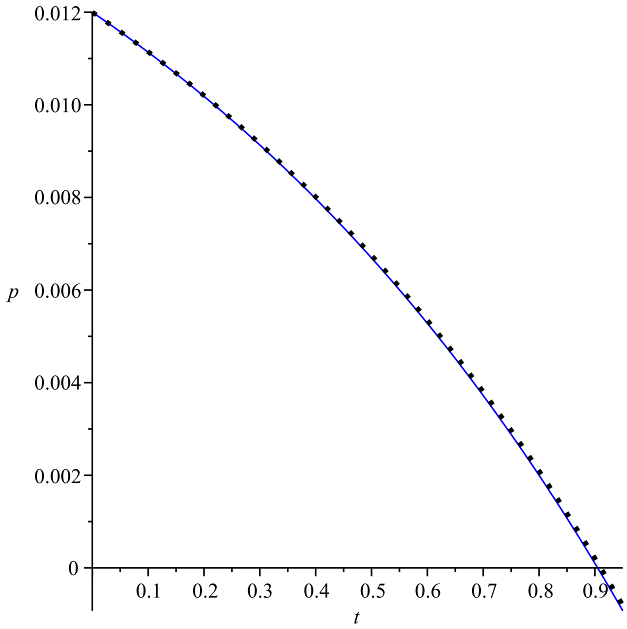

Figure 2.11: Evolution of the population subject to the subcritical harvesting given by (2.46) of data of Figure 2.10, but now with starting population $\mu = 0.012$ satisfies (2.50). The dashed curve shows 10δ , where δ is the indicator from (2.25)

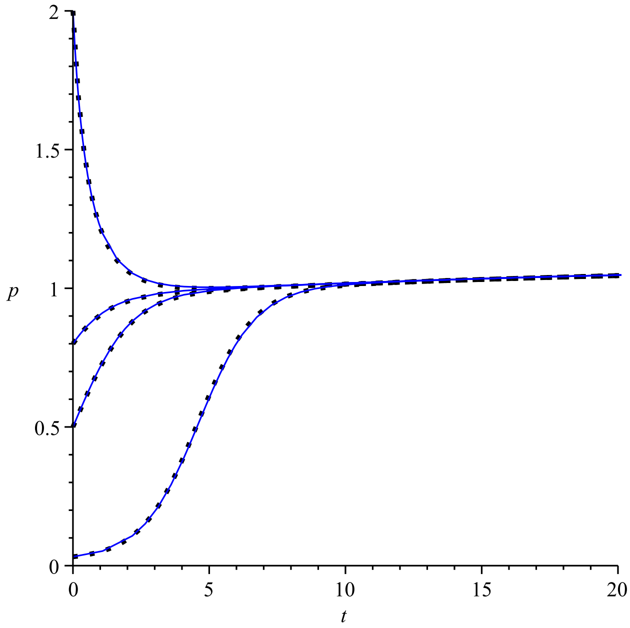

Figure 2.12: Evolution of the population given by (2.35) for the time varying data of Figure 2.10, but with $\sigma = 0.02$, $\epsilon = 0.05$ and various $\mu = 0.032, 0.5, 0.8, 2.0$

In Figure 2.11, the same r, k, h and σ apply, but now the starting population, $\mu \approx 0.02$ is so small that the population declines to zero.

In this case, $\mu = 0.012 < \frac{1}{2}k(0)\,(1 - \chi(0)) \approx 0.02$; that is, the initial population satisfies the criterion (2.50) and extinction occurs, even though the harvesting is subcritical.

Figure 2.12 shows population evolution under subcritical harvesting for the same r, k, h and σ, $\epsilon = 0.05$ and a range of μ values satisfying (2.44). In each case, there is a rapid initial transition region (where t_0 dominates) to a periodic limiting state where t_1 variation dominates. The agreement between approximation and numerical solution is excellent, implying that accuracy of the approximations is independent of initial conditions.

Figure 2.13 shows population evolution under subcritical harvesting from an initial $\mu = 0.032$ for periodic slowly varying r, k and h and a range of ϵ values. In each case, the agreement between approximation and numerical solution is very good, although, for

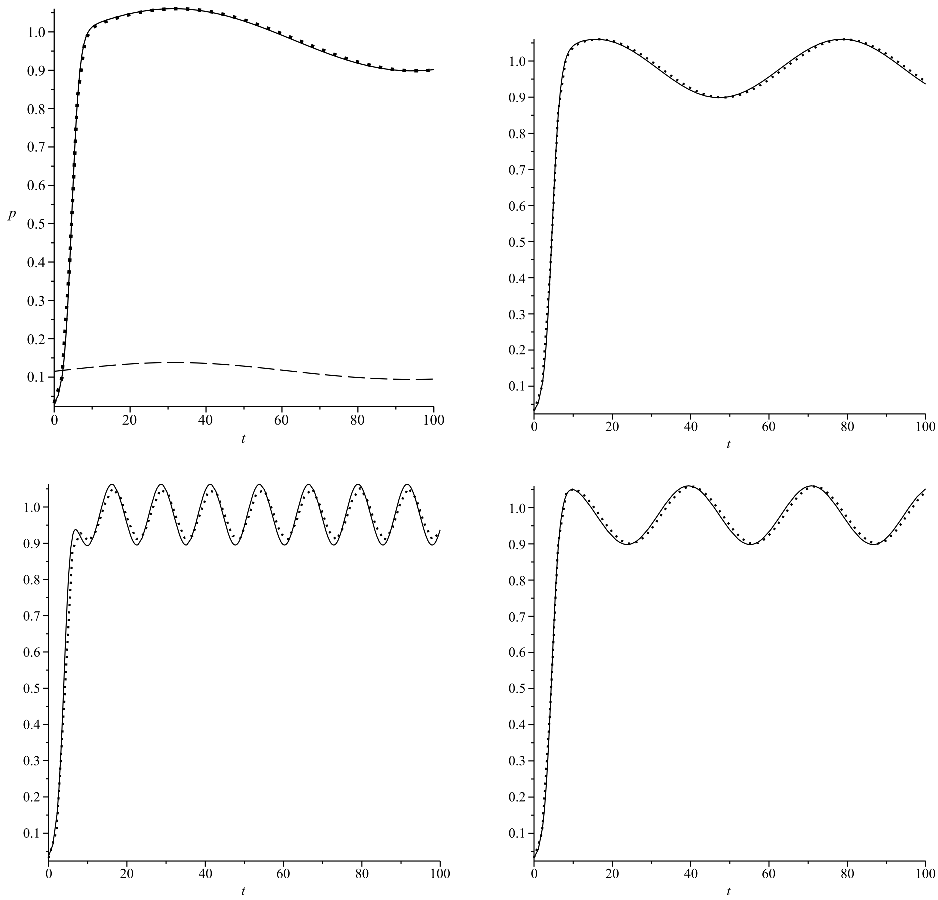

Figure 2.13: *Evolution of the population subject to subcritical harvesting with survival to a slowly varying state, for data used in Figure 2.10, where* $\epsilon = 0.05, 0.1, 0.2, 0.5$ *in clockwise order starting from the top left corner.*

the largest value of ϵ, where $\epsilon = 0.5$ (which might not be regarded as small), there is a noticeable discrepancy.

2.4.4 Supercritical Harvesting with Slowly Varying Periodic Growth Rate, Carrying Capacity and Harvesting

Figure 2.14 shows results for supercritical harvesting for periodic slowly varying r, k and h where

$$r(\epsilon t) = 1 + 0.03\sin(\epsilon t), \qquad k(\epsilon t) = 1 + 0.03\sin(\epsilon t), \qquad h(\epsilon t) = 1 + 0.04\sin(\epsilon t). \quad (2.64)$$

For this data,

$$\begin{aligned} \delta(\epsilon\, t) &= 1 - \frac{4\sigma(1 + 0.04\sin(\epsilon t))}{(1 + 0.03\sin(\epsilon t))^2} \\ &< 1 - \frac{0.96\sigma}{(1.03)^2} \\ &< 1 - 3.92\sigma. \end{aligned}$$

Thus in this case $\delta(\epsilon\, t) < 0$ when $\sigma > 0.25$. So, for the value of $\sigma = 0.5$ used in Figure figure4, the harvesting is supercritical where $\delta(\epsilon\, t) < -0.96$.
Thus, for $\sigma = 0.5$, as expected, the population declines from an initial $\mu = 1$ to zero at $t_1 \approx 3$ (or $t \approx 60$).

This last result may also be obtained by replacing t_0 by (2.33) in (2.53), setting the resulting first two terms of (2.53) to zero and numerically solving the resulting transcendental equation in t_1.

2.5 Discussion

The expansions (2.35), (2.46) and (2.53) are two-term explicit easily applied approximations to the evolving population $p(t, \epsilon)$ in the subcritical and supercritical harvesting

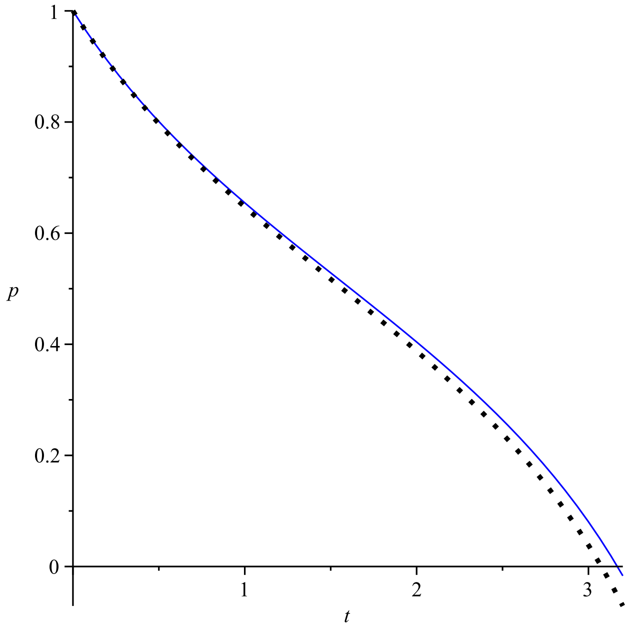

Figure 2.14: Evolution of the population subject to supercritical harvesting given by (2.53) with extinction population where $\mu = 1.0$, $\sigma = 0.5$, and $\epsilon = 0.05$

cases respectively. They apply for arbitrarily slowly varying functions r, k and h and parameters values ϵ, σ and μ, providing ϵ is small.

However, some simple restrictions on r, k and h are required for their construction. It seems sufficient that r, k and h be continuously differentiable on the domain of definition of the expansion. For (2.35) (subcritical survival) this is $t \geq 0$, while for (2.46) (subcritical extinction) and (2.53) (supercritical extinction) it will be $0 \leq t \leq t^*$, where t^* is where the population reaches zero, since from extinction onwards, the physical population problem is no longer relevant.

Note that these two term expansions all contain terms at the $O(\epsilon)$ level for which $\chi(t_1)$ is in the denominator. Thus, at points where $\chi(t_1) = 0$, these terms are undefined and so too are these expansions. More generally, at points where $\chi(t_1) = O(\epsilon)$, the second terms in these expansions become comparable with the leading order ones; i.e., the expansions become *disordered*, and fail as representations of the population. In particular, such disordering occurs in the neighbourhood of points where there is a change of harvesting from subcritical to supercritical or vice versa. In such neighbourhoods, the solution structure changes and so too do the expansions representing such solutions. Such transition regions will be analysed later in Chapter 4.

It must also be recognised that the process by which (2.35), (2.46) and (2.53) are constructed is purely formal, and is based on the assumption that the problem (2.3) has a solution $p(t, \epsilon)$ that is represented by these expansions where ϵ is sufficiently small. Thus, while the comparison with numerical solutions in Sections $2.4.1 - 4$ are encouraging, they do not prove the validity of these expansions, or the existence of a solution $p(t, \epsilon)$ behind them, This matter will be considered in Chapter 3.

The calculations of Sections 2.2, 2.3 have been published in Idlango et al [40].

Chapter 3

Existence and Uniqueness for the Harvesting Model

3.1 Introduction

The harvesting model, as considered in Chapter 2 takes a dimensionless form as the initial value problem for $p(t, \epsilon)$;

$$\frac{dp(t,\epsilon)}{dt} = r(\epsilon t)p(t,\epsilon)(1 - \frac{p(t,\epsilon)}{k(\epsilon t)}) - \sigma h(\epsilon t), \quad p(0,\epsilon) = \mu. \tag{3.1}$$

In that Chapter, multiscaling methods were used to obtain explicit approximations to the solution of the problem (3.1) for small positive ϵ, under appropriate conditions.
While this process was successful, and yielded expressions that agreed very well with the results of numerical solutions, the approach was purely formal, and rested on the assumption that the initial value problem (3.1) had a solution $p(t, \epsilon)$ that was unique on $t \geq 0$ and which was approximated well by the multiscale approximation as $\epsilon \to 0$.

In the present Chapter, we justify this formal process by proving rigorously that

- the problem (3.1) has a solution $p(t, \epsilon)$ (that is expressed by (2.35) or (2.46) and (2.53))

- this solution is unique, and

- the estimate $p(t, \epsilon) - p_0(t, \epsilon) = O(\epsilon)$

holds uniformly on an appropriate set of t-values, for all ϵ small enough, in a sense to be defined. Here, $p_0(t, \epsilon)$ is the leading order approximation obtained in Chapter 2.

We note, from Chapter 2, that this approximation takes different forms in three distinct cases:

(i) subcritical harvesting with survival as discussed in Section 2.3.1 ,

(ii) subcritical harvesting with extinction as discussed in Section 2.3.1,

(iii) supercritical harvesting as detailed in Section 2.3.2.

We will see in the following sections that the details of the existence of the solution $p(t, \epsilon)$ depends heavily on the properties of $p_0(t, \epsilon)$ displayed in each of these three forms

We begin by making the following basic assumptions about the functions r, k and h in (3.1):

- **A1:** The functions $r(\epsilon t)$, $k(\epsilon t)$ and $h(\epsilon t)$ are continuous and continuously differentiable functions of t for all $t \geq 0$ and all ϵ in a neighbourhood of zero.

  Note: in what follows, certain estimates will be found to hold for all ϵ small enough; that is, for all ϵ in $0 < \epsilon \leq \epsilon_0$, where ϵ_0 is some suitably small positive value, that will change with the context. We will indicate this property with the phrase f*or all ϵ in a neighbourhood of zero*, implying that such an ϵ_0 exists to make the estimate uniform in that interval of ϵ values.

- **A2:** There exist positive numbers α_1, α_2, β_1, β_2, β_3, γ_1 and γ_2 which are independent of ϵ such that

$$\begin{aligned} \alpha_1 \le r(\epsilon t) &\le \alpha_2 \\ \beta_1 \le k(\epsilon t) &\le \beta_2, \ |k^{'}(\epsilon t)| \le \beta_3 \\ \gamma_1 \le h(\epsilon t) &\le \gamma_2, \end{aligned} \tag{3.2}$$

for all $t \ge 0$ and ϵ in a neighbourhood of zero.

- **A3:** The function $h(\epsilon t)$ satisfies the condition [(2.27), or (2.28)] of the subcritical harvesting or supercritical harvesting case respectively, i.e.,

$$h(\epsilon t) < r(\epsilon t)\, k(\epsilon t)/4\sigma \text{ or } h(\epsilon t) > r(\epsilon t)\, k(\epsilon t)/4\sigma$$

on $t \ge 0$ and ϵ in a neighbourhood of zero.

From (2.25), we have

$$\delta(\epsilon t) = 1 - \frac{4\sigma h(\epsilon t)}{r(\epsilon t)\ k(\epsilon t)},$$

so the above is equivalent to assuming the existence of positive constants ρ_1, ρ_2 independent of ϵ such that either

$$\delta(\epsilon t) \ge \rho_1 > 0 \ \text{ or } \ \delta(\epsilon t) \le -\rho_2 < 0 \tag{3.3}$$

for all $t \ge 0$ and ϵ in a neighbourhood of zero, in the subcritical harvesting or supercritical harvesting cases, respectively.

We note that the hypotheses above means that $\chi(\epsilon t) = \sqrt{|\delta(\epsilon t)|}$ satisfies

$$0 < \ \rho_3 \ \le \ \chi(\epsilon t) \le \ \rho_4,$$

for all $t \ge 0$ and ϵ in a neighbourhood of zero, for positive constants ρ_3, ρ_4 independent of ϵ.

3.2 Reformulation of the Initial Value Problem as an Integral Equation

We now rewrite the initial value problem (3.1) for $p(t,\epsilon)$, in terms of a new function $u(t,\epsilon)$ that represents the difference between $p(t,\epsilon)$ and $p_0(t,\epsilon)$, the leading order of approximate solution of (3.1); that is,

$$u(t,\epsilon) = p(t,\epsilon) - p_0(t,\epsilon). \tag{3.4}$$

By substituting (3.4) into (3.1), we obtain

$$\frac{du}{dt} + \beta(t,\epsilon)u = R(p_0(t,\epsilon)) + \gamma(u(t,\epsilon)), \;\; u(0,\epsilon) = 0, \tag{3.5}$$

where

$$\beta(t,\epsilon) = r(\epsilon t)\left(\frac{2p_0(t,\epsilon)}{k(\epsilon t)} - 1\right), \tag{3.6}$$

$$R(p_0(t,\epsilon)) = r(\epsilon t)p_0(t,\epsilon)\left(1 - \frac{p_0(t,\epsilon)}{k(\epsilon t)}\right) - \sigma h(\epsilon t) - \frac{dp_0(t,\epsilon)}{dt}, \tag{3.7}$$

$$\gamma(u(t,\epsilon)) = -\frac{r(\epsilon t)}{k(\epsilon t)}u^2(t,\epsilon). \tag{3.8}$$

where $R(p_0(t,\epsilon))$ is termed the residual.

Multiplying both sides of (3.5) by the integrating factor

$$I = e^{\omega(t,\epsilon)}, \text{ where } \omega(t,\epsilon) = \int_0^{\epsilon t} \beta(s,\epsilon)ds \tag{3.9}$$

converts the differential equation in the initial value problem (3.5) to

$$\frac{d}{dt}(e^{\omega(t,\epsilon)}u) = e^{\omega(t,\epsilon)}\left[R(p_0(t,\epsilon)) + \gamma(u(t,\epsilon))\right].$$

Integrating both sides from 0 to t and applying the condition $u(0,\epsilon) = 0$ gives

$$u(t,\epsilon) = e^{-\omega(t,\epsilon)}\int_0^t e^{\omega(s,\epsilon)}\, R(p_0(s,\epsilon))ds + e^{-\omega(t,\epsilon)}\int_0^t e^{\omega(s,\epsilon)}\,\gamma(u(s,\epsilon))ds. \tag{3.10}$$

The equation (3.10) is a nonlinear integral equation, that is equivalent to the initial value problem (3.5).

For, clearly, as we have constructed above, (3.5) implies (3.10).

Conversely, if $u(t,\epsilon)$ is a continuous solution of (3.10), it is differentiable, and evaluation at $t = 0$ shows that $u(t,\epsilon)$ meets the initial condition of (3.5). Moreover, differentiating both sides of (3.10) gives

$$\begin{aligned} \frac{d}{dt}u(t,\epsilon) &= R(p_0(t,\epsilon)) + \gamma(u(t,\epsilon)) - \omega'(t,\epsilon)\, e^{\omega(t,\epsilon)} \left[\int_0^t e^{\omega(s,\epsilon)}\, R(p_0(s,\epsilon))ds \right. \\ &+ \left. e^{-\omega(t,\epsilon)} \int_0^t e^{\omega(s,\epsilon)}\, \gamma(u(s,\epsilon))ds \right] \end{aligned} \tag{3.11}$$

and then substituting (3.10) into the third term of the right hand side of (3.11) gives the differential equation of (3.5).

We may express equation (3.10) symbolically in the form

$$u(t,\epsilon) = T\, R(p_0(t,\epsilon) + T\gamma(u(t,\epsilon)), \tag{3.12}$$

where the map T is defined by

$$Tf(t,\epsilon) = e^{-\omega(t,\epsilon)} \int_0^t e^{\omega(s,\epsilon)} f(s,\epsilon)ds. \tag{3.13}$$

Clearly, T is linear, and is defined on the set of functions continuous on any subset of $t \geq 0$ for each ϵ in a neighbourhood of zero. Further, under assumption **A1**, T maps continuous functions of t into continuous functions of t.

Note, however, that due to the quadratic nonlinearity in u of γ, (3.12) is a nonlinear operator equation.

3.3 Existence of a Unique Solution

In this section, we show the existence of a solution $u(t,\epsilon)$ of the operator equation (3.12) in an appropriate subspace, X, of the continuous functions of t defined on $t \geq 0$ or some appropriate subset.

We further show that this $u(t,\epsilon)$ is unique and satisfies the estimate

$$u(t,\epsilon) = O(\epsilon) \tag{3.14}$$

uniformly on an interval of t values to be defined. This implies that $p(t,\epsilon)$ exists as a unique solution of the problem (3.1) and satisfies the condition

$$p(t,\epsilon) - p_0(t,\epsilon) = O(\epsilon), \tag{3.15}$$

on $t \geq 0$ or some appropriate subset.

Firstly, we consider the set X of functions $f(t,\epsilon)$ which are bounded continuous function of t on an interval $[0,a]$, for each ϵ in a neighbourhood of zero. This set is a Banach space with the norm

$$\|f\| = \sup_{t\in[0,a]} |f(t,\epsilon)|. \tag{3.16}$$

Note that $\|f\|$ depends on ϵ.

(For basic properties of Banach spaces, see [9], Section1.3).

Then, the nonlinear equation (3.12) is a nonlinear equation on X.

Further, if we define the nonlinear map $N : u \to \widetilde{u}$ on X by

$$\tilde{u}(t,\epsilon) = N\,u(t,\epsilon) = TR(p_0(t,\epsilon)) + T\gamma(u(t,\epsilon)) \tag{3.17}$$

for all ϵ in a neighbourhood of zero, then N maps the space X into itself.

Since the properties of the nonlinear map N are governed by the properties of the linear map T, we make the following assumptions **A4, A5** about the map T:

- **A4:** There exists $c > 0$ and independent of ϵ for all ϵ in a neighbourhood of zero, such that for any $f(t,\epsilon)$ in X,

  $$|Tf(t,\epsilon)| \;\leq\; c\|f\|, \tag{3.18}$$

  for all $t \in [0,a]$.

- **A5:** There exist a positive constant b independent of ϵ as above, such that

  $$\|TR(p_0(t,\epsilon))\| \leq b\,\epsilon \tag{3.19}$$

  for all $t \in [0,\, a]$.

In terms of N, equation (3.12) becomes

$$u = Nu, \tag{3.20}$$

i.e., the solution of (3.12) is a fixed point in X of the nonlinear map N. We thus seek to show the existence of this fixed point.

To prove this we will use the following Theorem :

Theorem (*Contractive Mapping Theorem*[9, 14]). *Let $S(\bar{x}, \rho)$ be the ball of radius ρ and centre $\bar{x}$ in a Banach space X. Suppose the map A maps S into itself and satisfies the condition that*

$$\|A(x) - A(y)\| \leq \lambda \|x - y\| \text{ for all } x, y \in S$$

where λ is a positive constant less than 1; i.e. A is contractive on S. Then A has one and only one fixed point in $S(\bar{x}, \rho)$.

We begin by showing that for ϵ in a neighbourhood of zero, N maps the ball

$$B = \{u : \|u\| \leq m\,\epsilon\} \tag{3.21}$$

in the space X into itself for some $m > 0$ independent of ϵ.
From (3.17), we have

$$\begin{aligned} \|\tilde{u}(t,\epsilon)\| &= \|TR(p_0(t,\epsilon)) + T\gamma(u(t,\epsilon))\| \\ &\leq \|TR(p_0(t,\epsilon))\| + \|T\gamma(u(t,\epsilon))\| \end{aligned} \tag{3.22}$$

and from **A4** and **A5** we get

$$\|\tilde{u}(t,\epsilon)\| \leq b\,\epsilon + c\,\|\gamma(u(t,\epsilon))\|. \tag{3.23}$$

Also,

$$\begin{aligned} \|\gamma(u(t,\epsilon))\| &= \|-\frac{r(\epsilon t)}{k(\epsilon t)} u^2(t,\epsilon)\| \\ &\leq \|-\frac{r(\epsilon t)}{k(\epsilon t)}\|\,\|u^2\| \\ &\leq d\|u\|^2 \end{aligned} \tag{3.24}$$

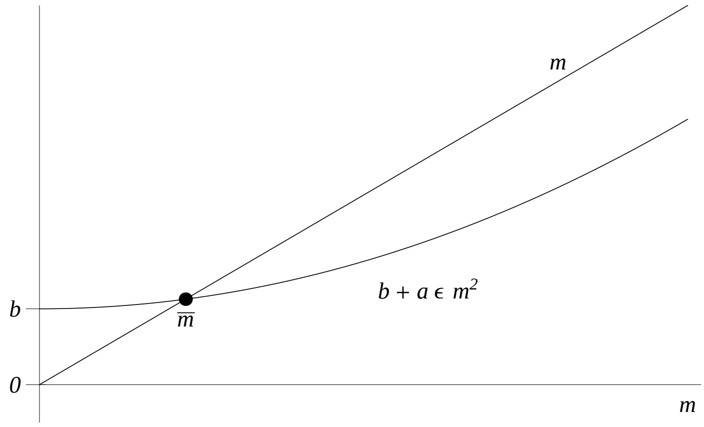

Figure 3.1: Graphical representation.

using assumption **A1**, where d is a positive finite number independent of ϵ.

For any $u \in B$, (3.23) gives, using (3.21) and (3.24)

$$\|\tilde{u}(t,\epsilon)\| \leq b\,\epsilon\, +\, m^2\epsilon^2$$

so, $\tilde{u}$ will be in the ball B if

$$b\,\epsilon + m^2\ d\,\epsilon^2 \leq m\,\epsilon; \tag{3.25}$$

i.e.,

$$b + m^2\ d\,\epsilon \leq m. \tag{3.26}$$

As it is shown in Figure 3.1, for small enough ϵ there will be a crossing point $\bar{m}$ (finite and positive) such that $m > \bar{m}$ for some finite choice of m independent of ϵ. Thus, N maps the ball B into itself, for all ϵ in a neighbourhood of zero.

We now show that for small enough ϵ, N is a contraction on the ball (3.21). First, suppose u_1, u_2 lie in B with images under N $\tilde{u}_1$, $\tilde{u}_2$; i.e., from (3.17) we have

$$\tilde{u}_1 = \,Nu_1\, = \,TR(p_0(t,\epsilon)) + T\gamma(u_1(t,\epsilon)), \tag{3.27}$$

and

$$\tilde{u}_2 = \,Nu_2\, = \,TR(p_0(t,\epsilon)) + T\gamma(u_2(t,\epsilon)). \tag{3.28}$$

Then

$$
\begin{aligned}
\tilde{u}_2 - \tilde{u}_1 &= [TR(p_0(t,\epsilon)) + T\gamma(u_2(t,\epsilon))] - [TR(p_0(t,\epsilon)) + T\gamma(u_1(t,\epsilon))] \\
&= T(\gamma(u_2(t,\epsilon)) \,-\, \gamma(u_1(t,\epsilon))) \qquad (3.29)
\end{aligned}
$$

so that,

$$\|\tilde{u}_2 - \tilde{u}_1\| \leq \|T\| \, \|\gamma(u_2) - \gamma(u_1)\|, \qquad (3.30)$$

from **A1** and (3.22).

Then,

$$
\begin{aligned}
\|\tilde{u}_2 - \tilde{u}_1\| &\leq c \, \| - \frac{r(\epsilon t)}{k(\epsilon t)} u_2^2 + (-\frac{r(\epsilon t)}{k(\epsilon t)}) u_1^2 \| \\
\|\tilde{u}_2 - \tilde{u}_1\| &\leq d \, c \, \|u_2^2 - u_1^2\| \\
&\leq d^* \, \|(u_2 - u_1)(u_2 + u_1)\| \\
&= d^* \, \|u_2(u_2 - u_1) + u_1(u_2 - u_1)\| \\
&\leq d^* \, \|u_2 - u_1\| \, [\|u_2\| + \|u_1\|] \qquad (3.31)
\end{aligned}
$$

But $u_2, u_1 \in B$, so $\|u_2\|, \|u_1\| \leq m \, \epsilon$ and so

$$\|\tilde{u}_2 - \tilde{u}_1\| \leq 2 \, d^* \, m \, \epsilon \|u_2 - u_1\|. \qquad (3.32)$$

Thus for ϵ positive and small enough, the map $N : u \to \tilde{u}$ is a contractive map on the ball B, i.e.,

$$\|Nu_2 - Nu_1\| \leq \lambda \, \|u_2 - u_1\|, \qquad (3.33)$$

where $\lambda = 2 \, d^* \, m \, \epsilon$, with $0 < \lambda < 1$ for all ϵ in a neighbourhood of zero.

So, N satisfies the conditions of the Contractive Mapping Theorem and N has a unique fixed point in the ball B. This means that the integral equation (3.10) has a unique solution $u(t, \epsilon)$ satisfying (3.14).

Thus, there is a unique solution $p(t, \epsilon)$ of the initial value problem (3.1) and the condition (3.15), holds, i.e.,

$$p(t, \epsilon) - p_0(t, \epsilon) = O(\epsilon)$$

for all $t \in [0, a]$.

Moreover, (3.5) implies that, for the solution $u(t, \epsilon)$ of (3.5),

$$\begin{aligned} \|u'(t,\epsilon)\| &\leq |\beta(t,\epsilon)|\|u\| + |R(p_0(t,\epsilon))| + |\frac{r(\epsilon t)}{k(\epsilon t)}|\|u\|^2 \\ &= O(\epsilon), \end{aligned}$$

since $\|u\| = O(\epsilon)$, then

$$p'(t,\epsilon) - {p'}_0(t,\epsilon) = O(\epsilon)$$

i.e., for small enough ϵ it is not only true that $p(t,\epsilon)$ is a unique solution close to the leading approximation $p_0(t,\epsilon)$, but also the derivative of $p(t,\epsilon)$, $p'(t,\epsilon)$ is close to the derivative of $p_0(t,\epsilon)$, $p_0'(t,\epsilon)$. This closeness improves as $\epsilon \to 0$.

This proof is based on the Assumptions **A4**, **A5** holding. The validity of these depend, in turn, on the properties of the linear operator T and the choice of the space $X = C[0, a]$; i.e., the choice of a. These, in turn, depend on the properties of the leading approximation $p_0(t,\epsilon)$, as constructed in Chapter 2. In the following sections, we will demonstrate how these properties, leave as is rise to the three forms of limiting behaviour described in Section 3.1, enable choices of X ensuring that the Assumptions **A4** and **A5** hold.

3.4 Existence for Subcritical Harvesting: Surviving Case

When the harvesting is subcritical and either (2.35) or (2.46) holds, the leading order term of the expansion for the solution of (3.1), $p_0(t,\epsilon)$, as given by (2.36) or (2.47) tends to a finite limit as $t \to \infty$. i.e., the first leading term; $p_0(t,\epsilon)$ is bounded, since from assumptions **A1** – **A3** where $0 < \chi(\epsilon t) \leq 1$, $p_0(t,\epsilon)$ given by (2.36), (2.47),

$$p_0(t,\epsilon) \to \frac{1}{2}k(\epsilon t)(1 + \chi(\epsilon t)) \text{ as } t \to \infty$$

or

$$\frac{2p_0}{k(\epsilon t)} - 1 \to \chi(\epsilon t) \tag{3.34}$$

and so,

$$0 < p_0(t,\epsilon) < \beta_2 < \infty, \tag{3.35}$$

where $\beta_2 > 0$ and is independent of ϵ, for all ϵ in a neighbourhood of zero.

In this case, we choose our Banach space X of Section 3.3 to be $C[0,\infty)$, i.e., we choose $a = \infty$. We show that, with this choice, the assumptions **A4** and **A5** hold for all $f(t,\epsilon) \in X$, where

$$\|f\| = \sup_{t\in[0,\infty)} |f(t,\epsilon)|. \tag{3.36}$$

3.4.1 Assumption A4 is Valid

In order to demonstrate the validity of Assumption **A4** , we require the following Lemma:

Lemma 3.4.1.1. *For all ϵ in a neighbourhood of zero, there exist finite $\delta > 0$ and $\bar{t} \geq 0$, independent of ϵ, such that $\beta(t,\epsilon) \geq \delta$ for all $0 \leq \bar{t} \leq t < \infty$.*

We first show that this Lemma ensures that the map T given by (3.13) satisfies the assumption **A4**, and delay the proof of Lemma 3.4.1.1 until later.

Firstly, we assume $\bar{t} > 0$. From (3.13) we can break up the integral into two parts, so that

$$Tf(t,\epsilon) = e^{-\omega(t,\epsilon)} \int_0^{\bar{t}} e^{\omega(s,\epsilon)} f(s,\epsilon)ds + e^{-\omega(t,\epsilon)} \int_{\bar{t}}^{t} e^{\omega(s,\epsilon)} f(s,\epsilon)ds, \tag{3.37}$$

where $\bar{t} \nrightarrow \infty$ as $\epsilon \to 0$.

(a). If $t \in [0,\bar{t}]$ then

$$|Tf(t,\epsilon)| \leq e^{-\omega(t,\epsilon)} \int_0^{\bar{t}} |e^{\omega(s,\epsilon)}|\,|f(s,\epsilon)|ds, \tag{3.38}$$

from Cauchy-Schwarz inequality

$$|Tf(t,\epsilon)| \leq \sup_{t\in[0,\bar{t}]} e^{-\omega(t,\epsilon)} \int_0^{\bar{t}} |e^{\omega(s,\epsilon)}|ds\,\|f\| \tag{3.39}$$

but, $\sup_{t\in[0,\bar{t}]} e^{-\omega(t,\epsilon)} \displaystyle\int_0^{\bar{t}} |e^{\omega(s,\epsilon)}|ds$ has finite number given by $\bar{A}$, independent of ϵ, so that

$$|Tf(t,\epsilon)| \leq \bar{A}\,\|f\| \quad \text{for all } 0 < t \leq \bar{t}. \tag{3.40}$$

(b). If $t \in [\bar{t}, \infty]$ then taking differentials in (3.9), we get

$$d\omega(t,\epsilon) = \beta(t,\epsilon)dt;$$

and by changing the limits of integration in the definite integral on $[\bar{t}, t]$ the map T in (3.37) becomes

$$\begin{aligned} Tf(t,\epsilon) &= e^{-\omega(t,\epsilon)} \int_0^{\bar{t}} e^{\omega(s,\epsilon)} f(s,\epsilon)ds \\ &+ e^{-\omega(t,\epsilon)} \int_{\omega(\bar{t},\epsilon)}^{\omega(t,\epsilon)} e^{\omega(s(\omega),\epsilon)} f(s(\omega),\epsilon) \frac{d\omega}{\beta(s(\omega),\epsilon)}. \end{aligned} \tag{3.41}$$

Thus

$$\begin{aligned} |Tf(t,\epsilon)| &\leq |e^{-\omega(t,\epsilon)}|\,|\int_0^{\bar{t}} e^{\omega(s,\epsilon)} f(s,\epsilon)ds| \\ &+ |e^{-\omega(t,\epsilon)}|\,|\int_{\omega(\bar{t},\epsilon)}^{\omega(t,\epsilon)} e^{\omega(s(\omega),\epsilon)} f(s(\omega),\epsilon) \frac{d\omega}{\beta(s(\omega),\epsilon)}| \\ &\leq |e^{-\omega(t,\epsilon)}| \int_0^{\bar{t}} e^{\omega(s,\epsilon)} |f(s,\epsilon)|ds \\ &+ |e^{-\omega(t,\epsilon)}|\,|\int_{\omega(\bar{t},\epsilon)}^{\omega(t,\epsilon)} e^{\omega(s(\omega),\epsilon)} |\frac{1}{\beta(s(\omega),\epsilon)}||f(s(\omega),\epsilon)|d\omega \end{aligned} \tag{3.42}$$

From Lemma 3.4.1.1, then there is $\delta > 0$ such that $\beta(t(\omega),\epsilon) \geq \delta$ for small positive ϵ, and so

$$
\begin{aligned}
|Tf(t,\epsilon)| &\leq e^{-\omega(t,\epsilon)} \int_0^{\bar{t}} e^{\omega(s,\epsilon)} ds \|f\| \\
&+ \frac{1}{\delta} e^{-\omega(t,\epsilon)} \int_{\omega(\bar{t},\epsilon)}^{\omega(t,\epsilon)} e^{\omega(s(\omega),\epsilon)} d\omega \, \|f\| \\
&\leq e^{-\omega(t,\epsilon)} \int_0^{\bar{t}} e^{\omega(s,\epsilon)} ds \|f\| \\
&+ \frac{1}{\delta} e^{-\omega(t,\epsilon)} \left[e^{\omega(t,\epsilon)} - e^{\omega(\bar{t},\epsilon)} \right] \|f\| \\
&= e^{-\omega(t,\epsilon)} \int_0^{\bar{t}} e^{\omega(s,\epsilon)} ds \|f\| + \frac{1}{\delta} \left(1 - e^{\omega(\bar{t},\epsilon) - \omega(t,\epsilon)} \right) \|f\| \qquad (3.43)
\end{aligned}
$$

and since the term $e^{\omega(\bar{t},\epsilon)-\omega(t,\epsilon)}$ has finite limit, so (3.43) becomes

$$
|Tf(t,\epsilon)| \leq e^{-\omega(t,\epsilon)} \int_0^{\bar{t}} e^{\omega(s,\epsilon)} ds \|f\| + \alpha \|f\| \qquad (3.44)
$$

where α is a positive constant independent of ϵ.

Now we can reformulate (3.9) as

$$
\omega(t,\epsilon) = \int_0^{\bar{t}} \beta(s,\epsilon) ds + \int_{\bar{t}}^{\epsilon t} \beta(s,\epsilon) ds
$$

where $\int_0^{\bar{t}} \beta(s,\epsilon) ds$ has finite value l_1 is independent of ϵ, and so

$$
e^{-\omega(t,\epsilon)} = l_1 \, e^{-\int_{\bar{t}}^{\epsilon t} \beta(s,\epsilon) ds}, \qquad (3.45)
$$

Substituting (3.45) into (3.44) gives

$$
|Tf(t,\epsilon)| \leq L \int_0^{\bar{t}} e^{\omega(s,\epsilon)} ds \|f\| + \alpha \|f\|. \qquad (3.46)
$$

where L is finite value independent of ϵ. From (3.40) we have

$$
\begin{aligned}
|Tf(t,\epsilon)| &\leq \bar{A} \|f\| + \alpha \|f\| \\
&\leq c \|f\|, \qquad (3.47)
\end{aligned}
$$

where $c = \alpha + \bar{A}$ for all $c > 0$, independent of ϵ for all ϵ in a nighbourhood of zero.

Thus, (a) and (b) show that for all $t \in [0, \infty)$ T is bounded i.e.,

$$\|T(f(t, \epsilon))\| \leq c\|f\|,$$

and Assumption **A4** is valid,

Note: when $\bar{t} = 0$, i.e., when $\beta(t, \epsilon)$ is strictly positive on $0 \leq t < \infty$, the proof follows as above. However, the first integral term of (3.37) is absent.

We now turn to the proof of the Lemma 3.4.1.1.

Proof of Lemma 3.4.1.1

Since there is no conditions on $\beta(t, \epsilon)$ as defined in (3.6) then it could be a positive or negative valued function, or have at least one zero on $t \geq 0$.

Substituting $t = 0$ into (3.6) gives

$$\beta(0, \epsilon) = r(0)(\frac{2\mu}{k(0)} - 1), \tag{3.48}$$

and noting (2.36) and (2.38), where the initial population μ satisfies the condition

$$\frac{k(0)}{2}(1 - \chi(0)) < \mu < \frac{k(0)}{2}(1 + \chi(0)).$$

we see, from (3.48), that $\beta(0, \epsilon)$ will be positive if

$$\frac{k(0)}{2} < \mu < \frac{k(0)}{2}(1 + \chi(0))$$

and negative if

$$\frac{k(0)}{2}(1 - \chi(0)) < \mu < \frac{k(0)}{2}.$$

Similarly, from (2.44) where μ satisfies the condition

$$\mu > \frac{k(0)}{2}(1 + \chi(0)),$$

$\beta(0, \epsilon)$ is always positive.

Now, from assumption **A1**, (3.6) and (3.34), as $t \to \infty$

$$\beta(t, \epsilon) \to r(\epsilon t)\, \chi(\epsilon t) > 0,$$

so that the function $\beta(t,\epsilon)$ may start with a positive or negative value, but for large times t will tend to a positive limiting value.

Now, consider the case where $\beta(t,\epsilon)$ has at least one zero on $0 < t < \infty$. Since, from the definition, (3.6), $\beta(t,\epsilon)$ contains $p_0(t,\epsilon)$, we divide our analysis into two cases:

- $p_0(t,\epsilon)$ defined as in (2.36):

  Let us suppose that t^* is the zero defined above, then from (3.6) we get

$$r(\epsilon t^*)(\frac{2\,p_0(t^*,\epsilon)}{k(\epsilon t^*)} - 1) = 0. \tag{3.49}$$

  Replacing t with t^* in (2.36) and substituting in (3.49) gives

$$r(\epsilon t^*)\,\chi(\epsilon t^*)\tanh(\frac{1}{2}(t_0(t^*) + A(\epsilon t^*) + c_0)) = 0,$$

  and from $\mathbf{A1} - \mathbf{A3}$,

$$\tanh\left[\frac{1}{2}(t_0(t^*) + A(\epsilon t^*) + c_0)\right] = 0, \tag{3.50}$$

  which implies

$$t_0(t^*) + A(\epsilon t^*) + c_0 = 0, \tag{3.51}$$

  where c_0 is a constant value defined in (2.38).

  The definition of t_0 (2.33) and assumptions $\mathbf{A1} - \mathbf{A3}$ give

$$t_0(t^*) \geq \alpha t^*, \tag{3.52}$$

  where $\alpha > 0$ is independent of small enough ϵ. Further, from (2.34)

$$A(\epsilon t^*) = -\int_0^{\epsilon t^*} \frac{k'(s^*)}{k(s^*)\chi(s^*)} ds^* \geq -\epsilon\,\omega t^*, \tag{3.53}$$

  where $\omega > 0$ and is independent of small positive ϵ. Thus,

$$\begin{aligned} \tfrac{1}{2}[t_0(t^*) + A(\epsilon t^*) + c_0] &\geq \tfrac{1}{2}[(\alpha - \epsilon\,\omega)t^* + c_0] \\ &\geq \tfrac{1}{4}(\alpha t^* + c_0) \end{aligned} \tag{3.54}$$

for ϵ small enough.

Thus, we have

$$\tfrac{1}{4}(\alpha t^* + c_0) \leq \tfrac{1}{2}[t_0(t^*) + A(\epsilon t^*) + c_0] \tag{3.55}$$

and

$$\tanh[\tfrac{1}{4}(\alpha t^* + c_0)] \leq \tanh[\tfrac{1}{2}(t_0(t^*) + A(\epsilon t^*) + c_0)]. \tag{3.56}$$

Thus, if $t^* \to \infty$ as $\epsilon \to 0$, we must have $\tanh(t_0(t^*) + A(\epsilon t^*) + c_0) \neq 0$ i.e., there is no zero of $\tanh\left[\tfrac{1}{2}(t_0(t^*) + A(\epsilon t^*) + c_0)\right]$ that will tend to ∞ as $\epsilon \to 0$; i.e., any t^* (independent of small ϵ) determined by (3.51) is bounded for ϵ small enough. Thus, there is a finite time such that $\bar{t}$ is independent of ϵ where $\bar{t} > t^*$ such that $\beta(t, \epsilon) \geq \delta$ where $\delta > 0$ (independent of ϵ) for small positive ϵ in a nighbourhood of zero. i.e., there exists a finite $\bar{t}$ independent of ϵ that makes $\beta(t, \epsilon)$ is always positive for all $t \geq \bar{t}$.

- $p_0(t, \epsilon)$ defined as in (2.37):

Again, let assume that t^* is the zero of (3.6) and using (2.37), we have

$$r(\epsilon t^*)\, \chi(\epsilon t^*) \coth(\frac{1}{2}(t_0(t^*) + A(\epsilon t^*) + c_{0R})) = 0, \tag{3.57}$$

where $c_{0R} > 0$.

Again, from (3.52) and (3.53), for small ϵ we get

$$t_0(t^*) + A(\epsilon t^*) + c_{0R} > \tfrac{1}{4}(\alpha t^* + c_{0R}) > 0.$$

Thus

$$\coth(\frac{1}{2}(t_0(t^*) + A(\epsilon t^*) + c_{0R})) > 0$$

and so the property (3.57) is not hold. This is means there is a finite time such $\bar{t} > t^*$ independent of ϵ, so that $\beta(t, \epsilon) \neq 0$. Thus, there exists $\delta > 0$ (independent of small enough ϵ) such that $\beta(t, \epsilon) \geq \delta$ for all $t \geq \bar{t}$.

3.4.2 Assumption A5 is Valid

- Firstly, we consider the case where $p_0(t,\epsilon)$ has the form (2.36).

  By (3.7) and (2.36) we have

$$\frac{dp_0(t,\epsilon)}{dt} = \frac{\epsilon}{2}\{k'(\epsilon t) + (k(\epsilon t)\,\chi(\epsilon t))'\tanh[\frac{1}{2}(t_0(\epsilon t) + A(\epsilon t) + c_0)]$$
$$+k(\epsilon\, t)\,\chi(\epsilon\, t)\left[1 - \tanh^2(\frac{1}{2}(t_0 + A(\epsilon t) + c_0))\right](\frac{t_0' + A'(\epsilon t)}{2})\}, \qquad (3.58)$$

where from (2.33),

$$t_0' = \frac{1}{\epsilon}r(\epsilon t)\chi(\epsilon t)$$

and from (2.34)

$$A'(\epsilon t) = \frac{k'(\epsilon\, t)}{k(\epsilon\, t)\,\chi(\epsilon\, t)}.$$

Substituting (2.36) and (3.58) into (3.22) gives

$$R(p_0(t,\epsilon)) = \tfrac{1}{4}\,r(\epsilon\, t)\,k(\epsilon\, t) - \sigma\, h(\epsilon\, t)$$
$$-\tfrac{1}{4}r(\epsilon\, t)k(\epsilon\, t)\chi^2(\epsilon\, t) + \frac{\epsilon}{4}k'(\epsilon\, t)\left[1 - \tanh^2[\tfrac{1}{2}(t_0 + A(\epsilon t) + c_0)]\right]$$
$$-\frac{\epsilon}{2}\{k'(\epsilon t) + (k(\epsilon t)\,\chi(\epsilon t))'\tanh[\frac{1}{2}(t_0(\epsilon t) + A(\epsilon t) + c_0)]\}. \qquad (3.59)$$

On using (2.25) and (2.32), we find that the leading order terms cancel, leaving the $O(\epsilon)$ terms, so that (3.59) gives

$$\begin{aligned} R(p_0(t,\epsilon)) &= \frac{\epsilon}{2}\{\frac{k'(\epsilon t)}{2}(1 - \tanh^2(\frac{1}{2}(t_0 + A(\epsilon t) + c_0)) \\ &- (k(t)\,\chi(t))'\,\tanh[\frac{1}{2}(t_0(\epsilon t) + A(\epsilon t) + c_0)])\}. \end{aligned} \qquad (3.60)$$

Using the assumptions **A1-A3** and fact that $-1 \le \tanh(..) \le 1$, we see that there exists a positive number λ independent of ϵ for all ϵ in a nighbourhood of zero, such that

$$\|R(p_0(t,\epsilon))\| \le \lambda\,\epsilon. \qquad (3.61)$$

Thus,

$$\|TR(p_0(t,\epsilon))\| \;\le\; \|T\|\,\|R(p_0(t,\epsilon))\|, \qquad (3.62)$$

and from **A4** and (3.61), for all $0 \leq t < \infty$ we have

$$\|TR(p_0(t,\epsilon))\| \;\; \leq \;\; b\,\epsilon, \quad b = c\lambda \tag{3.63}$$

for some positive b independent of ϵ for all ϵ in a nighbourhood of zero.

- Now, we consider the case where $p_0(t,\epsilon)$ has the form (2.47).

From (3.7) and using (2.47), we have

$$\begin{aligned} R(p_0(t,\epsilon)) = {} & \tfrac{1}{4}\, r(\epsilon\, t)\, k(\epsilon\, t) - \sigma\, h(\epsilon\, t) - \tfrac{1}{4} r(\epsilon\, t) k(\epsilon\, t) \chi^2(\epsilon\, t) \coth^2[\tfrac{1}{2}(t_0 + A(\epsilon t) + c_0)] \\ & + \frac{\epsilon}{2} k(\epsilon\, t) \chi(\epsilon t) \left(\frac{t_0' + A'(\epsilon t)}{2} \right) \left(\coth^2[\tfrac{1}{2}(t_0 + A(\epsilon t) + c_0)] - 1 \right) \\ & - \frac{\epsilon}{2}\, (k(\epsilon t)\, \chi(\epsilon t))' \coth[\frac{1}{2}(t_0(\epsilon t) + A(\epsilon t) + c_0)]. \end{aligned} \tag{3.64}$$

Again with the same steps as above, we obtain

$$\begin{aligned} R(\tilde{p}_0(t,\epsilon)) \;\; = \;\; & \frac{\epsilon}{2} \{ \frac{k'(\epsilon t)}{2} (\coth^2(\frac{1}{2}(t_0 + A(\epsilon t) + c_0) - 1) \\ - \;\; & (k(t)\, \chi(t))' \coth[\frac{1}{2}(t_0(\epsilon t) + A(\epsilon t) + c_0)])\}. \end{aligned} \tag{3.65}$$

Under the assumptions **A1-A3** and the properties that either $\coth(..) > 1$ or $\coth(..) < -1$, there exists a positive number λ, (independent of ϵ), such that

$$\|TR(p_0(t,\epsilon))\| \;\; \leq \;\; b\,\epsilon, \tag{3.66}$$

for some positive b independent of ϵ for all ϵ in a nighbourhood of zero.

Thus, Sections 3.4.1 and 3.4.2 together with the results of Section 3.3 show the existence of a unique solution of the problem (3.1) as stated on $[0,\infty)$, when the harvesting is subcritical and survival applies. However, in the next section, we will consider the case where the subcritical harvested population dies out in a short period of time and also the assumptions **A4**, **A5** are satisfied.

3.5 Existence for Subcritical Harvesting: Extinction Case

In this case, the leading approximation, $p_0(t,\epsilon)$ as constructed in Section 2.3.1, reaches zero (i.e., is extinguished) at $t=\tilde{t}$, i.e., in finite time. This is due to the initial population μ being too low, and in spite of harvesting being subcritical. Further, beyond $t=\tilde{t}$ where p_0 is negative, $p_0(t,\epsilon)\to-\infty$ as $t\to\hat{t}$, for some $\hat{t}>\tilde{t}$; that is, p_0 asymptotes to $-\infty$ as $t\to\hat{t}$.

Thus, if we select our a in $C[0,a]$ such that $\tilde{t}<a<\hat{t}$, with a independent of ϵ and finite for all ϵ in a neighbourhood of zero, we may expect to show that the assumptions **A4** and **A5** hold on $X=C[0,a]$.

Thus, if at time $\tilde{t}$ the population p_0 reaches zero, we have

$$p_0(\tilde{t},\epsilon)=\frac{k(\epsilon\,\tilde{t})}{2}(1+\chi(\epsilon\,\tilde{t})\coth(\frac{1}{2}(t_0(\epsilon\,\tilde{t})+A(\epsilon\,\tilde{t})+c_{0R})))=0. \tag{3.67}$$

Similarly, as $t\to\hat{t}$ (where $0<\tilde{t}<\hat{t}$), we have $p_0(t,\epsilon)\to-\infty$ if

$$\tanh\left[\frac{1}{2}(t_0(\epsilon t)+A(\epsilon t)+c_{0R})\right]\to 0, \tag{3.68}$$

which means that

$$\frac{1}{2}(t_0(\epsilon\hat{t})+A(\epsilon\hat{t})+c_{0R})=0. \tag{3.69}$$

From (3.67), we have

$$1+\chi(\epsilon\tilde{t})\coth(\frac{1}{2}(t_0(\epsilon\tilde{t})+A(\epsilon\tilde{t})+c_{0R}))=0, \tag{3.70}$$

or,

$$\frac{1}{2}\left(t_0(\widetilde{\epsilon t})+A(\epsilon\tilde{t})+c_{0R}\right)=\operatorname{arccoth}\left[\frac{-1}{\chi(\widetilde{\epsilon t})}\right] \tag{3.71}$$

and since $0<\chi(\widetilde{\epsilon t})<1$, $\operatorname{arccoth}(\frac{1}{\chi(\widetilde{\epsilon t})})$ is defined and finite, so we can rewrite (3.71) as

$$t_0(\epsilon\tilde{t})+A(\epsilon\tilde{t})+c_{0R}=-2\operatorname{arccoth}(\frac{1}{\chi(\epsilon\tilde{t})}). \tag{3.72}$$

Subtracting (3.72) from (3.69) gives

$$t_0(\epsilon\hat{t}) - t_0(\epsilon\tilde{t}) + A(\epsilon\hat{t}) - A(\epsilon\tilde{t}) = 2\text{arccoth}(\frac{1}{\chi(\epsilon\tilde{t})}). \tag{3.73}$$

Now, from the definition (2.33) of t_0,

$$t_0(\epsilon\hat{t}) - t_0(\epsilon\tilde{t}) = \frac{1}{\epsilon}\int_{\epsilon\tilde{t}}^{\epsilon\hat{t}} r(\epsilon\, s)\,\chi(\epsilon\, s)\, ds \tag{3.74}$$

and from assumption **A1**, there exist λ_1, $\lambda_2 > 0$ and independent of ϵ, such that

$$\lambda_1\,(\hat{t} - \tilde{t}) \leq t_0(\epsilon\hat{t}) - t_0(\epsilon\tilde{t}) \leq \lambda_2\,(\hat{t} - \tilde{t}). \tag{3.75}$$

Also, by boundedness properties (assumption $\mathbf{A1}, \mathbf{A2}$), there exist positive σ_1, $\sigma_2 > 0$ and independent of ϵ in a neighbourhood of zero, such that

$$-\sigma_1\,\epsilon\,(\hat{t} - \tilde{t}) \leq A(\epsilon\hat{t}) - A(\epsilon\tilde{t}) \leq \sigma_2\,\epsilon\,(\hat{t} - \tilde{t}). \tag{3.76}$$

Now, (3.73) leads to

$$t_0(\epsilon\hat{t}) - t_0(\epsilon\tilde{t}) = 2\text{arccoth}(\frac{1}{\chi(\epsilon\tilde{t})}) - [A(\epsilon\hat{t}) - A(\epsilon\tilde{t})]. \tag{3.77}$$

From the right hand side of (3.76), we have

$$-[A(\epsilon\hat{t}) - A(\epsilon\tilde{t})] \geq -\sigma_2\,\epsilon\,(\hat{t} - \tilde{t}),$$

so that (3.77) becomes

$$t_0(\epsilon\hat{t}) - t_0(\epsilon\tilde{t}) \geq 2\text{arccoth}(\frac{1}{\chi(\epsilon\tilde{t})}) - \sigma_2\,\epsilon\,(\hat{t} - \tilde{t}). \tag{3.78}$$

Using the right hand side of (3.75) gives

$$\lambda_2\,(\hat{t} - \tilde{t}) \;\; \geq \;\; 2\text{arccoth}(\frac{1}{\chi(\epsilon\tilde{t})}) - \sigma_2\,\epsilon\,(\hat{t} - \tilde{t}), \tag{3.79}$$

which implies that

$$\hat{t} - \tilde{t} \;\; \geq \;\; \frac{2\text{arccoth}(\frac{1}{\chi(\epsilon\tilde{t})})}{\lambda_2 \,+ \sigma_2\,\epsilon}, \tag{3.80}$$

since $0 < \chi(\epsilon\tilde{t}) < 1$ and so $\coth^{-1}(\frac{1}{\chi(\epsilon\tilde{t})})$ has an upper limit independent of ϵ and as $\epsilon \to 0$. Thus, there exists a positive number κ independent of ϵ for all ϵ in a neighbourhood of zero such that

$$\hat{t} - \tilde{t} \;\; \geq \;\; \kappa, \tag{3.81}$$

i.e., $\hat{t}$ and $\tilde{t}$ remain separated for all ϵ small enough.

The property (3.81) is an independent of μ, since c_{0R} is not involved. i.e., it holds for all μ less than the critical value producing extinction.

Also, from (3.69),

$$t_0(\epsilon\hat{t}) = -A(\epsilon\hat{t}) - c_{0R},$$

which leads to

$$\gamma\,\hat{t} \leq \epsilon\,\delta\,\hat{t} + |c_{0R}|$$

for some γ, $\delta > 0$ independent of ϵ.

Thus,

$$\hat{t} \;\; \leq \;\; \frac{|c_{0R}|}{\gamma - \epsilon\,\delta} \tag{3.82}$$

where c_{0R} is independent of ϵ (but we may have c_{0R} getting larger as μ tends to a critical value). Thus $\hat{t}$ is bounded above as $\epsilon \to 0$ for each given μ that satisfies (2.50).

Thus, in this case, the leading approximation $p_0(t, \epsilon)$ reaches zero at $t = \tilde{t}$, and tends to $-\infty$ as $t \to \hat{t} > \tilde{t}$. For all ϵ in a neighbourhood of zero, $\hat{t}$ is bounded above, and the interval $[\tilde{t}, \hat{t}]$ remains finite and non-empty. Thus, we may select our a to lie in $[\tilde{t}, \hat{t}]$, i.e., we choose

$$\tilde{t} \; < \; a \; < \; \hat{t}.$$

(For example we may choose $a = (\tilde{t} + \hat{t})/2$, the midpoint of $[\tilde{t}, \hat{t}]$.)

Then, $[0, a]$ is a finite bounded interval containing the zero of $p_0(t, \epsilon)$ and on which $p_0(t, \epsilon)$ is a continuous bounded function of t for all ϵ in a neighbourhood of zero.

Thus, for this a we choose $X = C[0,a]$, with $[0,a]$ a finite interval, and seek to show that the assumptions **A4** and **A5** hold for this choice.

Note that $p_0(t,\epsilon)$ is negative on that part of $[0,a]$ past the value $\tilde{t}$. We expect that the exact solution $p(t,\epsilon)$, lying in a neighbourhood of p_0, will also display this property. This is, of course, not relevant to the physical context of the model (3.1).

3.5.1 Assumptions A4 and A5 are Valid.

With our choice of a as described above, $[0,a]$ is a finite interval, independent of ϵ for all ϵ in a neighbourhood of zero, and $X = C[0,a]$, with

$$\|f\| = \sup_{t\in[0,a]} |f(t,\epsilon)|$$

for any $f \in X$.

We now show that, for this choice, **A4** and **A5** hold.

Assumption A4 is Valid:

T is bounded linear map on $C[0,a]$.

For, since $t \in [0,a]$ then

$$\begin{aligned} |Tf(t,\epsilon)| &\le e^{-\omega(t,\epsilon)} \int_0^a |e^{\omega(s,\epsilon)}| \, \|f(s,\epsilon)\| ds \\ &\le \sup_{t\in[0,a]} e^{-\omega(t,\epsilon)} \int_0^a |e^{\omega(s,\epsilon)}| ds \, \|f\| \end{aligned} \tag{3.83}$$

where $e^{-\omega(t,\epsilon)} \int_0^a |e^{\omega(s,\epsilon)}| ds$ is bounded, independent of ϵ.

Thus, there exists a $c > 0$ independent of ϵ such that

$$|Tf(t,\epsilon)| \le c\,\|f\|, \tag{3.84}$$

for all $t \in [0,a]$.

Assumption A5 is Valid:

Since in Section 3.4.2 we have proved that the Assumption **A5** is satisfied for all $t \ge 0$, consequently it is satisfied on the bounded interval $t \in [0, a]$. Thus, there exists $b > 0$, (independent of ϵ), such that

$$\|TR(p_0(t,\epsilon))\| \leq b\,\epsilon, \;\text{ for any } \quad t \in [0,a].$$

Thus, from above results, this proves the existence of a unique solution of the problem (3.1) as stated on $[0, a]$, when the harvesting is subcritical but when extinction, occurs.

3.6 Existence for Supercritical Harvesting: Extinction Case

In this case, where the harvesting is so high so that the population declines to extinction (zero) for any initial value. Using Assumptions **A1-A3**, and as in Section 3.5, we will show that this extinction occurs in finite time. Further, as in Section 3.5, we show how a choice of $0 < a < \infty$ may be made so that we have $X = C[0, a]$ for our function space as introduced in Section 3.2.

Let suppose that the population p_0 that presented by (2.54) reaches zero at time $t_1^*, (t_1^* > 0)$, i.e.,

$$p_0(t^*,\epsilon) = \frac{k(\epsilon\, t^*)}{2}(1 - \chi(\epsilon\, t^*)\tan(\frac{1}{2}(t_0(\epsilon\, t^*) - A(\epsilon\, t^*) + d_0))) = 0, \tag{3.85}$$

while $p_0 \to -\infty$ as $t \to \hat{t}_1 > t_1^*$ where

$$\cos\left[\tfrac{1}{2}(t_0(\epsilon t) - A(\epsilon t) + d_0)\right] = 0, \tag{3.86}$$

so that

$$t_0(\epsilon\,\hat{t}) - A(\epsilon\,\hat{t}) + d_0 = \pi, \tag{3.87}$$

where $\hat{t} > t^*$.

From (3.85) , we have

$$1 - \chi(\epsilon\, t^*)\tan(\frac{1}{2}(t_0(\epsilon\, t^*) - A(\epsilon\, t^*) + d_0)) = 0 \tag{3.88}$$

and so,

$$t_0(\epsilon\, t^*) - A(\epsilon\, t^*) + d_0 = 2\arctan(\frac{1}{\chi(\epsilon\, t^*)}) \tag{3.89}$$

where $\chi(\epsilon\, t^*) \geq 1$ and therefore $\frac{\pi}{4} \leq \arctan(\frac{1}{\chi(\epsilon\, t^*)}) < \frac{\pi}{2}$.

Subtracting (3.89) from (3.87) gives

$$t_0(\epsilon\, \hat{t}) - t_0(\epsilon\, t^*) - [A(\epsilon\, \hat{t}) - A(\epsilon\, t^*)] = \pi - 2\arctan(\frac{1}{\chi(\epsilon t)}). \tag{3.90}$$

Again, using (2.33), for all t^*, $\hat{t}1$, we obtain

$$t_0(\epsilon\, \hat{t}) - t_0(\epsilon\, t^*) = \int_{\epsilon\, t^*}^{\epsilon\, \hat{t}} r(\epsilon\, s)\, \chi(\epsilon\, s)\, ds. \tag{3.91}$$

From assumption **A1**, there exist $\tilde{\lambda}, \tilde{\sigma} > 0$ and independent of ϵ, such that

$$t_0(\epsilon\, \hat{t}) - t_0(\epsilon\, t^*) \leq \tilde{\lambda}\, (\hat{t} - t^*), \tag{3.92}$$

and

$$A(\epsilon\, \hat{t}) - A(\epsilon t^*) \geq -\tilde{\sigma}\, \epsilon\, (\hat{t} - t^*). \tag{3.93}$$

Now, (3.90) leads to

$$\begin{aligned} t_0(\epsilon\hat{t}) - t_0(\epsilon t) &= \pi - 2\arctan(\frac{1}{\chi(\epsilon t)}) + [A(\epsilon t) - A(\epsilon t)], \\ &\geq \pi - 2\arctan(\frac{1}{\chi(\epsilon\tilde{t})}) - \tilde{\sigma}\, \epsilon\, (\hat{t} - t^*), \end{aligned} \tag{3.94}$$

and from (3.92), we have

$$\tilde{\lambda}\, (\hat{t} - t^*) \geq \pi - 2\tan^{-1}(\frac{1}{\chi(\epsilon t)}) + \tilde{\sigma}\, \epsilon\, (\hat{t} - t^*), \tag{3.95}$$

which implies that

$$\hat{t} - t^* \geq |\frac{\pi - 2\arctan(\frac{1}{\chi(\epsilon t^*})}{\tilde{\lambda} + \tilde{\sigma}\, \epsilon}| \tag{3.96}$$

where the right hand side is bounded independent of ϵ, so for small enough ϵ, there exists a positive number such ζ independent of ϵ such that

$$\hat{t} - t^* \geq \zeta, \tag{3.97}$$

we note from (3.97) that $\hat{t} - t^*$ is independent of μ. Also, from (3.87), we have

$$t_0(\epsilon\,\hat{t}) = \pi - d_0 + A(\epsilon\,\hat{t})$$

where $-\frac{\pi}{2} < |d_0| < \frac{\pi}{2}$ and so for some $\tilde{\sigma}$, $\tilde{\lambda} > 0$ independent of ϵ then

$$\tilde{\delta}\,\hat{t} \leq \pi - d_0 + \tilde{\sigma}\,\epsilon\,\hat{t}$$

i.e.,

$$\hat{t} \;\; \leq \;\; |\frac{\pi - d_0}{\tilde{\delta} - \epsilon\,\tilde{\sigma}}| \tag{3.98}$$

where d_0 given by (2.56).

Thus $\hat{t}$ has an upper limit as $\epsilon \to 0$, for a given μ. As in Section 3.5, we may choose a such that

$$t^* \; < \; a < \; \hat{t}$$

where the interval $[t^*, \hat{t}]$ remains non-empty for all ϵ in a neighbourhood of zero. Then, $[0, a]$, is a finite interval, with a bounded above for all such ϵ. Then, we can choose $X = C[0, a]$ as our function space, for such an a. The proof that assumptions **A4** and **A5** are valid then follows as in Section 3.5.

3.7 Discussion

The contraction mapping proof used in this Chapter has the two-fold benefit of delivering existence and uniqueness of solution results for the initial value problem (3.1), for all ϵ in a neighbourhood of zero. Even more, the smoothness of the kernel in the integral equation (3.10) (ensured by the conditions **A1** and **A2** imposed on the model parameters), means that the solution $p(t, \epsilon)$ and its derivative $p'(t, \epsilon)$ are approximated closely as $\epsilon \to 0$ by the leading approximation $p_0(t, \epsilon)$ and its derivative ${p'}_0(t, \epsilon)$, respectively.

The conditions **A1** and **A2** imposed on the model parameters $r(\epsilon t), k(\epsilon t)$ and $h(\epsilon t)$ are sufficient to make the proof proceed.

It is important to note that the proof depends critically on the assumption **A3** holding for all relevant t. This assumption governs whether the level of harvesting is below or above a (slowly varying) critical value. Thus, points $\bar{t}$ where $\delta(\epsilon\bar{t}) = 0$ (transition points, as described in Chapter 2) are ruled out. The effects of transitions will be dealt with using a formal process in Chapter 4, following.

A consequence of the contraction mapping construction is the existence of an iterative scheme generating a sequence $u_n(t, \epsilon)$ of approximations to the fixed point u of the nonlinear map N, that takes the form

$$u_{n+1} = N\, u_n, \qquad n = 0, 1, 2 \ldots$$

which converges (in the norm $\| \,.\, \|$) to the fixed point u for any initial iterate $u0$ in the ball B.

In terms of the linear map T, this becomes

$$u_{n+1} = TR(p_0(t, \epsilon)) + T\gamma(u_n(t, \epsilon)). \tag{3.99}$$

If we choose $u_0(t, \epsilon) = 0$ (which lies in B), we get, from (3.99), a first approximation

$$u_1 = TR(p_0(t, \epsilon)), \tag{3.100}$$

which we know is $O(\epsilon)$ whatever our choice of the space X. This just reinforces the estimates obtained in the proof. In practice, calculation of u_1, and further terms in the sequence of iterates u_n from (3.99) would be near impossible. Hence, we construct our approximating sequence by the direct multi-scaling method used here, safe in the knowledge that it does represent an existing solution to the problem. Thus, the proof of this chapter puts this formula process on a firm basis.

In each of the cases considered in Chapter 2, we have constructed two term approximation of the form

$$p_0(t, \epsilon) + \epsilon\, p_1(t, \epsilon) + \ldots \tag{3.101}$$

for the solution of the problem (2.5). It seems reasonable that, at the expense of some analysis, the results of the present chapter could be obtained replacing the leading order approximation $p_0(t,\epsilon)$ by (3.101). Thus, we expect to find that **A5** would be replaced by

$$\|T\,R(p_0 + \epsilon\, p_1)\| \leq b\,\epsilon^2, \tag{3.102}$$

where the properties of T would be obtained, for ϵ small enough.

Then, the existence-uniqueness proof should proceed as before, with the ball B replaced by

$$B = \{u : \|u\| \leq m\,\epsilon^2\}.$$

We would then obtain a validation of the two-term approximation (3.101); i.e., we would replace the estimate (3.15) by

$$p(t,\epsilon) - p_0(t,\epsilon) - \epsilon\, p_1(t,\epsilon) = O(\epsilon^2).$$

The proof put forward in this chapter is being prepared for publication in Idlango et al [38].

Chapter 4

Analysis of Transitions in a Harvested Logistic Model

4.1 Introduction

In Chapter 2 and in Idlango et al. [40], we investigated the evolution of a single species population $p(t, \epsilon)$ subject to a slowly varying logistic harvesting law (2.3), which can be characterized in dimensionless form as the initial value problem

$$\frac{dp(t,\epsilon)}{dt} = r(t_1)p(t,\epsilon)\left(1 - \frac{p(t,\epsilon)}{k(t_1)}\right) - \sigma h(t_1), \qquad p(t = 0, \epsilon) = \mu, \tag{4.1}$$

where $p(t, \epsilon)$ is the population at times $t \geq 0$, $\mu > 0$ is an arbitrary initial population, while $t_1 = \epsilon\, t$, represents the 'slow' time scale and t is a 'normal' time scale, (see Chapter 2, Section 2.2.1). The dimensionless parameter ϵ is the ratio of the normal to slow time scales, so that $0 < \epsilon \ll 1$.

Applying a multiscaling technique based on the limit $\epsilon \to 0$, we obtained an approximate expansion for $p(t, \epsilon)$ and used this to examine the behaviour of the solutions of (4.1) as time progresses, (see Chapter 2, Sections 2.3.1, 2.3.2 and [40]).

The behaviour of these solutions was found to be dependent on the function $\delta(t_1)$ defined in (2.25) as

$$\delta(t_1) = 1 - \frac{4\,\sigma h(t_1)}{r(t_1)\,k(t_1)}, \quad \text{where } \; t_1 = \epsilon\, t. \tag{4.2}$$

When $\delta(t_1) > 0$ the harvesting is subcritical; and in this case, we showed that, providing $\mu > \frac{1}{2}k(0)\{1 - \chi(0)\}$, the leading terms of the expansions (2.35) and (2.46) tend to a slowly varying limiting state (2.41), i.e.,

$$p(t, \epsilon) \to \tfrac{1}{2}k(t_1)\,\{1 + \chi(t_1)\} - \epsilon\,\left\{\frac{(k(t_1)\,\chi(t_1))' + k'(t_1)}{2\,r(t_1)\chi(t_1)}\right\} + O(\epsilon^2) \tag{4.3}$$

as $t \to \infty$; that is, the population *survives* to this state. Note that this slowly varying limiting state is independent of the initial population, μ. When $0 < \mu < \frac{1}{2}k(0)\{1-\chi(0)\}$, the leading term (2.47) of the expansion (2.46) tends to zero in finite time; i.e., the population is *extinguished*.

When $\delta(t_1) < 0$, so that the harvesting is supercritical, the leading term (2.54) of the expansion (2.53) tends to zero in finite time; i.e., the population is *extinguished*.

We note that the both of the leading term expansions (2.36) (or (2.47)) and (2.54) are valid in regions where $\delta(t_1)$ is strictly positive or negative, respectively; i.e., $\delta(t_1)$ is bounded away from zero as $\epsilon \to 0$. However, in a neighbourhood of points where $\delta(t_1) = 0$, these expansions become *disordered*, in the sense that at points t_1 where $\delta(t_1) = O(\epsilon)$, the second terms in these expansions will be comparable with the leading order ones. This disordering occurs, in particular, in a neighbourhood of points where subcritical harvesting converts to the supercritical case. Thus, to represent the solution of (2.3) in a neighbourhood of points where $\delta(t_1) = 0$, we need to reconsider (4.1) in more detail, following a similar line of analysis to that of [30, 31, 56].

As well, as we note above (Chapter 2, Section 2.3.1), survival to the limiting state (4.3) occurs only for a specific range of values of the parameter μ. For smaller μ values, even with subcritical harvesting, the population dies out in finite time. Thus, in the following analysis, we will focus specifically on the case where we initially have subcritical harvesting with survival to the state (4.3) which converts to supercritical harvesting, with subsequent extinction. This phenomena may occur as a result of increasing the harvesting rate over time (i.e., the sign of $\delta(t_1)$ changes from positive to negative) leading the population to decline and ultimately extinction.

As we have noted above, this change is characterized by a transition from a region where $\delta(t_1) > 0$ (survival) to a region where $\delta(t_1) < 0$ (extinction). So, this transition point will occur at a zero of $\delta(t_1)$.

Specifically, we assume that at the *transition time*, expressed by $t_1 = \bar{t}_1$, the function $\delta(\bar{t}_1)$ satisfies

$$\delta(\bar{t}_1) = 0 \quad \text{and} \quad \delta'(\bar{t}_1) < 0, \tag{4.4}$$

the second inequality ensuring that the zero of $\delta(t_1)$ is simple. We will thus subdivide the region $t \geq 0\,(t_1 \geq 0)$, that is indicated in Figure 4.1 as follows:

Region 1: Harvesting is subcritical, with survival, and $0 \leq t_1 < \bar{t}_1$;

Region 2: A transition region surrounding $\bar{t}_1$;

Region 3: Harvesting is supercritical, with extinction and $\bar{t}_1 < t_1 < T_e$, where T_e the time at which extinction occurs.

In the following Sections, we will obtain an approximate representation for the solution of (4.1) over all $t \geq 0$, by combining the approximate solutions of the initial vale problem (4.1) in Regions 1 and 3 with an approximate solution in the transition region (Region 2), using a matching technique. We will then create an approximate solution for (4.1), uniformly valid on $t \geq 0$ by a composition process.

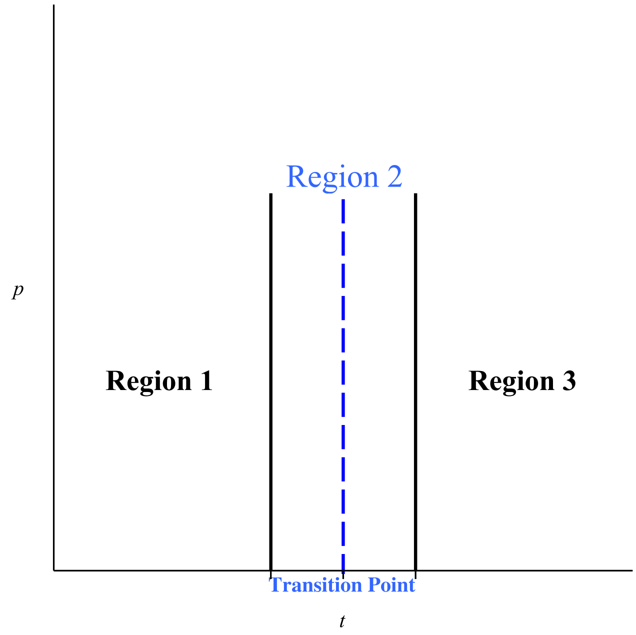

Figure 4.1: Graphical representation of the Regions 1, 2 and 3.

4.2 Solutions Away From the Transition Region (Region 1 and 3)

In this Section, we consider the solutions of (4.1) in Region 1 where the harvesting is subcritical, (i.e., from $t = 0$ to a neighbourhood to the left of the transition point) and in Region 3 where the harvesting is supercritical (i.e., from a neighbourhood to the right of the transition point to extinction time T_e).

In Region 1 (as analysed in Section 2.3.1), where $\delta(t_1) > 0$, as discussed above, the solution of (4.1) is represented by (2.40) or (2.44), and tends to the limiting expansion (4.3).

In Region 3, the solution is approximated by the leading order term of the expansion (2.54), redefined here by

$$p_0(t, \epsilon) = \tfrac{1}{2}k(t_1)\left\{1 - \sqrt{-\delta(t_1)}\,\tan\left[\tfrac{1}{2}(\tilde{t}_0 - \tilde{A}(t_1) + d)\right]\right\} + O(\epsilon), \tag{4.5}$$

where

$$\tilde{t}_0 = \frac{1}{\epsilon}\int_{\bar{t}_1}^{t_1} r(s)\sqrt{-\delta(s)}\,ds,$$
$$\tilde{A}(t_1) = -\int_{\bar{t}_1}^{t_1} \frac{k'(s)}{k(s)\sqrt{-\delta(s)}}\,\mathrm{d}s \tag{4.6}$$

are appropriatly modified forms of t_0, $A(t_1)$ while d is a constant to be determined.

4.3 Solution within the Transition Region (Region 2)

Here, we construct an approximate solution of the differential equation in the initial value problem (4.1) valid in Region 2. In order to investigate this transition region solution, we reformulate (4.1) to include the $\delta(\epsilon t)$ term, i.e., from (2.25) we have

$$\delta(t_1) = 1 - \frac{4\,\sigma h(t_1)}{r(t_1)\,k(t_1)},$$
$$\sigma\, h(t_1) = \frac{r(t_1)\,k(t_1)}{4}(1-\delta(t_1)), \tag{4.7}$$

so that the differential equation in (4.1) becomes

$$\frac{dp(t,\epsilon)}{dt} = r(t_1)p(t,\epsilon)\left(1 - \frac{p(t,\epsilon)}{k(t_1)}\right) - \frac{r(t_1)\,k(t_1)}{4}\left(1-\delta(t_1)\right). \tag{4.8}$$

We now examine solutions of (4.8) in a neighbourhood of the transition point, at which $\delta(\bar{t}_1) = 0$. To analyse this transition region in more detail, we define a *local variable* τ by

$$t_1 = \bar{t}_1 + \epsilon^\alpha \tau, \;\; -\infty < \tau < \infty, \tag{4.9}$$

where α is an undetermined positive constant.

In terms of τ, the solution of (4.8) is denoted by

$$\tilde{p}(\tau,\epsilon) \equiv p(\bar{t}_1 + \epsilon^\alpha\tau, \epsilon).$$

In terms of the new variables $\tilde{p}$ and τ defined in above, the differential equation (4.8) becomes

$$\frac{d\tilde{p}}{d\tau} = \epsilon^{\alpha-1}\, r(\bar{t}_1+\epsilon^\alpha\tau)\left(\tilde{p} - \frac{\tilde{p}^2}{k(\bar{t}_1+\epsilon^\alpha\tau)} - \frac{k(\bar{t}_1+\epsilon^\alpha\tau)}{4}(1-\delta(\bar{t}_1+\epsilon^\alpha\tau))\right). \tag{4.10}$$

Expanding (4.10) in powers of ϵ gives

$$\frac{d\tilde{p}}{d\tau} = \left(r(\bar{t}_1)\tilde{p}\left(1-\frac{\tilde{p}}{k(\bar{t}_1)}\right) - (1/4)\,r(\bar{t}_1)k(\bar{t}_1)\right)\epsilon^{\alpha-1}$$
$$+[\,r'(\bar{t}_1)\,\tilde{p}\left(1-\frac{\tilde{p}}{k(\bar{t}_1)}\right) + \frac{r(\bar{t}_1)\,k'(\bar{t}_1)}{k(\bar{t}_1)^2}\,\tilde{p}^2 + \frac{1}{4}r(\bar{t}_1)k(\bar{t}_1)\delta'(\bar{t}_1)$$
$$-\frac{1}{4}\frac{d}{d\bar{t}_1}\left(r(\bar{t}_1)k(\bar{t}_1)\right)\,]\tau\,\epsilon^{2\alpha-1} + O(\epsilon^{3\alpha-1}). \tag{4.11}$$

Now, we seek a value of α that will create a balance of orders as $\epsilon \to 0$ in this differential equation.

To obtain a value for α, we look back to the first two terms of the Region 1 expansion, (4.3), in order to gain greater insight.

Substituting (4.9) into (4.3) and expanding the terms for small ϵ gives an expansion for the limiting population (4.3) in terms of τ as

$$\tfrac{1}{2}k(\bar{t}_1) + \tfrac{1}{2}k(\bar{t}_1)\sqrt{\delta'(\bar{t}_1)\,\tau}\,\epsilon^{\frac{\alpha}{2}} + \tfrac{1}{2}\,\tau\,k'(\bar{t}_1)\,\epsilon^{\alpha} + O(\epsilon^{3\alpha/2}). \tag{4.12}$$

Based on this, we propose that the solution in the transition region has the form

$$\tilde{p}(\tau,\epsilon) = \tfrac{1}{2}k(\bar{t}_1) + \epsilon^{\frac{\alpha}{2}}\,u_0(\tau) + \epsilon^{\alpha}\,u_1(\tau) + O(\epsilon^{3\alpha/2}), \tag{4.13}$$

where u_0, u_1 are as yet undetermined functions.

We see that the first term of (4.13) now is $O(1)$ and the second term is $O(\epsilon^{\frac{\alpha}{2}})$ as $\epsilon \to 0$. Substituting (4.13) into the transition equation (4.11) gives the leading terms as

$$\epsilon^{\frac{\alpha}{2}}\frac{du_0(\tau)}{d\tau} + \epsilon^{\alpha}\frac{du_1(\tau)}{d\tau} = \left(-\frac{r(\bar{t}_1)}{k(\bar{t}_1)}\,u_0^2(\tau) + \frac{1}{4}\,r(\bar{t}_1)\,k(\bar{t}_1)\,\delta'(\bar{t}_1)\,\tau\right)\epsilon^{2\alpha-1}$$
$$+\; O(\epsilon^{5\alpha/2-1}), \tag{4.14}$$

where now, the first term of the right hand side of (4.14) is $O(\epsilon^{2\alpha-1})$. Thus, there is a balance between the $O(\epsilon^{\frac{\alpha}{2}})$ term on the left side of (4.14) with $O(\epsilon^{2\alpha-1})$ on the right when $\alpha = 2/3$.

Choosing $\alpha = 2/3$, we have (4.13) as

$$\tilde{p}(\tau,\epsilon) = \tfrac{1}{2}k(\bar{t}_1) + \epsilon^{1/3}\,u_0(\tau) + O(\epsilon^{2/3}), \tag{4.15}$$

while (4.12) becomes

$$\tfrac{1}{2}k(\bar{t}_1) + \tfrac{1}{2}k(\bar{t}_1)\,\sqrt{\delta'(\bar{t}_1)\,\tau}\,\epsilon^{1/3} + \tfrac{1}{2}\,\tau\,k'(\bar{t}_1)\,\epsilon^{2/3} + O(\epsilon),\ \ \tau < 0. \tag{4.16}$$

Also, with $\alpha = 2/3$, the differential equation for $u_0(\tau)$, (4.14), becomes

$$\frac{du_0(\tau)}{d\tau} = -\frac{r(\bar{t}_1)}{k(\bar{t}_1)}\,u_0^2(\tau) + \frac{1}{4}\,r(\bar{t}_1)\,k(\bar{t}_1)\,\delta'(\bar{t}_1)\,\tau,\ \ \delta'(\bar{t}_1) < 0, \tag{4.17}$$

which is a Riccati differential equation. Now, for large and negative τ, we expect the left hand side of (4.17), $du_0(\tau)/d\tau$, tends to zero, hence

$$\frac{r(\bar{t}_1)}{k(\bar{t}_1)}\,u_0^2(\tau) - \frac{1}{4}\,r(\bar{t}_1)\,k(\bar{t}_1)\,\delta'(\bar{t}_1)\,\tau \to 0$$

and so, for large and negative τ, the solution $u_0(\tau)$ of (4.17) is

$$u_0(\tau) \to \pm\frac{1}{2}\,k(\bar{t}_1)\,\sqrt{\delta'(\bar{t}_1)\,\tau},\ \ \tau < 0\ \delta'(\bar{t}_1) < 0. \tag{4.18}$$

If $\tau < 0$ and $u_0 > \frac{1}{2}\,k(\bar{t}_1)\,\sqrt{\delta'(\bar{t}_1)\,\tau}$ then the right hand side of (4.17) is negative. Thus, $du_0(\tau)/d\tau$ is negative and $u_0(\tau)$ decreases as τ increases. The solution moves towards the positive branch in (4.18).
If $\tau < 0$ and $-\frac{1}{2}\,k(\bar{t}_1)\,\sqrt{\delta'(\bar{t}_1)\,\tau} < u_0 < \frac{1}{2}\,k(\bar{t}_1)\,\sqrt{\delta'(\bar{t}_1)\,\tau}$ then the right hand side of (4.17) is positive. Thus, $du_0(\tau)/d\tau$ is positive and $u_0(\tau)$ increases, As τ increases, the solution moves towards the positive branch in (4.18).

If $\tau < 0$ and $u_0 < -\frac{1}{2}\,k(\bar{t}_1)\,\sqrt{\delta'(\bar{t}_1)\,\tau}$ then the right hand side of (4.17) is negative and so, $u_0(\tau)$ decreases and becomes more negative as τ increases, (i.e., the solution moves away from the negative solution in (4.18).

Thus, for large and negative τ the solution of (4.17) approaches the positive solution of (4.18)

Also, comparing (4.18) with (4.12), we see that to match terms in Region 2 with corresponding terms in Region 1, the correct solution branch as $\tau \to -\infty$ to be selected must satisfy

$$u_0(\tau) \to +\frac{1}{2}\,k(\bar{t}_1)\,\sqrt{\delta'(\bar{t}_1)\,\tau}. \tag{4.19}$$

Now, as we have noted, (4.17) is a Riccati equation; and this can be solved using the Cole-Hopf transformation (see [63]), by which $u_0(\tau)$ is related to $\phi(\tau)$ by

$$u_0(\tau) = \frac{k(\bar{t}_1)}{r(\bar{t}_1)} \frac{1}{\phi(\tau)} \frac{d\,\phi(\tau)}{d\tau} = \frac{k(\bar{t}_1)}{r(\bar{t}_1)} \frac{d\,(\ln \phi(\tau))}{d\tau}. \tag{4.20}$$

Substituting (4.20) into (4.17) produces a second order linear equation for $\phi(\tau)$;

$$\frac{d^2\,\phi(\tau)}{d\tau^2} - a^3\,\tau\,\phi(\tau), = 0 \tag{4.21}$$

where

$$a^3 = \frac{1}{4}\,r(\bar{t}_1)^2\,\delta'(\bar{t}_1)\, < 0. \tag{4.22}$$

Equation (4.21) is an Airy differential equation which has the general solution

$$\phi(\tau) = C_1 \mathrm{Ai}(a\,\tau) + C_2\,\mathrm{Bi}(a\,\tau). \tag{4.23}$$

where $\mathrm{Ai}(x)$ and the related function $\mathrm{Bi}(x)$, are Airy functions see [1].

Combining (4.19) and (4.20), we require that

$$\frac{d\,(\ln \phi(\tau))}{d\tau} \to \frac{r(\bar{t}_1)}{2} \sqrt{\delta'(\bar{t}_1)\,\tau} \ \text{ as } \tau \to -\infty,$$

and so

$$\ln \phi(\tau) \to \frac{1}{3}\,r(\bar{t}_1)\,\tau\,\sqrt{\delta'(\bar{t}_1)\,\tau} \ \text{ as } \tau \to -\infty$$

which leads to

$$\phi(\tau) \ \to\ e^{\frac{1}{3}\,r(\bar{t}_1)\,\tau\,\sqrt{\delta'(\bar{t}_1)\,\tau}},\ \ \delta'(\bar{t}_1) < 0,\ \text{ as } \tau \to -\infty. \tag{4.24}$$

Consequently, as $\tau \ \to\ -\infty$,

$$\phi(\tau) \ \to\ 0. \tag{4.25}$$

However, [1]- Chapter 10, as $\tau \ \to\ -\infty$

$$\mathrm{Ai}(a\,\tau) \to 0 \ \text{ and } \ \mathrm{Bi}(a\,\tau) \to \infty.$$

Thus, in (4.23) we choose $C_2 = 0$ (to satisfy condition (4.25)) and hence

$$\phi(\tau) = C_1 \mathrm{Ai}(a\,\tau),$$

so that the solution (4.20) becomes

$$u_0(\tau) = \frac{k(\bar{t}_1)}{r(\bar{t}_1)} \frac{a\,\mathrm{Ai}'(a\,\tau)}{\mathrm{Ai}(a\,\tau)}. \tag{4.26}$$

We note that $u_0(\tau)$ contains no arbitrary constant.

Substituting (4.26) into (4.15) gives the solution in the transition region (Region 2) as

$$\tilde{p}(\tau, \epsilon) = \tfrac{1}{2}k(\bar{t}_1) + \epsilon^{1/3} \frac{k(\bar{t}_1)}{r(\bar{t}_1)} \frac{a\,\mathrm{Ai}'(a\,\tau)}{\mathrm{Ai}(a\,\tau)} + O(\epsilon^{2/3}). \tag{4.27}$$

As $\tau \to -\infty$ (i.e., as we move into Region 1), the solution (4.27) has the property

$$\tilde{p}(\tau, \epsilon) \to \tfrac{1}{2}k(\bar{t}_1) + \epsilon^{1/3}\tfrac{1}{2}k(\bar{t}_1)\sqrt{\delta'(\bar{t}_1)\,\tau} + O(\epsilon^{2/3}) \tag{4.28}$$

Thus, by comparing the first two leading order terms of (4.16) and (4.28), we see that matching is automatically achieved. Moreover, the common terms (or *common part*) in these two expansions is given by

$$C_{1\,2}(\tau, \epsilon) = \tfrac{1}{2}k(\bar{t}_1) + \epsilon^{1/3}\tfrac{1}{2}k(\bar{t}_1)\sqrt{\delta'(\bar{t}_1)\,\tau}. \tag{4.29}$$

Thus, in the combined Region 1 and Region 2, (2.36) represents the leading term approximate solution of (4.1) in Region 1, while (4.27) represents the leading term approximate solution in Region 2. These approximate solutions merge in going from Region 1 to Region 2, with a common part being given by (4.29).

We now consider the case of the transition of the solutions from Region 2 to Region 3, typified by the limit $\tau \to \infty$.

From (4.26), as time $\tau \to \infty$ (and using $\mathrm{Ai}(-z)$ and $\mathrm{Ai}'(-z)$ expansions(see [1]-Chapter 10)), we get for large positive τ,

$$u_0(\tau) = \frac{k(\bar{t}_1)}{2}\sqrt{-\delta'(\bar{t}_1)}\,\sqrt{\tau}\left[-\tan(\zeta - \tfrac{\pi}{4}) + \frac{1}{\zeta}[a_1\,\tan^2(\zeta - \tfrac{\pi}{4}) + b_1] + O\left(\frac{1}{\zeta^2}\right)\right], \tag{4.30}$$

where

$$\zeta = \frac{r(\bar{t}_1)}{3}\sqrt{-\delta'(\bar{t}_1)}\,\tau^{3/2}, \;\; a_1 = \tfrac{5}{72}, b_1 = \tfrac{-7}{5}a_1 = \tfrac{-7}{5}(\tfrac{5}{72}) = \tfrac{-7}{72}.$$

Thus, as $\tau \to \infty$, from (4.27),

$$\tilde{p}(\tau, \epsilon) \to \tfrac{1}{2}k(\bar{t}_1) - \epsilon^{1/3}\frac{k(\bar{t}_1)}{2}\sqrt{-\delta'(\bar{t}_1)}\,\sqrt{\tau}\{\tan(\zeta - \tfrac{\pi}{4}) + \frac{1}{\zeta}[a_1 \tan^2(\zeta - \tfrac{\pi}{4}) + b_1] + O\left(\frac{1}{\zeta^2}\right)\} + O(\epsilon^{2/3}). \tag{4.31}$$

This is the expansion for the Region 2 solution on moving into Region 3.

We now consider the expansion (4.5), (4.6) valid in Region 3, and its behaviour as we move from Region 3 to Region 2. From (4.9), with $\alpha = 2/3$ we have,

$$t_1 = \bar{t}_1 + \epsilon^{2/3}\tau, \tag{4.32}$$

and so

$$\tau = \frac{(t_1 - \bar{t}_1)}{\epsilon^{2/3}}. \tag{4.33}$$

The substitution

$$\upsilon = \frac{1}{\epsilon^{2/3}\,\tau}(s - \bar{t}_1), \tag{4.34}$$

so that

$$ds = \epsilon^{2/3}\,\tau\, d\upsilon,$$

converts the first of (4.6) to

$$\tilde{t}_0 = \frac{1}{\epsilon}\int_0^1 r(\bar{t}_1 + \epsilon^{2/3}\,\tau\,\upsilon)\,\sqrt{-\delta(\bar{t}_1 + \epsilon^{2/3}\,\tau\,\upsilon)}\,\epsilon^{2/3}\,\tau\, d\upsilon.$$

Expanding the functions $r(\bar{t}_1 + \epsilon^{2/3}\,\tau\,\upsilon)$, $\delta(\bar{t}_1 + \epsilon^{2/3}\,\tau\,\upsilon)$ in powers of small ϵ gives

$$r(\bar{t}_1 + \epsilon^{2/3}\,\tau\,\upsilon) = r(\bar{t}_1) + \epsilon^{2/3}\,\tau\,\upsilon\, r'(\bar{t}_1) + O(\epsilon^{4/3})$$

and

$$\sqrt{-\delta(\bar{t}_1 + \epsilon^{2/3}\,\tau\,\upsilon)} = \sqrt{-\delta'(\bar{t}_1)} + O(\epsilon), \;\; \delta(\bar{t}_1) = 0,$$

so that, integrating with respect to υ gives

$$\tilde{t}_0 = r(\bar{t}_1)\,\sqrt{-\delta'(\bar{t}_1)}\,\tau^{3/2} + O(\epsilon). \tag{4.35}$$

Similarly, from (4.6) and using (4.34) we have

$$\tilde{A}(t_1) = -\int_{\upsilon=0}^{1} \frac{k'(\bar{t}_1 + \epsilon^{2/3}\,\tau\,\upsilon)}{k(\bar{t}_1 + \epsilon^{2/3}\,\tau\,\upsilon)\,\sqrt{-\delta'(\bar{t}_1 + \epsilon^{2/3}\,\tau\,\upsilon)}}\,\mathrm{d}\upsilon.$$

Expanding the integrand for small ϵ and then integrating with respect to υ gives

$$\tilde{A}(t_1) = -2\,\epsilon^{1/3} \frac{k'(\bar{t}_1)}{k(\bar{t}_1)\,\sqrt{-\delta'(\bar{t}_1)}} \sqrt{\tau} + O(\epsilon^{2/3}). \tag{4.36}$$

where τ is defined by (4.33). Thus, by using (4.32), substituting the leading terms of (4.35) and (4.36) into the expansion (4.5), and expanding in powers of ϵ, we obtain

$$p_0(t,\epsilon) = \frac{k(\bar{t}_1)}{2} - \epsilon^{1/3}\,\frac{k(\bar{t}_1)}{2}\sqrt{-\delta'(\bar{t}_1)}\sqrt{\tau}\,\tan\left[\frac{1}{3}\,\sqrt{-\delta(\bar{t}_1)}\,r(\bar{t}_1)\,\tau^{3/2} + \frac{d}{2}\right] + O(\epsilon^{2/3}). \tag{4.37}$$

Comparing the leading terms of (4.37), which represents the solution of Region 3 with the transition solution (4.31) for positive τ, we see that these agree on choosing

$$d = -\frac{\pi}{2}. \tag{4.38}$$

With this choice, the common terms (*common part*) between (4.31) and the expansion (4.37) becomes

$$C_{2\,3}(\tau,\epsilon) = \frac{k(\bar{t}_1)}{2} - \epsilon^{1/3}\,\frac{k(\bar{t}_1)}{2}\sqrt{-\delta'(\bar{t}_1)}\sqrt{\tau}\,\tan\left[\frac{1}{3}\,\sqrt{-\delta(\bar{t}_1)}\,r(\bar{t}_1)\,\tau^{3/2} - \frac{\pi}{4}\right] + O(\epsilon^{2/3}), \tag{4.39}$$

where τ is defined by (4.33).

4.4 A Uniform Approximation

We have constructed approximations to the solution of the problem (4.1) on separate subregions Region 1, Region 2 and Region 3 as above. Now, we formulate uniformly valid approximations to this solution on the entire interval $t \geq 0$. We begin by constructing an approximation valid uniformly on $[0, \bar{t}_1]$, that is, up to the transition point, by using the *additive composition technique.* This involves adding the leading terms of the approximate solution on Region 1 ((2.36) or (2.47)) to (4.27) and then subtracting the common part (4.28).

Thus, this uniform solution $p_{12}(t,\epsilon)$ in this case becomes

$$p_{12}(t,\epsilon) = p_0(t,\epsilon) + \tilde{p}(\tau,\epsilon) - C_{12}(\tau,\epsilon), \tag{4.40}$$

where the leading term of $p_0(t,\epsilon)$ is given by (2.36) or (2.47), so that

$$\begin{aligned} p_{12}(t,\epsilon) = \tfrac{1}{2}k(t_1) + \frac{k(t_1)\,\sqrt{\delta(t_1)}}{2}\tanh\left[\tfrac{1}{2}(t_0 + A(t_1) + c_0)\right] + \frac{k(\bar{t}_1)}{r(\bar{t}_1)}\,\frac{a\,\mathrm{Ai}'(a\,\tau)}{\mathrm{Ai}(a\,\tau)} \\ -\tfrac{1}{2}k(\bar{t}_1)\,\sqrt{\delta'(\bar{t}_1)\,\tau} + O(\epsilon), \ \ 0 \le t_1 < \bar{t}_1, \ \tau < 0, \end{aligned} \tag{4.41}$$

where a, τ are given by (4.22) and (4.33) respectively; or

$$\begin{aligned} p_{12}(t,\epsilon) = \tfrac{1}{2}k(t_1) + \frac{k(t_1)\,\sqrt{\delta(t_1)}}{2}\coth\left[\tfrac{1}{2}(t_0 + A(t_1) + c_{0R})\right] + \frac{k(\bar{t}_1)}{r(\bar{t}_1)}\,\frac{a\,\mathrm{Ai}'(a\,\tau)}{\mathrm{Ai}(a\,\tau)} \\ -\tfrac{1}{2}k(\bar{t}_1)\,\sqrt{\delta'(\bar{t}_1)\,\tau} + O(\epsilon), \ \ 0 \le t_1 < \bar{t}_1, \end{aligned} \tag{4.42}$$

where t_0, and $A(t_1)$ are given by (2.33) and (2.34) respectively, τ is defined by (4.33), while c_0, c_{0R} are defined by (2.38) and (2.45) respectively, (see the discission on c_0 values in Chapter 2 - Section 2.3.1).

We can write (4.41) and (4.42) in terms of t, $\bar{t}$ using (4.32) and (4.33) as

$$\begin{aligned} p_{12}(t,\epsilon) = \tfrac{1}{2}k(\epsilon\,t) + \frac{k(\epsilon\,t)\,\sqrt{\delta(\epsilon\,t)}}{2}\tanh\left[\tfrac{1}{2}(t_0 + A(\epsilon\,t) + c_0)\right] \\ +\epsilon^{1/3}\frac{k(\epsilon\,\bar{t})}{r(\epsilon\,\bar{t})}\,\frac{a\,\mathrm{Ai}'(a\,\epsilon^{1/3}\,(t-\bar{t}))}{\mathrm{Ai}(a\,\epsilon^{1/3}\,(t-\bar{t}))} - \tfrac{1}{2}\epsilon^{1/2}\,k(\epsilon\,\bar{t})\,\sqrt{-\delta'(\epsilon\,\bar{t})}\,(t-\bar{t})^{1/2} + O(\epsilon^{2/3}), \quad 0 < \epsilon t < \epsilon\bar{t}. \end{aligned} \tag{4.43}$$

or

$$\begin{aligned} p_{12}(t,\bar{t},\epsilon) = \tfrac{1}{2}k(\epsilon\,t) + \frac{k(\epsilon\,t)\,\sqrt{\delta(\epsilon\,t)}}{2}\coth\left[\tfrac{1}{2}(t_0 + A(\epsilon\,t) + c_{R0})\right] \\ +\frac{k(\epsilon\,\bar{t})}{r(\epsilon\,\bar{t})}\,\frac{a\,\mathrm{Ai}'(a\,\epsilon^{1/3}\,(t-\bar{t}))}{\mathrm{Ai}(a\,\epsilon^{1/3}\,(t-\bar{t}))} - \tfrac{1}{2}\epsilon^{1/2}\,k(\epsilon\,\bar{t})\,\sqrt{-\delta'(\epsilon\,\bar{t})}\,(t-\bar{t})^{1/2} + O(\epsilon^{2/3}) \quad 0 < \epsilon t < \epsilon\bar{t}. \end{aligned} \tag{4.44}$$

Again by applying the same composition technique, we create a uniform approximation on $[\bar{t}, \infty)$, that is, beyond the transition point.

From (4.5), (4.31) and (4.39) we have a general expansion for all $t_1 \geq \bar{t}_1$ as

$$p_{23}(t,\epsilon) = \tilde{p_0}(t,\epsilon) + \tilde{p}(\tau,\epsilon) - C_{23}(\tau,\epsilon), \tag{4.45}$$

and so

$$\begin{aligned} p_{23}(t,\epsilon) = {}& \tfrac{1}{2}k(t_1) - \frac{k(t_1)\,\sqrt{-\delta(t_1)}}{2}\tan\left[\tfrac{1}{2}(\tilde{t}_0 + \tilde{A}(t_1) - \frac{\pi}{2})\right] + \frac{k(\bar{t}_1)}{r(\bar{t}_1)}\,\frac{a\,\mathrm{Ai}'(a\,\tau)}{\mathrm{Ai}(a\,\tau)} \\ & + \epsilon^{1/3}\frac{k(\bar{t}_1)}{2}\,\sqrt{-\delta'(\bar{t}_1)}\,\sqrt{\tau}\,\tan(\frac{1}{3}\,\sqrt{-\delta(\bar{t}_1)}\,r(\bar{t}_1)\,\tau^{3/2} - \tfrac{\pi}{4}),\;\; t_1 > \bar{t}_1. \end{aligned} \tag{4.46}$$

Rewriting (4.46) in terms of t and $\bar{t}$ gives

$$\begin{aligned} p_{23}(t,\epsilon) = {}& \tfrac{1}{2}k(\epsilon t) - \frac{k(\epsilon t)\,\sqrt{-\delta(\epsilon t)}}{2}\tan\left[\tfrac{1}{2}(\tilde{t}_0 + \tilde{A}(\epsilon\, t) - \frac{\pi}{2})\right] + \frac{k(\epsilon\bar{t})}{r(\epsilon\bar{t})}\,\frac{a\,\mathrm{Ai}'(a\,\epsilon^{1/3}\,(t-\bar{t}))}{\mathrm{Ai}(a\,\epsilon^{1/3}\,(t-\bar{t}))} \\ & + \epsilon^{1/2}\frac{k(\epsilon\bar{t})}{2}\,\sqrt{-\delta'(\epsilon\bar{t})}\,(t-\bar{t})^{1/2}\,\tan(\frac{\epsilon^{1/2}}{3}\,\sqrt{-\delta(\epsilon\bar{t})}\,r(\epsilon\bar{t})\,(t-\bar{t})^{3/2} - \frac{\pi}{4}),\;\; \epsilon t > \epsilon\bar{t}. \end{aligned} \tag{4.47}$$

Thus, the combination of (4.43) (or (4.44)) and (4.47) provide two component composite expansions for the population $p(t,\epsilon)$ which cover Regions 1, 2 and 3; i.e.,over all $t \geq 0$.

4.5 An Example: Periodic Growth with Saturating Carrying Capacity and Linear Harvesting

In the previous Section, we have constructed an explicit expansion approximating the solution of the problem (4.1) that gives the population $p(t,\epsilon)$ as a function of time over the Regions 1, 2 and 3; i.e., over all $t \geq 0$. In the case considered, the population has made a transition from a subcritical harvesting with survival situation via the transition region to a supercritical harvesting situation resulting in extinction. This expansion represents the solution of (4.1) on $t \geq 0$ for any arbitrary slowly varying parameters $r(\epsilon\, t)$, $k(\epsilon\, t)$ and $h(\epsilon\, t)$, and small and positive ϵ.

We now consider the application of these results of this Section to specific examples of the functions r, k and h.

We choose the parameters r, k and h as slowly varying functions given by

$$\begin{aligned} r(\epsilon\, t) &= 1 - 0.3\,\sin(2\,\epsilon\, t), \\ k(\epsilon\, t) &= 1 + 0.2\,\tanh(\epsilon\, t), \\ h(\epsilon\, t) &= 1 + 2\,\epsilon\, t. \end{aligned} \tag{4.48}$$

where $\epsilon = 0.05$ and $\sigma = 0.05$ and from (4.2), we have

$$\delta(\epsilon\, t) = 1 - 4\frac{0.2(1 + 2\,\epsilon\, t)}{(1 - 0.3\,\sin(2\,\epsilon\, t))\,(1 + 0.2\,\tanh(\epsilon\, t))}. \tag{4.49}$$

Here, the growth rate $r(\epsilon\, t)$ has a slow periodic variation while the carrying capacity $k(\epsilon\, t)$, increases slowly from and monotonically 1 to a saturation value of 1.2. The harvesting rate, $h(\epsilon\, t)$, is a slowly increasing linear function of t. These choices might correspond to a population that has a slow periodic growth rate, r, perhaps due to seasonal variation, being harvested at a slowly increasing rate in an environment providing an increasing but limited carrying capacity, k.

Figure 4.2 shows that for these choices, (with $\sigma = 0.05$ and $\epsilon = 0.05$), $\delta(\epsilon t)$ given by (4.2) has a single transition (i.e., $\delta(\epsilon t)$ changes from positive to negative) at the transition point $\bar{t} \approx 58$.

On the other hand, Figure 4.3 shows that, with the choices (4.48) and $\sigma = 0.05$ and $\epsilon = 0.01$, $\delta(\epsilon t)$ given by (4.2) has a single transition at $\bar{t} \approx 33$.

Figure 4.4 displays the behaviour of the numerical solution of (4.1) incorporating (4.2) with $\sigma = 0.05$, $\epsilon = 0.05$ and initial population $\mu = 0.3$, obtained using a fourth order explicit Runge-Kutta method with step size $h = 0.001$. This clearly shows the evaluation of the population from the starting value 0.3, through a relatively rapid transient region (where t_0 variation dominates) to a slowly varying limiting state. As t approaches the transition point, the population diverges from this limiting state, and on passing

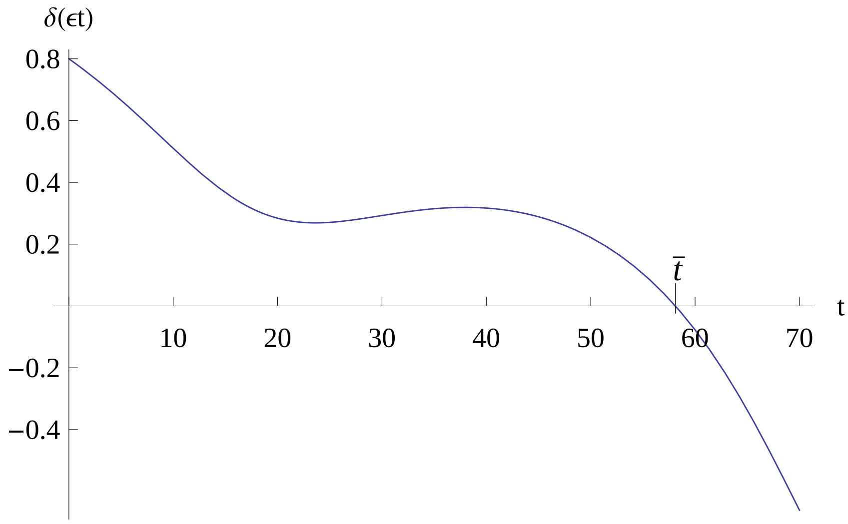

Figure 4.2: Plot of $\delta(\epsilon t)$ given by (4.2) with $\sigma = 0.05$ $\epsilon = 0.05$ using data of (4.48).

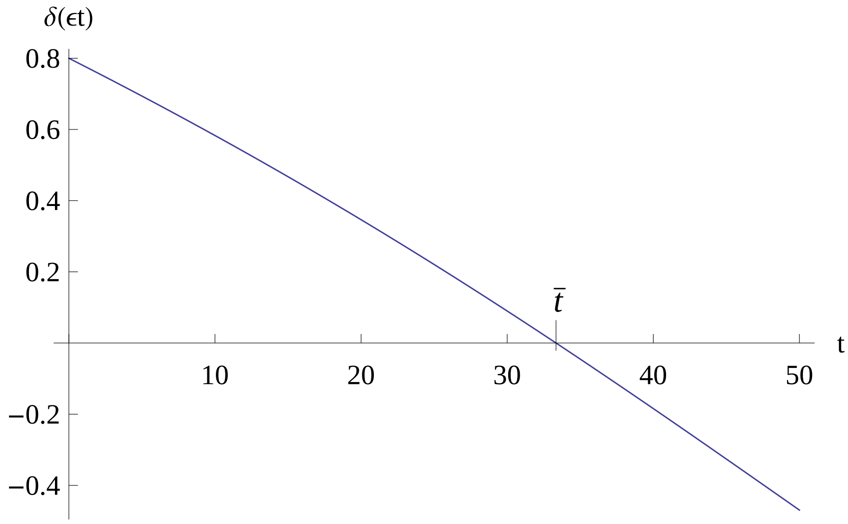

Figure 4.3: Plot of $\delta(\epsilon t)$ given by (4.2) with $\sigma = 0.05$ $\epsilon = 0.01$ using data of (4.48).

transition, reduces monotonically to zero at $t \approx 66.2$.

In Figures 4.5 to 4.10, we compare the behaviour of this numerical solution with the various components of the expansions of the expression (4.43) and (4.47), which, for $\mu = 0.3$, apply here.

Figure 4.5 shows that the leading term of (4.43), corresponding to (2.36) agrees very well with the numerical solution, up to a neighbourhood of the transition point (where $\delta(t_1) \approx 0$) where they diverge.

Figure 4.6 considers the transition region to the left of transition point $t < \bar{t}$, and compares the numerical solution with the leading terms (2.36), the transition solution (4.27) and the common part C_{12} given by (4.29). Clearly, close to $\bar{t}$, C_{12} (the dashed line) and the leading term expansion (2.36)(continuous curve) are virtually identical; so the composition (4.40) is, effectively, the transition solution (4.27).

On the other hand, away from $\bar{t}$, C_{12} is close to the transition solution (4.27), so (4.40) gives the leading term (2.36) to leading order. Thus, in either direction (for $t < \bar{t}$), the composition (4.40) approximates the numerical solution well.

In Figure 4.7 we see the uniformly valid expansion (4.43) based on (4.40) up to the transition point, i.e., in $t < \bar{t}$, compared with the numerical solution. The agreement is very good.

Figure 4.8 considers the transition region just past the transition point, i.e., in $t > \bar{t}$, where the harvesting is now supercritical. Here, the leading order supercritical term (4.37) and common term C_{23} given by (4.39) are very close throughout. Near to the transition point (with $t > \bar{t}$), then, the compositions (4.45) gives the transition solution; and this closeness continues up to the point of extinction, with small correction.

Figure 4.9 shows that the composite expansion (4.47) and numerical solution remain close up to the point of extinction.

Figure 4.10 compares the union of the two composite expansions (4.43) and (4.47) on

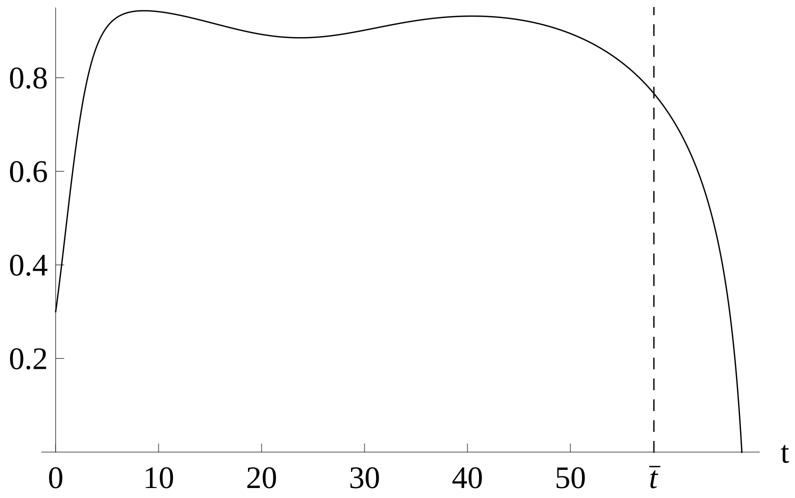

Figure 4.4: Plot of the numerical solution of (4.1) using (4.48), where $\mu = 0.3, \epsilon = 0.05$, $\sigma = 0.05$ and $\bar{t} \approx 58$.

$t \geq 0$, with the numerical solution. We see that they are clearly very close.

In Figures 4.11 to 4.15, we see a repeat of the above sequence of comparisons for the case where $\epsilon = 0.01$, $\sigma = 0.05$ and $\mu = 1.3$. Here, the transition value is at $\bar{t} \approx 33$, and the composite expansion (4.44) applies on the subcritical region.

Note that for both of the initial values considered ($\mu = 0.3$, 1.3), the population is in a surviving state in $t < \bar{t}$, but becomes extinct on $t > \bar{t}$, after a finite time.

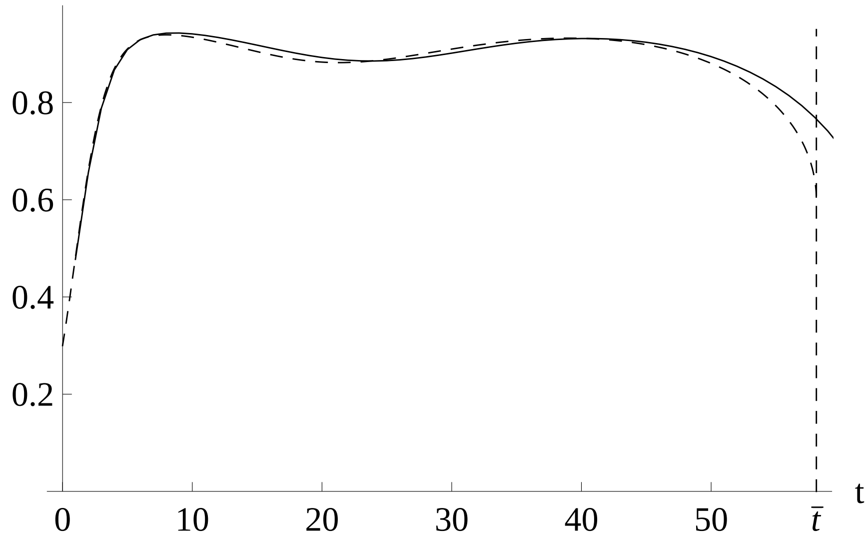

Figure 4.5: Plot of the numerical solution of (4.1)(continuous curve) and the leading term asymptotic approximation (4.43) up to the transition point (dashed curve) using (4.48), with $\mu = 0.3, \epsilon = 0.05$, $\sigma = 0.05$ and $\bar{t} \approx 58$.

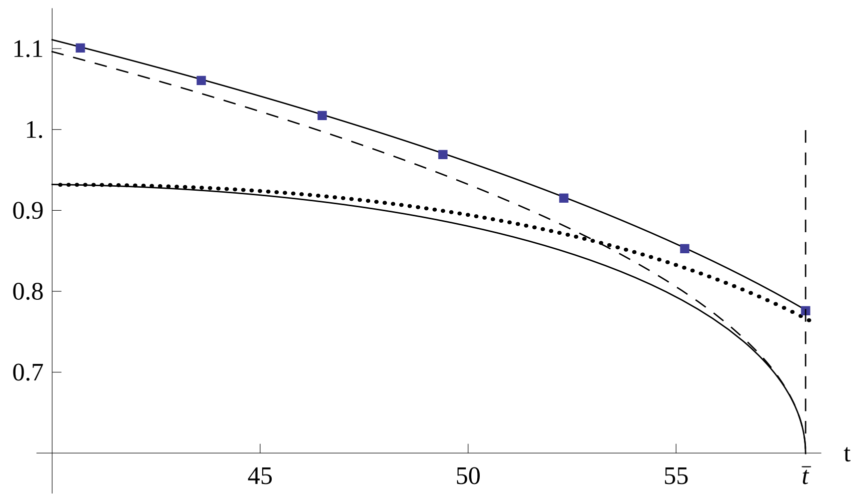

Figure 4.6: Plot of the leading term expansion (2.36) (continuous curve) and transition solution (4.27) (solid with filled squares) with common terms (4.29) (dashed line) and Subcritical numerical solution of (4.1)(dotted), using the parameters (4.48) $\mu = 0.3, \epsilon = 0.05$, $\sigma = 0.05$.

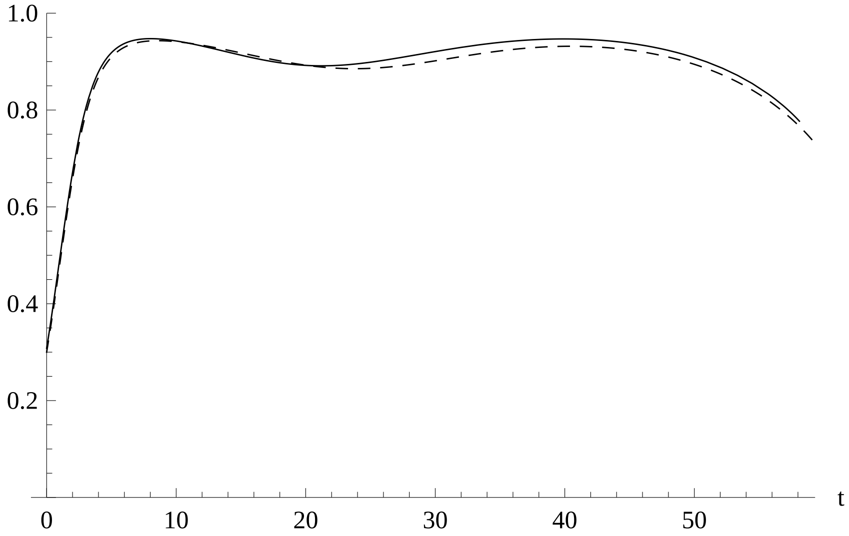

Figure 4.7: Plot of the uniformly valid subcritical expansion (4.43)(continuous curve) and compared with numerical solution (dashed) using the parameters (4.48) with $\mu = 0.3, \epsilon = 0.05$, and $\sigma = 0.05$.

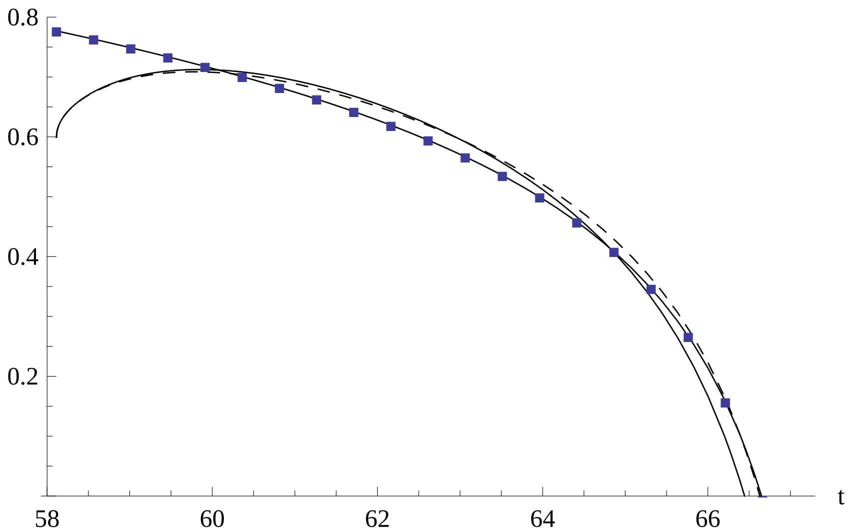

Figure 4.8: Plot of the two terms of supercritical expansion (4.5)(continuous curve) using (4.48) where $\mu = 0.3, \epsilon = 0.05$ and transition solution (4.27) (solid with filled squares), with common terms (dashed).

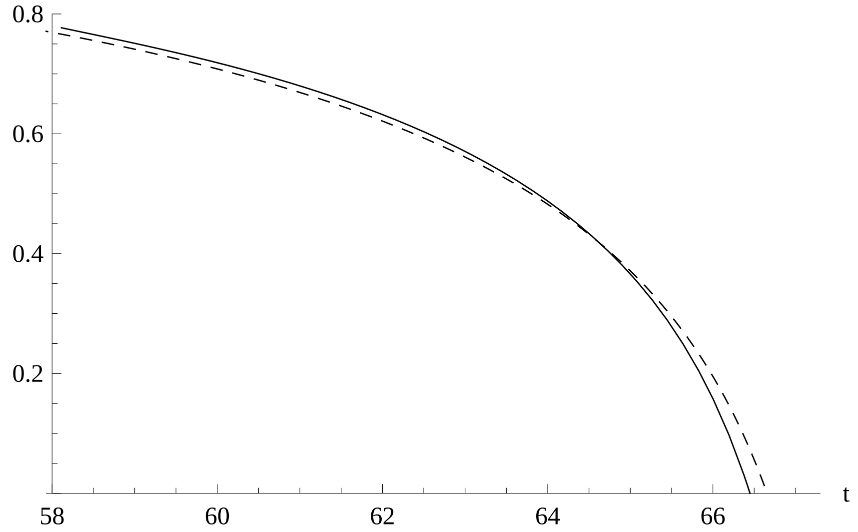

Figure 4.9: Plot of the valid approximate expansion (4.47) (continuous curve) and the numerical solution (dashed) for the choice (4.48) where $\mu = 0.3, \epsilon = 0.05$, $\sigma = 0.05$.

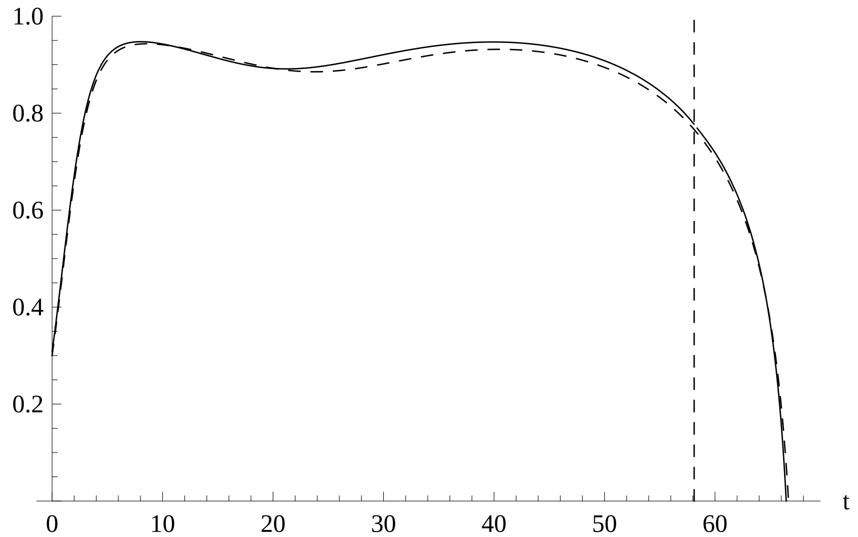

Figure 4.10: Plot of the composite expansions (4.43), (4.47) (solid) with the numerical solution of (4.1)(dashed), considering the choice (4.48) with $\mu = 0.3, \epsilon = 0.05$, $\sigma = 0.05$ and $\bar{t} \approx 58$..

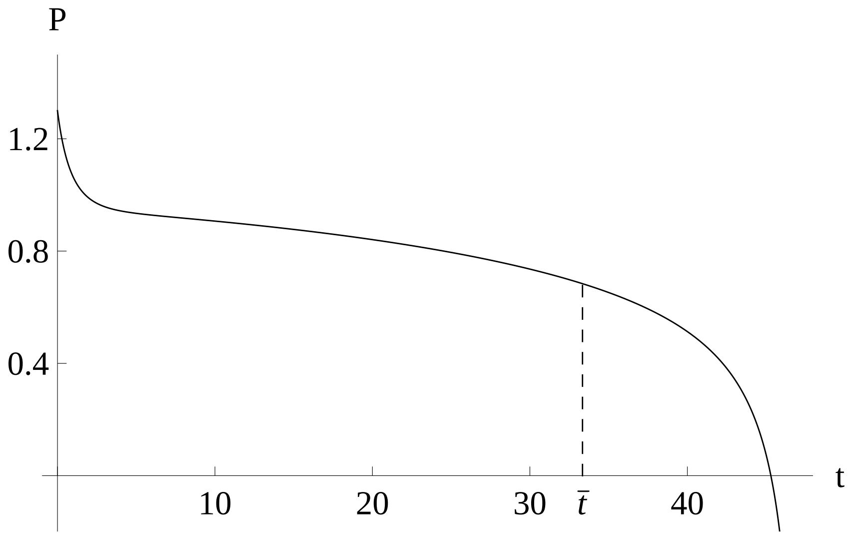

Figure 4.11: Plot of the numerical solution of (4.1) using (4.48), where $\mu = 1.3, \epsilon = 0.01$, $\sigma = 0.05$

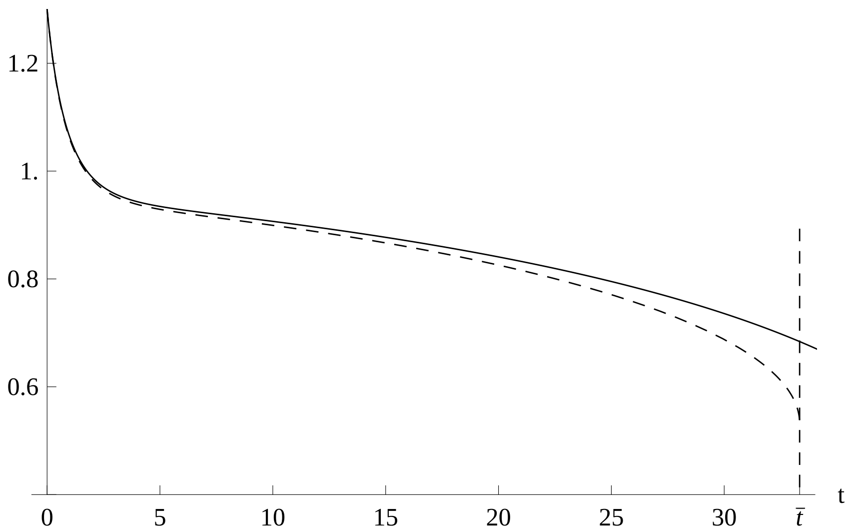

Figure 4.12: Plot of the numerical solution of (4.1)(continuous curve) and the two term asymptotic approximation up to the transition point (2.47) (dashed curve) using (4.48), where $\mu = 1.3, \epsilon = 0.01$, $\sigma = 0.05$ and $\bar{t} \simeq 33$

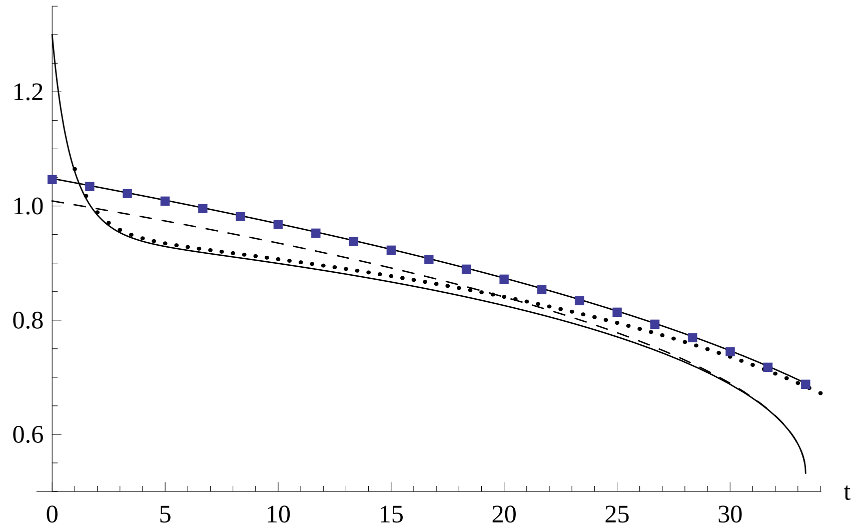

Figure 4.13: Plot of leading term asymptotic expansion (2.47) (continuous curve) and transition solution (4.27) (solid with filled squares) with common terms (4.29)(dashed line) and Subcritical numerical solution of (4.1)(dotted), using the parameters (4.48) with $\mu = 1.3, \epsilon = 0.01$, and $\sigma = 0.05$.

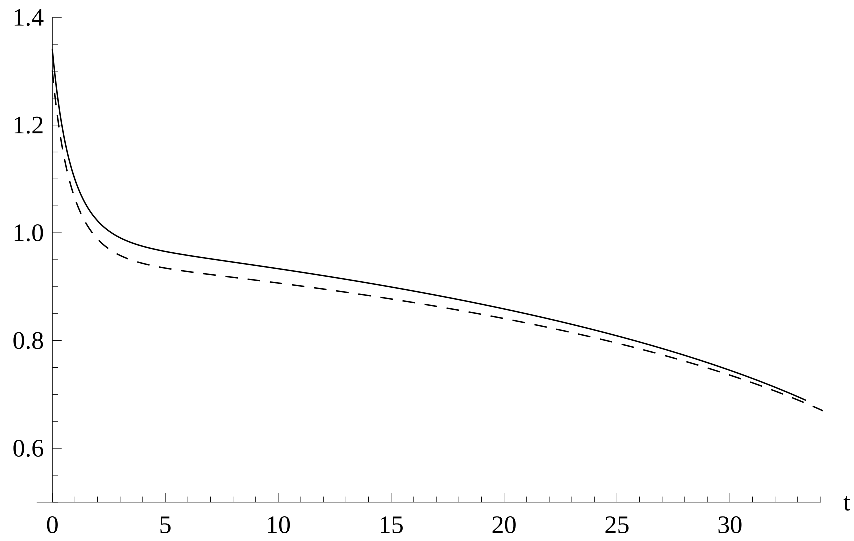

Figure 4.14: Plot of uniform subcritical expansion (4.44) (continuous curve), considering the parameters (4.48) with $\mu = 1.3, \epsilon = 0.01$, and $\sigma = 0.05$, and compared with numerical solution (dashed).

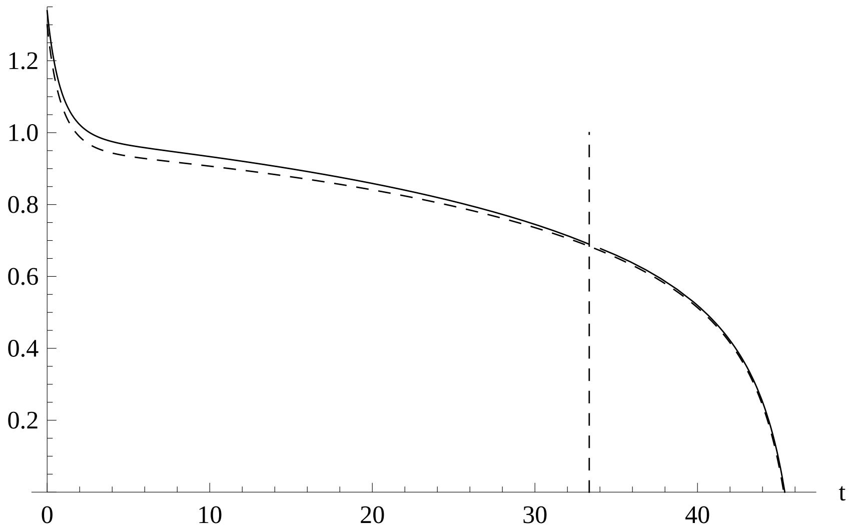

Figure 4.15: Plot of composite expansions (4.44), (4.47)(solid) and numerical solution of (4.1)(dashed) using (4.48) where $\mu = 1.3, \epsilon = 0.01$, $\sigma = 0.05$ and $\bar{t} \simeq 33.34$..

4.6 A Harvested Logistic Model with Exact Solution: Comparison with the Asymptotic Approximations

In this section, we compare an exact solution of the problem (4.1) with the composite approximations constructed in Section 4.4.

To do this, we consider a special case of the functions $r(t_1)$, $k(t_1)$ and $h(t_1)$, namely

$$h(t_1) = \frac{1}{4\,\sigma}[1 - (\bar{t}_1 - t_1)] = \frac{1}{4\,\sigma}[1 - \epsilon(T - t)]$$
$$r(t_1) = k(t_1) = 1. \tag{4.50}$$

Substituting (4.50) into (4.2) gives

$$\delta(t_1) = 1 - 4\,\sigma\,h(\epsilon t) = \bar{t}_1 - t_1 = \epsilon(T - t). \tag{4.51}$$

and applying the condition (4.4) gives

$$\delta(\bar{t}_1) = 0 \Leftrightarrow \bar{t}_1 = t_1 \text{ and } \delta'(\bar{t}_1) = -1 < 0.$$

Thus the transition point occurs at $t_1 = \bar{t}_1 = \epsilon T$. Then, (4.1) becomes

$$\frac{dp(t,\epsilon)}{dt} = p(t,\epsilon)\,(1 - p(t,\epsilon)) - \frac{1}{4}[1 - \epsilon(T - t)], \qquad p(0,\epsilon) = \mu. \tag{4.52}$$

4.6.1 The Exact Solution

We now solve (4.52) exactly. By putting

$$p(t,\epsilon) = \frac{1}{2} + \frac{\phi'}{\phi}, \tag{4.53}$$

where ϕ is a function of time t, substituting (4.53) into (4.52) gives

$$\phi'' = \frac{\epsilon}{4}(T - t)\,\phi. \tag{4.54}$$

Letting

$$z = \left(\frac{\epsilon}{4}\right)^{1/3}(T - t) \tag{4.55}$$

converts the equation (4.54) for ϕ in terms of z as

$$\frac{d^2\phi}{dz^2} - z\,\phi = 0. \tag{4.56}$$

Equation (4.56) has the general solution

$$\phi = d_1\,\mathrm{Ai}(z) + d_2\mathrm{Bi}(z) \tag{4.57}$$

where $d_1\, d_2$ are constants, and $\mathrm{Ai}(z), \mathrm{Bi}(z)$ are Airy functions see [1].

This leads to

$$\frac{\phi'}{\phi} = -\left(\frac{\epsilon}{4}\right)^{1/3}\frac{d_1\,\mathrm{Ai}'(z) + d_2\mathrm{Bi}'(z)}{d_1\,\mathrm{Ai}(z) + d_2\mathrm{Bi}(z)}. \tag{4.58}$$

From (4.52) and (4.55), at $t = 0$ we have $p(0) = \mu$ and $z(t=0) = \left(\frac{\epsilon}{4}\right)^{1/3} T$, this leads to

$$\mu - \frac{1}{2} = -\left(\frac{\epsilon}{4}\right)^{1/3}\frac{d_1\mathrm{Ai}'(z_0) + d_2\mathrm{Bi}'(z_0)}{d_1\mathrm{Ai}((z_0) + d_2\mathrm{Bi}(z_0)}, \quad z_0 = \left(\frac{\epsilon}{4}\right)^{1/3} T \tag{4.59}$$

and so,

$$d_1 = (\tfrac{\epsilon}{4})^{1/3}\ \mathrm{Bi}'((\tfrac{\epsilon}{4})^{1/3}T) + (\mu - 1)\,\mathrm{Bi}((\tfrac{\epsilon}{4})^{1/3}T) \tag{4.60}$$

and

$$d_2 = -\left[(\tfrac{\epsilon}{4})^{1/3}\,\mathrm{Ai}'((\tfrac{\epsilon}{4})^{1/3}\,T) + (\mu-1)\,\mathrm{Ai}((\tfrac{\epsilon}{4})^{1/3}\,T)\right]. \tag{4.61}$$

and so (4.53) gives

$$p(t,\epsilon) = \tfrac{1}{2} - (\tfrac{\epsilon}{4})^{1/3}\frac{d_1\,\mathrm{Ai}'[(\tfrac{\epsilon}{4})^{1/3}\,(T-t)] + d_2\,\mathrm{Bi}'[(\tfrac{\epsilon}{4})^{1/3}\,(T-t)]}{d_1\,\mathrm{Ai}[(\tfrac{\epsilon}{4})^{1/3}\,(T-t)] + d_2\,\mathrm{Bi}[(\tfrac{\epsilon}{4})^{1/3}\,(T-t)]}. \tag{4.62}$$

Thus, (4.62) provides the exact solution of (4.1) for the special case of the functions defined by (4.50).

4.6.2 The Asymptotic Approximations

We now construct the composite approximation to the solution of the problem (4.52) using the approximations of Section 4.4.

- **Asymptotic Approximation in the Subcritical Region.**

In this case where $t < T$, where T is transition point as discussed above. From (2.33), we have

$$t_0 = \tfrac{2}{3}\epsilon^{1/2}\,[T^{3/2} - (T-t)^{3/2}], \tag{4.63}$$

and so, $0 \le t_0 \le \epsilon^{1/2}\,T^{3/2}$. Thus, the leading terms of subcritical harvesting expansion (2.36) or (2.47) become

$$p(t_0,t_1) = \tfrac{1}{2}\,\left[1 + \sqrt{\bar{t}_1 - t_1}\,\tanh[\tfrac{1}{2}(t_0 + c_0)]\right] + O(\epsilon) \tag{4.64}$$

or

$$p_0(t_0,t_1) = \tfrac{1}{2}\,\left[1 + \sqrt{\bar{t}_1 - t_1}\,\coth[\tfrac{1}{2}(t_0 + c_{0R})]\right] + O(\epsilon) \tag{4.65}$$

where

$$\begin{aligned} c_0 &= 2\,\mathrm{arctanh}\left(\frac{2\mu-1}{\sqrt{\epsilon\,T}}\right), \\ c_{0R} &= 2\,\mathrm{arccoth}\left(\frac{2\mu-1}{\sqrt{\epsilon\,T}}\right), \end{aligned} \tag{4.66}$$

with t_0 given by (4.63). The common term of (4.29) becomes

$$C_{12}(t_1, \bar{t}_1) = \tfrac{1}{2}(1 + \sqrt{\bar{t}_1 - t_1}) + O(\epsilon)$$

where $\tau = \epsilon^{1/3}(t - T)$ and the transition solution given by (4.27) is

$$p(t, \epsilon) = \frac{1}{2} + \epsilon^{1/3} a \, \frac{\mathrm{Ai}'(a\,\epsilon^{1/3}(t - T))}{\mathrm{Ai}(a\,\epsilon^{1/3}(t - T))} + O(\epsilon^{2/3}) \tag{4.67}$$

where $a = -(\frac{1}{4})^{1/3}$.

Thus the uniformly valid solution (4.43) up to the transition becomes

$$\begin{aligned} p_{12}(t, \epsilon) = \frac{1}{2} + \Big[1 + \sqrt{\epsilon\,\bar{t} - \epsilon\,t}\,\tanh[\tfrac{1}{2}(t_0 + c_0)]\Big] \\ -(\frac{\epsilon}{4})^{1/3}\,\frac{\mathrm{Ai}'((\frac{\epsilon}{4})^{1/3})[T - t]}{\mathrm{Ai}((\frac{\epsilon}{4})^{1/3})[T - t])} - \tfrac{1}{2}\sqrt{\epsilon\,\bar{t} - \epsilon\,t}. \end{aligned} \tag{4.68}$$

Alternatively, in the case of (4.44), we get

$$\begin{aligned} p_{12}(t, \epsilon) = \frac{1}{2} + \Big[1 + \sqrt{\epsilon\,\bar{t} - \epsilon\,t}\,\coth[\tfrac{1}{2}(t_0 + c_{0R})]\Big] \\ -(\frac{\epsilon}{4})^{1/3}\,\frac{\mathrm{Ai}'((\frac{\epsilon}{4})^{1/3})[T - t]}{\mathrm{Ai}((\frac{\epsilon}{4})^{1/3})[T - t])} - \tfrac{1}{2}\sqrt{\epsilon\,\bar{t} - \epsilon\,t} \end{aligned} \tag{4.69}$$

where t_0 , c_0 and c_{0R} are given by (4.63) and (4.66) respectively.

- **Asymptotic Approximation in the Supercritical Region**:

In this case where $t > T$, using (4.50), (4.6) gives

$$\tilde{t}_0 = \tfrac{2}{3}\epsilon^{1/2}\,[(T - t)^{3/2}] \tag{4.70}$$

while the leading term supercritical harvesting expansion (4.5) becomes

$$p(\tilde{t}_0, t) = \tfrac{1}{2}\,\Big[1 - \epsilon^{1/2}\,(t - T)\tan[\tfrac{1}{2}(\tilde{t}_0 - \frac{\pi}{2})]\Big] + O(\epsilon), \tag{4.71}$$

while again the solution in the transition region is as given by (4.67). The common terms, (4.39), in this case are

$$C_{23}(t, \epsilon) = \tfrac{1}{2}\,\Big[1 - \epsilon^{1/2}\,(t - T)\tan[\zeta - \frac{\pi}{4}]\Big], \tag{4.72}$$

where

$$\zeta = \frac{\tilde{t}_0}{2}.$$

Thus the uniformly valid approximation in the supercritical region is

$$p_{23}(t,\epsilon) = \frac{1}{2} - (\frac{\epsilon}{4})^{1/3}\,\frac{\mathrm{Ai}'((\frac{\epsilon}{4})^{1/3})[T-t]}{\mathrm{Ai}((\frac{\epsilon}{4})^{1/3})[T-t])},\ \text{ for }\ t \geq T. \tag{4.73}$$

Hence, for small ϵ, (4.68) or (4.69) and (4.73) provide approximate composition for the population of (4.52) that move from subcritical to supercritical state respectively through the transition point T for special parameters given by (4.50).

For particular values of parameters ϵ and initial condition μ of (4.52), Figure 4.16 displays both the exact solution (4.62) and additive composition of asymptotic approximation of the subcritical region (4.68) that is moved to the asymptotic approximation of the supercritical region (4.73) trough a transition point $T = 40$ for special parameters given by (4.50) where $\epsilon = 0.05$, $\mu = 0.2$. These lines are visually identical.

For a different initial condition $\mu = 2$, $\epsilon = 0.5$ and same choices of (4.50), Figure 4.17 also indicates a very close agreement between the exact solution (4.62) and both uniform approximate expansions (4.69) and (4.73) respectively.

4.7 Discussion

The calculations of this chapter show how the global multiscaling approach can be combined with local asymptotic analysis at points where the former fails, to construct overall representations of the evolving population. Even though the analysis has been carried out in a specific case (i.e., a survival-extinction transition), the techniques are readily adaptable to other circumstances- e.g., extinction-survival. The comparisons with numerical computations for the particular example chosen in Section 4.5 show excellent agreement between the two methods. Moreover, the general uniform approximations (4.43, 4.44)

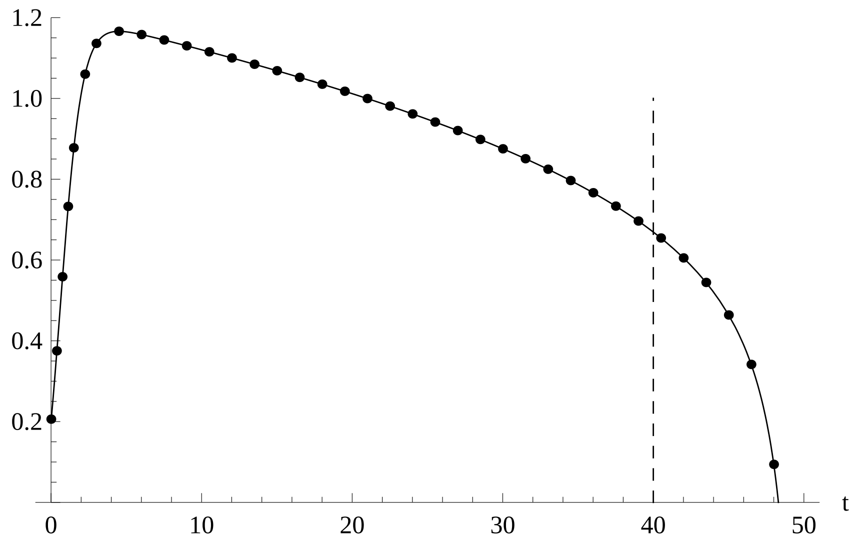

Figure 4.16: Comparing the exact solution (4.62) (solid) with the asymptotic approximations (4.68) and (4.73) (dotted) where $\epsilon = 0.05$, $\mu = 0.2$ and $T = 40$ (dashed vertical).

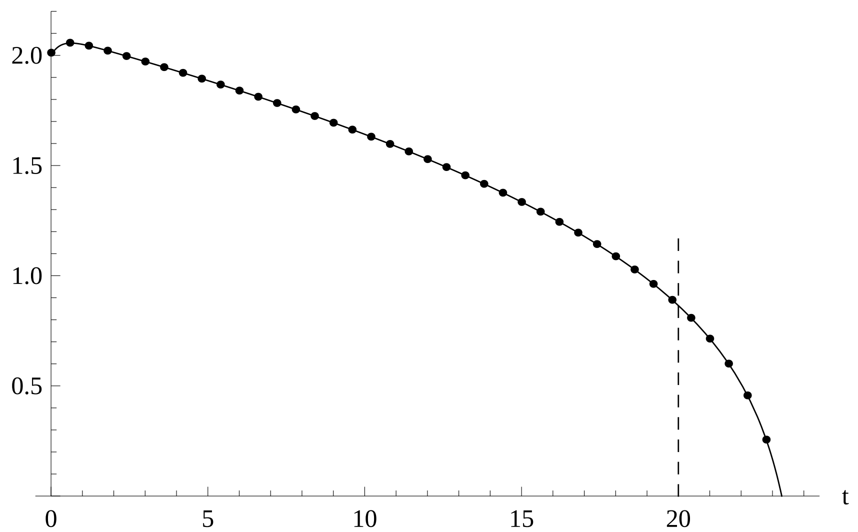

Figure 4.17: The exact solution (4.62) (solid) and gathering the two asymptotic approximations (4.69) and (4.73)(dotted) where $\epsilon = 0.5, \mu = 2$ and $T = 20$(Transition point).

and (4.46, 4.47) are quite explicit, and may be used with any appropriate functions r, k and h.

The example of Section 4.6 where an exact solution of the problem (4.1) can be found for specific choices of r, k and h, further reinforces the value of these approximations. It is clear from Figures 4.16 and 4.17 that there is excellent agreement between the asymptotic expansion and the exact solution. This increases our confidence in the analysis.
Transitions in the harvesting model are being prepared for publication in Idlango et al [36].

Chapter 5

A Single Species Model Exhibiting an Allee Effect

5.1 Introduction

In general, the Allee effect (see Gonzalez-Olivares, et al [22], Courchamp, et al. [17]) in a single species population model can be simply described as the phenomenon where the population grows faster when the species are at high population density than it would if the population were at low population density. That is, the Allee effect reduces the population growth at low densities.

While there is a range of single species population models that exhibit this effect, we consider one here that is simple in structure, but which displays the Allee effect quite clearly.

This model (see, for example, [17]) can be formulated as the initial value problem for population P:

$$\frac{dP}{dT} = R\,P\left(1-\frac{P}{K}\right)\left(\frac{P}{M}-1\right), \quad P(0)=P_0, \tag{5.1}$$

where R, K, M and P_0 are positive constants, with P the population size at time $t \geq 0$, R the growth rate, K the carrying capacity, and M is the critical population size, $0 <$

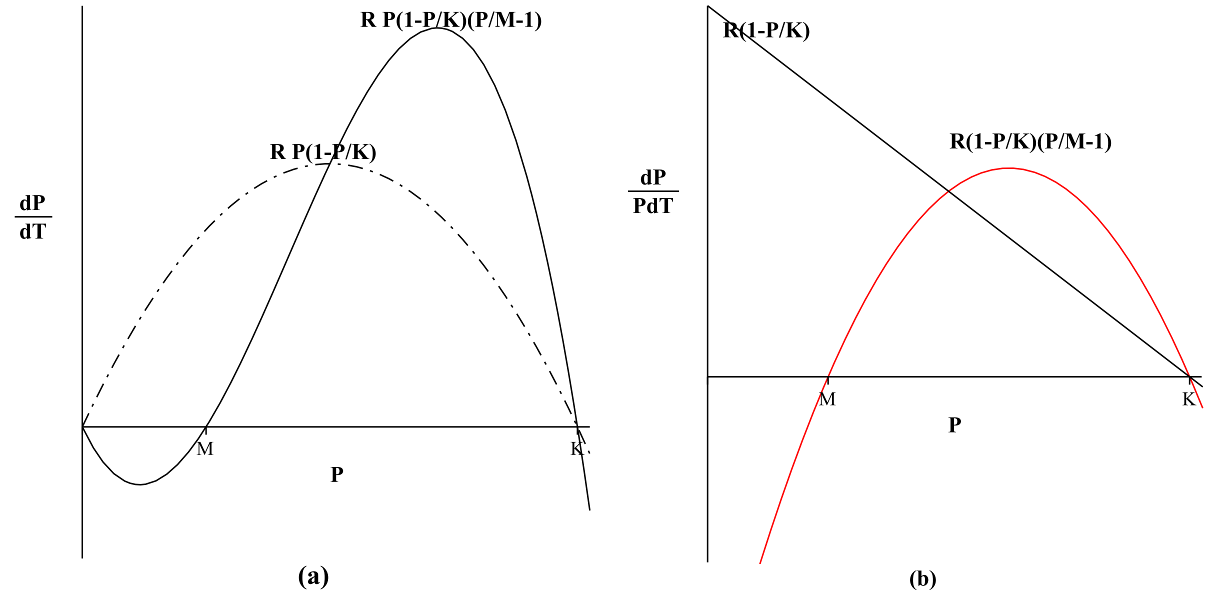

Figure 5.1: Growth curves of Logistic model and Allee model

$M < K$.

This presents a modified form of logistic growth function by expressing the Allee effect. In Figure 5.1 it is clear that, in the logistic model, for any initial population the population endures to its carrying capacity K and the per capita population growth rate is a maximum when P is small.

However, with the appearance of the Allee effect in (5.1) compared with logistic growth term, Figure 5.1 shows that for any initial population above M, $P_0 > M$ the population survives to K and encounters extinguishment when the population is below its critical size, M. That is, the per capita growth rate is negative below M and is positive otherwise.

However, when this model is applied to real world situations, the model parameters R, K and M in (5.1) may not be constants, but may, in fact, be functions of time. Such variation may naturally arise from varying environmental and/or physiological effects.

In such cases, (5.1) takes the form

$$\frac{dP}{dT} = R(T)P\left(1 - \frac{P}{K(T)}\right)\left(\frac{P}{M(T)} - 1\right), \quad P(0) = P_0, \tag{5.2}$$

where now $R(T), K(T)$ and $M(T)$ are positive real valued functions on $T \geq 0$ satisfying

$$0 < \; M(T) \; < K(T) \;\; \text{for all} \;\; T \geq 0. \tag{5.3}$$

Following the approach to the harvesting problem of Chapter 2, we simplify the initial value problem (5.2) and assume that $R(T)$, $K(T)$, $M(T)$ may be written in terms of dimensionless functions r, k, m as

$$R(T) = R_0 r(T/T^*),$$

$$K(T) = K_0 k(T/T^*),$$

$$M(T) = M_0 m(T/T^*)$$

where R_0, K_0 and M_0 are representative values of these functions and T^* is an assumed common intrinsic time scale for the variation of R, K, M. Note that the parameters R, K, M may vary on the same the time scale or possibly on different time scales. We choose here the simplest case, where they have the same time scale.

With dimensionless time scale and population scale given by

$$t = R_0 T,$$

$$p = P/K_0$$

respectively, we can rewrite (5.2) as

$$\frac{dp(t,\epsilon)}{dt} = r(\frac{1}{R_0 T^*}t)p(t,\epsilon)\left(1 - \frac{p(t,\epsilon)}{k(\frac{1}{R_0 T^*}t)}\right)\;\left(\frac{\alpha\; p(t,\epsilon)}{m(\frac{1}{R_0 T^*}t)} - 1\right),\; p(0,\epsilon) = \mu. \tag{5.4}$$

Thus we define $\epsilon = 1/(R_0 T^*)$ as a ratio of the intrinsic population variation time scale, $1/R_0$, to T^*. Then, (5.4) becomes

$$\frac{dp(t,\epsilon)}{dt} = r(\epsilon t)p(t,\epsilon)\left(1 - \frac{p(t,\epsilon)}{k(\epsilon t)}\right)\;\left(\frac{\alpha\; p(t,\epsilon)}{m(\epsilon t)} - 1\right),\; p(0,\epsilon) = \mu, \tag{5.5}$$

where α and μ are positive constant parameters, defined by

$$\alpha = K_0/M_0,\;\; \mu = P_0/K_0. \tag{5.6}$$

In (5.6) we assume that $K_0/M_0 > 1$, so that $\alpha > 1$. Note that when any of R, K or M is constant, its dimensionless counterpart is unity. Thus, if $R(T)$ is constant, $r(\epsilon t) \equiv 1$ in the problem (5.5). Similarly for $K(T)$ or $M(T)$.

In terms of dimensionless quantities, the condition (5.3) becomes

$$0 < m(\epsilon t) < \alpha k(\epsilon t) \text{ for all } t \geq 0. \tag{5.7}$$

For general varying coefficients r, k and m, the problem (5.5) may not be solved exactly, and must be solved using numerical techniques. However, where the time scale of variation of these parameters is relatively large compared with p itself, i.e., ϵ is small, the problem (5.5) involves two time scales; t, the normal time, and ϵt, the slow time. As in a parallel analysis (see Chapter 2, [29]), we can apply a multiscaled perturbation method based on these time scales to construct analytic approximations to the solutions of the problem (5.5).

In the next section, we will investigate the basic dimensionless Allee model when the parameters are positive constants. In this case, (5.5) is autonomous and separable, but we show that we can construct an implicit representation for the exact solution for an arbitrary μ, as well as expansions for the limiting states of this solution as $t \to \infty$.

In subsequent sections, we use this analysis as a basis for a multitiming analysis of the problem (5.5) as describe above, when the coefficients r, k and m are slowly varying.

5.2 Constant Model Parameters

In this section, we consider R, K, M as positive constants. Thus we we put $r(\epsilon t) \equiv k(\epsilon t) \equiv m(\epsilon t) \equiv 1$ and this gives the dimensionless initial value problem

$$\frac{dp}{dt} = p(1-p)(\alpha\, p - 1), \;\; p(0) = \mu, \tag{5.8}$$

where μ is a dimensionless initial population value, and $\alpha = K/M$.

Before we seek the solution of (5.8), let us look at the analysis of the critical points $p = 0$, $1/\alpha$ and 1 of the differential equation in (5.8). The stability of these points can be determined by the stability index of this equation, that is

$$\sigma(p) = [2\,(\alpha + 1) - 3\,\alpha\, p]\, p - 1.$$

Thus, the critical point $p = 0$ is a stable state since $\sigma(0) = -1 < 0$, while $\sigma(1) = 1 - \alpha$, $\alpha > 1$, so that $p = 1$ is a stable state. However, since $\sigma(1/\alpha) = 1 - 1/\alpha > 0$, $p = 1/\alpha$ is an unstable point.

Thus, the population described by (5.8) has stable equilibria at $p = 0$, $p = 1$ and an unstable equilibrium at $p = 1/\alpha$.

We note the initial value problem (5.8) cannot be solved explicitly, but an implicit solution of (5.8) is given by

$$\left(\frac{p}{\mu}\right)^{\alpha-1} \left(\frac{1-p}{1-\mu}\right) \left(\frac{\alpha\,\mu - 1}{\alpha\, p - 1}\right)^{\alpha} = e^{(1-\alpha)\, t}, \quad \alpha > 1 \tag{5.9}$$

The behaviour of the solution (5.9) is dependent on the value of μ. Thus, if the population is initially in a stable state, i.e., $\mu > 1/\alpha > 0$, then $p(t)$ survives to the limiting value of the stable state 1 (i.e., $p(t) \to 1$) from below when $1/\alpha < \mu < 1$ and from above when $\mu > 1$ as $t \to \infty$. However, in the case where $0 < \mu < 1/\alpha$, the population dies out i.e., $p(t) \to 0$ as $t \to \infty$. The respective curves in Figure 5.2 illustrates these previous cases of the solution (5.9) where $\alpha = 2$ and $\mu = 0.2, 0.7, 2.0$. For given values of t, μ, and α, the solution of equation (5.9) (for p) is found using a root finding method.

5.2.1 The Behaviour of the Solution as $t \to \infty$.

Now, from (5.9) as $t \to \infty$, the term $e^{(1-\alpha)t}$, $\alpha > 1$ on the right side of (5.9) tends to zero. This allows us to represent the solution (5.9) as a power series in terms of the small term $e^{(1-\alpha)t}$ as $t \to \infty$. Defining

$$p(t) = f_0 + f_1\, e^{(1-\alpha)\, t} + f_2\, e^{2(1-\alpha)\, t} + O(e^{3(1-\alpha)\, t}), \tag{5.10}$$

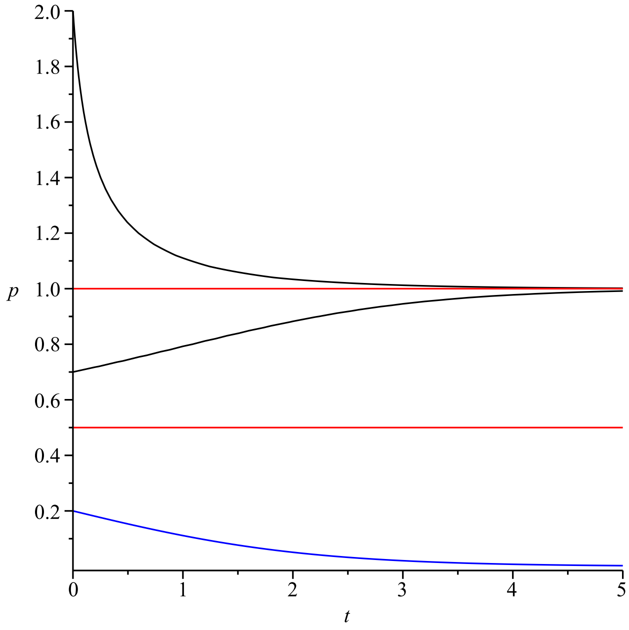

Figure 5.2: Typical solutions for the system (5.8) as given by (5.9), for $\alpha = 2$ and $\mu = 0.2, 0.7, 2.0$.

substituting (5.10) into (5.9) and expanding in powers of $e^{(1-\alpha)t}$ give the coefficients f_i, $i = 0, 1, 2 \ldots$ by equating like powers of $e^{(1-\alpha)t}$. These are

$$f_0 = 0,\ 1, \quad f_1 = \beta, \quad f_2 = \frac{2\alpha - 1}{\alpha - 1}\beta^2, \ldots,$$

where β is a real constant given by

$$\beta = \left(\frac{\mu(\alpha - 1)}{\alpha\mu - 1}\right)^{\alpha} \left(\frac{\mu - 1}{\mu}\right), \quad \mu > \frac{1}{\alpha}. \tag{5.11}$$

By choosing $f_0 = 1$, this gives the expansion (5.10) for $p(t)$ when $p(t) \to 1$ as $t \to \infty$ by

$$p(t) = 1 + \beta\ e^{(1-\alpha)\ t} + \frac{2\alpha - 1}{\alpha - 1}\beta^2\, e^{2(1-\alpha)t} + O(e^{3(1-\alpha)t}). \tag{5.12}$$

where β is given by (5.11).

We note that series (5.12), converges at a rate which is dependent on the value of α, through the exponential terms.

Figure 5.3 shows a comparison between the expansion (5.12) and the solution of the problem (5.8) given by (5.9) as $t \to \infty$, for typical values of $\alpha = 1.5$ and $\mu = 0.8$, 1.2.

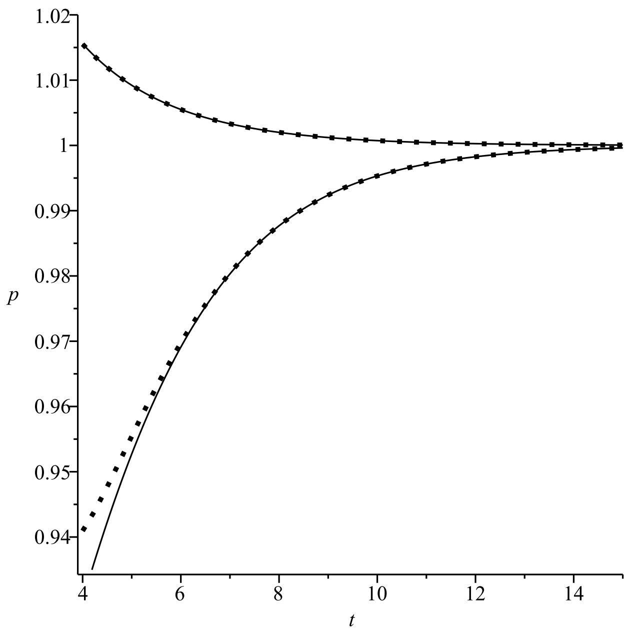

Figure 5.3: The asymptotic expansion (5.12) (dotted) and the solution from (5.9) (solid) where $\alpha = 1.5$, $\mu = 0.8,\, 1.2$, and $t \geq 4$.

Since the expansion (5.12) is calculated for large t, we see that, from Figure 5.3, when t is small, there is a difference between exact solution and asymptotic expansion; however (as expected) we see that the exact solution is close to the approximate expansion (5.12) as t increases.

By applying a similar analysis as above, but now in the case where $p(t) \to 0$ as $t \to \infty$, we define $p(t)$ as Taylor series by

$$p(t) = g_0 + g_1\, e^{-t} + g_2\, e^{-2t} + O(e^{-3\,t}). \tag{5.13}$$

resubstituting (5.13) into (5.9), expanding and equating like powers of e^{-t}, we obtain the coefficients $g_0, g_1, \ldots$ and the expansion (5.13) for $p(t)$ then becomes

$$p(t) = \delta\; e^{-t} - (\alpha + 1)\delta^2\, e^{-2t} + O(e^{-3t}), \tag{5.14}$$

where δ is a positive constant given by

$$\delta = \mu \left(\frac{1 - \mu}{(1 - \alpha\, \mu)^{\alpha}} \right)^{\frac{1}{\alpha - 1}}, \;\; \mu < \frac{1}{\alpha}. \tag{5.15}$$

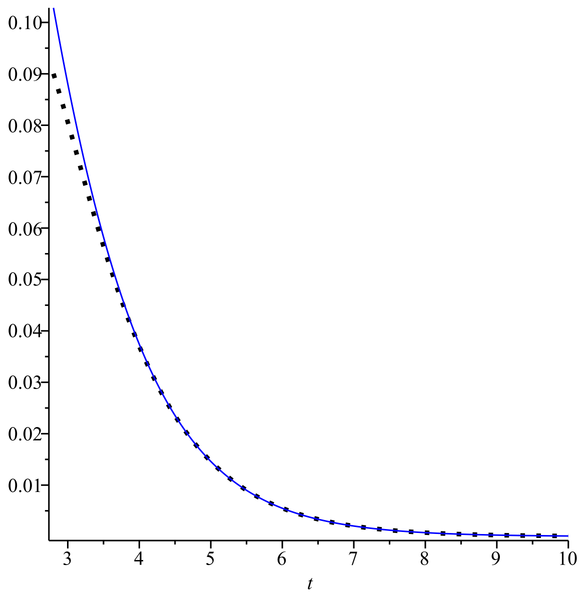

Figure 5.4: Comparison of asymptotic expansion (5.14) (dotted) the solution from (5.9) (solid) where $\alpha = 1.5$, $\mu = 0.4$ and $t \geq 3$.

We note that in (5.14), the rate of convergence to zero is independent of α. Figure 5.4 compares the exact solution (5.9) with the expansion (5.14) using the same parameters as those of Figure 5.3, but now with $\mu = 0.4$. Again, as in Figure 5.3, this shows a small difference between both solutions for small t, but this disappears as time t progresses.

5.3 The Slowly Varying Allee Model

As discussed previously in Section 5.1, from here on, we will consider the more general case (5.5), where the parameters r, k and m show slow variation; i.e., ϵ is small. Thus, the problem (5.5) now involves two time scales; a slow time, ϵt and a normal time t. As in a similar discussion in the Chapter 2 - Section 2.2.1, and [26, 57], we will employ a multiscale perturbation approach to construct approximate representations for the solutions of this problem for small values of the parameter ϵ.

Since the differential equation in (5.5) is now nonautonomous, the critical point analysis of the previous section no longer applies. However, we can expect that, for ϵ small, solutions of (5.5) might tend to either zero (an extinguished solution) or $k(\epsilon t)$ (a surviving solution to slowly varying limiting state) in some sense and not the other slowly varying state $m(\epsilon t)/\alpha$.

5.3.1 The Multiscale Equation

For the purpose of solving the problem (5.5), we reformulate the differential equation (5.5) as a multiscale equation for small ϵ.

Referring to successful applications of a multi-scaling technique that are based on two time scales (see Chapter 2 - Section 2.2.1 , [40]), we aim to find an approximate solution for (5.5) using an analogous method, by defining the normal time, t_0 and slow time, t_1, as introduced in Chapter 1 by

$$t_0 = \frac{1}{\epsilon}\, g(t_1) \qquad \text{and} \qquad t_1 = \epsilon t, \tag{5.16}$$

where $g(t_1)$ is a positive valued function on all $t_1 > 0$ satisfying the same conditions as discussed in Chapters 1

Following the approach of Chapter 2, we regard $p(t, \epsilon)$, the solution of (5.5), as a function $\tilde{p}(t_0, t_1, \epsilon)$ of both t_0 and t_1, i.e.,

$$\tilde{p}(t_0, t_1, \epsilon) \equiv p(t, \epsilon).$$

From (5.16), by applying the chain rule and substituting into (5.5), we convert the differential equation in (5.5) to the multiscaled partial differential equation for $\tilde{p}(t_0, t_1, \epsilon)$

$$g'(t_1) D_0\, \tilde{p} + \epsilon\, D_1\, \tilde{p} = r(t_1)\, \tilde{p}\, \left(1 - \frac{\tilde{p}}{k(t_1)}\right)\, \left(\frac{\alpha\, \tilde{p}}{m(t_1)} - 1\right), \tilde{p}(0, 0, \epsilon) = \mu,$$

where D_0 and D_1 represent partial derivatives taken with respect to t_0 and t_1 respectively. Note that (5.17) displays ϵ explicitly rather than implicity as in (5.5) and this allows us to employ a perturbation technique which constructs an approximate solution of (5.5) that is valid for all $t > 0$.

5.3.2 Perturbation Analysis

We now express the unknown function $\tilde{p}(t_0, t_1, \epsilon)$ as a Poincaré expansion in ϵ, namely

$$\tilde{p}(t_0, t_1, \epsilon) = \tilde{p}_0(t_0, t_1) + \epsilon \tilde{p}_1(t_0, t_1) + \epsilon^2 \tilde{p}_2(t_0, t_1) + \dots . \tag{5.17}$$

Substituting (5.17) into (5.17), expanding in powers of ϵ and equating coefficients of like powers of ϵ, give partial differential equations for $\tilde{p}_0$ as

$$g'(t_1) D_0 \tilde{p}_0 = r(t_1)\ \tilde{p}_0 \left(1 - \frac{\tilde{p}_0}{k(t_1)}\right) \left(\frac{\alpha \tilde{p}_0}{m(t_1)} - 1\right) \tag{5.18}$$

and for $\tilde{p}_1$

$$g'(t_1) D_0 \tilde{p}_1 - r(t_1) \left[2\tilde{p}_0 \left(\frac{\alpha k(t_1) + m(t_1)}{k(t_1) m(t_1)}\right) - 1 - \frac{3\ \alpha \tilde{p}_0^3}{k(t_1) m(t_1)}\right] \tilde{p}_1 = -D_1 \tilde{p}_0. \tag{5.19}$$

The partial differential equation (5.18) may be solved as in Section 5.2 for $\tilde{p}_0(t_0, t_1)$ in implicit form as

$$\left(\frac{\tilde{p}_0}{\alpha\, \tilde{p}_0 - m(t_1)}\right)^{\frac{\alpha k(t_1)}{m(t_1)}} \left(1 - \frac{k(t_1)}{\tilde{p}_0}\right) - A(t_1)\, e^{-\gamma(t_1)\, t_0} = 0, \tag{5.20}$$

where $A(t_1)$ is an arbitrary function of t_1 and

$$\gamma(t_1) = \frac{r(t_1)}{g'(t_1)} \left(\frac{\alpha k(t_1)}{m(t_1)} - 1\right) > 0. \tag{5.21}$$

From (5.21), we see that for each $t_1 \geq 0$, $e^{-\gamma(t_1)\, t_0} \to 0$ as $t_0 \to \infty$; and consequently, either the population survives to limiting state, $k(t_1)$, that is $\tilde{p}_0(t_0, t_1) \to k(t_1)$ or goes to extinction, that is $\tilde{p}_0(t_0, t_1) \to 0$.

We now consider the expansion behaviour of $\tilde{p}_0(t_0, t_1)$ for the survival situation (where $p_0 > m(t_1)/\alpha$) and the extinction situation (where $p_0 < m(t_1)/\alpha$) as $t_0 \to \infty$, separately.

5.3.3 The Survival Case

On considering solving (5.19) for $\tilde{p}_1(t_0, t_1)$, we note that the original multiscale equation (5.17) and consequently, (5.19), are first order partial differential equations. In the usual way for this process, (see [40, 26]), we seek only a particular solution $\tilde{p}_1$ of (5.19); i.e., one that doesn't include any further arbitrary functions of t_1. Clearly, if $\tilde{p}_0$ is known explicitly, the linear equation (5.19) for $\tilde{p}_1$ can be easily solved to provide such a particular solution, using an integrating factor. Unfortunately, here $\tilde{p}_0$ is only known implicitly from (5.20). So, this approach fails.

Here, we take an analogous approach to that of Section 5.2; i.e., we can find the asymptotic expansion for $\tilde{p}_0$ the implicit solution of (5.20) that tends to $k(t_1)$ (survives) as $t_0 \to \infty$, in this more general case where the parameters have slow variations. We can then use this expansion for large t_0 to obtain an expression describing the behaviour of $\tilde{p}_1(t_0, t_1)$ as $t_0 \to \infty$.

Following the procedure of Section 5.2.1, we express (5.20) in terms of the following approximation

$$\tilde{p}_0 = f_0(t_1) + f_1(t_1)\, e^{-\gamma(t_1)\, t_0} + f_2(t_1)\, e^{-2\gamma(t_1)\, t_0} + O(e^{-3\gamma(t_1)\, t_0}),$$

as $t_0 \to \infty$ and equating like powers of $e^{-\gamma(t_1)\, t_0}$ where, $\gamma(t_1) > 0$, we evaluate the coefficients $f_i(t_1), i = 0, 1 \ldots$, to obtain the expansion for $\tilde{p}_0(t_0, t_1)$ as $t_0 \to \infty$ in the form

$$\tilde{p}_0(t_0, t_1) \quad = \quad k(t_1) + B(t_1) e^{-\gamma(t_1)\, t_0} + O(e^{-2\gamma(t_1)\, t_0}), \tag{5.22}$$

where

$$B(t_1) = A(t_1)\, k(t_1) \left(\frac{\alpha k(t_1) - m(t_1)}{k(t_1)} \right)^{\frac{\alpha\, k(t_1)}{m(t_1)}}. \tag{5.23}$$

From (5.22), we see that the leading term of the expansion (5.17) of $\tilde{p}_0$, for the solution of (5.17) that tends to $k(t_1)$ as $t_0 \to \infty$ converges at the exponential rate, $e^{-\gamma(t_1) t_0}$; and we would expect that the solution $\tilde{p}_1$ of (5.19) to show this rate of convergence too.

Substituting leading terms of the approximate expansion for $\tilde{p}_0$ given by (5.22) into the differential equation (5.19) for $\tilde{p}_1$ gives an approximate equation for $\tilde{p}_1$ as

$$D_0\tilde{p}_1 - \frac{r(t_1)}{g'(t_1)}\left(1 - \frac{\alpha k(t_1)}{m(t_1)} + \ldots\right)\tilde{p}_1 \;=\; \frac{-D_1}{g'(t_1)}[k(t_1) + B(t_1)e^{-\gamma(t_1)\,t_0} + \ldots]. \quad (5.24)$$

Retaining these leading terms, and solving the resultant linear equation for $\tilde{p}_1$ as $t_0 \to \infty$ gives

$$\tilde{p}_1(t_0, t_1) = \frac{k'(t_1)}{r(t_1)(1 - \frac{\alpha k(t_1)}{m(t_1)})} - \frac{e^{-\gamma(t_1)t_0}}{g'(t_1)}\{B'(t_1)\,t_0 + B(t_1)\gamma'(t_1)\frac{t_0^2}{2}\} + \ldots \quad (5.25)$$

which is an expression as $t_0 \to \infty$ for $\tilde{p}_1$ the solution of (5.17) that corresponds to the solution of (5.17) that tends to $k(t_1)$ as $t_0 \to \infty$.

Noting from above that we expect convergence of $\tilde{p}_1$ to its limit to be at least as fast as $\tilde{p}_0$ is to k, we see that from (5.25), in order to satisfy the condition of convergence at a rate of $e^{-\gamma(t_1)t_0}$ for $\tilde{p}_1$, we need to eliminate the terms involving t_0 and t_0^2. Thus, we choose

$$B'(t_1) = 0 \quad (5.26)$$

and

$$\gamma'(t_1) = 0. \quad (5.27)$$

The choice (5.26) and (5.23) leads to

$$A(t_1)\,k(t_1)\left(\frac{\alpha k(t_1) - m(t_1)}{k(t_1)}\right)^{\frac{\alpha\,k(t_1)}{m(t_1)}} = c_s,$$

giving

$$A(t_1) = c_s\; a_s(t_1), \quad (5.28)$$

where

$$a_s(t_1) = \left[k(t_1)(\frac{\alpha k(t_1) - m(t_1)}{k(t_1)})^{\frac{\alpha k(t_1)}{m(t_1)}}\right]^{-1}, \quad (5.29)$$

with c_s an arbitrary constant that depends on the initial condition. Here, subscript s refers to the case where the solution of (5.5) survives; i.e., tends to $k(\epsilon t)$ as $t \to \infty$.

The condition (5.27), leads us to choose $\gamma(t_1)$ = constant = 1, and so (5.21) gives

$$g'(t_1) = r(t_1)(\frac{\alpha k(t_1)}{m(t_1)} - 1),$$

from which, with (5.16), we get

$$t_{0s} = \frac{1}{\epsilon}\int_0^{t_1} r(s)(\frac{\alpha k(s)}{m(s)} - 1)ds, \tag{5.30}$$

where the subscript s has the meaning as given above.

Applying the choices (5.16), (5.28), (5.30) for t_1, $A(t_1)$ and t_{0s} respectively, with $\gamma(t_1) = 1$ to (5.20), we arrive at a leading order approximation $p_{0s}(t,\epsilon)$ to that solution of the differential equation of (5.5) that tends to $k(\epsilon\, t)$ as $t \to \infty$, represented implicitly by

$$F(p_{0s}, t) - c_s\, a_s(t_1)\, e^{-t_{0s}} = 0, \tag{5.31}$$

where

$$F(u, \epsilon\, t) = \left(\frac{u}{\alpha\, u - m(t_1)}\right)^{\frac{\alpha k(t_1)}{m(t_1)}} \left(1 - \frac{k(t_1)}{u}\right). \tag{5.32}$$

where $t_1 = \epsilon\, t$, and t_{0s} is given by (5.30).

The expression (5.31) involves the constant c_s which can be determined by applying the initial condition of (5.5) at $t = t_0 = t_1 = 0$.

Applying this initial condition to (5.31) gives

$$c_s \;=\; k(0)\,(1 - \frac{k(0)}{\mu})\left(\frac{1 - \frac{m(0)}{\alpha k(0)}}{1 - \frac{m(0)}{\alpha\mu}}\right)^{\frac{\alpha k(0)}{m(0)}} \tag{5.33}$$

where

$$\mu > \frac{m(0)}{\alpha}. \tag{5.34}$$

Thus, (5.31) provides an implicit representation for p_{0s}, a leading order approximation to the evolving population $p_s(t,\epsilon)$ in the surviving situation, when ϵ is small, and for arbitrary slowly varying model parameters $r(\epsilon t)$, $k(\epsilon t)$ and $m(\epsilon t)$ satisfying (5.7) and initial population μ.

Note: When r, k and m are positive constants, so that $r(\epsilon t) \equiv k(\epsilon t) \equiv m(\epsilon t) \equiv 1$, a straight forward calculation shows that the leading term approximation (5.31) with (5.29) and (5.33) can be reduced to the implicit solution (5.9).

5.3.4 The Extinction Case

Again, for large t_0, the task of determining the approximate expansion of $\tilde{p}_0$ that tends to zero as $t_0 \to \infty$ using (5.20) and applying this to obtain an expansion for $\tilde{p}_1$ from (5.19) as $t_0 \to \infty$ follows a similar process that led to the results obtained in Section 5.3.3.

The asymptotic expansion for the solution of (5.20) that tends to zero as $t_0 \to \infty$ ($\tilde{p}_0$ is extinguished) is

$$\tilde{p}_0(t_0, t_1) \quad = \quad C(t_1)\, e^{-\frac{r(t_1)}{g'(t_1)} t_0} + O(e^{-2\frac{r(t_1)}{g'(t_1)} t_0}), \tag{5.35}$$

where

$$C(t_1) = \left(\left(\frac{-A(t_1)}{k(t_1)} \right)^{m(t_1)} (-m(t_1))^{\alpha\, k(t_1)} \right)^{1/(\alpha k(t_1) - m(t_1))}, \tag{5.36}$$

and $A(t_1)$ is an undetermined arbitrary function of t_1.

As in Section 5.3.3, we expect from (5.35) that $\tilde{p}_0$ and $\tilde{p}_1$ for the expansion (5.17) for the solution that tends to zero as $t_0 \to \infty$ displays a typical convergence rate of $e^{-(r(t_1)/h'(t_1))t_0}$.

Substituting the leading terms of (5.35) into (5.19), we obtain

$$D_0 \tilde{p}_1 + \left(\frac{r(t_1)}{g'(t_1)} + \ldots \right) \tilde{p}_1 = -(g'(t_1))^{-1} D_1 \left[C(t_1) e^{-\frac{r(t_1)}{g'(t_1)} t_0} + \ldots \right] \tag{5.37}$$

Solving (5.37) gives

$$\tilde{p}_1(t_0, t_1) \quad = \quad \frac{e^{-\frac{r(t_1)}{g'(t_1)} t_0}}{g'(t_1)} \{ C'(t_1)\, t_0 - \left(\frac{r(t_1)}{g'(t_1)} \right)' C(t_1) \frac{t_0^2}{2} + \ldots \} \tag{5.38}$$

which is an expression for $\tilde{p}_1$ as $t_0 \to \infty$ corresponding to $\tilde{p}_0$, the solution of (5.17) that tends to zero as $t_0 \to \infty$.

However, as $t_0 \to \infty$, the appearance of terms $C'(t_1)\,t_0$ and $\left(\frac{r(t_1)}{g'(t_1)}\right)' C(t_1)\frac{t_0^2}{2}$ lead $\tilde{p}_1(t_0, t_1)$ to converge slower to its limit compared to $\tilde{p}_0(t_0, t_1)$. Thus, to avoid this behavior we need to remove these terms.

Thus, we set

$$C'(t_1) = 0 \tag{5.39}$$

and

$$\left(\frac{r(t_1)}{g'(t_1)}\right)' = 0, \tag{5.40}$$

and in a like manner to the process of Section 5.3.3, we obtain

$$A(t_1) = -c_e\, a_e(t_1), \tag{5.41}$$

defining the arbitrary function $A(t_1)$, where

$$a_e(t_1) = -k(t_1)(-m(t_1))^{\frac{-\alpha k(t_1)}{m(t_1)}}, \tag{5.42}$$

and c_e is an arbitrary constant.

Then (5.40) and (5.16) lead us to define the time scale t_{0e} as

$$t_{0e} = \frac{1}{\epsilon}\int_0^{t_1} r(s)ds, \tag{5.43}$$

where the subscript e denotes association with the solution of (5.5) that is extinguished; i.e., tends to *zero* as $t \to \infty$.

Similarly, the choices (5.16), (5.41), (5.43) for t_1, $A(t_1)$ and t_0 respectively, lead to a leading order approximation $p_{0e}(t, \epsilon)$ to that solution of (5.5) that tends to *zero* as $t \to \infty$:

$$F(p_{0e}, t) + c_e\, a_e(t_1)\, e^{-(\frac{\alpha k(t_1)}{m(t_1)} - 1)t_{0e}} = 0, \tag{5.44}$$

where $F(u, t)$ is given by (5.32). Substituting the initial condition (5.5) into (5.44) gives

$$c_e = -(\frac{\mu}{1 - \frac{\alpha\mu}{m(0)}})^{\frac{\alpha k(0)}{m(0)}} \left(\frac{\mu - k(0)}{\mu k(0)}\right) \tag{5.45}$$

where

$$\mu < \frac{m(0)}{\alpha}. \tag{5.46}$$

Thus, (5.44) represents an implicit expression for p_{0e}, the leading order approximation to the evolving population in the extinguishing case, where constant c_e is given by (5.45). This is valid for small ϵ, and for arbitrary slowly varying model parameters $r(\epsilon t)$, $k(\epsilon t)$ and $m(\epsilon t)$ satisfying (5.7) and initial population μ.

Note: Again, when r, k and m are constants ($r(t_1) \equiv k(t_1) \equiv m(t_1) \equiv 1$), a straight forward calculation brings down the implicit solution (5.9) of the leading term approximation (5.44) with considering (5.42) and (5.45).

As we have noted above, the implicit expressions for leading term p_{0s} and p_{0e}, that make it impossible to obtain explicit representations for higher order terms p_{1s} and p_{1e}. However, from (5.22) and (5.25) we see that as $t \to \infty$,

$$\begin{aligned} p_{0s} &\to k(\epsilon t) \\ p_{1s} &\to \frac{k'(\epsilon t)}{r(\epsilon t)(1 - \frac{\alpha k(\epsilon t)}{m(\epsilon t)})}; \end{aligned} \tag{5.47}$$

and so, as $t \to \infty$

$$p_s \to k(\epsilon t) + \epsilon \frac{k'(\epsilon t)}{r(\epsilon t)\left(1 - \frac{\alpha k(\epsilon t)}{m(\epsilon t)}\right)} + \dots. \tag{5.48}$$

(5.35) and (5.38) give

$$p_e \to 0, \quad \text{as } t \to \infty. \tag{5.49}$$

5.4 Comparison of the Implicit Expansions with Numerical Solutions.

Although (5.31) and (5.44) are implicit approximate expansions for the solutions of (5.5), their forms are relatively simple and can be compared with the results of numerical solutions of (5.5), by choosing appropriately slowly varying functions $r(\epsilon t)$, $k(\epsilon t)$ and $m(\epsilon t)$.

Because of the complexity of the expansions (5.31) and (5.43) we will not make the result of subetituting the chosen forms of r, k and m explicit.

We thus introduce model parameters as periodic functions on a slow time ϵt (as used in Figures 5.5-5.11) as

$$\begin{aligned} m(\epsilon t) &= 1.0 + 0.05\ \sin(\epsilon t), \\ k(\epsilon t) &= 1.0 + 0.13\ \sin(\epsilon t), \\ r(\epsilon t) &= 1.0 + 0.01\ \sin(\epsilon t) \end{aligned} \tag{5.50}$$

with $\alpha = 3$ and $\epsilon = 0.01$.

Figure 5.5 shows the surviving case (5.31), and compares the leading term expansion (5.31) with the numerical solution of the initial value problem (5.5) for a relatively short period of time where μ satisfies the criterion (5.34)(i.e., $\mu > m(0)/\alpha = 0.33$). This clearly displays the initial transient interval, where t_{0s} is the dominate time scale, together with slowly varying state corresponding to (5.44). It is clear that these solutions display a very good agreement indeed.

In Figure 5.6, with the same choice (5.52) but for much a larger time interval, we can see the agreement remains, with the periodic limiting state given by (5.48) where $r(\epsilon t)$, $k(\epsilon t)$ and $m(\epsilon t)$ are given by (5.52), well-developed.

Figures 5.7 and 5.8 show the situation for a different initial starting population satisfies (5.34). Again, the leading term expansion (5.31) agrees well with the numerical representation of (5.5) whether the time t is small or large.

Figure 5.9 makes similar comparisons for a range initial population values μ satisfies (5.34); and, again, good agreement results.

Figure 5.10 shows the effects of increasing ϵ on the accuracy of our approximation compared with the numerical result. It shows that agreement is very good up to $\epsilon = 0.5$ which is probably not be regarded as small.

Figure 5.11 displays analogous results when the starting population μ holds satisfies

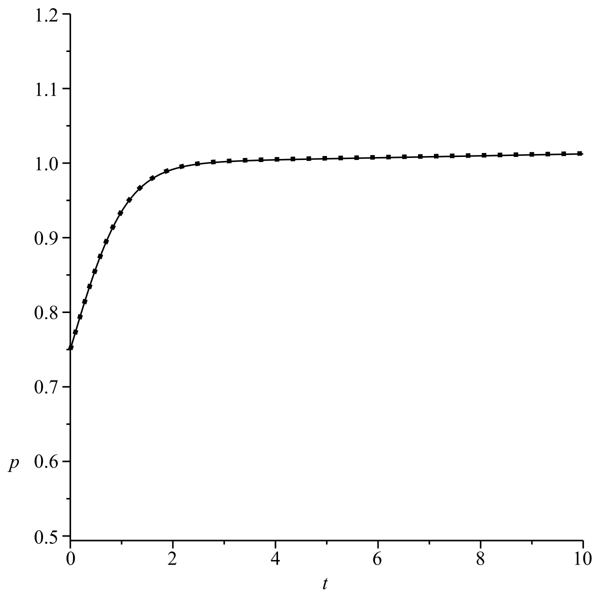

Figure 5.5: The leading term approximation (5.31) (dotted curve) and the numerical solution of (5.5) (solid curve) for a surviving population with $r(\epsilon t)$, $k(\epsilon t)$ and $m(\epsilon t)$ given by (5.52), and $\mu = 0.75$.

the criterion (5.46) (i.e., $\mu = 0.1 < m(0)/\alpha = 0.33$), and the leading term expansion is that of the extinction situation (5.45) (where the population is declining to zero).

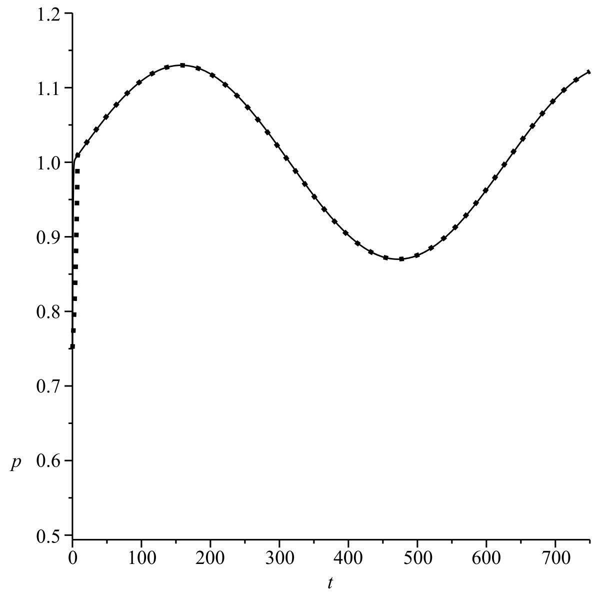

Figure 5.6: The leading term approximation for a surviving population (5.31) (dotted line) and the numerical solution (solid line) for the surviving population case with the choices of Figure 5.5, but for large times t.

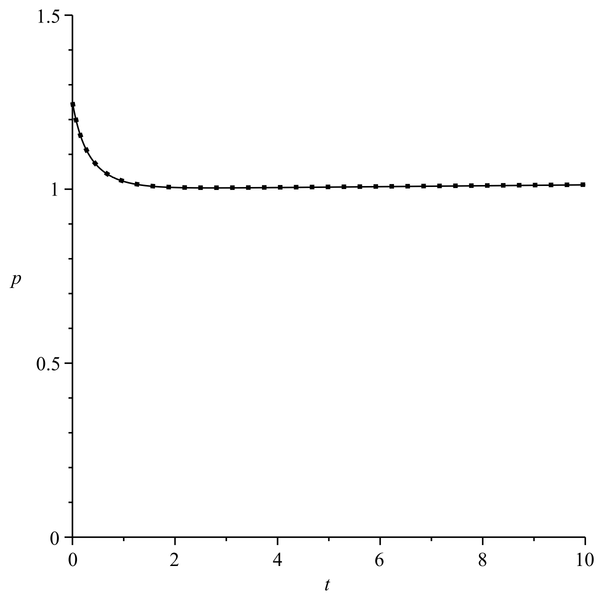

Figure 5.7: Evolution of the leading term approximation for a surviving population (5.31)(dotted) compared with the numerical solution (solid) of (5.52), with $\mu = 1.14$ for a small time period.

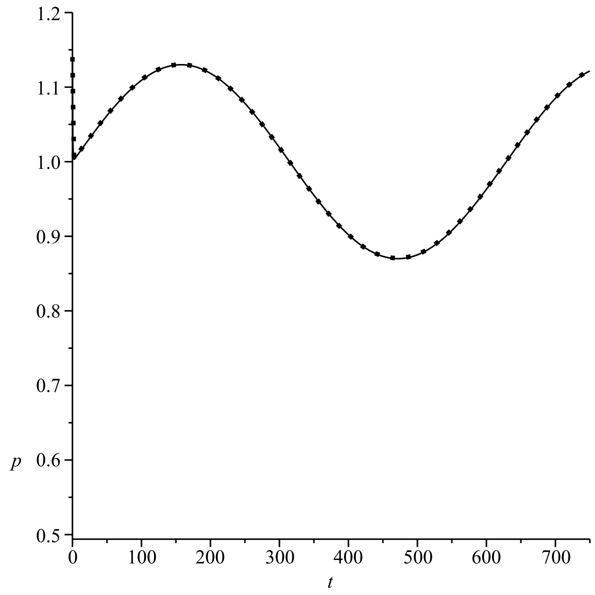

Figure 5.8: The leading term approximation for a surviving population (5.31) (dotted) compared with the numerical solution of (5.5) (solid) for data of Figure 5.7 and large times t.

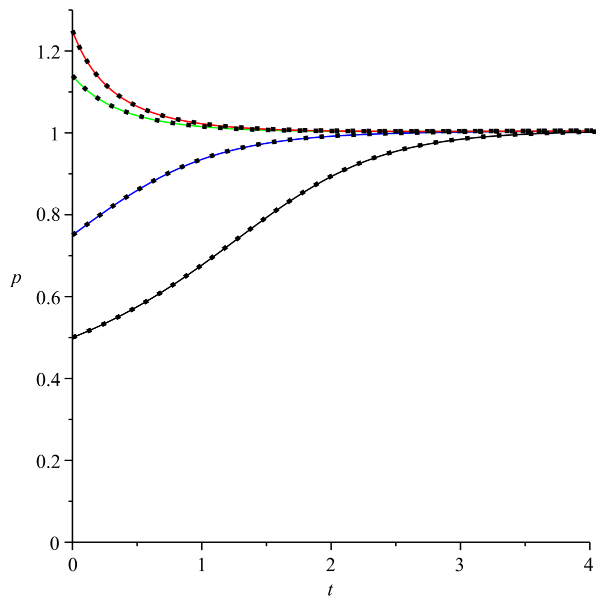

Figure 5.9: The leading term approximation for a surviving population (5.31) (dotted) compared with the numerical solution of (5.5) (solid) for data of Figure 5.7, but with different initial values $\mu = 0.5,\ 0.75,\ 1.25,\ 1.5$.

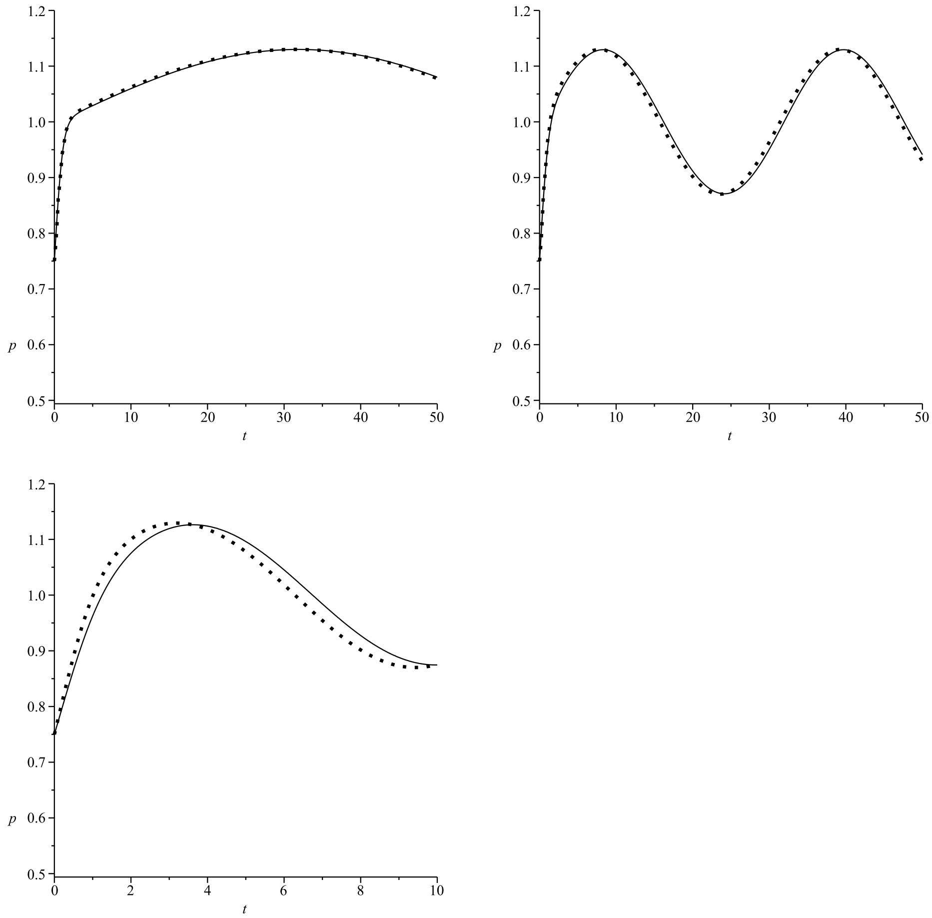

Figure 5.10: *Evolution of a surviving population* (5.31) *(dotted) compared with the numerical solution of* (5.5) *for data used in Figure 5.5, where* $\epsilon = 0.05, 0.2, 0.5$ *in clockwise order starting from the top left corner.*

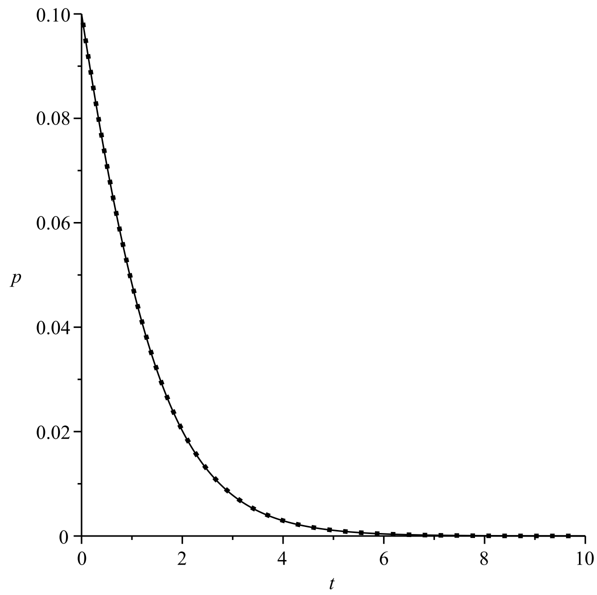

Figure 5.11: Comparison of the leading approximation (5.44) (dotted) for an extinguished population with the numerical solution of (5.5) (solid) for $r(\epsilon t)$, $k(\epsilon t)$ and $m(\epsilon t)$ as defined by (5.52) but now with $\mu = 0.1$.

5.5 Transition Analysis

In the preceding sections, we have constructed leading order approximations for the solutions of the problem (5.5) when the model parameters r, m and k are slowly varying; i.e., ϵ, the ratio of time scales, is small.

Using a multiscaling method, we found leading order expressions for the surviving and extinguishing solutions of (5.5) when the condition (5.7) holds. In particular, we found that the surviving population, approximated by p_s defined implicitly in (5.31), tended to a limiting value given by (5.48). However, as (5.48) shows, in any neighborhood of any point $\bar{t}_1$ where

$$\alpha k(\bar{t}_1) - m(\bar{t}_1) = 0, \;\; \bar{t}_1 = \epsilon \bar{t} \tag{5.51}$$

disordering will occur in this expansion, and the limiting state (5.48) fails to represent the evolving surviving population. i.e., at points t_1 where $\alpha k(t_1) - m(t_1) = O(\epsilon)$ the second term in (5.48) becomes comparable with the leading term. Although (5.48) represents a limiting state, it is clear that the same would occur in the full expansion representing the surviving population on $t \geq 0$. (Recall that the next term, $\epsilon \tilde{p}_1$ in (5.17) could not be calculated explicitly on $t \geq 0$. However, this limiting problem points to a similar failure of any overall expansion.)

Equation (5.51) will certainly hold at a point $t = \bar{t}$ for which

$$0 \;<\; \frac{m(\epsilon t)}{\alpha} < k(\epsilon t) \quad \text{for} \quad 0 \leq t < \bar{t}, \tag{5.52}$$

and

$$0 \;<\; k(\epsilon t) < \frac{m(\epsilon t)}{\alpha} \quad \text{for} \quad t > \bar{t}. \tag{5.53}$$

At such points, $t = \bar{t}$, termed *transition points*, the roles of $k(\epsilon t)$ and $m(\epsilon t)$ are interchanged. Up to $t = \bar{t}$, one solution of (5.5) tends to a neighborhood of $k(\epsilon t)$; but beyond $t = \bar{t}$, it may tend to $m(\epsilon t)/\alpha$, or may decay to zero. This depends critically on the nature of the solution transition through $t = \bar{t}$. Thus, a local analysis of the solutions of (5.5) in a neighborhood of any transition point is required. In what follows, we consider the case where there is a single transition point $\bar{t}_1$ on $\bar{t} > 0$, so that (5.52) and (5.53) apply.

To analyse the transition behaviour of (5.5) and to simplify the calculation, we consider the special case when $r(t_1) \equiv m(t_1) \equiv 1$ (i.e., $R(T)$ and $M(T)$ are constants) and $k(t_1)$ is a slowly varying function.

Then, (5.5) becomes

$$\frac{dp(t, \epsilon)}{dt} = p\left(1 - \frac{p}{k(t_1)}\right) \; (\alpha\, p - 1)\,; \qquad p(0, \epsilon) = \mu, \tag{5.54}$$

while (5.52) and (5.53) become

$$0 \;<\; \frac{1}{\alpha} < k(\epsilon t) \quad \text{for} \quad 0 \leq t < \bar{t} \tag{5.55}$$

and

$$0 \;<\; k(\epsilon t) < \frac{1}{\alpha} \quad \text{for} \quad t > \bar{t}. \tag{5.56}$$

Further, we impose the condition that

$$k'(\epsilon\ \bar{t}) < 0; \tag{5.57}$$

i.e., the zero of $\alpha\ k(\epsilon t) - 1$ is *simple.*

5.5.1 Subregions

Now, as in Chapter 4, in dealing with this situation, we define the following three time subregions of $t \geq 0$:

- In Region 1 (as discussed previously in Section 5.3.3) where the condition (5.55) holds, the approximate expansion of the solution (5.5) is represented to leading order by the leading term p_{0s}, represented implicitly by (5.31), that tends to the limiting state (5.48).

- Region 2 is a transition region surrounding $\bar{t}_1 = \epsilon\bar{t}$ where the condition $k(\epsilon\bar{t}) = 1/\alpha$ holds.

- In Region 3 where $t_1 > \bar{t}_1 \quad (t > \bar{t})$, (5.56) holds and $1/\alpha$ and $k(t_1)$ exchange roles $(1/\alpha \leftrightarrow k(t_1))$ in (5.31), while from (5.30), the initial point of $\tilde{t}_{0s}$ in this region is taken to be $\bar{t}_1$, so that, in Region 3,

$$\tilde{t}_{0s} = \frac{1}{\epsilon}\int_{\bar{t}_1}^{t_1} \left(\frac{1}{\alpha\ k(s)} - 1\right) ds. \tag{5.58}$$

Thus the leading term of the expansion for the solution in Region 3 becomes

$$F(p_{0s}, t) - D\,\tilde{a}_s(t_1)\,e^{-\tilde{t}_{0s}} = 0, \tag{5.59}$$

where

$$F(p_{0s}, t) = \left(\frac{p}{\alpha(p - k(t_1))}\right)^{\frac{1}{\alpha\,k(t_1)}} \left(1 - \frac{1}{\alpha p}\right), \tag{5.60}$$

$$\tilde{a}_s(t_1) = \left[\frac{1}{\alpha}\left(\frac{1 - \alpha\,k(t_1)}{1/\alpha}\right)^{\frac{1}{\alpha\,k(t_1)}}\right]^{-1} \tag{5.61}$$

and D is an undetermined constant.

5.5.2 Solution in the Transition Region

Following the procedure of Section 4.3, , we introduce a new local variable τ defined by

$$t_1 = \bar{t}_1 + \epsilon^{\sigma}\tau, \quad -\infty < \tau < \infty. \tag{5.62}$$

where σ is positive and unknown constant. Thus the solution of (5.54) (in terms of τ) has the form

$$\tilde{p}(\tau, \epsilon) \equiv p(\bar{t}_1 + \epsilon^{\sigma}\tau, \epsilon). \tag{5.63}$$

At the transition point $\bar{t}_1$, we have

$$k(\bar{t}_1) = \frac{1}{\alpha} \text{ and } k'(\bar{t}_1) < 0. \tag{5.64}$$

and so

$$k(t_1 + \epsilon^{\sigma}\tau) = \frac{1}{\alpha} + \epsilon^{\sigma}\tau\, k'(\bar{t}_1) + \dots. \tag{5.65}$$

Substituting (5.63) and (5.65) into (5.54) and expanding in powers of ϵ, the differential equation (5.54) in terms of τ becomes

$$\frac{d\tilde{p}}{d\tau} = -\epsilon^{\sigma-1} p \left(1 - \alpha\tilde{p}\right)^2 - \epsilon^{2\sigma-1}\alpha^2\, \tilde{p}^2 \left(1 - \alpha\tilde{p}\right) k'(\bar{t}_1)\, \tau + O\left(\epsilon^{3\sigma-1}\right), \tag{5.66}$$

where $\sigma > 0$. Since there is no balancing of powers of ϵ that can determine the value of σ in (5.66), we thus reconsider the limiting state (5.48) in Region 1 in terms of τ using (5.62) and (5.64), so we have

$$p(\tau, \epsilon) \to \frac{1}{\alpha} + \epsilon^{\sigma} k'(\bar{t}_1)\tau - \epsilon^{1-\sigma}\frac{1}{\alpha\tau} + O(\epsilon). \tag{5.67}$$

Thus (5.67) shows that the transition equation could be in the form

$$p(\tau, \epsilon) = \frac{1}{\alpha} + \epsilon^{\sigma} u_0(\tau) + \dots \tag{5.68}$$

Substituting (5.68) into (5.66) and balancing powers of ϵ for both sides, we choose

$$\sigma = \frac{1}{2}, \tag{5.69}$$

and so we obtain the equation for u_0 as

$$\frac{du_0}{d\tau} = \alpha\, u_0(\tau)\,(k'(\bar{t}_1)\tau - u_0(\tau))\,. \tag{5.70}$$

Solving (5.70) for $u_0(\tau)$ gives

$$u_0(\tau) = \frac{\sqrt{-2\ \alpha\ k'(\bar{t}_1)}\, e^{\alpha k'(\bar{t}_1)\,\tau^2/2}}{\alpha\sqrt{\pi}\ \mathrm{erf}\left(\frac{1}{2}\sqrt{-2\alpha\ k'(\bar{t}_1)}\tau\right) + C\sqrt{-2\alpha\ k'(\bar{t}_1)}}; \quad k'(\bar{t}_1) < 0 \tag{5.71}$$

where

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt,$$

is the error function, (see [1]) and C is a constant. Expanding (5.71) as $\tau \to -\infty$ and comparing with (5.67) (where $\sigma = 1/2$), shows there is matching between expansions in Regions 1 and 2 if we choose

$$C = \frac{\alpha\sqrt{\pi}}{\sqrt{-2\ \alpha\ k'(\bar{t}_1)}}.$$

Thus with this choice, the approximate leading terms of the expansions for the transition region solution (5.68) become

$$p(\tau,\epsilon) = \frac{1}{\alpha} + \epsilon^{1/2}\frac{\sqrt{-2\alpha\ k'(\bar{t}_1)}\, e^{\alpha k'(\bar{t}_1)\,\tau^2/2}}{\alpha\sqrt{\pi}\left(\mathrm{erf}\left(\frac{1}{2}\sqrt{-2\ \alpha\ k'(\bar{t}_1)}\tau\right)+1\right)} + O(\epsilon), \quad k'(\bar{t}_1) < 0, \tag{5.72}$$

and so for large and negative τ, (i.e.,$\tau \to -\infty$) the expansion of the solution (5.72) that represents the common part in the overlap between Region 1 and Region 2 is

$$C_{1,2}(\tau,\epsilon) = \frac{1}{\alpha} + \epsilon^{1/2}k'(\bar{t}_1)\tau + O(\epsilon), \quad k'(\bar{t}_1),\ \tau < 0. \tag{5.73}$$

Now we look at the matching of the expansions in Regions 2 and 3. The transition from Region 2 to Region 3 corresponds to the considering large and positive values of τ. From (5.72), the asymptotic series expansion of the $O(\epsilon^{1/2})$ term for large τ gives

$$\lim_{\tau\to\infty}\frac{\sqrt{-2\ \alpha\ k'(\bar{t}_1)}}{\alpha\sqrt{\pi}}\left(\frac{e^{1/2\alpha k'(\bar{t}_1)\,\tau^2}}{\left(\mathrm{erf}\left(\frac{1}{2}\sqrt{-2\alpha\ k'(\bar{t}_1)}\tau\right)+1\right)}\right) = \frac{\sqrt{-2\alpha\ k'(\bar{t}_1)}}{\alpha\sqrt{\pi}}\left(\frac{0}{1+1}\right) = 0.$$

Thus, as $\tau \to \infty$ the transition solution (5.72) tends to

$$\tilde{p}(\tau,\epsilon) \to \frac{1}{\alpha}. \tag{5.74}$$

Now we consider the leading term of the surviving solution (5.31) in the overlap region between Regions 2 and 3.

For small ϵ where the condition (5.56) holds, and by considering (5.59) and (5.58) and expanding for small ϵ and integrating, we obtain

$$\tilde{t}_{0s} = -\frac{1}{2}\alpha\, k'(\bar{t}_1)\ \tau^2, \ \ k'(\bar{t}_1) < 0. \tag{5.75}$$

Substituting (5.64) and (5.75), the leading order (5.59) becomes

$$F(p_{0s}, t) - D\,\tilde{a}_s(t_1)\, e^{\frac{1}{2}\alpha\, k'(\bar{t}_1)\tau^2} = 0, \qquad k'(\bar{t}_1) < 0, \tag{5.76}$$

where $0 < k(t_1) < 1/\alpha$ and $F(p_{0s}, t)$ and $\tilde{a}_s(t_1)$ defined by (5.60) and (5.61) respectively. Now, for large τ the term $e^{\frac{1}{2}\alpha\, k'(\bar{t}_1)\ \tau^2}$ in (5.76) converges to zero and by following the procedure used in Section 5.3.3, and also considering that $k(t_1)$ switches with $m(t_1)/\alpha$ (or particularly in this case $k(t_1) \leftrightarrow 1/\alpha$), we see that (5.22) becomes

$$\tilde{p}_{0s}(\tau, \epsilon) \ \ = \ \ \frac{1}{\alpha} + D\ \tilde{a}_s(t_1)\ e^{\frac{1}{2}\alpha\, k'(\bar{t}_1)\ \tau^2} + O(e^{\alpha\, k'(\bar{t}_1)\ \tau^2}), \tag{5.77}$$

Rewriting (5.77) in terms of τ using (5.62) and (5.65) gives

$$\tilde{p}_{0s}(\tau, \epsilon) = \frac{1}{\alpha} + D\ \tilde{a}_s(\bar{t}_1 + \sqrt{\epsilon}\tau)\ e^{\frac{1}{2}\alpha\, k'(\bar{t}_1)\ \tau^2} \tag{5.78}$$

expanding (5.77) for small ϵ gives

$$\tilde{p}_{0s}(\tau, \epsilon) \to \frac{1}{\alpha} + D\ \tilde{a}_s(\bar{t}_1)\, e^{\frac{1}{2}\alpha\, k'(\bar{t}_1)\ \tau^2} + O(\sqrt{\epsilon})\ldots. \tag{5.79}$$

By comparing (5.79) with (5.74), there is only a common term $1/\alpha$ if we choose

$$D = 0. \tag{5.80}$$

Thus for this particular case, the expansion solution for Region 3 is given, to leading order, by

$$\tilde{p}_{0s}(\tau, \epsilon) = \frac{1}{\alpha}, \tag{5.81}$$

and the common part is

$$C_{23} = \frac{1}{\alpha}. \tag{5.82}$$

It is important to note that the choice $D = 0$ in (5.80) means that the implicit leading order approximation in Region 3, given by (5.57) contains no $\tilde{t}_{0s}$ dependence; i.e., the p_{0s} obtained is completely slowly varying to the level of approximation considered. Thus, the slowly varying transition solution continues on to represent the evolving population. This phenomenon has been noted in an analogous case [30]; and, even with an improved level of approximation, ([56]), the transition behaviour was found to have no marked effect.

5.5.3 A Uniform Approximation

Following the same technique used in Section 4.4, [30, 31] we aim to create a composite expansion that describes the behaviour of the solution throughout three Regions R_1, R_2 and R_3. That is achieved by adding valid expansions in Regions 1 and 2 or Regions 2 and 3 then subtracting the common term.

For the interval $[0, \bar{t}_1]$, the uniform valid solution is

$$\begin{aligned} p_{12} = p_{0s} + \epsilon^{1/2} & \frac{\sqrt{-2\alpha\ k'(\epsilon\bar{t})}\, e^{1/2\alpha k'(\epsilon\bar{t})\,\tau^2}}{\alpha\sqrt{\pi}\left(\operatorname{erf}\left(1/2\sqrt{-2\alpha\ k'(\epsilon\bar{t})}\tau\right)+1\right)} \\ & -\epsilon^{1/2}\left(k'(\epsilon\bar{t})\tau\right) \end{aligned} \tag{5.83}$$

where $\tau = \epsilon^{1/2}(t - \bar{t})$ and from (5.32) p_{0s} satisfies

$$\left(\frac{p_0}{\alpha\, p_0 - 1}\right)^{\alpha k(\epsilon t)} \left(1 - \frac{k(\epsilon t)}{p_0}\right) - A(\epsilon t)\, e^{-t_0} = 0.$$

From (5.72) and (5.81), (5.82) the uniform valid expansion for interval $[\bar{t}_1, \infty]$ in this particular case is represented only by the transition solution (5.72) that is

$$p_{23}(\tau,\epsilon) = \frac{1}{\alpha} + \epsilon^{1/2}\frac{\sqrt{-2\alpha\ k'(\epsilon\bar{t})}\, e^{\frac{1}{2}\alpha k'(\bar{t}_1)\,\tau^2}}{\alpha\sqrt{\pi}\left(\operatorname{erf}\left(\frac{1}{2}\sqrt{-2\alpha\ k'(\epsilon\bar{t})}\tau\right)+1\right)} + O(\epsilon), \qquad k'(\epsilon\bar{t}) < 0. \tag{5.84}$$

Figures 5.12 and 5.13 display the behaviour of the union of the two uniformly valid

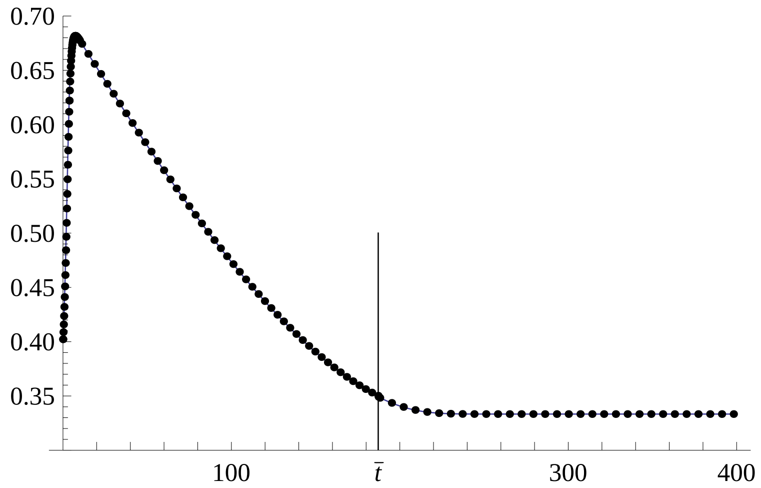

Figure 5.12: Plot of the composite expansion (5.83) (dotted) and the numerical solution of (5.54)(solid), where $k(\epsilon t) = 0.2 + (1 + e^{\epsilon t})^{-1}$ with $\mu = 0.4, \epsilon = 0.01$, $\alpha = 3$ and $\bar{t} \approx 187$.

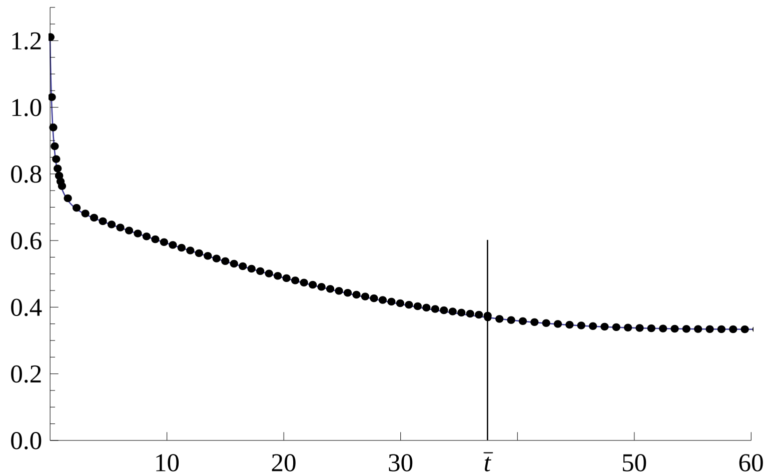

Figure 5.13: Plot of the composite expansion (5.84) (dotted) and the numerical solution of (5.54)(solid) using the same choice of $k(t_1)$ as in Figure 5.12, where $\mu = 1.2, \epsilon = 0.05$, $\alpha = 3$ and $\bar{t} \approx 37$ $\mu = 1.2$.

expansions (5.83) and (5.84) through the transition point where $\bar{t_1}$, compared with the numerical solution of (5.54) with considering two different initial values $\mu = 0.4,\ 1.2$ and different values of ϵ. It is shown a very good agreement between the numerical solution and the uniform valid solutions through the transition point.

5.6 Discussion

With the values (5.33), (5.45) for the constants c_s and c_e, (5.31) and (5.43) provide implicit representations for p_{0s} and p_{0e}, leading order approximations to the evolving populations p_s and p_e in the surviving and extinguishing situations, respectively, when ϵ is small, and for arbitrary slowly varying model parameters $r(\epsilon t)$, $k(\epsilon t)$ and $m(\epsilon t)$ satisfying (5.7) and initial population μ.

Although these approximations are represented as implicit functions, their form is relatively simple and they are easily represented graphically (and compared with the results of using a numerical ODE solver) using such packages as Maple or Mathematica. The comparisons with numerical solutions in Section 5.4 show that in three specific cases, these approximations are very good indeed.

Unfortunately, in contrast to the harvesting problem of Chapter 2 the implicit representations for p_{0s} and p_{0e} make it impossible to obtain explicit representations for their higher order counterparts p_{1s} and p_{1e}. Our only higher order expansion is for the surviving population;

$$p_s \to k(\epsilon t) + \epsilon \frac{k'(\epsilon t)}{r(\epsilon t)\left(1 - \frac{\alpha k(\epsilon t)}{m(\epsilon t)}\right)} + \dots \tag{5.85}$$

as $t \to \infty$, and, trivially, for the extinguished population, as $t \to \infty$

$$p_e \to 0. \tag{5.86}$$

Regardless of this, the limiting form (5.85) enables us to note that the surviving population tends to the carrying capacity $k(\epsilon t)$ as $t \to \infty$. Moreover, in view of (5.7), this approaches

k from the below when $k'(\epsilon t) < 0$ and above when $k'(\epsilon t) < 0$. The limiting form (5.86) just demonstrates the exponential decay of the extinguished population.

The implicit leading order approximations (5.31) and (5.43) do provide much more information about the behaviour of the populations in the initial "transient" region, where t_0 variation dominates. The comparisons of Section 5.5, where the involved parameters $k(\epsilon t)$ and $m(\epsilon t)$ interchange roles (in a particular case), demonstrate how this multiscaling analysis can be coupled with local asymptotic analysis to obtain a representation of the evolution of the population throughout the transition process. Thus, they reflect the approach to such transitions as are considered in Chapter 4. Again, the results, as shown in Figures 5.12 and 5.13 show the very good agreement between the asymptotic results and the numerical solutions.

The calculations of Section 5.3 have been published in Idlango et al [39].

Chapter 6

A Single Species Logistic Model Subject to Saturating Holling II Harvesting

6.1 Introduction

In this chapter, we consider a single species logistic population model that is harvested at a rate determined by a Holling type II functional. Mathematically, this can be expressed as an initial value problem for the population $P(T)$ at time $T \geq 0$:

$$\frac{dP}{dT} = RP\left(1 - \frac{P}{K}\right) - \frac{HP}{A+P}, \quad P(0) = P_0, \tag{6.1}$$

where R, K, H, A and P_0 are positive constants. Note that the first term on the right hand side represents logistic growth with R the growth rate and K the carrying capacity; while the term $HP/(A+P)$ is a Holling type II function harvesting term, that increases monotonically with P to a maximum ('saturation') value of H [34],(see Figure 6.1). Here, H is the maximal rate of harvesting, and A is the half capturing saturation constant. When there is no harvesting (i.e., $H = 0$), the differential equation of (6.1) reduces to that of simple logistic growth. Note that the harvesting term might also be interpreted as predation, with a predation rate given by $HP/(A+P)$.

However, in reality these parameters may actually change as time changes, as a result of surrounding environmental variations. In particular, they may vary slowly with time. Therefore, in this section, we reconsider the problem and investigate the effect of slow parameter variation.

Then, the initial value problem (1.4) is replaced by

$$\frac{dP}{dT} = R(T)P\left(1 - \frac{P}{K(T)}\right) - \frac{H(T)\,P}{A(T)+P}, \qquad P(0) = P_0, \tag{6.2}$$

where $R(T)$, $K(T)$, $H(T)$ and $A(T)$ are positive functions of T on $T \geq 0$.

6.2 Dimensionless Model

We begin by converting the problem (6.2) to a dimensionless form. To express (6.2) in dimensionless form we consider that the parameters above may be expressed in the form

$$\begin{aligned} R(T) &= R_0\, r(T/T^*), \\ K(T) &= K_0\, k(T/T^*), \\ H(T) &= H_0\, h(T/T^*), \\ A(T) &= A_0\, a(T/T^*), \\ t &= R_0 T, \\ P &= A_0\, p, \end{aligned}$$

where R_0, K_0, H_0 and A_0 represent characteristic values of these functions and T^* is a time scale for variation of these parameters, assumed the same for all. This gives the problem (6.2) as

$$\frac{dp}{dt} = r(\epsilon\, t)p\left(1 - \frac{p}{\eta\; k(\epsilon\, t)}\right) - \sigma h(\epsilon t)\frac{p}{a(\epsilon\, t)+p},$$

$$p(0,\epsilon) = \mu, \tag{6.3}$$

where $\epsilon = 1/(R_0 T^*)$ is the ratio of the intrinsic population variation time scale, $1/R_0$, to T^*, while σ, η and μ are constant positive dimensionless parameters, defined by

$$\sigma = \frac{H_0}{R_0\, A_0}, \qquad \eta = \frac{K_0}{A_0} \qquad \mu = \frac{P_0}{A_0}. \tag{6.4}$$

6.3 The Constant Parameter Model

Here, we consider the situation where R, K, H and A are positive constants, which leads to writing $r(\epsilon t) \equiv k(\epsilon t) \equiv h(\epsilon t) \equiv a(\epsilon t) \equiv 1$. This gives us the dimensionless problem

$$\frac{dp}{dt} = p\left(1 - \frac{p}{\eta}\right) - \sigma \frac{p}{1+p}, \quad p(0) = \mu, \tag{6.5}$$

which involves three constant positive dimensionless parameters; η, σ and μ.

Here, we can see that as p becomes large (that is, $p \gg 1$ or $P \gg A$), the harvesting factor tends to a constant; i.e.,

$$\lim_{p\to\infty} \sigma \frac{p}{1+p} = \sigma.$$

However, when $p \ll 1$ (or $P \ll A$) the value of this function approaches $\sigma\, p$ and the harvesting is now a function of the current population; i.e., it is density dependent. Figure 6.1 displays the behaviour of this function.

Now, before we investigate the solutions of (6.5), we consider the properties of the critical points (CPs) of the differential equation in (6.5).

6.3.1 Critical Points and Stability

To study the critical points (CPs) of (6.5), we first find these points by setting the right hand side of the differential equation in (6.5) to zero, that is, we set

$$F(p) = p\left(1 - \frac{p}{\eta}\right) - \sigma \frac{p}{1+p} = 0. \tag{6.6}$$

The stability index is given by

$$F'(p) = \left(1 - \frac{2\,p}{\eta}\right) - \frac{\sigma}{(1+p)^2}. \tag{6.7}$$

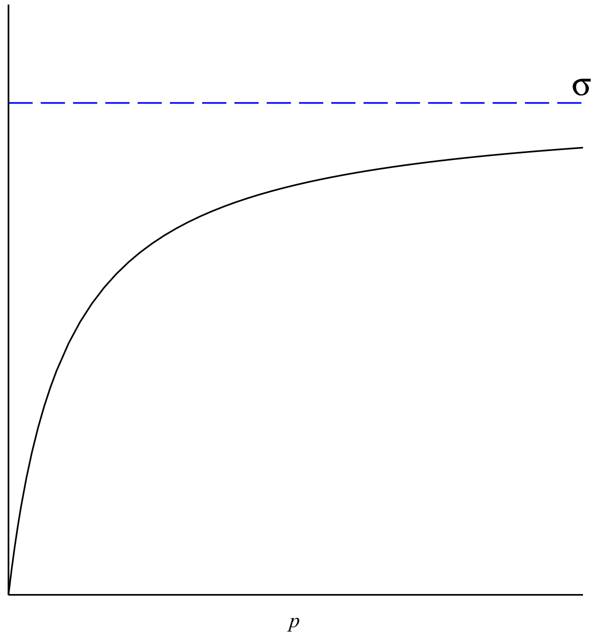

Figure 6.1: Plot of the harvesting function $\frac{\sigma\, p}{1+p}$.

Equation (6.6) may be written as

$$p\, G(p) = 0 \tag{6.8}$$

where

$$G(p) = 1 - \frac{p}{\eta} - \frac{\sigma}{1+p}; \tag{6.9}$$

while (6.7) becomes

$$F'(p) = p\, G'(p) + G(p). \tag{6.10}$$

So, CPs of (6.5) are $p = 0$ or points $p > 0$ where $G(p) = 0$, i.e., points where the line

$$1 - \frac{p}{\eta} \tag{6.11}$$

intersects the hyperbola

$$\frac{\sigma}{1+p}. \tag{6.12}$$

Now, from (6.10),

$$F'(0) = G(0) = 1 - \sigma$$

so the CP $p = 0$ which exists and is real for all $\sigma > 0$, is unstable when $0 < \sigma < 1$, and stable if $\sigma > 1$.

For CPs $p > 0$ given by intersections of (6.11) and (6.12), we have $G(p) = 0$ and, for such points,

$$F'(p) = p\,G'(p)$$

and so, at such points (when they occur), the sign of $F'(p)$ is given by that of $G'(p)$; i.e., the sign of

$$\left\{\text{slope of line } 1 - \frac{p}{\eta}\right\} - \left\{\text{slope of hyperbola } \frac{\sigma}{1+p}\right\}. \tag{6.13}$$

From the above, nonzero CPs; i.e., nonzero solutions of (6.6) are given by

$$p = Q_1 = \tfrac{1}{2}\left(\eta - 1 - \sqrt{(\eta+1)^2 - 4\,\eta\,\sigma}\right)$$

$$p = Q_2 = \tfrac{1}{2}\left(\eta - 1 + \sqrt{(\eta+1)^2 - 4\,\eta\,\sigma}\right), \tag{6.14}$$

where the quantity under the square root is positive if $\sigma < \frac{(\eta+1)^2}{4\,\eta}$ and then $Q_2 > Q_1$ and both are real. As we have noted above, the intersections between the straight line $(1 - p/\eta)$ and hyperbola $\sigma/(1+p)$ will determine the critical points and their nature. For example, in Figure 6.2, we can see that there may be one or two nonzero critical points, depending on the values of η and σ. When there are two, with $Q_1 < Q_2$, the criterion (6.13) shows that Q_1 is unstable.

To get more insight, we discuss the stability boundary of CPs. This can be obtained from (6.6) and (6.7) as the solution of the simultaneous nonlinear equations

$$\left(1 - \frac{p}{\eta}\right) = \frac{\sigma}{1+p}, \tag{6.15}$$

$$\left(1 - \frac{2p}{\eta}\right) = \frac{\sigma}{(1+p)^2}. \tag{6.16}$$

Solving (6.15) and (6.16) for η and σ in terms of p gives

$$\eta = 1 + 2\,p, \tag{6.17}$$

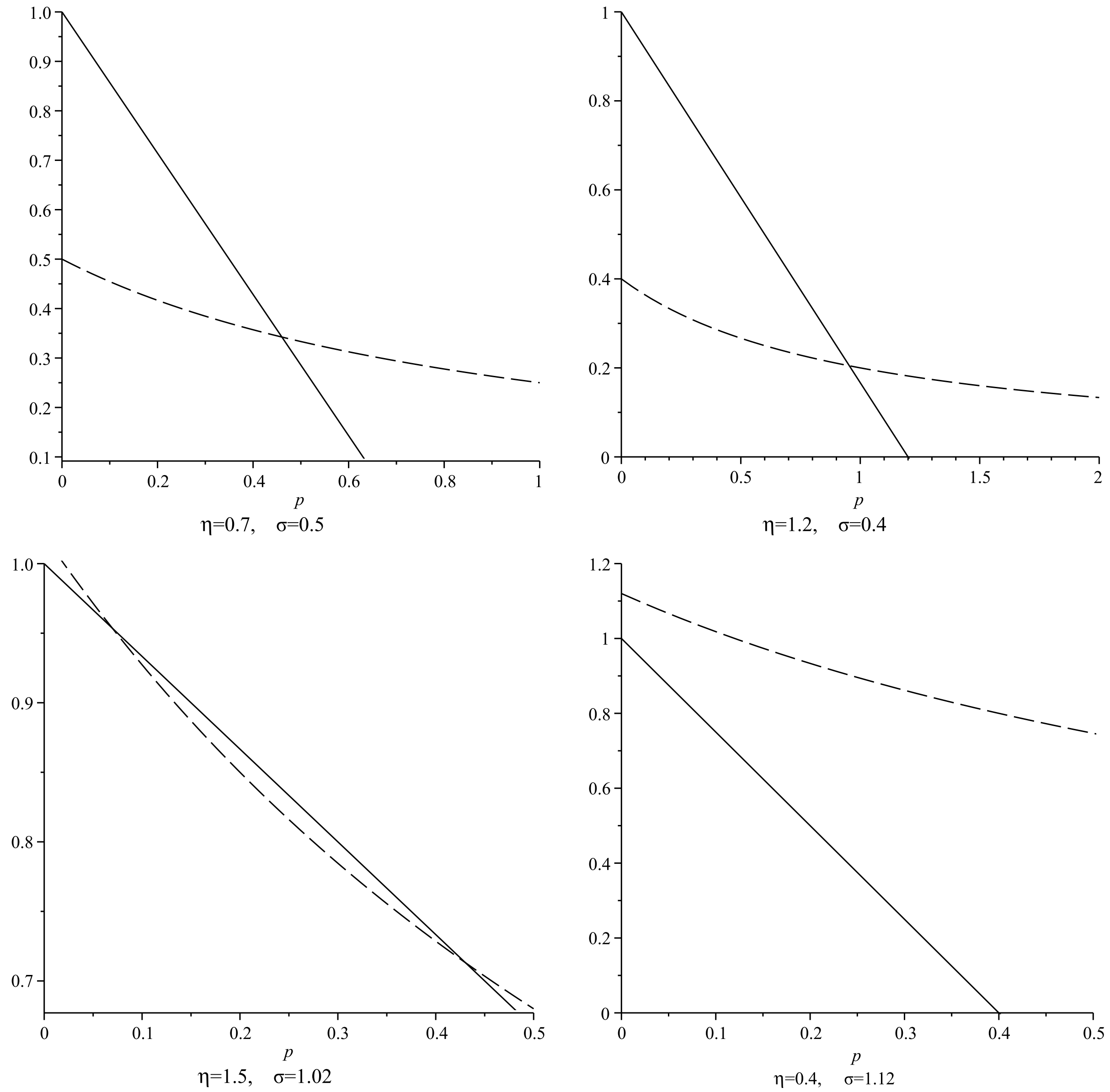

Figure 6.2: The graphs of $1-p/\eta$ and $\sigma/(1+p)$ for various values of η and σ.

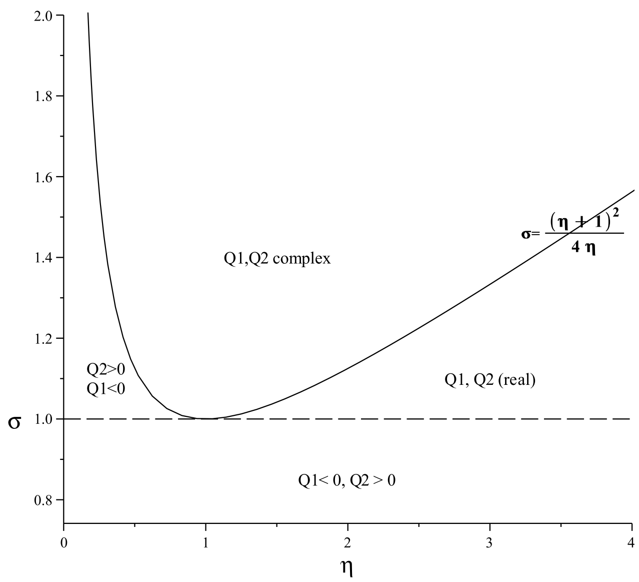

Figure 6.3: Plot of boundaries of parameter σ as defined by (6.19).

$$\sigma = \frac{(1+p)^2}{1+2p}. \tag{6.18}$$

Substituting (6.17) into (6.18) gives the stability boundary of CPs in the (η, σ) plane as

$$\sigma = \frac{(1+\eta)^2}{4\eta}. \tag{6.19}$$

By considering $p = 0$ as one of the equilibria, the stability index of other points Q_1, Q_2 given by (6.14) where $(Q_2 > Q_1)$ can be detailed as follows:

[**1**]. when

$$1 < \sigma < \frac{(\eta+1)^2}{4\eta} \quad \text{while} \quad \eta > 1, \tag{6.20}$$

Q_1, Q_2 are positive equilibria. Applying the condition (6.20), gives $F'(Q_1) > 0$ and $F'(Q_2) < 0$ and so Q_1 is an unstable point and Q_2 is a stable point. Thus, populations having starting population $\mu < Q_1$ will tend to extinction as $t \to \infty$; while those with $\mu > Q_1$ will tend to the limiting population Q_2; they will survive.

[**2**]. when

$$\sigma < \frac{(\eta+1)^2}{4\eta} \qquad \text{and} \qquad \eta < 1 \tag{6.21}$$

the CPs Q_1, Q_2 are real and negative .

[**3**]. when

$$\sigma > \frac{(\eta+1)^2}{4\eta} \qquad \text{and} \qquad \eta > 0 \tag{6.22}$$

then Q_1, Q_2 are complex. There is no stable nonzero equilibrium and the value of σ (number of predators) is sufficiently large to force the population p into extinction.

[**4**]. when $\eta = \sigma = 1$ then $p = Q_2 = Q_1 = 0$.

[**5**]. when

$$0 < \sigma \leq 1, \; \eta > 0 \tag{6.23}$$

then $p = Q_2$ is positive and stable, while $p = Q_1$ is negative and stable. In this case, all populations tend to the limiting population $Q_2 > 0$.

This analysis, and Figure 6.3 in particular, make it clear that there are a range of ultimate states for the solutions of (6.3), depending on the values of η and σ. Holding η constant (i.e., holding the ratio K_0/A_0 fixed) constitutes a vertical line in Figure 6.3; and allowing σ to increase from zero takes us through a range of possible end-states for these solutions. Thus, fixing η and letting σ increase leads us through 'survival' states, where $0 < \sigma < 1$, and the population tends to the state Q_2, and to 'survival' and 'extinction' states, $1 < \sigma < (\eta+1)^2/4\eta$, where Q_2 and the zero state $p = 0$ are stable options. Finally, for $\sigma > (\eta+1)^2/4\eta$, only the zero state $p = 0$ is available, and extinction always occurs. Analogous behaviour occurs when $\eta > 1$.
For particular situation where $\eta = 1$ ($K_0 = A_0$), there is one basic transition, from survival to Q_2 in $0 < \sigma < 1$ and extinction to zero in $\sigma > 1$.

In the following sections, we consider this situation; i.e., we set $\eta = 1$. This simplifies the analysis, somewhat. Thus, the problem (6.5) becomes

$$\frac{dp}{dt} = p\,(1-p) - \sigma \frac{p}{1+p}, \quad p(0) = \mu, \tag{6.24}$$

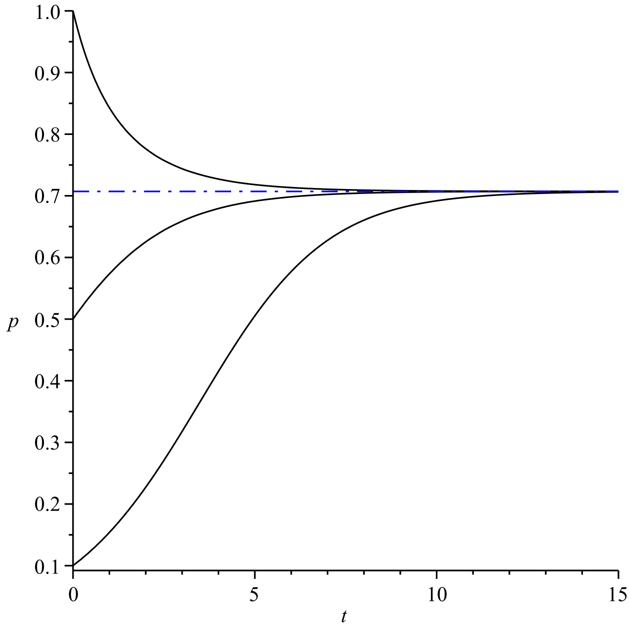

Figure 6.4: Plot of subcritical exact solution for (6.24) as given by (6.25), where $\sigma = 0.5$ and $\mu = 0.1,\ 0.5,\ 1$.

6.3.2 The Implicit Solution for the Subcritical Case: $0 < \sigma < 1$

The initial value problem (6.24) cannot be solved explicitly, so we solve it in implicit form as

$$\left(\frac{p}{\mu}\right)^{\frac{-2}{1+Q}} \left(\frac{p-Q}{\mu-Q}\right) \left(\frac{p+Q}{\mu+Q}\right)^{\frac{1-Q}{1+Q}} = e^{-2\frac{1-\sigma}{1+Q}t}, \tag{6.25}$$

where Q is a constant, given by

$$Q = \sqrt{1-\sigma}. \tag{6.26}$$

Figure 6.4 indicates the behaviour of the exact solution (6.25) for a range of initial values $\mu = 0.1,\ 0.5,\ 1$ when $\sigma = 0.5 < 1$. Here, the solution for any initial value below Q ($\mu = 0.1, 0.5$) or the solution for initial values above Q ($\mu = 1$) each converge to the stable equilibrium Q as expected.

6.3.3 Behaviour of the Subcritical Solution as $t \to \infty$

Here, we parallel similar analysis used in Chapter 5, Section 5.2. As $t \to \infty$, the term $e^{-2\frac{1-\sigma}{1+Q}t}$ on the right side of (6.25) tends to zero. Hence, we propose an approximate expansion for the solution (6.25), when the population survives to the limiting state, as $t \to \infty$, as a series in terms of power of $e^{-2\frac{1-\sigma}{1+\lambda}t}$, in the form

$$p(t) = f_0 + f_1\, e^{-2\frac{1-\sigma}{1+Q}t} + O(e^{-4\frac{1-\sigma}{1+Q}t}). \tag{6.27}$$

Substituting (6.27) into (6.25), and expanding in powers of $e^{-2\frac{1-\sigma}{1+Q}t}$ and equating like powers give the coefficient $f_0 = Q$.

Substituting $f_0 = Q$ into (6.27) to evaluate the other coefficients, we obtain the expansion for $p(t)$ as $p(t) \to Q$ as $t \to \infty$, as

$$p(t) = Q + (\mu - Q)\left[\left(\frac{\mu}{Q}\right)^2 \left(\frac{2Q}{\mu + Q}\right)^{1-Q}\right]^{\frac{-1}{1+Q}} e^{-2\frac{1-\sigma}{1+Q}t} + O(e^{-4\frac{1-\sigma}{1+Q}t}). \tag{6.28}$$

As is shown in Figures 6.5 and 6.6, the expansion (6.28) of the surviving population that tends to a limiting state is in good agreement with the numerical solution of (6.24) only for large t. This numerical solution was obtained by using a Runge-Kutta-Fehlberg fourth-fifth order method with degree four interpolant.

6.3.4 The Implicit Solution of the Supercritical Case: $\sigma > 1$

Again, the initial value problem (6.24) can be solved implicitly as

$$\left(\frac{p^2}{p^2 + \sigma - 1}\right) e^{\omega(p)} - c\, e^{-2\,(\sigma-1)\ t} = 0 \tag{6.29}$$

where

$$\omega(p) = 2\sqrt{\sigma - 1}\,\arctan\left(\frac{p}{\sqrt{\sigma - 1}}\right) \tag{6.30}$$

and c is given by

$$c = \frac{\mu^2}{\mu^2 + \sigma - 1}\ e^{\omega(\mu)}. \tag{6.31}$$

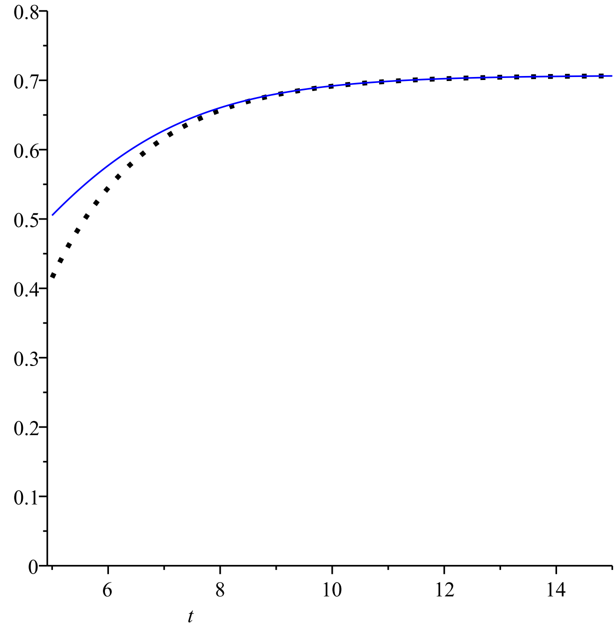

Figure 6.5: The subcritical asymptotic expansion (6.26)(dotted) with numerical solution of (6.24) (solid line) where $\sigma = 0.5$ and $\mu = 0.1, t \geq 5$.

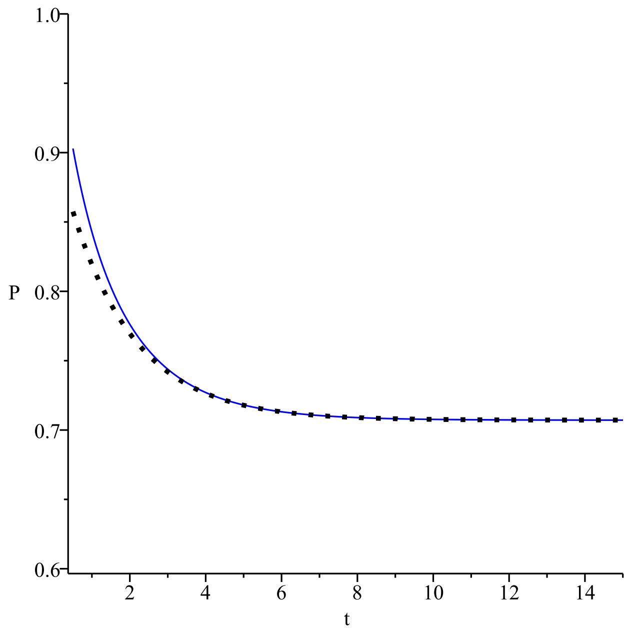

Figure 6.6: The subcritical asymptotic expansion (6.16)(dotted line) with numerical solution of (6.24) (solid line) where σ is as in Figure 6.5 and $\mu = 1, t \geq 1$.

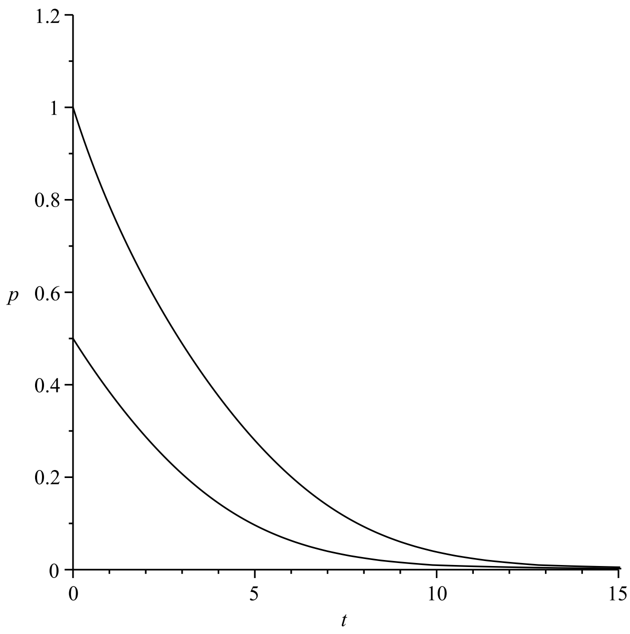

Figure 6.7: Plot of exact supercritical solution (6.29), where $\sigma = 1.5$ and $\mu = 0.5,\ 1$.

In Figure 6.7, where $\sigma = 1.5 > 1$, the harvesting is supercritical. It can be seen that for the values of $\mu = 0.5$, 1, populations always die out.

Similarly, we can find the approximate expansion for $p(t)$ when $p(t) \to 0$ (dies out) as $t \to \infty$ by defining

$$p(t) = f_0 + f_1\, e^{-(\sigma-1)\,t} + O(e^{-2\,(\sigma-1)t}).$$

Evaluating the coefficients f_i, $i = 0, 1, 2\ldots$ gives the expansion for $p(t)$ as $p(t) \to 0$ as $t \to \infty$, in the form

$$p(t) = (c(\sigma-1))^{1/2}\ e^{-(\sigma-1)\,t} + c(\sigma-1)\ e^{-2\ (\sigma-1)\,t} + O(\ e^{-3(\sigma-1)\,t}), \tag{6.32}$$

where c is given by (6.31). Figure 6.8 exhibits a comparison of the numerical solution with this approximate expansion in the extinction situation. It is clear that there is a close agreement as $p(t) \to 0$.

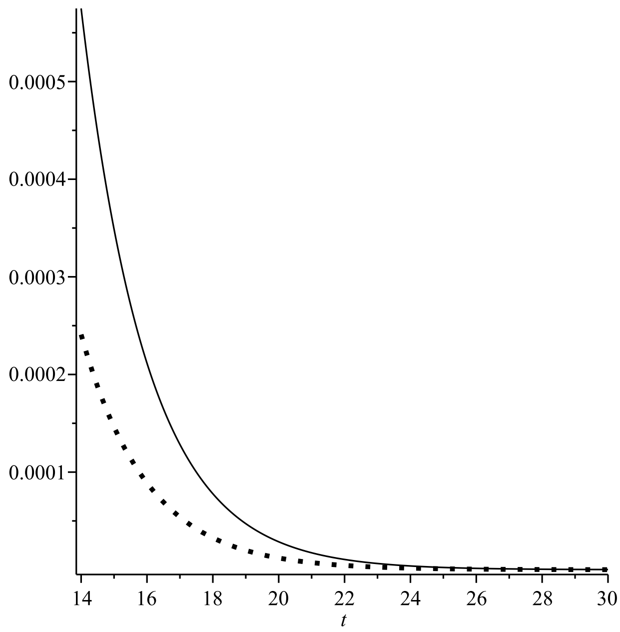

Figure 6.8: The asymptotic expansion (6.32)(dotted line) for supercritical behaviour and the numerical solution of (6.5) (solid line) where $\sigma = 1.5$ and $\mu = 0.5, t \geq 14$.

6.4 Slowly Varying Model Parameters

In previous sections, we have analysed the basic model with constant parameters.
Here, from (6.3) and for simplicity, we assume that only the harvesting rate is varying slowly on a time scale T^*, i.e., $(r(\epsilon\, t) \equiv k(\epsilon\, t) \equiv a(\epsilon\, t) \equiv 1)$. Further, we assume $\eta = 1$ as in the constant coefficient case.
Thus, (6.3) becomes

$$\frac{dp}{dt} = p\,(1-p) - \sigma h(\epsilon t)\frac{p}{1+p},$$
$$p(0,\epsilon) = \mu. \tag{6.33}$$

Note that the differential equation in (6.33) is in a form such that an approximate method such as the multi-scaling technique can be applied to yield an approximate solution.
As discussed in previous sections, in what follows our analysis will be separated into two

distinct cases

$$\text{when} \qquad 0 < \sigma\ h(t_1) < 1 \qquad \text{for all} \quad t_1 \geq 0; \tag{6.34}$$

termed the *subcritical surviving* case; and

$$\text{when} \qquad \sigma\ h(t_1) > 1 \qquad \text{for all} \quad t_1 \geq 0; \tag{6.35}$$

termed the*supercritical* case.

6.4.1 The Multiscale Equation

We now consider the generalized problem (6.33) where h is a slowly varying function of t, corresponding to a small positive ϵ.

While t and ϵt can be viewed as two possible time scales for the problem (6.33), to be more general we reconsider the slow time, $t_1 = \epsilon t$ and the normal time, t_0 as given in Chapter 1 by (1.14). We now regard $p(t, \epsilon)$, the solution of (6.33), as a function $\tilde{p}(t_0, t_1, \epsilon)$ of both t_0 and t_1, that is $\tilde{p}(t_0, t_1, \epsilon) \equiv p(t, \epsilon)$. Applying the chain rule gives

$$\frac{d\tilde{p}}{dt} = g'(t_1)\, D_0\tilde{p} + \epsilon D_1\tilde{p}. \tag{6.36}$$

Substituting (6.36) into (6.33), gives the multiscaled equation:

$$g'(t_1)D_0\tilde{p} + \epsilon D_1\tilde{p} = \tilde{p}\,(1 - \tilde{p}) - \sigma\, h(t_1)\frac{\tilde{p}}{(1 + \tilde{p})}, \tag{6.37}$$

for the unknown function $\tilde{p}(t_0, t_1, \epsilon)$, where D_0 and D_1 denote partial derivatives taken with respect to t_0 and t_1. Note that ϵ is displayed explicitly in (6.37) rather than implicitly as in (6.33) and by following the same strategy as in previous chapters we employ a perturbation technique to approximate the solution of (6.33) that is valid for all $t > 0$.

In terms of $\tilde{p}$, the initial condition $p(0, \epsilon) = \mu$ becomes

$$\tilde{p}(0, 0, \epsilon) = \mu. \tag{6.38}$$

6.4.2 Perturbation Analysis

Defining $\tilde{p}(t_0, t_1, \epsilon)$ as a Poincaré expansion in ϵ gives

$$\tilde{p}(t_0, t_1, \epsilon) = \tilde{p}_0(t_0, t_1) + \epsilon \tilde{p}_1(t_0, t_1) + \epsilon^2 \tilde{p}_2(t_0, t_1) + \dots . \tag{6.39}$$

Substituting (6.39) into (6.37), expanding in powers of ϵ and equating coefficients of like powers of ϵ yields a non-linear partial differential equation for $\tilde{p}_0(t_0, t_1)$ as

$$g'(t_1) D_0 \tilde{p}_0 = \tilde{p}_0 \; (1 - \tilde{p}_0) - \sigma \, h(t_1) \frac{\tilde{p}_0}{(1 + \tilde{p}_0)}, \tag{6.40}$$

and linear partial differential equation for $\tilde{p}_1(t_0, t_1)$ as

$$g'(t_1) D_0 \tilde{p}_1 - \left(1 - 2\tilde{p}_0 - \frac{\sigma \, h(t_1)}{(1 + \tilde{p}_0)^2} \right) \tilde{p}_1 = -D_1 \tilde{p}_0. \tag{6.41}$$

Solving (6.40) gives the implicit solution for $\tilde{p}_0(t_0, t_1)$ as

$$\tilde{p}_0^{\frac{-2}{1+Q(t_1)}} \; (\tilde{p}_0 - Q(t_1)) \; (\tilde{p}_0 + Q(t_1))^{\frac{1-Q(t_1)}{1+Q(t_1)}} - A(t_1) \, e^{-\beta(t_1) \, t_0} = 0, \tag{6.42}$$

where $A(t_1)$ is an arbitrary function depending only on t_1,

$$\beta(t_1) = \frac{2}{g'(t_1)} \frac{1 - \sigma \, h(t_1)}{1 + Q(t_1)}, \tag{6.43}$$

and

$$Q(t_1) \quad = \quad \sqrt{1 - \sigma \; h(t_1)}. \tag{6.44}$$

As we found in Chapter 5, Section 5.3.2, we have obtained an implicit representation (6.42) for $\tilde{p}_0(t_0, t_1)$, and this implicit expression cannot be used to find the solution of the linear differential equation (6.41) for $\tilde{p}_1(t_0, t_1)$. Since, as t_0 gets large, the term $e^{-\beta(t_1) \, t_0}$ converges to zero, it is reasonable to find the explicit approximate expansions for $\tilde{p}_0(t_0, t_1)$ as $t_0 \to \infty$ which can be used to estimate the limiting state of $\tilde{p}_1(t_0, t_1)$. To do this, we will consider two separate cases in the following sections.

6.4.3 The Subcritical Surviving Case, $\sigma\, h(t_1) < 1$

In a similar way to the calculations of Section 5.3.2, we expand the implicit solution (6.42) in terms the small term of $e^{-\beta(t_1)\, t_0}$, $\beta(t_1) > 0$, to obtain an expression that tends to the surviving state $Q(t_1)$ as $t_0 \to \infty$. We now use this expansion to find the limiting form of $\tilde{p}_1(t_0, t_1)$ as $t_0 \to \infty$.

A suitable asymptotic expansion is

$$\tilde{p}_0(t_0, t_1) = f_0(t_1) + f_1(t_1)\, e^{-\beta(t_1)t_0} + f_2(t_1)\, e^{-2\beta(t_1)t_0} + O(e^{-3\beta(t_1)t_0}). \tag{6.45}$$

Substituting (6.45) into (6.42) and expanding and equating the coefficients of like powers of $e^{-\beta(t_1)\, t_0}$ gives the expansion for $\tilde{p}_0(t_0, t_1)$ as $t_0 \to \infty$ in the form

$$\tilde{p}_0(t_0, t_1) = Q(t_1) + A(t_1) \left[\frac{(2\, Q(t_1))^{1-Q(t_1)}}{(Q(t_1))^2} \right]^{-1/(1+Q(t_1))} e^{-\beta(t_1)t_0} + O(e^{-2\ \beta(t_1)t_0}), \tag{6.46}$$

where $A(t_1)$ is the undetermined function of t_1 of (6.42).

For the solutions of (6.41) that tend to $Q(t_1)$ as $t_0 \to \infty$, we attempt to obtain an approximation p_1^* to $\tilde{p}_1$ by replacing $\tilde{p}_0$ in the coefficients of $\tilde{p}_1$ on the left hand side of (6.41) by $Q(t_1)$, the first term of (6.46); and by the first two terms of (6.46) on the right hand side. These changes convert (6.41) to the linear differential equation

$$\begin{aligned} &D_0 p_1^* + \beta(t_1)\, p_1^* \\ &\qquad = \frac{-1}{g'(t_1)}\, D_1 \left[Q(t_1) + \left(A(t_1) \left(\frac{(2Q(t_1))^{1-Q(t_1)}}{(Q(t_1))^2} \right)^{-1/(1+Q(t_1))} e^{-\beta(t_1)t_0} \right) \right]. \end{aligned} \tag{6.47}$$

Solving (6.47) leads to

$$\begin{aligned} p_1^*(t_0, t_1) = &-\frac{Q'(t_1)}{g'(t_1)\, \beta(t_1)} - t_0 \left(A(t_1) \left[\frac{(2Q(t_1))^{1-Q(t_1)}}{(Q(t_1))^2} \right]^{-1/(1+Q(t_1))} \right)' e^{-\beta(t_1)t_0} \\ &+ \frac{t_0^2}{2} \beta'(t_1)\, A(t_1) \left[\frac{(2\, Q(t_1))^{1-Q(t_1)}}{(Q(t_1))^2} \right]^{-1/(1+Q(t_1))} e^{-\beta(t_1)t_0}. \end{aligned} \tag{6.48}$$

Following analogous arguments to those used in Section 5.3.3, we eliminate the coefficients of t_0 and t_0^2 by choosing

$$\left(A(t_1)\,\left[\frac{(2Q(t_1))^{1-Q(t_1)}}{(Q(t_1))^2}\right]^{-1/(1+Q(t_1))}\right)' = 0, \tag{6.49}$$

$$\beta'(t_1) = 0. \tag{6.50}$$

From (6.49), we get

$$A(t_1) = c_s\; a_s(t_1), \tag{6.51}$$

where

$$a_s(t_1) = \left(\frac{(Q(t_1))^2}{(2Q(t_1))^{1-Q(t_1)}}\right)^{-1/(1+Q(t_1))}, \tag{6.52}$$

and c_s is an undetermined constant, while subscript 's' refers to survival (or subcritical). From (6.50), we choose $\beta(t_1) = 1$. Thus the time scale t_{0s} is then given by

$$t_{0s} = \frac{2}{\epsilon}\int_0^{t_1} \frac{Q^2(s)}{Q(s)+1}ds. \tag{6.53}$$

Thus the leading term expansion for the surviving population included in (6.42) is given by

$$F(p_{0s}, t) - c_s\; a_s(t_1)\, e^{-t_{0s}} = 0, \tag{6.54}$$

where

$$F(u,t) = u^{\frac{-2}{1+Q(t_1)}}\,(u - Q(t_1))\;\left(\frac{u + Q(t_1)}{2\,Q(t_1)}\right)^{\frac{1-Q(t_1)}{1+Q(t_1)}}, \tag{6.55}$$

and $a_s(t_1)$ is as defined by (6.52).

Thus, for large t_0, with the choices (6.49) and (6.50), the second leading term (6.48) tends to

$$p_1(t,\epsilon) \to -\epsilon\,\frac{Q'(\epsilon t)\,(1 + Q(\epsilon t))}{2\,Q(\epsilon t)^2},\;\; t_1 = \epsilon\; t. \tag{6.56}$$

Applying the initial condition (6.3) into (6.54) gives the constant c_s as

$$c_s = (\mu - Q(0))\left(\frac{\mu}{Q(0)}\right)^{\frac{-2}{1-Q(0)}}\;\left(\frac{\mu + Q(0)}{2\,Q(0)}\right)^{\frac{1-Q(0)}{1+Q(0)}}. \tag{6.57}$$

Thus, for small and positive ϵ, we construct an implicit solution (6.54) of the leading term $p_{0s}(t_0, t_1)$ and as $t_0 \to \infty$, the population $p_s(t, \epsilon)$ survives to the limiting state

$$p_s(t, \epsilon) \to Q(\epsilon t) - \epsilon \, \frac{Q'(\epsilon t)\,(1 + Q(\epsilon t))}{2\,Q(\epsilon t)^2}; \tag{6.58}$$

where the function $Q(\epsilon t)$ is defined by (6.44), for arbitrary parameter $h(\epsilon t)$ satisfying (6.34).

- **A Comparison Between the Leading Term Expansion and a Numerical Solution** :

Here, to test the accuracy of our approximate expansion we compare it with the numerical solution of (6.42), where the carrying capacity is taken as a constant and the harvesting function $h(\epsilon t)$ varies periodically with time and is given by

$$h(\epsilon t) = 1 + 0.02 \sin(\epsilon t), \tag{6.59}$$

where $\epsilon = 0.02, \;\; \sigma = 0.5$ and the condition (6.34) holds. The close agreement between the approximate solution for the surviving population and the numerical solution of (6.3) can be seen in all of Figures 6.9-6.12 for short or long time frames. In these Figures the population with different initial values survives to the limiting state (6.58), with

$$Q(\epsilon t) = \sqrt{1 - \sigma h(\epsilon t)}$$

where $h(\epsilon t)$ is given by (6.59). Figure 6.13 shows the effects of increasing ϵ on the accuracy of our approximation compared with the numerical result. It shows that agreement is very good up to $\epsilon = 0.2$ which is probably not be regarded as small.

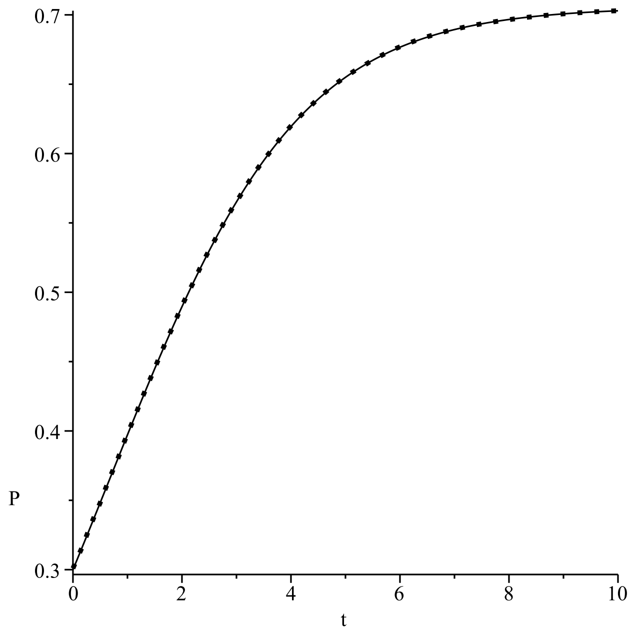

Figure 6.9: The leading term approximation of surviving population (6.54) (dotted) and the numerical solution of (6.3)(solid) using the parameters (6.59) where $\epsilon = 0.02$, $\sigma = 0.5$ with $\mu = 0.3$.

- **Logistically Varying Harvesting** :

We assume the harvesting term arises as the solution of a logistic problem

$$\frac{dh(t_1)}{dt_1} = r\, h(t_1)\left(1 - \frac{h(t_1)}{k}\right), \qquad h(0) = \lambda, \tag{6.60}$$

where r, k and λ are given positive constants.

Here, the slowly varying harvesting or predation arise from a slowly evolving external second population that is unaffected by the first population. Thus, (6.33) and (6.60) are weakly coupled.

Solving (6.60) for $h(t_1)$ gives

$$h(t_1) = \frac{k}{1 + (k/\lambda - 1)e^{-r\,t_1}}, \qquad t_1 = \epsilon\, t, \tag{6.61}$$

where

$$\lambda < h(t_1) < k.$$

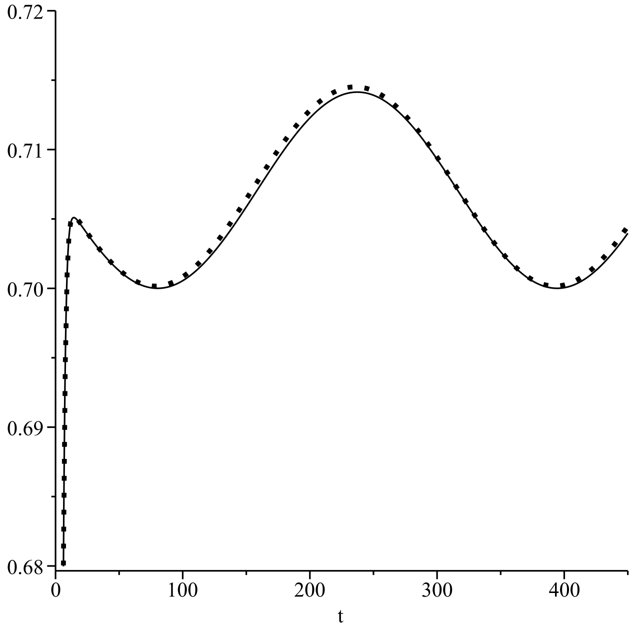

Figure 6.10: Using same choices as in Figure 6.9 but for large t, a comparison of the approximate solution (6.54) (dotted) and the numerical solution of (6.3)(solid).

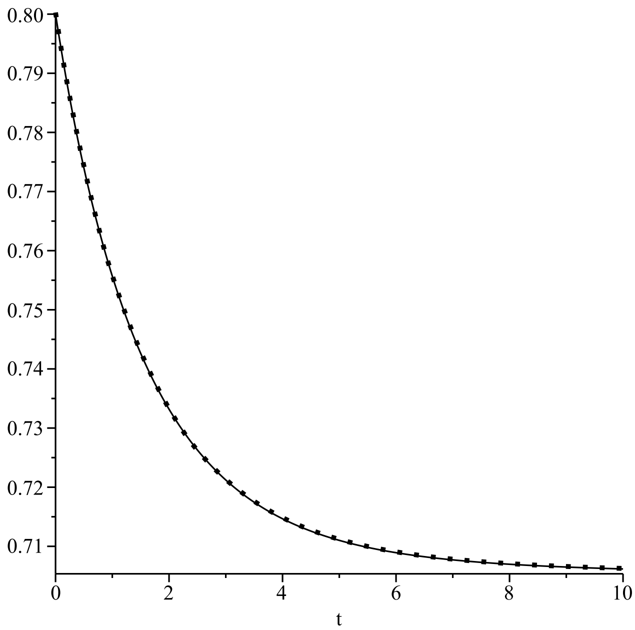

Figure 6.11: Comparison of the surviving population (6.54) (dotted line) with numerical solution of (6.3)(solid) where σ, $h(\epsilon t)$ as are in Figure 6.9, while $\mu = 0.8$.

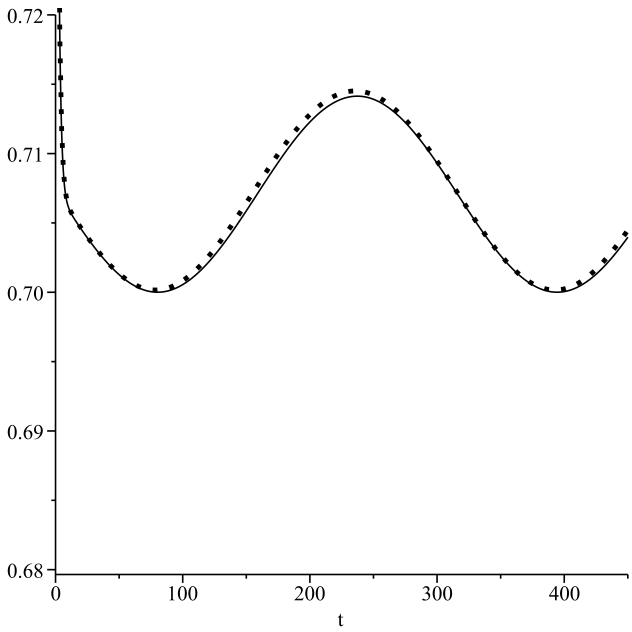

Figure 6.12: Comparison of the surviving population (6.54)(dotted) with numerical solution (solid) of (6.3) where σ, μ and $h(\epsilon t)$ are as in Figure 6.11 .

Then, the condition for subcritical harvesting (6.34)) is met if we choose $\sigma k < 1$ since then,

$$\sigma\lambda < \sigma\ h(t_1) < \sigma k, \quad t_1 \geq 0.$$

For simplicity, we choose $r = 1$, $k = 1$, $\lambda = 0.3$ and with $\sigma = 0.75$ we ensure that the (6.34)) condition holds.

The logistic harvesting function (6.61) is shown in Figure 6.14 with the above choices. In this figure the population with the initial value $\mu = 0.3$ survives to the limiting state (6.58), with

$$Q(\epsilon t) = \sqrt{1 - \sigma h(\epsilon t)}$$

where $h(\epsilon t)$ is given by (6.61). It demonstrates the accuracy of our leading term approximation which compares well with the numerical solution of the problem.

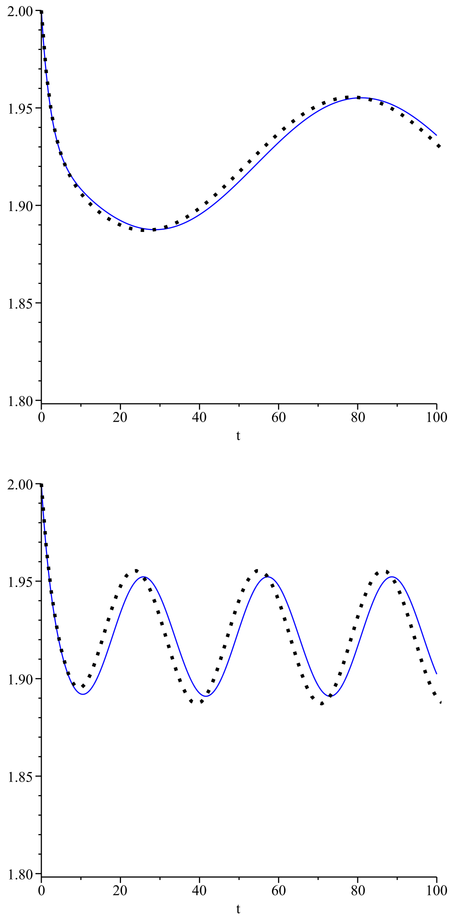

Figure 6.13: *Evolution of a surviving population* (6.54)*(dotted) compared with the numerical solution of* (6.3) *for data used in Figure 6.11, where* $\mu = 2.0$ *and* $\epsilon = 0.06, 0.08, 0.2$ *in clockwise order starting from the top left corner.*

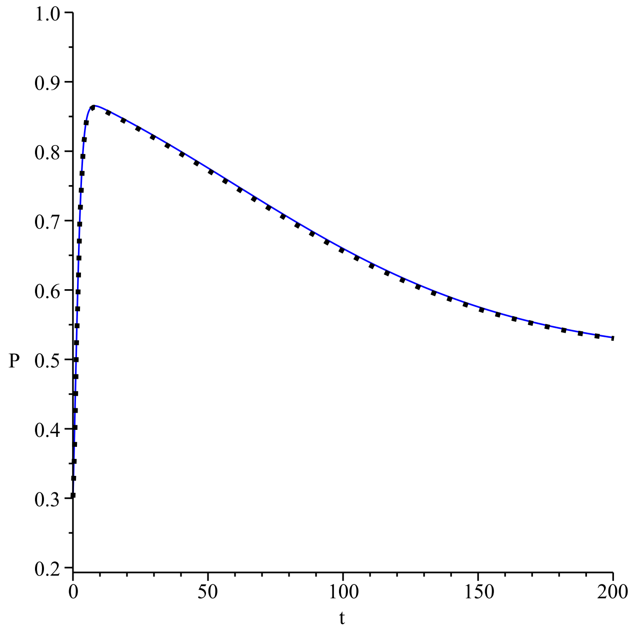

Figure 6.14: Comparison of the surviving population (6.54)(dotted) with numerical solution of (6.3) (solid) where $h(\epsilon t)$ satisfies (6.61)) with $r = 1$, $k = 1$, $\lambda = 0.75$, $\sigma = 0.3$ and $\mu = 0.3$.

6.4.4 The Extinction Case, $\sigma\, h(t_1) > 1$

By similar analysis to that of Section 6.4.2, solving the leading term equation (6.40) where the condition (6.35) holds, gives

$$\left(\frac{\tilde{p}_0^2}{\tilde{p}_0^2 + \sigma\, h(t_1) - 1}\right) e^{\omega(\tilde{p}_0)} - A(t_1)\, e^{-\beta(t_1)\ t} = 0, \tag{6.62}$$

where

$$\omega(\tilde{p}_0) = 2\sqrt{\sigma\ h(t_1) - 1}\, \arctan\left(\frac{\tilde{p}_0}{\sqrt{\sigma\ h(t_1) - 1}}\right), \tag{6.63}$$

$$\beta(t_1) = \frac{2}{g'(t_1)}\, (\sigma\, h(t_1) - 1) > 0, \tag{6.64}$$

and $A(t_1)$ is an arbitrary function of t_1.

By similar analysis to that of the previous section, an expansion for the solution $\tilde{p}_0$ in (6.62) that tends to zero as $t_0 \to \infty$ is given by

$$p_0(t_0, t_1) = [A(t_1)(\sigma\, h(t_1) - 1)]^{1/2}\ e^{-\frac{\beta(t_1)}{2}\ t_0} + O(e^{-\beta(t_1)\ t_0}). \tag{6.65}$$

Applying the same technique as in Section 6.4.3, to the solutions of (6.41) that tend to zero as $t_0 \to \infty$, we find an approximation $\tilde{p}_1^*$ to $\tilde{p}_1$ by considering $\tilde{p}_0 = 0$ (in the first term of (6.65)) in the coefficients of $\tilde{p}_1$ on the left hand side of (6.41), and by the first two terms of (6.65) on the right hand side. Thus, the differential equation (6.41) becomes

$$D_0\tilde{p}_1^* + \beta(t_1)\tilde{p}_1^* = \frac{-1}{g'(t_1)} D_1 \left[[A(t_1)(\sigma\, h(t_1) - 1)]^{1/2} \; e^{-\frac{\beta(t_1)}{2}\; t_0} \right] \tag{6.66}$$

Solving (6.66) for $\tilde{p}_1^*$, we get

$$\tilde{p}_1^* = -\frac{e^{-\frac{\beta(t_1)}{2} t_0}}{g'(t_1)} \left[([A(t_1)(\sigma\, h(t_1) - 1)]^{1/2})'\; t_0 - [A(t_1)(\sigma\, h(t_1) - 1)]^{1/2} \;\beta'(t_1)\; \frac{t_0^2}{4} \right]. \tag{6.67}$$

To remove the t_0 and t_0^2 terms, in the usual way, we choose

$$([A(t_1)(\sigma\, h(t_1) - 1)]^{1/2})' = 0, \tag{6.68}$$

$$(\beta(t_1))' = 0, \tag{6.69}$$

to give

$$A(t_1) = d_e \; a_e(t_1) \tag{6.70}$$

$$a_e(t_1) = (\sigma\, h(t_1) - 1)^{-1} \tag{6.71}$$

where d_e is an arbitrary constant and subscript 'e' denotes extinction.

From (6.69), we put

$$\beta(t_1) = 1,$$

that gives

$$t_{0e} = \tfrac{1}{\epsilon} \int_0^{t_1} (\sigma h(s) - 1)\, ds. \tag{6.72}$$

Thus, substituting (6.70) into (6.42), we obtain the leading term of an approximate solution for extinction case as

$$F(p_{0e}, t) - d_e \; a_e(t_1)\, e^{-t_{0e}} = 0, \tag{6.73}$$

where

$$F(u,t) = \frac{u^2}{u^2 + \sigma\ h(t_1) - 1}\ e^{\omega(u)}, \tag{6.74}$$

while $\omega(u)$ and $a_e(t_1)$ are defined by (6.63) and (6.71) respectively.

Substituting the initial condition (6.3) into (6.73) gives the arbitrary constant d_e as

$$d_e = (\sigma\, h(0) - 1)\ \left(\frac{\mu^2}{\mu^2 + \sigma\ h(0) - 1}\right)\ e^{\omega(\mu)} \tag{6.75}$$

Thus the implicit leading term (6.73) represents the behaviour of the population that is driven to extinction, that is $p_e(t,\epsilon) \to 0$ as $t \to \infty$.

- **A Comparison Between the Leading Term Expansion of Extinction Population and a Numerical Solution** :

The behaviour of the extinction population represented by (6.73), where $p(t,\epsilon) \to 0$, is shown in Figure 6.15. There is almost exact agreement with the numerical solution of (6.14) for given parameters as given in Figure 6.15. Also it can be observed, in this Figure, that the population will be driven to zero in infinite time.

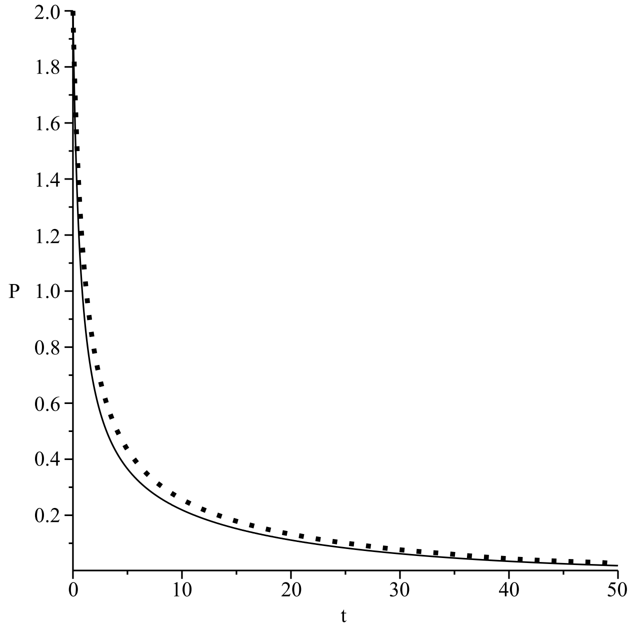

Figure 6.15: Evolution of the leading term solution of the extinguished population (dotted) given by (6.73) and numerical solution of (6.3)(solid) for $\sigma = 1.05$, $h(\epsilon t) = 1.0 + 0.01\sin(\epsilon\, t)$ with $\mu = 2$ and $\epsilon = 0.02$.

6.5 Transition Analysis

We have approximated the solutions of the problem (6.33) when only the harvesting term $h(t_1)$ was slowly varied. We found leading order expressions for the surviving and extinction cases of (6.33) when the condition (6.34) and (6.35) hold respectively. However, as in similar discussions in Chapters 4 and 5, the population may change its behaviour from survival to extinction through a transition region, i.e., in any neighborhood of any point $\bar{t}_1$ where

$$\sigma\ h(\bar{t}_1) = 1; \qquad \text{or} \qquad Q(\bar{t}_1) = 0, \tag{6.76}$$

for then $a_s(t_1)$ in (6.54), given by (6.52) is undefined, so that the surviving state solution p_{0s} fails to be defined. Thus, we reanalyse the solutions of (6.33) in a neighborhood of an isolated transition point $\bar{t}_1$ on $t \geq 0$.

Defining $\delta(t_1)$ by

$$\delta(t_1) = 1 - \sigma\ h(t_1), \tag{6.77}$$

we parallel the approach of Chapter 5 and divide $t \geq 0$ $(t_1 \geq 0)$, into three regions:

Region 1, where $0 \leq t_1 < \bar{t}_1$, or survival region, where $\delta(t_1) > 0\,(\sigma\ h(t_1) < 1)$, and the behaviour of the population is modelled by the implicit leading order solution p_{0s}, (6.54), that tends to the limiting state (6.58);

Region 2, a transition region, which surrounds the transition point $t_1 = \bar{t}_1$; where $\delta(\bar{t}_1) = 0$;

Region 3, $\bar{t}_1 < t_1 < \infty$, or extinction region, where $\delta(t_1) < 0$, $(\sigma\ h(t_1) > 1)$ and the leading term approximation to the population is modified to be

$$F(p_{0e}, t) - d\ a_e(t_1)\, e^{-t_{0e}} = 0, \tag{6.78}$$

where the initial point of $\tilde{t}_{0e}$ is replaced with $\bar{t}_1$, so that

$$\tilde{t}_{0e} = -\frac{1}{\epsilon}\int_{\bar{t}_1}^{t_1} \delta(s). \tag{6.79}$$

and d is undetermined constant.

We further assume that, while $\delta(\bar{t}_1) = 0$,

$$\delta'(\bar{t}_1) < 0, \tag{6.80}$$

so the zero of δ at $\bar{t}_1$ is simple.

6.5.1 Solution within the Transition Region

Following the analysis of Chapter 4, we define a new variable τ by

$$t_1 = \bar{t}_1 + \epsilon^{\alpha}\,\tau \tag{6.81}$$

for $\alpha > 0$, and write the solution of (6.33), as $\tilde{p}(\tau, \epsilon)$, where

$$\tilde{p}(\tau, \epsilon) = p(\bar{t}_1 + \epsilon^{\alpha}\tau, \epsilon). \tag{6.82}$$

Noting (6.77) we substitute into the differential equation (6.33), and expand in powers of ϵ to obtain

$$\frac{d\tilde{p}}{d\tau} = \epsilon^{\alpha-1}\left[\tilde{p} - \tilde{p}^2 - \frac{\tilde{p}}{1+\tilde{p}}\right] + \epsilon^{2\alpha-1}\left[\frac{(\delta'(\bar{t}_1)\tau)\tilde{p}}{1+\tilde{p}}\right]. \tag{6.83}$$

Similarly, we look at the expansion for the surviving limiting state (6.58) in terms of τ to estimate the α value. This gives

$$p(\tau,\epsilon) \to \sqrt{-\delta'(\bar{t}_1)\,\tau}\,\epsilon^{\frac{\alpha}{2}} + O(\epsilon), \tag{6.84}$$

and suggests that the solution of the transition equation has the form

$$p(\tau,\epsilon) = u_0(\tau)\,\epsilon^{\frac{\alpha}{2}} + O(\epsilon^{\alpha}). \tag{6.85}$$

Substituting (6.85) into (6.83) gives

$$\alpha = \frac{1}{2} \tag{6.86}$$

and the leading term of transition equation as

$$\frac{du_0(\tau)}{d\tau} = \delta'(\bar{t}_1)\ \tau\ u_0(\tau) - u_0(\tau)^3. \tag{6.87}$$

Solving (6.87) gives

$$u_0(\tau) = \left[\frac{\sqrt{-\delta'(\bar{t}_1)}\ e^{\delta'(\bar{t}_1)\ \tau^2}}{\mathrm{erf}(\sqrt{-\delta'(\bar{t}_1)}\tau)\ \sqrt{\pi} + C\ \sqrt{-\delta'(\bar{t}_1)}}\right]^{1/2}, \tag{6.88}$$

where erf(..) is the error function, (see [1]) .

Thus, to match Region 1 with 2, we choose

$$C = \frac{\sqrt{\pi}}{\sqrt{-\delta'(\bar{t}_1)}}$$

and expanding the solution (6.88) for large and negative τ gives

$$\epsilon^{1/4}\ u_0(\tau) \to \epsilon^{1/4}(\sqrt{-\delta'(\bar{t}_1)\ \tau} - \frac{1}{4\sqrt{-\delta'(\bar{t}_1)}\tau^{3/2}} + \ldots) \qquad \text{as} \qquad \tau \to -\infty \tag{6.89}$$

so that the common part of the solution between the Regions 1 and 2 is

$$\epsilon^{1/4}\sqrt{-\delta'(\bar{t}_1)\ \tau}, \qquad \delta'(\bar{t}_1) < 0. \tag{6.90}$$

Thus, by using matching technique, the uniformly valid expansion for the solution over Region 1 and Region 2 has the form

$$p_{1\,2} = p_{0s} + \epsilon^{1/4}\left[\frac{\sqrt{-\delta'(\bar{t}_1)}\ e^{\delta'(\bar{t}_1)\ \tau^2}}{\sqrt{\pi}(\mathrm{erf}(\sqrt{-\delta'(\bar{t}_1)}\tau)+1)}\right]^{1/2} - \epsilon^{1/4}(\sqrt{-\delta'(\bar{t}_1)\ \tau}) + \ldots \tag{6.91}$$

where p_{0s} is defined implicitly by (6.55).

We note that, as $\tau \to -\infty$ in (6.91), the $\epsilon^{1/4}$ terms cancel, and the composite solution is given by p_{0s}, the surviving implicit solution (6.55), as expected. Moreover, in the transition region (where $t_{0s} = \infty$), $p_{0s} \to \epsilon^{1/4}\sqrt{-\delta'(\bar{t}_1)\ \tau}$, and so the solution is represented by $\epsilon^{1/4}\ u_0(\tau)$, again, as expected.

The expansion (6.91) is assumed to hold up to $t_1 = \bar{t}_1$, the transition point. To obtain an approximation to the solution on $t_1 > \bar{t}_1$; i,e., Region 3, we note that in Region 3, the extinguishing solution is approximated, to leading order, by p_{0e}, defined by (6.78).

If we follow analysis that is completely analogous to that of Section 5.5.2, we find that to match this solution with $\epsilon^{1/4}\ u_0(\tau)$ as $\tau \to \infty$, we must choose $d = 0$. Thus, to leading order, solutions in Region 3 are represented by the slowly varying solutions of

$$f(p_{0e}, t) = 0.$$

One of these is $p_{0e} = 0$, so we choose the expansion in Region 3 to have the limiting value zero. This just the limiting value of the transition solution, $\epsilon^{1/4}\ u_0(\tau)$, so in Regions 2 and 3 the combined uniform expansion is

$$\epsilon^{1/4}\left[\frac{\sqrt{-\delta'(\bar{t}_1)}\ e^{\delta'(\bar{t}_1)\ \tau^2}}{\sqrt{\pi}(\mathrm{erf}(\sqrt{-\delta'(\bar{t}_1)}\tau)+1)}\right]^{1/2}, \tag{6.92}$$

that is, the transition solution itself. As noted in Chapter 5, this phenomenon has also been observed in a related transition problem [30].

6.6 Discussion

As in Chapters 2 and 5, we have been able to apply the multi-timing method to obtain approximations to the solutions of the harvested model problem (6.26); or, in fact, its dimensionless form (6.27). Because of the implicit nature of solutions for this, we could only obtain a leading order approximation. However, as the excellent comparisons with numerical solutions show, we may place considerable faith in the approximations.

A simplified version of the problem has been considered here - namely that where the "*harvesting coefficient*" $H(T)$ (or $h(t)$) was slowly varying. This is assumed to be most relevant to physical reality. However, the analysis could be extended to the case where some, or all of the other parameters R, K and A varied slowly in time, on the same (longer) time scale as H. This would simply result in more complex algebra, but the results would be comparable. The number and the definition of dimensionless parameters in (6.27) would change (see the discussion about nondimensionalisation at the beginning of Chapter 2).

Analysis of the dimensionless constant coefficient problem (6.5) in Section 6.3 showed that there is a variety of possible solution types (survival or extinction) as the parameters η and σ varied in the (η, σ) plane, with solution type changes at various points. For the related slowly varying problem (6.3), we expect these phenomena to again occur, with dramatically increased complexity in solution type and type changes. Because of the complexity of this situation, transitions in solution type have only been considered in the simplified slowly varying problem (6.33). The analysis of more general problem (6.3) has been left for future work.

In this chapter, we have concentrated on harvesting (or predation) characterised by a type II Holling term. However, the techniques used here would be relatively easily adapted to a type III Holling term, as applied in the analysis of the spruce budworm problem (see Murray [48]). Again, this points to future work.

The analysis presented in this chapter is being prepared for publication in Idlango et al [37].

Chapter 7

Conclusion

This dissertation has examined several single species population models involving ordinary differential equations where the model parameters are assumed to be functions of a single slow time. Analytic (explicit or implicit) approximate solutions have been obtained by employing the straightforward multi-scaling method by which the differential equation is reformulated as a partial differential equation involving a small positive parameter ϵ. This partial differential equation can then be solved using a perturbation method. The method as used here involved two different scales of time, the slow time scale and a normal one, with the small parameter ϵ measuring the ratio of the first to the second.

While the multiscaling perturbation method is a powerful and logical technique, sometimes the complexity of the mathematical equations involved and the algebraic steps needed in the solution process cause the application of this method to solving problems to become very complicated. In such cases, using software packages such as Maple and Mathematica can indeed reduce this complexity of formal calculations and eliminate errors in the algebra. This has been the case in the calculations needed to complete the analysis of this dissertation.

The analytic approximations obtained in all the problems studied in Chapters 2, 5 and 6 provide general representations for a general range of model parameters under certain reasonable conditions in each case. This contrasts with the results obtained using numerical methods of solution, that are only valid for particular (numerical) choices of data.

Although the approximations that were constructed using the multiscaling method failed to represent the solution of the the problem in regions termed *transition regions*, a separate analysis in such regions using the matched expansions method created valid composite approximations for the solution over the domain of definition of the problem. This combination of the "global" multiscaling method with the "local" matched expansions method is a powerful combination that has proved to be very successful.

The accuracy of our results have compared very well with the results of the numerical computations for different choices of slow time parametric variation. While these approximation techniques are very successful, we note that they are a formal process only, being based on the assumption of a unique exact solution of the original problem, for which the approximation constructed is indeed a good approximation. When the problem is not too complicated, powerful theoretical techniques can be used to establish the existence of a unique solution to the problem that is close to the formal approximation. This was done using contraction mapping proof in Chapter 3 for the harvesting model. While we note that such a proof might be more complex for the approximations of Chapters 5 and 6, it seems clear that the success of the proof of Chapter 3 depended on the characteristics of the linear map T given by (3.13). These, in turn, depended on the properties of the leading approximation, p_0. Thus, if similar properties of the corresponding linear maps relevant to the problems of the Chapters 5 and 6 could be established (based on the relevant p_0), a proof might be successfully executed. Moreover, the nature of the exact solution might depend heavily on the background data, as in Chapter 3.

Carrying out the construction of the approximations of Chapters 2, 5 and 6 leads us to the significant observation that the multiscaling method will yield useful results whenever the corresponding constant parameter problem can be solved, either explicitly or implicitly. For then, when slowly varying parameters are involved, a leading order approximation can be constructed, directly, as in Chapter 2, by eliminating undesirable terms; or indirectly, as in Chapters 5 and 6, by eliminating terms generated by the limiting form of the (implicit) leading term in the expansion.

This leads to the extension of this technique to models where the constant parameter form has an exact solution, whether explicit or implicit. A comprehensive collection of such solutions can be found in Tsoularis and Wallace [61].

The only limitation is the complexity of the analysis involved. As in this thesis, local analysis using matched expansions could deal with any transitions arising.

One other extension to consider would be to look at the situation where the time scales of parameter variation were not all the same. Thus, for example, in the case of the harvesting model, Section 2.1, it would not be true that $T_R = T_K = T_H$. Then, if all these time scales were long relative to T^*, the intrinsic time scale of the the model (R^{-1} in (2.2)), there would be three small parameters,

$$\epsilon_R = (T_R\, R_0)^{-1} \quad \epsilon_K = (T_K\, K_0)^{-1} \quad \epsilon_H = (T_H\, H_0)^{-1}$$

and three slow times,

$$t_{1R} = \epsilon_R t \quad t_{1K} = \epsilon_K t \quad t_{1H} = \epsilon_H t$$

with approximate "normal″ times t_{0R}, $t_{0K}\, t_{0H}$ defined in terms of functions g_R, g_K and g_H. Clearly, the complexity of the analysis would increase dramatically. However, in some cases, a hierarchy might be established in terms of one small parameter, $\epsilon = \epsilon_R$, say, so that

$$t_{1R} = \epsilon\, t, \quad t_{1K} = \epsilon^2\, t, \quad t_{1H} = \epsilon^3\, t,$$

for example, giving a clear separation in the time scales of variation in R, K and H. In this case, the analysis, although complicated, might be tractable, yielding a solution depending on the four time scales t_0, t_{1R}, t_{1K} and t_{1H}.

While only one-dimensional problems have been considered in this thesis, it is clear that the multiscaling approach may be adapted to systems modelling population growth - for example, predator-prey and competing species. In such cases, the analysis would be of a higher order of magnitude in difficulty, but, in particular cases, could be profitable. Two distinct situations are possible. One arises when all the defining parameters of the system are slowly varying, due to external factors. This might be the more difficult to analyse.

The second occurs when the cross linking between the component model differential equations is weak, due, for example, to a slower growth rate of one species. This is seen in the second example of Section 6.4.3 (6.60), where a slowly varying harvesting (or predation) is introduced into a Holling type II evolving population from a slowly varying logistic source.

An analogous predator-prey model, with slowly growing predator has also been treated by multiscaling techniques - see Grozdanovski [26].

While the model considered in Chapter 6 in a single species one, it may be considered as a sub-case of a two dimensional system in which harvesting (or predation) arises from a second slowly varying source. Thus, (6.33) could be seen as one component of Holling-Taner type predator-prey model (see Tanner [60] , or Arrowsmith [4]). If the predator showed slow growth, the analysis of Chapter 6 would be relevant to a study of such model.

A similar approach could be applied to study the full spruce budworm problem [42] where the predation by birds could be viewed as relatively slowly changing.

Bibliography

[1] Abramowitz, M., and Stegun, I. *Handbook of Mathematical Functions.* National Bureau of Standards, Washington D.C., 1964.

[2] Allee, W. Animal aggregations: A study in general sociology. *University of Chicago Press* (1931). Chicago.

[3] Allee, W. C., Emerson, A, E., and Park, O. *Principles of Animal Ecology.* Saunders Co., Philadelphia, 1949.

[4] Arrowsmith, D. K., and Place, C. M. *Dynamical Systems: Differential Equations, Maps, and Chaotic Behaviour.* Chapman & Hall, 1992.

[5] Banks, R. *Growth and Diffusion Phenomena: Mathematical Frameworks and Application.* Springer-Verlag, Berlin, Germany, 1994.

[6] Bazykin, A., Khibnik, A., and Krauskopf, B. *Nonlinear Dynamics of Interacting Populations.* World Scientific, 1998.

[7] Beddington, J., and May, R. Harvesting natural populations in a randomly fluctuating environment. *Science 197(4302)* (1977), 463–465.

[8] Benardete, D. M., Noonburg, V. W., and Pollina, B. Qualitative tools for studying periodic solutions and bifurcations as applied to the periodically harvested logistic equation. *American Mathematical Monthly 115*, 3 (2008), 202–219.

[9] Berger, M. S. *Nonlinearity and Functional Analysis.* Springer-Verlag, New York, 1983.

[10] Berryman, A. A. The orgins and evolution of predator-prey theory. *Ecology* (1992), 1530–1535.

[11] BOUKAL, D. S., and BEREC, L. Single-species models of the Allee effect: Extinction boundaries, sex ratios and mate encounters. *Journal of Theoretical Biology 218*, 3 (2002), 375 – 394.

[12] Brauer, F., and Castillo-Chavez, C. *Mathematical Models in Population Biology and Epidemiology.* Springer, 2001.

[13] Brauer, F., and Snchez, D. A. Constant rate population harvesting: Equilibrium and stability. *Theoretical Population Biology 8*, 1 (1975), 12–30.

[14] Braun, M. S. Coleman, C. S., Drew, D., and Lucas, W. *Modules in Applied Mathematics: Volume 1: Differential Equation Models.* Academic press, New York, 1977.

[15] Bush, A. *Perturbation Methods for Engineers and Scientists.* CRC Press, USA, 1992.

[16] Cooke, K. L., and Witten, M. One-dimensional linear and logistic harvesting models. *Mathematical Modelling 7* (1986), 301–340.

[17] Courchamp, F., Clutton-Brock, T., and Grenfell, B. Inverse density dependence and the Allee effect. *Trends in Ecology and Evolution 14*, 10 (1999), 405–410.

[18] Cromer, T. Harvesting in a seasonal environment. *Mathematical and Computer Modelling 10*, 6 (1988), 445–450.

[19] Edelstein-Keshet, L. *Mathematical Models in Biology.* SIAM, USA, 2005.

[20] Gause, G. *The Struggle for Existance.* Hafner Publishing, New York, USA, 1969.

[21] Ghansah, B., de Almeida, P., Palmeira, C., and Faria, L. An introduction to dynamical systems applied to exploited populations. *Oecologia Australis 16*, 1 (2012), 63–80.

[22] Gonzalez-Olivares, E., Gonzalez-Yeaz, B., Mena-Lorca, J., and Romos-Jiliberto, R. Modelling the Allee effect: are the different mathematical forms proposed equivalents? *Procceedings of the 2006 International Symposium on Mathematical and Computational Biology: BIOMAT* (2007), 53–71.

[23] Gordon, S. P. Modeling population growth and extinction. *PRIMUS 19*, 6 (2009), 548–560.

[24] Graef, J., Padhi, S., and Pati, S. Periodic solutions of some models with strong Allee effects. *Nonlinear Analysis: Real World Applications* (2011).

[25] Grozdanovski, T. *Slow Variation in the Single Species Gompertz Population Model.* Honours thesis, RMIT University, Melbourne, 2005.

[26] Grozdanovski, T. *Multi-Scaling Methods Applied to Population Models.* Ph.D. thesis, School of Mathematical and Geospatial Sciences, RMIT University, Melbourne, Australia, 2009.

[27] Grozdanovski, T., and Shepherd, J. Slow variation in the Gompertz model. *ANZIAM Journal 47* (2007), C541–554.

[28] Grozdanovski, T., Shepherd, J., and Stacey, A. Density dependent harvesting of a logistic population in a slowly varying environment. *ANZIAM Journal 51* (2009), C32–C47.

[29] Grozdanovski, T., Shepherd, J., and Stacey, A. Multi-scaling analysis of a logistic model with slowly varying coefficients. *Applied Mathematics Letters 22*, 7 (2009), 1091–1095.

[30] Grozdanovski, T., Shepherd, J., and Stacey, A. Transitions in density dependent harvesting of a logistic population in a slowly varying environment. *ANZIAM Journal 51* (2009), C393–C408.

[31] Haberman, R. Slowly varying jump and transition phenomena associated with algebraic bifurcation problems. *SIAM Journal on Applied Mathematics 37*, 1 (1979), 69–106.

[32] Haberman, R. *Mathematical Models: Mechanical Vibrations, Population Dynamics and Traffic Flow*. SIAM, USA, 1998.

[33] Hastings, A. *Population Dynamics*. Elsevier, New York, 2001.

[34] Holling, C. S. The functional response of predators to prey density and its role in mimicry and population regulation. *Memoirs of the Entomological Society of Canada 97* (1965), 5–60.

[35] Holmes, M. *Introduction to Perturbation Methods*. Springer-Verlag, New York, 1995.

[36] Idlango, M., Gear, J., and Shepherd, J. Transition in a slowly varying harvested logistic population model. *Applied Mathematics Letters* (**accepted**).

[37] Idlango, M., Shepherd, J., and Gear, J. Slow variation in a single speciec logistic population model with harvesting Holling typeII predation. *to be submitted to Mathematical Methods in the Applied Sciences*.

[38] Idlango, M., Shepherd, J., and Gear, J. Validation of a multiscale approximation for a slowly varying harvested logistic population model. *to be submitted to Mathematical Methods in the Applied Sciences*.

[39] Idlango, M., Shepherd, J., and Gear, J. Multiscaling analysis of a slowly varying single species population model displaying an Allee effect. *Mathematical Methods in the Applied Sciences* (In press).

[40] Idlango, M., Shepherd, J., Nguyen, L., and Gear, J. Harvesting a logistic population in a slowly varying environment. *Applied Mathematics Letters 25* (2012), 81–87.

[41] Legovic, T., and G., P. Harvesting population in a periodic environment. *Ecological Modelling 24* (1984), 221 – 229.

[42] Ludwig, D., Jones, D. D., and Holling, C. S. Qualitative analysis of insect outbreak systems: The spruce budworm and forest. *Journal of Animal Ecology 47* (1978), 315–332.

[43] Malthus, T. *An Essay on the Principle of Population*. Printed for J. Johnson, St. Paul's Church-Yard, London, 1798.

[44] May, R. Models for single populations. *Theoretical Ecology* (1981), 5–29.

[45] Maynard-Smith, J. *Models in Ecology*. Cambridge University Press, Cambridge, 1974.

[46] Meyer, P., and Ausubel, J. Carrying capacity: a model with logistically varying limits. *Technological Forecasting and Social Change 61*, 3 (1999), 209–214.

[47] Murdock, J. *Perturbations Theory and Methods*. Society for Industrial and Applied Mathematices, 1999.

[48] Murray, J. *Mathematical Biology. I. An Introduction.* Springer-Verlag, Heidelberg, 2002.

[49] Nayfeh, A. *Perturbation Methods.* John Wiley and Sons, USA, 1973.

[50] Nguyen, L. *Slowly Varying Culling in a Single-Species Population Model.* Honours thesis, School of Mathematical and Geospatial Sciences, RMIT University, Melbourne, Australia, 2006.

[51] Pearl, R., and Reed, L. On the rate of growth of the United States population since 1790 and its mathematical representation. *Proceeding of the National Academy of Sciences 6* (1920), 275–288.

[52] Petersen, W. *Malthus.* Harvard University Press, Massachusetts, USA, 1979.

[53] Rizaner, F. B., and Rogovchenko, S. P. Dynamics of a single species under periodic habitat fluctuations and Allee effect. *Nonlinear Analysis: Real World Applications 13* (2012), 141–157.

[54] Rosenblat, S. Population models in a periodically fluctuating environment. *Journal of Mathematical Biology* (1980), 23–36.

[55] Sanchez, D. A. Populations and harvesting. *SIAM Review 19*, 3 (1977), 551–553.

[56] Shepherd, J., Stacey, A., and Grozdanovski, T. Analysis of the power law logistic population model with slowly varying coefficients. *Mathematical Methods in the Applied Sciences 35*, 2 (2012), 238–248.

[57] Shepherd, J., and Stojkov, L. The logistic population model with slowly varying carrying capacity. *ANZIAM Journal 47* (2007), C492–506.

[58] Stephens, P., Sutherland, W., and Freckleton, R. What is the Allee effect? *Oikos 87* (1999), 185–190.

[59] STOJKOV, L. *Population Modelling with Slowly Varying Carrying Capacities.* Honours thesis, School of Mathematical and Geospatial Sciences, RMIT University, Melbourne, Australia, 2003.

[60] TANNER, J. T. The stability and the intrinsic growth rates of prey and predator populations. *Ecology 56*, 4 (1975), 855–867.

[61] TSOULARIS, A., AND WALLACE, J. Analysis of logistic growth models. *Mathematical Biosciences 179*, 1 (2002), 21 – 55.

[62] VERHULST, P. F. *Notice sur la loi que la populations suit dans son accroissement*, vol. 10. Correspondence Mathematique et Physique, Guetelet, 1838.

[63] WHITHAM, G. B. *Lectures on Wave Propagation.* Springer-Verlag, Berlin, 1979.