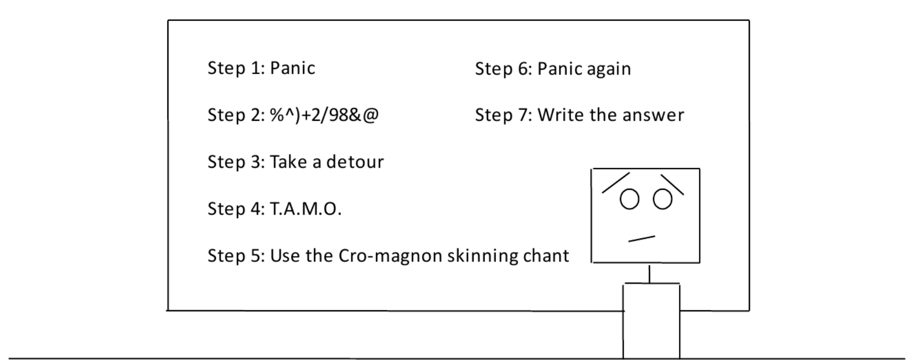
Washington State University

MATH 315

Fall 2009 Lecture Notes

Differential Equations

Authors:
Travis Mallett — Josh Fetbrandt



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Dear Reader

These notes were written by two students (Travis Mallett and Josh Fetbrandt) at Washington State University in Fall 2009 and are intended to be used with Differential Equations with Boundary Value Problems (Second Edition) by John C. Polking. As we were studying differential equations using this book, we began to find it extremely dysfunctional in many respects. Its poor explanations and confusing structure made it very difficult to learn anything. In the end, we decided to write our own material (based on online lectures and other books) to make up for Polking's book. Although these notes were a direct response to this specific textbook, our work should be applicable to any standard differential equations class.

We have abandoned many of the formal aspects of the traditional textbook but all of the information in these notes should be technically correct without the bore of mathematical formalities. We have also tried to obtain a good balance between the practical and theoretical. While the authors of our textbook waved their hands a lot and did not explain concepts, we have tried to rectify this by telling you not only how to use the equations but give explanations about why they work where appropriate. It was our intention to relate many of the concepts to things that should be intuitive to the reader. We feel that it is important to understand (from a gut-feeling level) what is going on. Otherwise many of these concepts drift into obscurity in our minds if they have no practical ramifications. Whether we actually accomplished all of these goals in our notes is another matter entirely. But we did our best under the circumstances (being in school and all).

The notes should also be fun to read and we hope you enjoy them. We incorporated many wise-cracks and jokes into the material for entertainment. Although the humor really has no redeeming value, we hope it at least puts a smile on your face a few times to make up for the fact that you are doing differential equations of all things. We have certainly learned a lot from writing these notes and hope you find them useful as you embark on your study of differential equations. If you have any questions or comments, please feel free to look us up.

Sincerely, Travis Mallett and Josh Fetbrandt

Chapter 1

Introduction to Differential Equations

Chances are, that in you journey through mathematics you have heard the term "Differential Equation." You may even vaguely remember something called a "Separable Differential Equation" from your highschool or college first semester calculus courses. If you don't know what a differential equation is, then this section is for you.

The world we live in contains many interrelated, changing entities. The earth moves with time as it orbits around the sun, tides are a result of tidal forces created by the moon and attributes of a gas such as pressure, volume, and temperature are all interrelated and dependent on one another. These and many other aspects of science speak of changing quantities. We call changing quantities *variables* and their rate of change their *derivatives*. Often we will want to express some relationship between a variable and its derivative to model some physical system or another. For example, we might want to express the speed of a roller coaster in terms of its position. We could somehow mathematically determine that the speed of a (very boring) roller coaster might decrease by some amount for every 100 feet it traveled:

$$\frac{dx}{dt} = -k\frac{x}{100}$$

And this, my friends, is what we call an Ordinary, Separable Differential Equation! We'll get to the ordinary and separable part later. For now, it is enough to know that whenever we write an equation involving a variable *and* its derivative, we call it a *differential* equation (for reasons that are obvious enough). In the above equation we see the *speed* of the roller coaster (the derivative) is some function of the *position* (designated by the usual x variable) with a weird k factor floating around in there which will tell us how much the speed is decreasing (and some other things). Because we have *both* the variable x and its derivative

$\frac{dx}{dt}$ in the same equation, it is a differential equation.

Another example of a differential equation in nature is Newton's Law of Cooling. This is a model that describes, mathematically, the change in temperature of an object in a given environment. The law states that the rate of change (in time) of the temperature is proportional to the difference between the temperature T of the object and the temperature T_e, of the environment surrounding the object. When we write this law in fancy math symbols we get a differential equation:

$$\frac{dT}{dt} = -k(T - T_e)$$

Other examples include exponential growth problems and radioactive decay, modeling vibrating springs and resistive-inductive (RL) circuits, and even taking into account air-resistance for objects falling towards the earth.

1.1 What This Means to You

Differential Equations weren't simply invented by theoretical mathematicians as another means to torture undergrads who just want to get through school......it is much more than that. The study of not only the physical world, but of economics and other social sciences are deeply indebted to the invention of the differential equation and in fact, differential equations are a necessary outcome of physical applications. Scientists who apply calculus based methods to model systems which involve changing, interrelated variables almost always, necessarily employ differential equations in some form. It is important throughout this course that you don't get lost in all the symbols. Of course since this is a course in mathematics and not science, you will inevitably be dragged into the realm of theoretical mathematics to develop some mathematical tools which you will later use to model physical systems.

The key idea to remember is that although the mathematics can become quite abstract in some places, it is simply a *means* to model real-life systems. The definition of a differential equation is simple enough, but it is helpful to visualize what the differential equations represent, especially when it comes to setting up problems and modeling systems. We will do our best to guide you through this process and help you understand where these things are coming from.

Chapter 2

First-Order Equations

So here's the deal. If you read Chapter 1, you know what a differential equation is, and if you skipped it, you hopefully already knew what it is. Simply an equation that has a variable *and* its derivative related to each other.

This chapter will help you solve *simple* differential equations that are in very specific forms. We will develop a few methods that may seem quite obvious, and a few tricky ones that are hard to figure out how they work. We will present you with the *how* and *why* these equations work and then give a step-by-step method for solving those types of equations. It's your choice if you want to study the inner workings of the methods, but we recommend it—it's well worth it!

2.1 Solutions to Separable Equations

Some of you may have seen these types of equations in your high school calculus course, perhaps near the end of the semester your teacher briefly mentioned them. But even if you haven't seen Separable Differential Equations before, don't worry, the concept is quite simple. Sometimes actually solving them can be tricky, but usually it's just a matter of algebra.

So here's an example equation. You'll quickly see why it's called "separable."

$$\frac{1}{y}\frac{dy}{dx} = x + 1$$

Now how in the world are we going to solve this thing? And what are we solving for?? For starters, we want a "normal" equation, one that doesn't have stupid derivatives in it; we want some function of y in terms of x, and that's it. End of story. Look at the equation and think about what would happen if we multiplied both sides by dx. I know, I know. It's not the most "mathematically" correct thing to do, since $\frac{dy}{dx}$ isn't *really* a plain old fraction, but a weird limit thing we call a derivative. But if we ignore the formalities, we get something like this:

$$\frac{1}{y}dy = (x+1)dx.$$

Notice how we have everything on the left hand side in terms of y and everything on the right hand side in terms of x. This is why it's called a *separable* equations—because we can separate the variables on each side of the equal sign. Now, how do we actually solve this? Simple, integrate both sides:

$$\int \frac{1}{y}dy = \int (x+1)dx$$

$$ln|y| + C_1 = x^2 + x + C_2$$

Notice that although we have two arbitrary constants of integration, we can simply combine them into one. Our answer is thus:

$$ln|y| = x^2 + x + C.$$

Well, if we want to finish solving for something that looks like $y = f(x)$, we'd exponentiate both sides by e to cancel the natural log and get

$$y = e^{x^2+x+C}$$

Which we can rewrite (pulling the e^C out and making it yet another generic constant:

$$y = Ce^{x^2+x}.$$

2.1.1 Summary of Separable Differential Equations

Separable ordinary differential equations are those that can be solved by the process of variable separation. This involves getting each side of the equation to consist of one (and only one) of the variables. Separable differential equations are of the form:

$$\frac{dy}{dx} = f(y)g(x)$$

or

$$\frac{dy}{dx} = \frac{f(y)}{g(x)}.$$

Solutions to separable differential equations are obtained as follows:

1. Separate the variables: $f(y)dy = g(x)dx$ or $\frac{dy}{f(y)} = g(x)dx$.
2. Integrate both sides: $\int f(y)dy = \int g(x)dx$ or $\int \frac{dy}{f(y)} = \int g(x)dx$.
3. Solve for the solution $y(x)$, if possible.

As long as the separating step is correct, you integrate without errors, don't forget the integration constant and solve for y with flawless algebra (remembering not to drop signs), you *will* get the right answer. It's that easy.

2.2 Linear Equations and the Integrating Factor

Well, hopefully you thought separable differential equations were quite easy. And you would think that we'd move on to something that's only a little bit harder. But no. Unfortunately the next thing we cover is how to solve linear equations using a trick called an "integrating factor," which is pretty unintuitive at first. We'll go through what this trick is and then give you the step-by-step guide on how to solve equations using this method. As you will see throughout this course, Differential Equations is really just giving you a big bag of tricks to solve various forms and types of equations. So we'll be covering many more tricks and special cases as we continue.

So let's take a look at a differential equation that is of the form:

$$y' + p(x)y = q(x).$$

Don't get scared by the $p(x)$ and $q(x)$ stuff. All that means is that you have a differential equation that has y' by itself, y is multiplied by some function of x, and $q(x)$ is another function of x. An example of this type of equation would be

$$y' + x^2 y = sin(x).$$

Now if you look at this form of equation and try to get it in a separable form like we did before, you should just stop trying 'cause it ain't going to happen. Now take a look at the LHS of the equation, $y' + p(x)y$. It *almost* could look like the product rule of differentiation. When we take the derivative of $y \cdot u$, we get $y'u + u'y$, right? So what if we found some sort of factor, u such that when we multiply whole differential equation with it, we get the LHS to simplify to $(uy)'$? How would this be useful? Well, we would have the equation

$$(uy)' = q(x)u,$$

which we could integrate both sides and solve for y. So let's try and figure out what this u factor would have to be. We *have* $y' + p(x)y = q(x)$. And we *want* $y'u + p(x)yu = (uy)' = q(x)u$. So let's relate the things we need to make true:

$$y'u + p(x)yu = (uy)' = y'u + u'y.$$

If we make the stuff multiplying y' equal on both sides and the stuff multiplying plain old y also equal, then we'll have what we want. So

$$y'u = y'u$$

and,

$$p(x)yu = u'y.$$

Well the first one isn't exactly very helpful, so we'll go with the second one. If we take

$$p(x)yu = u'y$$

and cancel out the y on each side, we have

$$p(x)u = u'.$$

Wait a second! We have a variable *and* its derivative in the same equation. This is another differential equation! So...you're telling me that in order to solve the original differential equation, we have to solve *another* one, just to get a stupid integrating factor? Yes, that's what's happening here. As you will find out, solving differential equations often means taking difficult equations to solve and converting them into simpler problems. If we write this in Leibniz notation, we see that it's separable[1]

$$\frac{du}{dx} = p(x)u.$$

Separating the variables gives:

$$\frac{1}{u}du = p(x)dx.$$

Now we integrate both sides,

$$\int \frac{1}{u}du = \int p(x)dx$$

and solve for u:

$$u = e^{\int p(x)dx}.$$

So this is our *integrating factor*.

2.2.1 Summary of Integrating Factor Technique

So after all that mathematical hoopla, you probably forgot why we were trying to find this mysterious integrating factor. To recap, we are tying to solve a differential equation of the form:

$$y' + p(x)y = q(x).$$

When we're actually solving these, you need to make your equation *look* like this form. Many times you'll have to divide things, etc, to get y' on it's own. Then you find the integrating factor u and multiply both sides by it. Since we

[1] Duh! Since we only know how to solve one kind of differential equation so far, it wouldn't make much sense if it were anything else.

chose the integrating factor such that multiplying $y' + p(x)y$ by u would give $(uy)'$, you can simply write down that down and set it equal to $q(x)u$. It is then trivial to integrate both sides of the equation and divide by u. The following steps will help you solve linear differential equations with the integrating factor method:

1. Rewrite the differential equation in the form: $y' + p(x)y = q(x)$.
2. Calculate the integrating factor: $u = e^{\int p(x)dx}$.
3. Multiply both sides by the resulting integrating factor so that the equation becomes: $(uy)' = q(x)u$.
4. Integrate this equation and solve for y: $y = \frac{1}{u}\int q(x)udx + \frac{C}{u}$ (note that the integration constant is explicitly stated here.

In the end, as long as you correctly identify $p(x), q(x)$, and calculate u, you can just skip to step 4 and do the integral to get your answer. But knowing *how* you get to that equation is helpful since the concept of the integrating factor is useful for many other types of problems. It's just another tool in your box.

2.3 Mixing Problems

Well now we've learned how to solve a couple types of differential equations. So now we'll actually use our skills to solve real-life problems! Unfortunately, the type of problems, mixing problems, we will be considering are often regarded as horribly annoying by differential equations students. And surprisingly, the difficulty doesn't even have anything to do with the differential equations themselves. So here's what a mixing problem might sound like:

> A 50 gal Tank initially contains 10 gal of fresh water. At t=0, a brine solution containing 1 lb of salt per gallon of water is poured into a tank at a rate of 4 gal/min, while the well stirred mixture leaves the tank at a rate of 2 gal/min. Find the amount of time required for overflow to happen, and the amount of salt in the tank at the time of overflow.

Now before you freak out about how impossible this sounds, let me assure you, the difficulty is only in setting up the problem, and *not* solving the differential equation. Essentially the main idea is to write down *everything* you know about the problem, and writing a governing differential equation which you can easily solve for the required answers. Rather than solve this *specific* problem, we'll show you how to set up the differential equation for a general-looking problem.

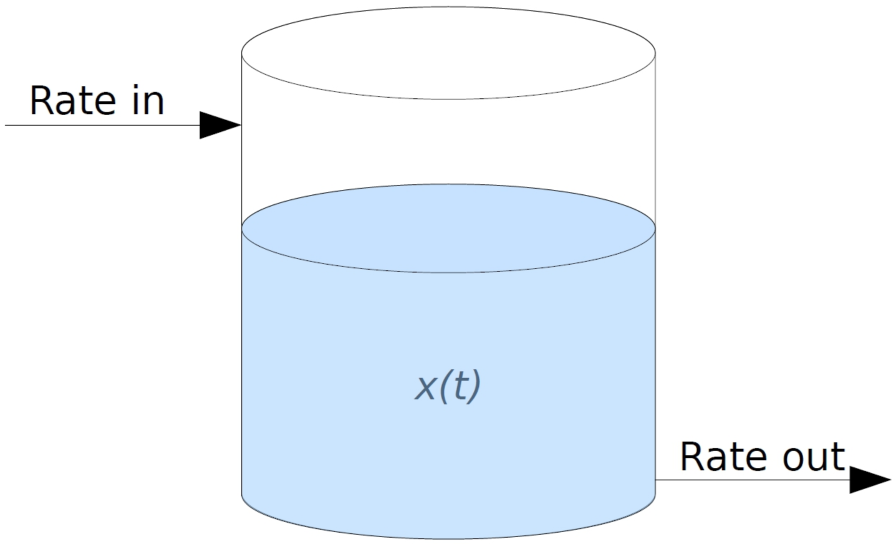


Figure 2.1: Mixing Problems

Shown in Figure 2.1 is a tank of some liquid which is possibly diluted by some concentration of salt. Flowing into the tank at the rate of $x(t)$ is a salt-water solution of a concentration C (possibly in lbs/gal). Also, there is a certain amount of liquid is being drained out. These mixing problems assume "perfect" mixing. That is, that when salt water enters the tank, the solution is mixed instantaneously. Shown in Table 2.1 is the list of mixing problem variables you might be given in a problem. We recommend the *first* step in solving these problems is to write down everything you've been given so you can setup the differential equation.

Table 2.1: Given Information in Mixing Problems

Symbol	**Description**
$x(t)$	Amount of salt in the tank at time t.
r_{in}	Rate of incoming water.
r_{out}	Rate of outgoing water.
C	Concentration of incoming salt water.
V	Volume of the tank.

Note that the volume of the tank may in some cases be a function of t depending on the rates in and out. The initial setup for the differential equation involves writing what the rate of change of salt content in the tank.

$$\frac{dx}{dt} = (\text{rate salt inflow}) - (\text{rate salt outflow})$$

Since the rates r_{in} and r_{out} are just that, rates, they do not quite represent the amount of salt in the tank. Therefore, we must multiply them by the *concentrations* which yields this differential equation.

$$\frac{dx}{dt} = Cr_{in} - r_{out}\frac{x}{V}$$

The first one is pretty obvious because we've been given the incoming concentration directly. But what's the outgoing concentration of salt? Well, a *concentration* is the amount of substance per volume. So we divide the amount of salt in the tank by the volume of the tank. This then a differential equation that can then be written in standard form and solved using an integrating factor since it is linear.

2.4 Exact Differential Equations

Exact differential equations can be confusing at first but are really invoked from the concept of *total differentials* from third semester calculus. Consider a function of two variables, $F(x, y)$. The so-called total differential of F is thought of as merely the total change in F, given small changes in x and y. The differential is defined to be:

$$dF = F_x dx + F_y dy.$$

The reason why this is relevant is because some differential equations can be written in the form of the total differential of another function. It is useful to find that original function as it will provide the solution to the differential equation. For example, the differential equation,

$$P(x, y) + \frac{dy}{dx}Q(x, y) = 0$$

can be rewritten as:

$$P(x, y)dx + Q(x, y)dy = 0.$$

This is done by simply by multiplying both sides by dx.

To solve exact differential equations, they must first be written in the form of a differential. It is then simply a matter of going *backwards* and finding the function it is the differential *for*. The following steps are used in this process:

1. Write the equation in differential form: $P(x, y)dx + Q(x, y)dy = 0$.
2. Verify that it is exact. (check that $P_y = Q_x$)

3. Solve $F_x = P$ for F. This simply involves integrating $P(x, y)$ with respect to x:
$$F(x, y) = \int P(x, y)dx + \phi(y).$$

4. Now take the partial derivative of your newly found F with respect to y and set it equal to Q:
$$F_y = \frac{\partial}{\partial y} \int P(x, y)dx + \phi'(y) = Q(x, y).$$

5. Solve for $\phi'(y)$, integrate it and insert it back into
$$F(x, y) = \int P(x, y)dx + \phi(y) = C.$$

2.5 Autonomous Equations and Stability

A useful technique for analyzing the solutions of certain differential equations without actually solving them is called *qualitative analysis* and works for what are called *autonomous equations*. Differential equations that are autonomous are those of the form:
$$\frac{dy}{dt} = f(y).$$
Notice that the independent variable, t, does not show up in the RHS of the equation. This is the definition of being autonomous.

The process of qualitatively analyzing these equations is something that we've been doing since the beginning of calculus (recall curve sketching). To sketch a curve, we take the derivative, set it equal to zero and solve for critical points. The process is the same here, except that we are *starting with the derivative*. Once the critical points have been determined, the portions of the curves $f(y)$ can be graphed and analyzed. It should be obvious that if the graph is above the horizontal axis, then the solution is increasing (since we are simply graphing and working with a derivative here), and if it is below the horizontal axis, then the solution is decreasing.

The process of drawing arrows on the horizontal axis is useful in determining the stability of the critical points (the places where $f(y)$ crosses the axis. This simply means that if arrows are pointing *towards* a critical point from both sides it is stable (because no matter whether the function is above or below that critical point, it will try to migrate towards it). And if the arrows point *away* from the critical point, it is unstable.

Follow these steps to analyze the solutions to an autonomous equation.

1. Find critical points by setting $f(y)$ to zero and solving for y.
2. Graph $f(y)$. This is where the analysis comes in.
   (a) If $f(y) > 0$ for a region, the solution is increasing there.
   (b) If $f(y) < 0$ for a region, the solution is decreasing there.
3. Draw arrows on the horizontal-axis pointing to the right if the solution is increasing and to the left it it is decreasing.
4. Graph the solution curves by placing horizontal lines with heights corresponding to the y vales obtained from the critical points. Then the curves above, below, and between these lines must conform to the behavior of increasing or decreasing that has already been determined.

2.5.1 Example of Qualitative Analysis

Given

$$\frac{dy}{dx} = -y^2 + 4,$$

find the equilibrium points (the critical points) and determine the stability of each.

1. The first step is to set this equation to zero and solve for the critical points: $-y^2 + 4 = 0 \rightarrow y = \pm 2$.
2. Then graph the function, $f(y)$, which in this case is $f(y) = -y^2 + 4$.
3. Draw the arrows along the horizontal axis. See Figure 2.2. Notice that when $y < 2$ or $y > 2$, the derivative curve is negative. This means that the function is decreasing in those areas. The only time that the derivative is positive is between -2, and 2. Notice that since the arrows are pointing away from the point -2 it is unstable there and since they point towards the point at 2, it is stable.
4. Finally you can graph the solution by drawing horizontal lines at $y = 2$ and $y = -2$ and then drawing the solution curves that are falling to the black depths of negative infinity below -2, rising between -2 and 2, and finally are falling from the heavens and asymptotically approaching 2. See Figure 2.3.

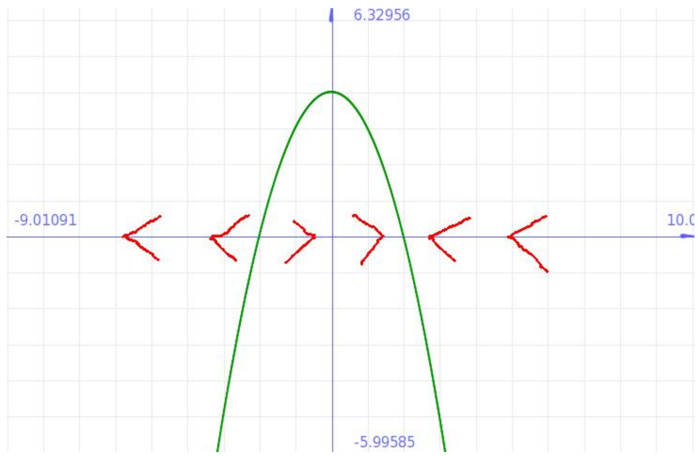


Figure 2.2: Qualitative Analysis for $f(y) = 0$

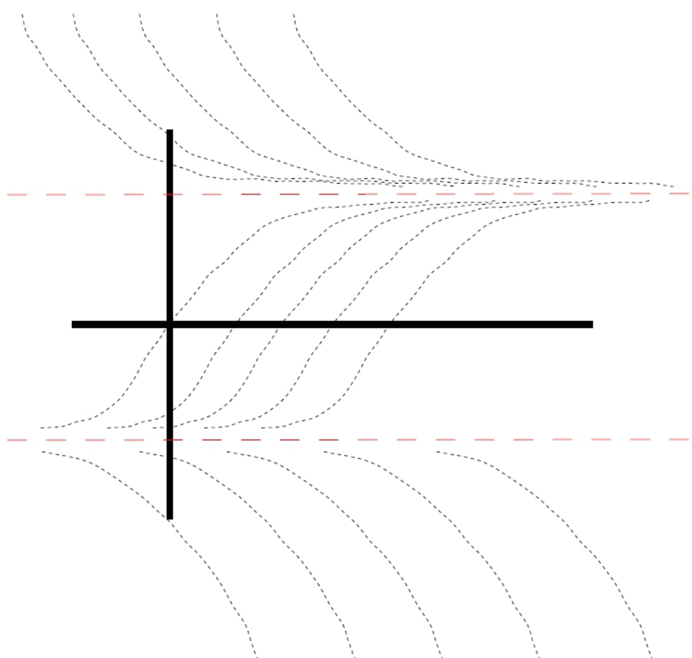


Figure 2.3: Qualitative Analysis of Solution

Chapter 3

Second-Order Equations

Well, now we get to move into a much more complicated arena of terror and mathematical gore. If you didn't pick it up previously, a *first-order* equation is one that only has the first derivative. Second-order equations, as you might guess, are those that contain the *second* derivative. Now the way higher-dimensional differential equations work is the same as with polynomials. You can have the highest derivative n *and* all the derivatives from 0 to n. Like this:

$$y^{(n)} + y^{(n-1)} + ... + y'' + y' + y = \text{stuff}$$

So for *second-order* equations, it might look like this:

$$a(x)y'' + b(x)y' + c(x)y = \text{more stuff}$$

As you might expect, solving second-order equations is harder than solving first-order ones. But unfortunately they don't seem to be twice as hard to solve, but some exponential increase in difficulty. This chapter won't explain *all* the nuances to solving these, but present some theory about what solutions look like, and how to solve them. The most annoying part of this chapter is the stuff on Harmonic Motion. So, buckle up, saddle up, or put on your moccasins, depending on your preferred mode of transportation, and we'll embark on this journey together!

3.1 General Solutions, Linearity, and Other Hoopla

Second-order differential equations might seem pretty straightforward, but it turns out there are at least three types of solutions to them, and hopefully I'll be able to motivate the reasoning behind this. Let's take a look at this equation:

$$y'' - 4y = 0.$$

Doesn't look too bad, right? As you will find shortly, solving second order differential equations is often a matter of "guessing" the solution and plugging it in. I know we haven't covered this yet, but believe me, it works. Let's guess a solution that looks like $y = e^{rt}$. If we plug it in to the equation (this means take the first and second derivatives and plugging them in), we get:

$$r^2e^{rt} - 4re^{rt}.$$

If we divide both sides by e^{rt}, we have:

$$r^2 - 4 = 0.$$

All we do now is solve this for r giving: $r = \pm 2$. Since r comes from our guessed solutions, we can say that one solution is:

$$y_1 = e^{2t}$$

and the other solution is:

$$y_2 = e^{-2t}.$$

But which is the *right* solution?!? After all, if we're modelling a real-life system, they can't *both* be right at the same time...can they? It turns out that the real answer isn't just one or the other, but *both*. Think about it this way, if you plug in the first solution y_1 to the original equation, it equals zero, right? (Because it's the solution to $y'' - 4y = 0$.) And if you plug in the second solution y_2, you also get zero. It turns out that if two solutions are linearly independent (review linear algebra for this), then the so-called *general solution* is some linear *combination* of the solutions.

$$y_g = c_1y_1 + c_2y_2$$

Now if you think about what would happen if you plug in the general solution, it should be clear that adding one solution (that equals zero when plugged in) to another solution (that also equals zero when plugged in) will give you zero. So the *sum* of the two solutions gives a more general solution that satisfies the equation. Basically what it comes down to is that the general solution is a way of combining all non-redundant information about the solutions into just *one* solution.

Okay, let's look at the same equation, but this time with a little twist:

$$y'' - 4y = cos(t).$$

Gross, right? Rather than go through it again, I'll just tell you that the results of our guessing game (only this time using some combination of $\sin(t)$ and $\cos(t)$) gives us a solution of

$$y = -\frac{1}{5}\cos(t).$$

And yes, it only came up with *one* solution. Now you might be thinking "Cool! Less work, and it was even more complicated looking!" But you are much to hasty. We need to have a little chat about those solutions we found above. What would happen if we combined all *three* solutions into one?

$$y = c_1 y_1 + c_2 y_2 - \frac{1}{5}\cos(t)$$

Why in the world would we want to do this? I mean, those solutions were for an entirely different problem! Actually, not quite. They were in fact for the *same* problem, just where the equation equalled zero. So if we plug those in to our new equation that equals $\cos(t)$ they should come out to be zero on the LHS, which *doesn't* change our solution. So if they don't *do* anything, why would we put them in? Well, those solutions actually do contain valuable information about the system that generates that differential equation.

There are two kinds of ordinary, linear, second-order differential equations: **homogeneous** and **inhomogeneous**. A *homogeneous* equation is a differential equation that *equals* zero and an *inhomogeneous* one equals what's called a *forcing term*. As you will see in the Harmonic Motion applications of second-order differential equations, we can think of these equations modelling real-life systems. For our two equations,

$$y'' - 4y = 0$$

and

$$y'' - 4y = cos(t),$$

they actually describe the same physical system. But the second one has some signal being *forced* into it, or it is being driven by some outside system that produces the $\cos(t)$. When we solve for the solutions to the *homogeneous* version of the system (without the $\cos(t)$) we are finding what's called the *natural response* of the system. That is, the solution to how the system acts *without* any driving force. The specific solution to $y'' - 4y = cos(t)$ is aptly called the *particular solution* and describes how the system responds to that *particular* forcing function. The next two sections will summarize how to find general solutions for both kinds of equations. Just keep in mind that *inhomogeneous* equations (the ones with a forcing term) *contain* the solutions to their equivalent homogeneous equations.

3.1.1 Homogeneous Equations

The homogeneous version of the equation is relatively straight forward as it is simply an application of the characteristic equation method which is presented in a following section. As shown in Figure 3.1, all one needs to do is find what is called a fundamental set of solutions that are linearly independent and combine them to get the general equation. Follow these steps to obtain the general solution for a linear, homogeneous equation.

1. For starters, the equation should look like this: $y'' + py' + qy = 0$. If not, you're reading the wrong section—go somewhere else.
2. Find a fundamental set of solutions using the Characteristic Equation (again, look at the next section).
3. Combine those solutions into a general solution which looks like this: $y_g = c_1y_1(t) + c_2y_2(t)$.

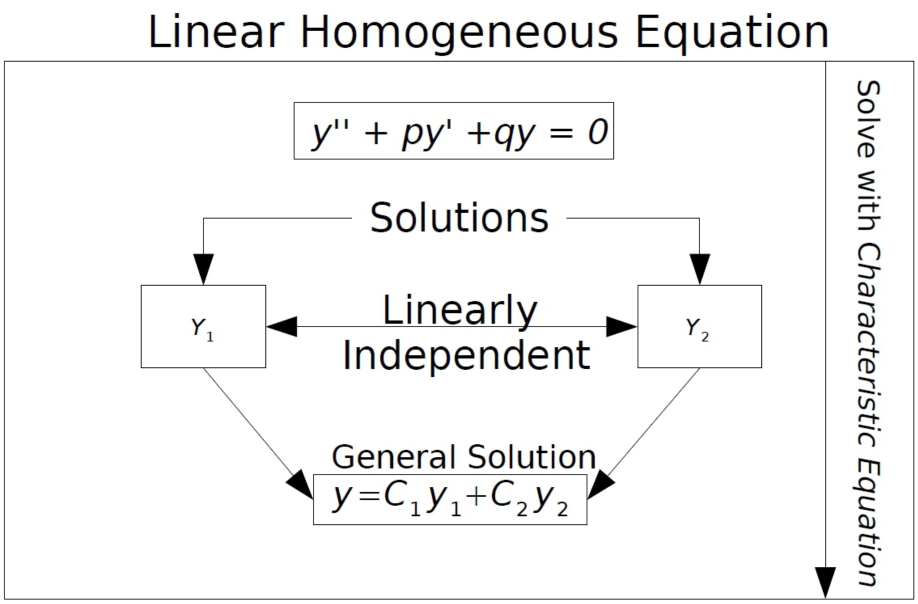


Figure 3.1: How to Solve Homogeneous Equations

3.1.2 Inhomogeneous Equations

Since the general solution to an inhomogeneous equation includes the solution for the homogeneous version, you must demolish the forcing term on the RHS and then find the general solution to the resulting homogeneous equation. You then have the great pleasure of finding the dreaded particular solution to the inhomogeneous equation. After a little sweat, blood and tears, you can simply add your general solution for the homogeneous version to the particular solution and you have the final answer, the general solution to the infamous inhomogeneous. No biggie right? The only problem is that the inhomogeneous equations make you solve the stupid thing twice! First you have to solve the homogeneous version and then solve the original inhomogeneous equation for a "particular solution," mix, add, subtract, and BAM!...a not so instant general equation in a box! Follow these steps to solve an inhomogeneous equation (you will have to keep reading in order to see how to do Undetermined Coefficients and Variation of Parameters). Figure 3.2 gives a flow-chart for solving these.

1. Make sure your equation is of the form $y'' + py' + qy = f$.
2. Convert your equation to the homogeneous equivalent: $y'' + py' + qy = 0$.
   (a) Solve the homogeneous equation (and keep track of your answer ($y_g = c_1y_1(t) + c_2y_2(t)$), this is a detour)
3. Return to the original equation, (this one: $y'' + py' + qy = f$. Need I remind you?)
4. Find a particular solution using Undetermined Coefficients, or Variation of Parameters.
5. Combine the particular solution with the general solution to the homogeneous version: $y_g = y_p + c_1y_1(t) + c_2y_2(t)$
6. You can skip step 6.

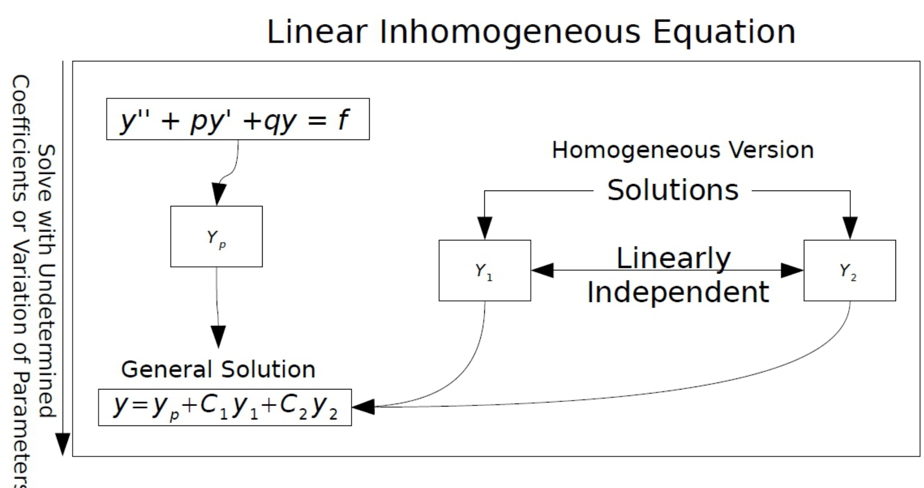


Figure 3.2: How to Solve Inhomogeneous Equations

3.2 Techniques for Solving Second-Order Equations

As we saw previously, there are several types of solutions to second-order differential equations. For homogeneous equations, we want to find a set of linearly independent solutions and for inhomogeneous equations we need those in addition to a particular solution. The main method for solving homogeneous equations is called the Characteristic Equation. There are several methods for finding particular solutions, and we will cover the Method of Undetermined Coefficients and Variation of Parameters.

3.2.1 The Characteristic Equation

The main method for solving second-order differential equations them turns out to be simply guessing a solution with some general parameters and then figuring out what the coefficients and such need to be. For *homogeneous* equations, we have something that looks like this:

$$y'' + Ay' + By = 0.$$

If you think about what kind of a solution this has to be, we come to the conclusion that is is probably some combination of powers of e. Sines and Cosines don't work because they always flip when you take derivatives and you need something that will "delete" itself when plugged into the differential equation. So if we simply *choose* a solution of

$$y = e^{rt},$$

we can take the derivatives and plug it in:

$$r^2 e^{rt} + Are^{rt} + Be^{rt} = 0.$$

If you notice that every term has an e^{rt}, we can divide all those out leaving:

$$r^2 + Ar + B = 0,$$

which is simply a quadratic equation. This is called the *Characteristic Equation*. We can usually just write this down directly from the differential equation since we know the e^{rt} terms will divide out. Then we can solve for the value of r that will make the solution work. Since this is a quadratic equation, there are three cases of combinations of r to worry about.

1. If the characteristic equation has two distinct, real roots, then a fundamental set of solutions is $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$.

2. If the characteristic equation has one repeated real root, then a fundamental set of solutions is $y_1(t) = e^{r_1 t}$ and $y_2(t) = te^{r_2 t}$.

3. If the characteristic equation has two complex conjugate roots, $a + bi$ and $a - bi$, then a fundamental set of solutions is $y_1(t) = e^{at}\cos(bt)$ and $y_1(t) = e^{at}\sin(bt)$

It turns out that there must be two linearly independent solutions to form the general solution which is in the form: $y_g = c_1 y_1(t) + c_2 y_2(t)$. To determine whether the two solutions are actually independent, we use the Wronskian determinant. The solutions are independent if the Wronskian is *never* equal to zero.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

If your Wronskian determinant is not equal to zero and the two solutions are really solutions, then you can legally form the general solution:

$$y_g = c_1 y_1(t) + c_2 y_2(t).$$

NOTE: This is the general form. If a problem asks for the fundamental set of solutions, it is really asking for something like this: $\{y_1(t), y_2(t)\}$.

3.2.2 Method of Undetermined Coefficients

Believe it or not, you've already seen the method of undetermined coefficients. I won't tell you where you've seen it just yet, you'll see later. To begin with, you are trying to find a "particular solution." This is as simple as it sounds. All you need to do is find any particular solution to an inhomogeneous equation. For a review of why you would ever want to do that anyway, look at the introduction to this section (with the diagram). Recall that you will see an equation that looks like:

$$y'' + py' + qy = \text{stuff}$$

The method of undetermined coefficients isn't really so much a "method" as it is a guessing game. You really don't care what the "stuff" is. You only care about what it looks like. Basically, you make an educated guess for a solution based on what the stuff looks like. For example, if "stuff" is some function like e^{rt}, your intuition will tell you that the solution to the differential equation probably does not involve sines, cosines, polynomials and other garbage. All that really needs to be done is make an educated guess make it work. The information in Table 3.1 should be self-explanatory (especially since I just explained it). //

Table 3.1: "Method" of Undetermined Coefficients

Forcing Function $f(t)$ (the "stuff")	Trial Solution $y_p(t)$	Comments (from the peanut gallery)
e^{rt}	ae^{rt}	
$cos(\omega t)$ or $sin(\omega t)$	$asin(\omega t) + bcos(\omega t)$	
$P(t)$	$p(t)$	P is a polynomial and lower case p is a polynomial of the same degree.
$P(t)cos(\omega t)$ or $P(t)sin(\omega t)$	$p(t)sin(\omega t) + q(t)sin(\omega t)$	P is a polynomial and lower case p and q are polynomials of the same degree
$e^{rt}cos(\omega t)$ or $e^{rt}sin(\omega t)$	$e^{r}t[acos(\omega t) + bsin(\omega t)]$	It keeps getting worse...
$e^{rt}P(t)cos(\omega t)$ or $e^{rt}P(t)sin(\omega t)$	$e^{r}t[p(t)cos(\omega t) + q(t)sin(\omega t)]$	P is a polynomial and lower case p and q are polynomials of the same degree

Once you've made your educated guess from the table, write:

$$y(t) = \text{(the educated guess)}.$$

Then find the first and second derivative:

$$y'(t) = \text{(the educated guess)}'$$

$$y''(t) = \text{(the educated guess)}''$$

Now plug all of your information (the guess and its derivatives) into the original differential equation and combine like terms. Great, now you have a really nasty looking differential equation that equals the original stuff (you remember the stuff right?). Look back up at the table. You see how there are these coefficients, a, b, $p(t)$, etc. that are...well...undetermined? If you did everything right, these coefficients will be in your new differential equation. This is most easily illustrated with an example:

---

1. Given this equation: $y'' + 3y' + 2y = \sin(t)$
2. Guess this: $y_p = a\sin(t) + b\cos(t)$ (from the table)
3. Calculate first and second derivatives: $y_p' = a\cos(t) - b\sin(t)$, $y_p'' = -a\sin(t) - b\cos(t)$

4. Put these back into the original equation: $(-a\sin(t)-b\cos(t))+3(a\cos(t)-b\sin(t))+2(a\sin(t)+b\cos(t))=\sin(t)$

5. Combine like terms: $(-a-b+2a)\sin(t)+(-b+3a+b)\cos(t)=\sin(t)$

Once you've reached this part, it's really easy. Simply compare what is on the LHS with the RHS. See how there are no cosines on the RHS? If you think long and hard, it stands to reason that the of cosine terms on the LHS must add up to zero. This means that the coefficient, $-b+3a+b$, (which by the way equals $3a$), must be equal to zero. It also follows that the term appended to the sine function on the LHS is equal to 1 (since that's how it is on the RHS). Now, you have a system of equations. (If you didn't understand that, look below).

$$(-a-b+2a)=1$$

$$3a=0$$

You already know how to solve this type of system. It could be put into a matrix or solved using Gauss-Jordan elimination (back solving). This leads to $a=0$ and $b=-1$. Now it is necessary to plug those (now determined) coefficients back into the original guess (handy little thing isn't it?). Like this: $y_p=0sin(t)-cos(t)=-cos(t)$. Now you've reached your destination of finding a *particular solution*!

At this point you should be saying to yourself, "this is a lot like partial fraction decomposition!" However, if you are sane, you'll be noticing that without the exclamation point (or not at all). Recall that in partial fractions you end up with a system of equations similar to that shown in the example above. It is the same idea: comparing coefficients of like powers of x and comparing the coefficients of like functions. Get the idea? Use the following step-by-step process to use the Method of Undetermined Coefficients:

1. Check that your equation is of the form: $y''+py'+qy=f$
2. Make an educated guess for the solution using Table 3.1, or your intuition.
3. Take the first and second derivatives of your guess: $y'(t)=$ (the educated guess)$'$, and $y''(t)=$ (the educated guess)$''$.
4. Plug those new derivatives back into the original differential equation.
5. Combine like terms (factor out all the undetermined coefficients)
6. Compare the coefficients on the LHS with the ones on RHS. Set the coefficients equal.
7. Solve for the coefficients and insert them back into the original guess.
8. Pass Go and collect $200.

3.2.3 Variation of Parameters

Now that you know how to find a particular solution using undetermined coefficients, you may be pleased (then again, maybe not) to know that there is another way. The method of variation of parameters also allows us to find the particular solution to a linear, second-order inhomogeneous differential equation. This is actually no big deal, just follow the formula.[1]

Given $y'' + p(t)y' + q(t)y = g(t)$, we look for a solution in the form of $y_p = v_1 y_1 + v_2 y_2$. By magic[2], the functions and are:

$$v_1 = \int \frac{-y_2(t)g(t)dt}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}$$

$$v_2 = \int \frac{y_1(t)g(t)dt}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}$$

3.3 Harmonic Motion (avoid this section)

Now we come to one of the most dreaded topics encountered thus far. Unfortunately there might be good reason to be scared (I certainly was). Before you stop reading and drift into depression, let me lend a word of comfort: harmonic motion is not *quite* as bad as it looks. Let me explain why. We simply need to keep one thing in mind, viz., that harmonic motion equations are simply second-order differential equations that model a physical phenomena such as a mass-spring system or an RLC circuit. The problem comes the fact that these inherently have a bunch of terminology tagging along which presents quite a confused muddle. However, if you realize that all it is is terminology, you can cancel your stay at the psychoward. All we are really doing is taking a second-order differential equation and giving names to the coefficients and various combinations of these coefficients so we can relate them to physical systems. Simple enough? If you are still confused, sorry. You just wasted five minutes of your life reading this paragraph.

3.3.1 Lots of Words

Let's begin by examining the equation for the motion of a vibrating spring (yummy) and relating the terms and coefficients to physical phenomena.

$$my'' + \mu y' + ky = F(t)$$

[1]Extensive research revealed that there is no *rational* reason for how or why variation of parameters works; it just does. Our findings show that it is the result of an ancient Cromagnon skinning chant. (Seriously though, we couldn't find out how it works!)

[2]One of the proof methods frequently invoked by mathematicians. Also relates to the Cromagnon skinning chant. This is often written in other texts as "TAMO" (Then A Miracle Occurs) and is a generally accepted proof technique.

Table 3.2: Spring-Mass Constants for Harmonic Motion Equations

Symbol	Name	Description
m	Mass of Weight	This one should be easy.
μ	Damping Constant	This is usually a non-negative constant that defines how "thick" the fluid is that slows down the motion. It acts as some resistance to the motion.
k	Spring Constant	This constant basically tells how stiff the spring is. The bigger the k value, the stiffer the spring.
$F(t)$	Forcing Function	The forcing function is not as bad as it sounds. If you actually *have* a forcing function (meaning the RHS is not zero), this can be equated to holding the spring from the top and moving it up and down with a certain frequency, force etc... Basically, you are "forcing" some motion into the system. If it is just zero, think of it as the top of the spring remaining stationary.

Now if you squint really hard and look at this equation upside down[3] you'll see that it looks very similar to the general equation we've been using all along for second-order differential equations. Namely, $y'' + py' + qy = f$. This means that all the previous techniques we've learned will still work! You should be at least slightly excited now since this means we really don't have to learn anything fundamentally new to work with harmonic motion. All we need to do is learn about some of the terminology that relates to it. So let's do it!

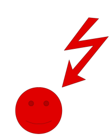
For starters, let's talk about what this is actually modeling. The easiest case is the mass-spring system and should be the most intuitive for you to understand what is going on. Imagine a spring that is being held at one end while the other end hangs down with a weight attached to it. The only other minor detail is that the weight is submersed in some heavy or thick liquid which slows down the rise and fall of the weight (when you actually let go of the weight and get things moving of course). Really that is all you need to keep in mind as we begin to talk about the terminology. Table 3.2 is a list of constants in the equation. There are a few extra equations listed in Table 3.3 that might be useful.

[3]Stop that! It was a joke!

Table 3.3: Specific Spring-Mass Constants

Equation	**Name**	**Description**
$k = \frac{mg}{x(0)}$	Spring Constant	g is our friend, the acceleration due to gravity near earth ($9.81 m/s^2$) and $x(0)$ is how far the spring is stretched naturally.
$\omega_o = \sqrt{\frac{k}{m}}$	Natural Freq.	This is the frequency at which the mass-spring system will move up and down by itself (with no forcing term).
$A = \sqrt{a^2 + b^2}$	Amplitude	This comes from the *solution* to the differential equation. Basically you will see a solution that might look like this: $y(t) = a\cos(\omega_0 t) + b\sin(\omega_0 t)$. It really comes down to what the sine and cosine are multiplied by.

3.3.2 Four Types of Harmonic Motion



Now that you know about some of the various terms that relate to harmonic motion we can talk a little about the different kinds of motion (yes there are more terms)[4] and what to with them. What do I mean by different *kinds* of motion? Well it's really quite simple. Let us begin with the simplest case that we already discussed, but make it even simpler. Already we've talked about the spring-mass system, that is a spring with a mass hanging at the end that is submerged in some liquid. What if we take away the liquid and have the mass just hanging free? This would mean that the damping constant is zero. Thus our equation (also assuming no forcing term) is:

$$my'' + ky = 0.$$

This type of motion has a name, **simple harmonic** or **free undamped**. This makes sense since it is really the simplest case of harmonic motion that we can think about.

The next type of motion we can look at is just a little tiny bit more complicated than the last one (but it does get worse from here). All we do is add a "forcing term" to the motion. To re-visualize this: imagine you are *holding* the top of the spring with the mass hanging freely from the end and moving the spring up

[4]Random smiley face gets hit by thunderbolt......you guessed it, harmonic motion.

and down (thus forcing it to move). The equation for this is the same as above but just with a forcing term:

$$my'' + ky = F(t)$$

This slightly more complicated type of motion also has a name, **forced undamped motion**. Really it is not any different than the previous example. However, if you want to expand your thinking a bit, you might realize that the previous motion type is a *homogeneous* differential equation while this one is *inhomogeneous*.

Now we will be coming a long way from the first type of motion. But really, this next type of harmonic motion is not much more difficult than the last. It just has another term (but no forcing term, so we even out right?). To again expand on this, the spring is stationary at the top. However, this time the mass at the end of the spring is not hanging freely. It is submersed in some liquid that causes a resistance to the movement (recall the damping constant). This makes the equation look like:

$$my'' + \mu y' + ky = 0$$

Not surprisingly, this type of motion also has a name. It is the **damped harmonic motion** or **free damped motion**.

Finally we come to the most complicated harmonic motion type. This involves all the factors being present that affect the motion of the mass. It is basically the same as the last type of motion only with a forcing term. To continue the tradition, let's paint the picture in our minds again. You are holding the spring from the top and moving it up and down in a way that can be defined with some forcing function (why are you doing this again?). From the end of the spring hangs a mass that is submersed in the liquid. You've probably already guessed what the equation looks like by now, but here it is anyway.

$$my'' + \mu y' + ky = F(t)$$

This type of motion is simply called **forced damped motion**, since it includes all three major factors that affect the motion of the spring-mass system.

3.3.3 Stable and Unstable Motion (Resonance)

Our discussion of harmonic motion would be incomplete if it did not pay some lip service to stable and unstable motion. I'm not going to beat around the bush here....no sir. I'm going to get straight to the point—straighter than a hungry pig's path to the trough[5]. Once I get right to the point, you'll know *exactly* what the difference between stable and unstable harmonic motion is...ehem. Basically

[5]It is now getting waaay to late at night/early in the morning to be writing notes for differential equations.........at least they will be interesting to read?

you just need to know about resonance. This is actually a simple concept once you wrap your brain around it.

First recall the natural frequency (the frequency at which the spring would bounce up and down if left alone). What happens if you add a forcing term to your motion that equals your natural frequency? Further explore in your mind what might happen if that same motion also had no damping. If you don't get a feel for what happens, let me help. Perhaps this illustration might help. Imagine if you will (and even if you won't), that you are holding a slinky[6] in one hand. You drop the bottom of the slinky as you hold onto the top. As the bottom of the slinky falls and it begins to stretch, you push your hand down, thus forcing the slinky to go further towards the ground. As it begins to recoil, you pull it up at the same time. It stretches up high above your head and as it begins to fall, you yank it down at the same time. At this point, the slinky hits the cabinet you are standing next to which holds your mothers china dishes and then smacks into the floor. You then spend the rest of your life grounded in your room to think about what you've done. You see what happened? Because the forcing function (your hand) had the same frequency as the slinky, the *amplitude* began to increase with each cycle (and you were grounded). This is because each time you are adding more force to the spring-mass system with exactly the right timing, thus forcing it to travel further each time. However, if the natural frequency is *not* equal to the forcing frequency, then sometimes the force will be pushing down a little when the spring is trying to pull up and vise versa—you will have a cancellation of forces. The motion will never get out of control.

This is the fundamental difference between stable and unstable motion. **Stable motion** will never continuously increase in amplitude but rather retain a steady and predictable oscillation pattern. **Unstable motion** (also called **resonance**) is where the frequency of the forcing term equals the natural frequency. The motion goes out of control as the amplitude increases all the time.

3.3.4 Summary of Harmonic Motion and Why you Shouldn't Care

Now that you know quite a bit about harmonic motion, you are probably wondering how to use it, specifically how to answer test questions using your knowledge. Mostly harmonic motion problem just involve some extra algebra on top of the familiar operations of differential equations that we've already learned (characteristic equation, undetermined coefficients, variation of parameters, etc...). What do I mean by extra algebra? Given a harmonic motion problem setup with initial conditions, you may be asked to:

1. Find the spring constant and natural frequency of the motion. *

[6] You remember those fascinating objects of entertainment from long ago don't you?

2. Formulate an initial-value problem that describes the motion of the mass.
3. Find the solution.
4. Find the amplitude of the harmonic motion. *

* Questions 1 and 4 are simply extra steps beyond what we've already encountered with differential equations. Simply put, we already know how to setup an initial value problem (look at the equations for the different kinds of motion) and we know how to solve it (think variation of parameters etc...). All that is different is just put some numbers into the formulas for the spring constant and calculate the amplitude of the harmonic motion once you find the solution. You see? It's really not that hard. All we did is add some terminology to what we already know.

3.3.5 Awesome Table That May Not Help

Figure 3.3 is a table of all the kinds of harmonic motion discussed so far.

Equation	Solution	Name of Motion	Amplitude	Type of Motion	Frequency
$\frac{d^2y}{dt^2} + \omega_0^2 y = 0$	$y_c = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$ $y_c = c \sin (\omega_0 t + \delta)$ $y_c = c \cos (\omega_0 t + \delta)$	Simple harmonic or free undamped motion.	$\sqrt{c_1^2 + c_2^2}$ c c	Perpetual motion. Stable.	ω_0 rad or $\frac{\omega_0}{2\pi}$ cycles per unit time.
$\frac{d^2y}{dt^2} + \omega_0^2 y = F \sin (\omega t + \beta)$	$y = y_c$ as given above $+ \frac{F}{\omega_0^2 - \omega^2} \sin (\omega t + \beta)$, $\omega_0 \neq \omega$	Forced undamped motion.	$c + \frac{F}{\omega_0^2 - \omega^2}$	Oscillatory. Stable.	
	$y = y_c - \frac{F}{2\omega_0} t \cos (\omega_0 t + \beta)$, $\omega_0 = \omega$		$c - \frac{F}{2\omega_0} t$	(Resonance) Unstable.	
$\frac{d^2y}{dt^2} + 2r \frac{dy}{dt} + \omega_0^2 = 0$	$y_c = c_1 e^{(-r+\sqrt{r^2-\omega_0^2})t} + c_2 e^{(-r-\sqrt{r^2-\omega_0^2})t}$, $r > \omega_0$.	Damped harmonic motion or free damped motion.	None	Non-oscillatory. Stable.	
	$y_c = c_1 e^{-rt} + c_2 t e^{-rt}$, $r = \omega_0$				
	$y_c = c e^{-rt} \sin (\sqrt{\omega_0^2 - r^2}\, t + \delta)$, $r < \omega_0$		ce^{-rt}	Oscillatory. Stable.	$\sqrt{\omega_0^2 - r^2}$ rad or $\frac{\sqrt{\omega_0^2 - r^2}}{2\pi}$ cycles per unit of time.
$\frac{d^2y}{dt^2} + 2r \frac{dy}{dt} + \omega_0^2 y = F \sin (\omega t + \beta)$	$y = y_c + \frac{F \sin (\omega t + \beta - \alpha)}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2r\omega)^2}}$	Forced damped motion.	$c + \frac{F}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2r\omega)^2}}$	Oscillatory. Stable.	Of steady state motion, ω rad or $\omega/2\pi$ cycles per unit time.

Figure 3.3: Table of Harmonic Motion

Chapter 4

Laplace Transform

4.1 Intuition, Gut-Feelings, and What this Means

The Laplace Transform is (yet again), one of those magical things in mathematics. Often-times we are presented with the "definition" and have no idea where it comes from. Here I will attempt to relate it to something we already know how to do. Recall from calculus the dreaded sequences and series topic. Do you remember that some discrete series can represent actual continuous functions? Even if you don't remember, just go with it; you'll see in a moment. Specifically, if you can think about the Taylor and Maclaurin series. These allowed you to represent continuous functions as an infinite series, and more importantly for our purposes, the converse. For example:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

If you reverse the equation (or read backwards, your choice), you can see that you are taking an infinite series and transforming it into a new function. Also notice that the index variable, n, "goes away" or gets "used up" in the series. Let's further explore another series:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$$

Again we will observe that we take a "function", in particular, $\frac{x^n}{n!}$, and insert it into an "operator," $\sum_{n=0}^{\infty} f(n)$, out comes a function that doesn't have the index operator, n, in it. Instead, it is a function of x. To state this bluntly, we took a function of n and operated on it to produce a function of x. This is the main idea behind the Laplace Transform. Although this is not an explanation of why the Laplace transform is so magical, it should offer some insight into what is actually going on. We can now extend our thinking about infinite series to our familiar friend, the integral. Recall that a limit of a Riemann Sum is the very definition of an integral, so we are only extending our thinking a little bit. All we will do is change the summation symbol to an integral and add a few more factors (to make things work). So here it goes, the definition of the Laplace transform:

$$\mathscr{L}\left(f(t)\right) = \int_0^\infty f(t)e^{-st}dt = F(s)$$

Don't panic just yet. If you look at this closely, you can see that the Laplace transform takes a function of t and "transforms" it into a function of s. Notice how the t also gets "used up" in the integral (since we are integrating with respect to t) the same way the n did in the series example. Here's a piece of terminology that you might want to know, the $f(t)$ is called the "kernel" of the function.

At this point you should have a *feel* for what the Laplace transform is. You don't need to know what it's used for just yet. But now it is time to address that scary looking factor of e^{-st}. For the purpose of understanding this, you must recall yet another thing from calculus (groan). Remember about the convergence and divergence of infinite series (and improper integrals for that matter)? Recall that the terms in an infinite series must be approaching zero to converge. Otherwise adding up an infinite number of items that aren't getting smaller goes to infinity right? Think about this, what happens if you wanted to do the Laplace transform where your kernel was $f(t) = t$. Simple enough, but notice that your terms aren't going to zero as t gets larger (without the e^{-st} factor). In fact, they are getting bigger and bigger. So the solution is to multiply the function that you want to transform by something that is getting smaller faster. Since the exponential function, specifically base e, is very friendly in terms of integration and differentiation, grows (or decreases depending on the sign of the exponent) extremely fast, and makes you look smart if you know it to more than a couple decimal places, it is the natural function that can make your function, $f(t)$, behave (i.e. go to zero).

But wait, doesn't that change your function!? We've gone from transforming our function $f(t)$, to transforming $f(t)e^{-st}$. If you think about it, it's actually not a problem. All we have to do is have the extra e factor be part of the transform. As long as we are consistent, all of our functions will be transformed the same. The benefit is that we could successfully transform functions that are growing very fast as long as they don't grow *faster* than e^{-st} decreases.

4.2 Groaning Forward: Examples of Laplace Transforms

Now that we kind of have an idea of what the Laplace transform is, we can compute some examples. All this means is that we can pick some sort of function (or kernel) and put it into the definition for the transform and compute

Table 4.1: Short Table of Laplace Transforms

No.	If $f(t) = \cdots$	Then $\mathscr{L}\left(f(t)\right) = F(s) = \cdots$
1	k	$\frac{k}{s}, s > 0$
2	t^n	$\frac{n!}{s^{n+1}}, s > 0, n = 1, 2, \cdots$
3	e^{at}	$\frac{1}{s-a}, s > a$
4	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}, s > a, n = 1, 2, \cdots$
5	$\sin(at)$	$\frac{a}{s^2+a^2}$
6	$\cos(at)$	$\frac{s}{s^2+a^2}$
7	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
8	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
9	$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
10	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
11	$\int_0^t f(t-a)g(t)dt$	$\mathscr{L}[f(t)] \cdot \mathscr{L}[g(t)] = F(s)G(s)$

the integral. For brevity, I will compute one example here for demonstration and then present the table of Laplace transforms. Let's compute the Laplace transform of some number, k.

$$\begin{aligned} \mathscr{L}(k) &= \int_0^\infty ke^{-st}dt = k\int_0^\infty e^{-st}dt \\ &= k\left[\frac{-e^{-st}}{s}\right] = \frac{k}{s}, \text{ if } s > 0 \end{aligned}$$

Don't worry too much about the $s > 0$ part. Just know right now that it is part of the deal with convergence and divergence. All it comes down to is if s were less than zero (negative), the would no longer be decreasing (because the negatives cancel) and we are now integrating from zero to infinity of a function that is getting very big very fast. The results are not pretty. So now that you've seen an example (although a very simple one), Table 4.1 contains a list of Laplace transforms that will be useful to you later. And by later, I mean very soon.

4.3 Inverse Laplace Transform

Now that you know what the Laplace transform it's time to start going backwards. The Laplace transform takes you (or more accurately, the function in question) into what is called the *s-domain*. At some point though, you'd like to get back to the reality of the time-domain. This turns out to be the hardest part. It is much like differentiation and integration. Think of differentiation as taking you forward a step and integration backward. As you probably very much know, integration is usually the hard part. It is the same way with the

Inverse Laplace transform.

To begin with, let me give some general hints about what is involved with the Inverse Laplace transform. As you will see in the next section when we solve differential equations using Laplace, you will have to find the Inverse Laplace transform of some rational function. This means that you will have to find the Inverse Laplace transform of a nasty fraction with polynomials on top and bottom. Thus, you will frequently need to employ the method of *partial fraction decomposition* (shoot you now you say?). I will not do a review of the technique of partial fractions here (you can find it in the book on pg. 205). However, before you decide to drop dead, let me tell you what is actually going on. In order to do the Inverse Laplace transform, you simply need to make what you have *look* like something in the table. Following is an example of this idea.

You are given this fraction to compute the Inverse Laplace transform:

$$\frac{s+9}{s^2-2s-3}.$$

Don't panic just yet. Remember, you just need to make it *look* like something in Table 4.1. This will require splitting up the fraction using partial fractions. You should end up with something like:

$$\frac{s+9}{s^2-2s-3} = \frac{-2}{s+1} + \frac{3}{s-3}.$$

If you look at the two fractions, you will see that they look quite similar to No. 3 in the table. If you factor out the constants from each term and use the table, you get:

$$\mathscr{L}^{-1}\left[\frac{s+9}{s^2-2s-3}\right] = -2e^{-t} + 3e^{3t}.$$

Remember that while the Laplace transform takes a function of t and turns it into a function of s, the Inverse Laplace transform goes the other way. Notice how we ended up with some t variables on the RHS. This is because the table has those variables in it.

Many times you will get to a function of s that looks *really* close to something on the table, but one of the numbers doesn't line up. In these cases you may have to multiply the entire thing by a creative 1 (meaning multiply top and bottom of the fraction by the same thing). For example, if you end up with a function like:

$$\frac{3}{s^2+4}.$$

You might see that this looks similar to the Laplace transform of $\sin(at)$. However, your constants (specifically a) are not right. You would really like to rewrite this as:

$$3\frac{2}{s^2+2}.$$

Unfortunately this is not mathematically correct since you essentially multiplied your whole function by 2. Instead, write it like this:

$$\frac{3}{2}\frac{2}{s^2+2},$$

and you're good! See how the 2's would cancel out and you still have the same fraction? But now you can write the Inverse Laplace transform like you wanted as

$$\frac{3}{2}\sin(2t).$$

Again let me say that the whole game of Inverse Laplace is making what you have look like something you already know (i.e. something on the table).

4.4 Solving Differential Equations with Laplace

Now that you have a lot of information about Laplace in general under your belt, it is probably time that we began to solve differential equations with it (after all, that is why you are here). Using Laplace to solve differential equations is a very powerful tool. But remember, with more power comes more responsibility. You must resist the temptation to enslave the weaker minds using the power you will gain from Laplace. Before we actually begin to solve these things, you need to know what the Laplace transform of a derivative is. The good news is that it turns out to include the Laplace transform of the original function (the one where the derivative came from). The bad news is that you must have initial conditions in order to take the Laplace transform of a derivative (which means you almost always have to have some kind of initial conditions to solve a differential equation using Laplace). I won't go through the proof here, but here is the Laplace transform of a first and second derivative:

$$\mathscr{L}(y') = s\mathscr{L}(y) - y(0)$$

$$\mathscr{L}(y'') = s^2\mathscr{L}(y) - sy(0) - y'(0)$$

We are almost ready to start solving. First let me present an outline of how we solve differential equations with Laplace:

1. Given a differential equation, take the Laplace transform of both sides.
2. Make sure the resulting equation does not have t in it (otherwise you missed something in your transform).
3. Solve for $\mathscr{L}(y)$ using algebra. (you should get: $\mathscr{L}(y) =$ nasty fraction junk)

4. Take the Inverse Laplace transform of both sides (the Inverse Laplace transform of $\mathscr{L}(y)$ is just y). This will usually involve partial fractions and other favorites.

5. Write your answer as $y =$ Inverse Laplace transform of RHS.

You will notice that it will take very little time to perform steps 1–3 and almost all of your time will be spent trying to unravel the Laplace transform and go back into the *t-domain*. So here we go. Let's do an example.

Given: $y'' - 2y' - 3y = 0$, and initial conditions: $y(0) = 1$, and $y'(0) = 0$.

1. Take the Laplace transform of both sides.
   $\mathscr{L}[y'' - 2y' - 3y] = \mathscr{L}(0)$ (Take a deep breath)
   $\mathscr{L}(y'') - 2\mathscr{L}(y') - 3\mathscr{L}(y) = 0$ (This is because Laplace is linear)
   $s^2\mathscr{L}(y) - sy(0) - y'(0)$ (Use Laplace of derivative)
   $-2\left[s\mathscr{L}(y) - y(0)\right] - 3\mathscr{L}(y) = 0$

2. There are no t variables left. So we are in good shape! This is a great place for a coffee break.

3. Solve for $\mathscr{L}(y)$.
   $\mathscr{L}(y)(s^2 - 2s - 3)$ (Factor out $\mathscr{L}(y)$)
   $-y(0)(s - 2) - y'(0) = 0$
   $\mathscr{L}(y)(s^2 - 2s - 3) = s - 2$ (Put in initial conditions and simplify)
   $\mathscr{L}(y) = \frac{s-2}{s^2-2s-3}$ (Solve for $\mathscr{L}(y)$)

4. Prepare to take the Inverse Laplace transform. As forewarned, this requires partial fractions.

$$\mathscr{L}(y) = \frac{s-2}{s^2 - 2s - 3} = \frac{1/4}{s-3} + \frac{3/4}{s+1}$$

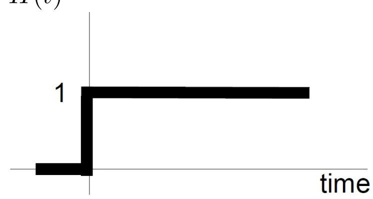
5. Inverse Laplace gives the answer:

$$y = \frac{1}{4}e^{3t} + \frac{3}{4}e^{-t}.$$

4.5 The Heaviside Function (Unit Step)

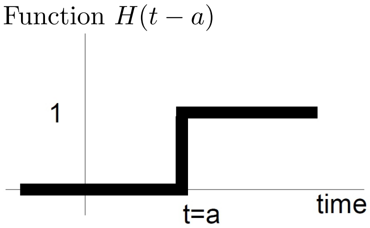
Now that you understand (or at least think you do) the Laplace transform, we can take a short detour through the Heaviside function. In other texts this is also commonly called the Unit Step Function (for reasons that will become apparent). Although the Heaviside function has some interesting side-effects when combined with the Laplace transform, the basic concept is really quite simple. The best explanation is relating the Heaviside function to a light switch. When you flip the switch, one of two things will happen: either the light bulb will turn on, or it will turn off. You can do whatever you like with the light switch. You can switch it on for a while and turn it back off whenever you choose. The only constraint though is that you are in one of two states, on or off (these could also be thought of 1 and 0). The same is true with the Heaviside function. This function $H(t)$ (which is quite boring by itself) is as shown in Figure 4.1. You need to know that the Heaviside function only turns on at $t = 0$.

Figure 4.1: Heaviside Function $H(t)$



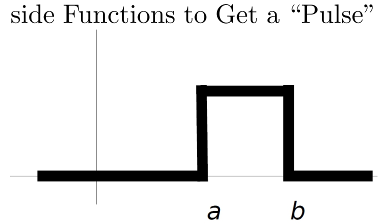
Because the Heaviside function only turns on at time zero, it is often more useful to turn it on at some other point. If you think about it, all that needs to be done is to perform a shift on the function. This is easily accomplished by replacing the t variable with $t - a$. If you recall from algebra this will indeed shift the function, and in fact, if a is positive, it will shift the function to the right. So from that thought process, you should gather that a shifted Heaviside function is written like this: $H(t - a)$. Refer to Figure 4.2. Remember, all this does is tells the Heaviside function where to turn on. If you want it to turn on at point a, just write it as $H(t - a)$.

Figure 4.2: Shifted Heaviside Function $H(t - a)$



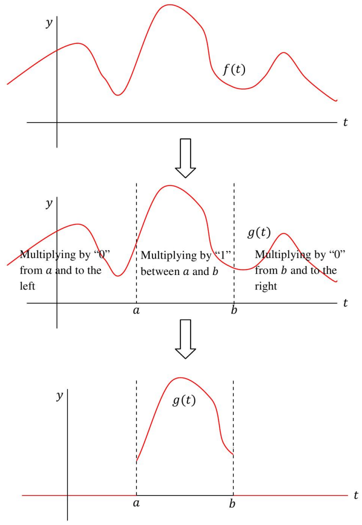
Now comes the most interesting Heaviside variation so far. What if you want to turn the light switch on and then off at some later time? Again, after a little thinking (you actually don't need to do the thinking part, just keep reading), you'll realize that this is a fairly simple modification. All you need to do is take a Heaviside function that turns on at point a and subtract from it another Heaviside function that turns off at point b. The

Figure 4.3: Subtracting Heaviside Functions to Get a "Pulse"



reason this works is that you are taking one Heaviside function which assumes a value of 1 at the point a, and subtracting another Heaviside function which becomes a 1 at the point b. Since both functions are a 1 for infinity after they turn on, it is easy to see that a $1-1=0$. This has the desired effect of turning on the Heaviside function at point a and then turning it off at point b. See Figure 4.3 to aid your understanding (you are reading this because you care right?). Mathematically this looks like: $H(t-a)-H(t-b)$. Often this is written as short hand: H_{ab}.

Figure 4.4: "Masking" Graphs with Heaviside



As you've probably noticed, if you haven't fallen asleep by now, the Heaviside function is really quite boring. All it does is turn on and off—zeros and ones. You would probably like a little more action. So here it goes, we will begin by multiplying various functions by the Heaviside. Sounds thrilling doesn't it? This isn't quite as exciting as it sounds (you did think it was going to be exciting didn't you?). Since the Heaviside function is zero from negative infinity until point a, and 1 thereafter, all it will do is "wipe away" the function up to the point a. This is simply because zero multiplied by anything is zero and anything multiplied by one is itself. So the function, we'll call it $f(t)$, will remain intact where the Heaviside is a 1. The same applies to the H_{ab} variation. Only in the interval $a < t < b$ will $f(t)$ remain untouched. Everything surrounding that will be zero. Check out the cool graphical representation in Figure 4.4.

4.5.1 Laplace Transform of the Heaviside

Our next exploration of the Heaviside function is to calculate its Laplace transform. But wait! Can you even do that? I mean, it's this strange function consisting of zeros and ones. Can you even calculate a Laplace transform of something like that? The answer is, yes you can (otherwise why would we have a whole section dedicated to it?). Before we actually calculate the Laplace transform, it is useful to define the Heaviside as a piecewise function:

$$H(t-a) = \begin{cases} 0 & : t < a \\ 1 & : t \geq a \end{cases}$$

Now that these are written as piecewise functions, it should be relatively painless to take the Laplace transform. Remember that the Laplace transform is merely

Table 4.2: Even Shorter Table of Laplace Transforms

No.	If $f(t) = \cdots$	Then $\mathscr{L}\left(f(t)\right) = F(s) = \cdots$
12	$H(t-a)$	$\frac{e^{-as}}{s}$
13	$H_{ab} = H(t-a) - H(t-b)$	$\frac{e^{-as}-e^{-bs}}{s}$
14	$H(t-a) \cdot f(t-a)$	$e^{-as}\mathscr{L}(f(t))$

a harmless integral. This means you are just calculating the area under that curve (keeping in mind the e^{-st} term). All this involves is splitting apart the integral like this:

$$\mathscr{L}(H_a) = \int_0^\infty H(t-a)e^{-st}dt = \int_0^a H(t-a)e^{-st}dt + \int_a^\infty H(t-a)e^{-st}dt$$

Do you see how we are simply integrating from 0 until the Heaviside function turns on and adding that to the integral from there to infinity? But recall that the Heaviside function is zero until it turns on. Therefore, the first integral is actually zero. This leads to:

$$\mathscr{L}(H_a) = \int_0^\infty H(t-a)e^{-st}dt = \frac{e^{-as}}{s}.$$

Using the result above, it is actually trivial to calculate the Laplace transform of H_{ab}. Since $H_{ab} = H_a - H_b$, all we need to do is subtract the transforms themselves like this:

$$\mathscr{L}(H_{ab}) = \mathscr{L}(H_a) - \mathscr{L}(H_b) = \frac{e^{-as}}{s} - \frac{e^{-bs}}{s} = \frac{e^{-as} - e^{-bs}}{s}.$$

Through a similar process it can be shown that:

$$\mathscr{L}(H(t-a) \cdot f(t-a)) = e^{-as}\mathscr{L}(f(t)).$$

Putting this all together, we should add Table 4.2 to our list of Laplace transforms.

4.5.2 Inverse Laplace Transform of the Heaviside (and the shift idea)

Now that you know about the Laplace transform of the Heaviside function (and that it does indeed exist), you should be ready for taking the Inverse Laplace transform. But there are a few things to notice about going back and forth between the *t-domain* and the *s-domain*. First of all, you should notice that the Laplace transform of $H(ta)$ looks awfully like the transform of "1" only with an extra e^{-as} factor[1]. Also notice that the Laplace transform of a shifted Heaviside, $H(ta)$, multiplied by a shifted function, $f(ta)$, results in the Laplace

[1]Recall that the Laplace transform of a number, k, is $\frac{k}{s}$ for $s > 0$.

transform of the *unshifted* version of the function multiplied by this e^{-as} term. Here's the deal. If you are taking the Inverse Laplace transform of something and you end up with an e^{-as} term multiplied by some function, chances are your transform will involve the Heaviside function. The confusing part is that when going back to the *t-domain*, you have to remember to shift the function. This can be really annoying and hard to keep straight. Do I shift it when going forward as well? How much do I shift it? How do I know if this is really a Heaviside function? Hopefully these are questions we can answer to ensure that you can work with the Laplace transforms of the Heaviside function *without* screwing up the whole shifting thing.

So let's do a couple examples that illustrate what is actually going on (in case you didn't follow the discussion above). Suppose that you are given this function to find the Inverse Laplace transform of:

$$\frac{e^{-s}}{s-2}$$

No matter how hard you try, you won't be able to get this function to look anything like a function in your regular Laplace transform table. But notice that it is only that pesky e^{-s} term causing all the problems. If you automatically begin to think of the Heaviside function when you see e^{-as} terms in a Laplace transform, you should be in good shape. To begin with, start with figuring out what exactly is the function $f(t)$ in this case. If you strip away the e^{-s} part, you are left with a function that looks very nice for doing Laplace transforms. Using the formula $\mathscr{L}(H(t-a)\cdot f(t-a)) = e^{-as}\mathscr{L}(f(t))$, you can see that you will need to shift this function by some quantity a. If you look at the RHS[2] of the equation, you can see that a in this case is whatever s is multiplied by in the exponential term. In this case, it is a harmless 1. So now you know how much you need to shift by, viz. one.

Now would be a good time to actually take the Inverse Laplace transform of the function. Because $\mathscr{L}(f(t)) = \frac{1}{s-2}$, it is easy to see from the table that the transform is $f(t) = e^{2t}$. Next you need to shift this function by -1. This gives: $f(t+1) = e^{2(t-1)}$. Remember from algebra that shifting is merely replacing anything that is just a t with $t-a$. Finally, if you are still following the formula, you need to multiply the shifted function by the Heaviside function. Your final answer will then look like this:

$$\mathscr{L}^{-1}\left(\frac{e^{-s}}{s-2}\right) = H(t-1)e^{2(t-1)}.$$

The shifting idea also applies when going *forward* by taking the Laplace transform. Sometimes you will find that the Heaviside is shifted by a different amount than $f(t-a)$. For example, Suppose you have to take the Laplace transform of

[2] The RHS is where we are at in this stage of the process. We really want to make it to the LHS for our answer.

$H(t-1)t^2$. You would *really* like it to look like this: $H(t-1)(t-1)^2$. Then you would be in good shape. Your only option then is to *force* the function to look like that. This could be accomplished by subtracting 1 and then adding right after: $H(t-1)(t-1+1)^2$. This can then be written as: $H(t-1)[(t-1)^2+2(t-1)+1]$. As gross as this looks, *this* method of writing $H(t-1)t^2$ will actually allow you to take the Laplace transform of the beast. All it comes down to is being a little creative and really just forcing the symbols into submission. If the function isn't shifted by the right amount, *make* it be shifted by whatever means possible. Just make sure that everything you do is a legal, algebraic operation.

4.6 Dirac Delta Function

The Dirac Delta Function (also simply called the Delta Function) is another weird mathematical function that somehow has many practical uses. It's a lot more strange than the Heaviside function, however. We thought of the Heaviside function as a switch of some kind that you turn on and off. The Delta function is more like going to the doctor. Let me explain. You are sitting on the doctor's table (what are those called again), the one covered with wax paper. He brings over an ominous looking triangle-shaped hammer. You grit your teeth and shut your eyes. "Relax!" He says. You try, but the anticipation of getting struck is too much for you. You feel a quick strike on your knee that doesn't hurt, but feels funny as your leg jerks up and almost hits him in the face. "Good," you think, "He won't be doing that again."

The Delta function is very similar and is used to model *impulses* as inputs to a system. There are many physical phenomena that can be modelled quite well as an impulse of a Delta function. So what exactly is this Delta function? Well, it's simply a spike of mathematical...er...stuff, that is infinitely high, and infinitely short in time, but the area of it has a value of 1. What kind of crazy horse-talk does that mean? Well look at Figure 4.5.

The way we define this function is using a small, positive, infinitesimal quantity usually denoted as ϵ. For all numerical purposes, $\epsilon = 0$. But in theoretical mathematics, we think of it as essentially being infinitely small, or at least as small of a number as a computer could possibly compute[3]. The Delta function is one ϵ wide and $\frac{1}{\epsilon}$ tall. You can see that the area $A = \epsilon \cdot \frac{1}{\epsilon} = 1$ as we advertised. The fact that it has an area of 1 is of both great and little importance. Basically we want it to have a finite, controlled area, but we can multiply the whole function by a constant to get a new area if we need to.

We denote this mystical Delta Function as δ.[4] So, to sum this up since it's really not that mind-blowing, here's the formal definition of the Dirac Delta Function:

[3]Technically, $\epsilon \to 0$.

[4]...the greek letter "delta"...surprise.

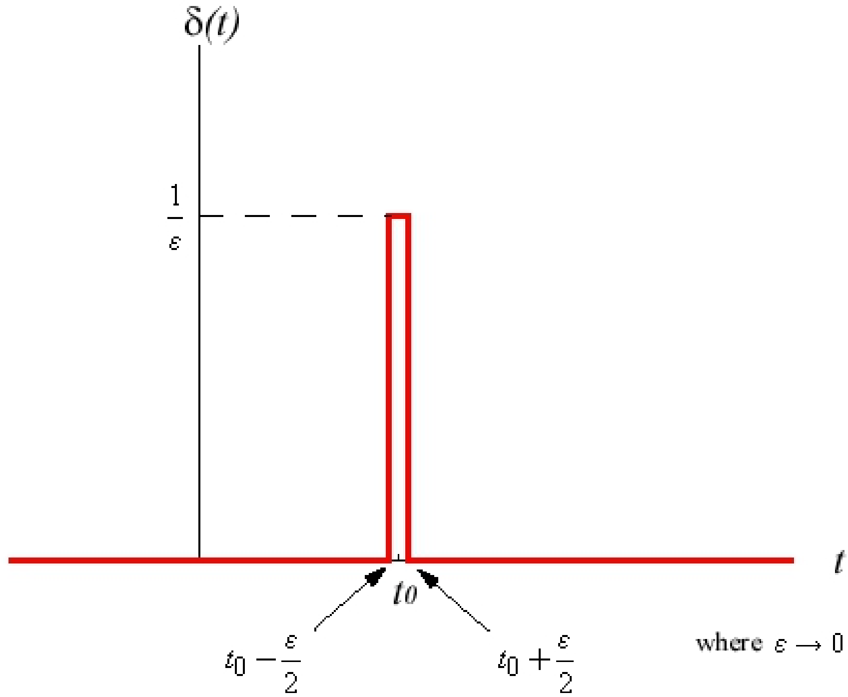


Figure 4.5: Dirac Delta Function

$$\delta(t) = \begin{cases} 0: & t \neq 0 \\ \infty: & t = 0 \\ \int_{-\infty}^{\infty} \delta(t) = 1 & \end{cases}$$

4.6.1 Laplace Transforms of the Delta Function

The Laplace transform of the Delta Function is actually quite simple. Recall that the Laplace transform is defined as an integral (which is simply the area under a curve). So if we want to find the Laplace Transform of the Delta Function, we simply need to know the area under it's "curve." Since we *defined* the area under the Delta function as exactly equal to 1, it kind of feels like cheating. But at least it's easy:

$$\mathscr{L}\left(\delta(t)\right) = \int_0^\infty \delta e^{-st} dt = e^{-s}$$

Yep, it's that simple. Well, okay. Maybe you'll have to think about that one for a few minutes. Just remember that delta function is zero for all of time except at $t \pm \epsilon$ and you are multiplying $\delta(t)$ by e^{-st}. If you don't really care, just accept it.

But here's the thing, we can use the Delta function to solve previously-unsolvable problems! Say you are solving a differential equation using the Laplace method and you end up with a function in the *s-domain* that looks like

$$F(s) = 3e^{-s}$$

You realize that there's nothing in your table of Laplace transforms that will help you...until now. If you realize that it's the delta function, you simply write the answer:

$$f(t) = 3\delta(t).$$

4.7 Discontinuous Forcing Terms

Suppose that before you is another harmonic motion problem. If you've read the section on harmonic motion you'll puff up your chest and after a long jungle wail and beating of your chest exclaim, "I can handle this!" But then you notice something a little odd about the question, something you didn't see before. You see a forcing term, but this time, the description in the problem says that the forcing term is zero until some point a where it turns on. This is similar to having an electric power source (which follows some sine wave pattern or such) but is disconnected from the harmonic circuit. Until the switch is pressed (connecting the power source to the circuit), the "forcing term" of the equation is zero. At this point your face pales and you run away from the problem screaming in mental agony.

However, the more observant reader will notice that a so-called discontinuous forcing term, could easily be written as a Heaviside function multiplied by the forcing function. If you then plan to solve the differential equation, you will see that you already have all the information to take the Laplace transform of both sides. So let's do an example:

Chapter 5

Linear Algebra Review

Before you ask, “Is this really necessary?” and skip this section, let me lend this word of advice: all the chapters from here on out depend heavily on your grasp of linear algebra concepts and your ability to work with matrices and such. So it is probably in your best interest to read this section and make sure you have a firm understanding of the linear algebra. There is a lot of random linear algebra review in the book, however, there are just a few concepts that you should be familiar with before continuing. So in the interest time, I will cover the most fundamental concepts that are required to get you going. For starters, we will begin with the simplest topic of putting systems of equations in a matrix and solving them. This should cover several topics that you need to know for later material. Then I will skip all the other material and cover eigenvalues and eigenvectors. The reason for this is that if you know how to calculate eigenvalues and eigenvectors, then you know all of the other material as well. So through the simple discussion of those two main topics, all the other material should come naturally.

5.1 Systems of Equations, the Matrix, and Solutions

This part of the review should be relatively painless since you *should* already be familiar with the concept of linear equations and rewriting a linear system of equations as a matrix. This section will assume that you have fundamental knowledge about systems of equations, matrices, and row-reducing. Here is an example. If you don’t know how this works, you will need to go back a little farther and review very basic linear algebra concepts since I will not cover them here. The following system of equations can be written as a matrix and row reduced like this:

$$\begin{matrix} x+y=3 \\ x+y=4 \end{matrix} \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right]$$

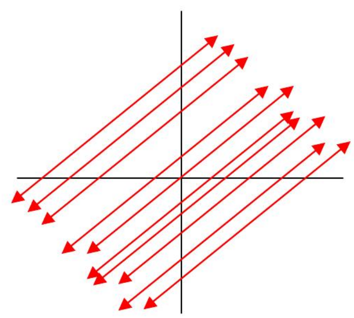
For starters, let's examine the types of solutions that can be obtained from systems of linear equations. It turns out that there are three possibilities. This comes from the fact that linear equations are only lines and there are only 3 ways to arrange a set of lines to form solutions.

1. No solution exists (you fail at life[1]).
2. One and only one solution (a smart cookie, you are).
3. Infinitely many solutions (you have a lot of potential but just can't decide what to do).

5.1.1 No Solution Exists

As an example of a system with no solution, look at the system of equations above. How do we know that this system has no solution? If you look at the bottom line of the reduced matrix, you will notice that this can be written as: $0x + 0y = 1$. Now think about it. Is there any possible combination of x and y that would make this equation true? Of course not. So this system of equations has no solution. Also, at a basic level, look at the equations themselves. Is there any way that two numbers when added together could equal 3 and then added together another time equal 4? Again, the answer is no. We can actually put pictures to these equations. Since they are linear equations, they end up being lines (surprise!). But you will see that the two lines do not intersect. If there was an intersection point, that point, (x, y), would be the solution (by definition). Figure 5.1 is not a picture of actual lines in the example. Instead, this is a picture of what multiple lines with no solutions look like. What would be required for a solution is all of the lines to intersect in one point. However, in the picture since they are all parallel, this cannot happen, hence no solution.

Figure 5.1: No Solution as Parallel Lines



5.1.2 One and Only One Solution

One solution occurs when a set of lines intersect in one point. This means all of the lines come to a single solution. For example, consider:

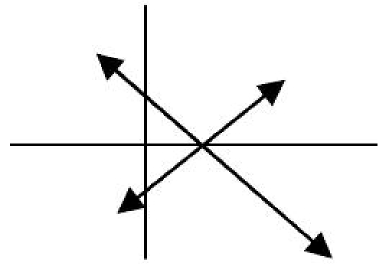
$$\begin{aligned} x + y &= 1 \\ x - y &= 1 \end{aligned}$$

[1]As an interesting exercise, have your math-savvy friends row-reduce a random matrix. Then tell them their fortune.

In this example the lines intersect in one place. Graphically, you get the picture in Figure 5.2. What does one and only one solution look like in a matrix? Putting those two equations into a matrix and row-reducing gives this:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right]$$

Figure 5.2: One and Only One Solution



Notice how each variable has a solution, there is a pivot in each row and column, you have an identity matrix on the left with augmented solutions on the right, your matrix was invertible to begin with.....however you want to look at it. Any of these ideas will get you there, namely, that this system of equations has one and only one solution.

5.1.3 Infinitely Many Solutions

The case of infinitely many solutions occurs when the lines from the equations intersect in infinitely many places. If you think about it (again, that thinking thing), the only way for this to happen is if the lines are in fact the same line. Although this seems kind of obvious and you might be wondering, "why don't we just look to see if they are the same line?" You should be aware that it is not always very easy to see this. Just to introduce the terminology, infinitely many solutions occur when your equations are linearly dependent. This means that one equation is simply a multiple of another. However, if you are working with larger systems of equations, it can be difficult to tell if they are so-called linearly dependent. This is where matrices and row-reduction comes in. So here is a simple case:

$$\begin{aligned} x + y &= 1 \\ -3x - 3y &= 1 \end{aligned}$$

Putting this into a matrix and performing row-reduction gives:

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Notice how there is no pivot in the y column. This means that there are infinitely many solutions. The variable y is not defined as anything so it can be anything it wants and still satisfy the equations. This is where the infinite solutions come from. Also remember that these equations are linearly dependent. If you look at the equations themselves, you can see that the second equation is simply -3 multiplied by the first. The fact that they are multiples of each other is the very definition of linear dependence. Keep in mind that all of this applies to higher-dimensional systems. I am only using 2×2 systems for the sake of making the concepts blatantly obvious without the complications of higher-dimensions.

5.2 Eigenvalues and Eigenvectors

Before we jump right into how to calculate eigenvalues and eigenvectors, it would be desirable to actually know what those things are to begin with. It's actually not too bad after you spend some time with it (or after you read this section hopefully). Let us begin with an eigenvector. Suppose you have a matrix $\mathbf{A}$, and a vector $\mathbf{u}$ which are being multiplied together. What if the result of that multiplication was another vector that was in fact some scaled version of $\mathbf{u}$? You reply, "yeah, so what if?" My answer to you is...if that's the case, then $\mathbf{u}$ is actually an eigenvector of $\mathbf{A}$. That's all? Big deal! That's actually all an eigenvector is. If you can find such a vector that when multiplied by $\mathbf{A}$ results in the same vector scaled by a constant, then you've accomplished your life's mission of finding the rare and valuable eigenvector. Here's the technical definition of an eigenvector:

> **DEFINITION:** An eigenvector of an $n \times n$ matrix $\mathbf{A}$ is a nonzero vector $\mathbf{x}$ such that $\mathbf{Ax} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an eigenvalue of $\mathbf{A}$ if there is a nontrivial solution $\mathbf{x}$ of $\mathbf{Ax} = \lambda\mathbf{x}$; such that $\mathbf{x}$ is called an eigenvector corresponding to λ.

Did you notice in the above definition that you actually got two definitions for the price of one? Now you know what an eigenvalue is as well as the eigenvector. Simply put, an eigenvalue is the scale factor that results from multiplying a particular eigenvector by $\mathbf{A}$. I won't go into all the details of why you want to find eigenvectors and the corresponding eigenvalues just yet. Instead, you need to know how to calculate them. Their uses will become apparent when we actually start solving systems of differential equations.

If you look at the equation $\mathbf{Ax} = \lambda\mathbf{x}$, you will see that it can be rewritten this way: $\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{0}$. Furthermore, if you factor out the $\mathbf{x}$ you can see that the equation looks like this: $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$. It turns out that the scalar λ is an eigenvalue of $\mathbf{A}$ if and only if the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$. has a nontrivial solution, that is, if and only if the equation has a free variable. If you recall anything about the invertible matrix theorem from linear algebra (and you probably don't), then you will remember that a matrix has a free variable only when it is not invertible which means its determinant is zero. So here's the reasoning, if we can find values of λ that make the determinant zero, then we are set and those values must be our eigenvalues. Finally I can show you the equation you will use to find eigenvalues.

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0}$$

Really all you need to do is subtract λ from the main diagonal of $\mathbf{A}$, take the determinant and set it equal to zero. Then solve for λ. Here is an example.

Given the following matrix, find the eigenvalues.

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

1. Find $\mathbf{A} - \lambda\mathbf{I}$ first.
$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 1-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix}$$

2. Find the determinant of that matrix, set it equal to zero and solve it for λ.
$$(1-\lambda)(3-\lambda) - (0)(0) = 0$$
$$(\lambda - 3)(\lambda - 1) = 0$$
$$\lambda = 1 \text{ and } \lambda = 3$$

Notice that the eigenvalues are actually the same as the entries along the diagonal. This is more than a coincidence. It turns out that the entries along the diagonal of a *triangular* matrix are actually equal to the eigenvalues. If you see a triangular matrix, the eigenvalues are on the diagonal. Don't try this with any matrix other than a *triangular* one though; it won't work.

Now it is about time for you to learn how to calculate an eigenvector that goes along with a particular eigenvalue. Although I won't go through the arguments here (that's for a linear algebra course), it turns out that the eigenvectors can be found by row-reducing the matrix $(\mathbf{A} - \lambda\mathbf{I})$ (for the particular λ that you want to use) and writing the solution in parametric vector form. Don't worry if this sounds intimidating to you. All it means is row reduce the augmented matrix and write solutions for each of the variables in terms of the free variable (you *will* have a free variable if λ is actually an eigenvalue). So let's review how to do that.

For a simple example, let us calculate an eigenvector for $\lambda = 3$ for the previous example. For starters, we need to calculate the matrix $(\mathbf{A} - \lambda\mathbf{I})$. This is as simple as subtracting 3 from each of the diagonals.

$$\mathbf{A} - \mathbf{3I} = \left[\begin{array}{cc|c} 1-3 & 0 & 0 \\ 0 & 3-3 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Suppose the variables in this case are x_1 and x_2. Take a look at what x_1 is equal to. It's obvious that it equals zero, but what does it equal in terms of x_2? If you just take the top row of the matrix and write the equation you get: $x_1 + 0x_2 = 0$. Now take a look at what x_2 is equal to. It looks kind of funny but you can write it like: $0x_2 = 0$. If you think about it, x_2 can be equal to anything and it will satisfy this equation. So a standard way of saying this is, $x_2 = x_2$. If you follow this process, you should be able to write the solution in parametric vector form like this:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Do you see what we did? All that happened was to rewrite the two equations we got for x_1 and x_2. If you were to actually multiply the x_2 into the vector on the right, you'd quickly notice that $x_1 = 0$ and $x_2 = x_2$ as desired. This is what is called parametric vector form. Now, as I said before, I won't go through the arguments here, but it turns out that the vector $[0, 1]$ is actually the eigenvector we have been looking for.

Follow these steps to calculate the eigenvalues and eigenvectors for a given matrix:

1. Calculate $\det(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0}$.
2. Set the resulting equation equal to zero and solve for λ. Save these as your eigenvalues.
3. Once you find the eigenvalues, calculate $\mathbf{A} - \lambda\mathbf{I}$ for one of the eigenvalues that you found (plug it into the equation and calculate it that way).
4. Augment the matrix that you get after plugging in your eigenvalue with the zero vector and row-reduce it.
5. Write your solution in parametric vector form.
6. Take the vectors multiplied by the free variable(s) from your parametric vector form and those become the eigenvectors.
7. Try to have a nice day after you probably screwed it up with eigenvectors and such.

Chapter 6

Introduction to Systems

Chapter 7

Solving Systems of Differential Equations

7.1 Graphing Solutions of Systems

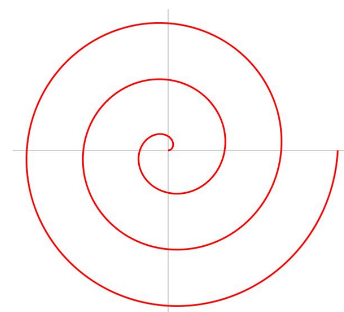
You have five possibilities to classify the type of solution under: saddle, nodal sink, nodal source, spiral sink, and spiral source. Each of the five categories has specific characteristics that make it unique and very different from the other four categories. It is important to notice that the picture (i.e. graph) of two different categories can look the same, but the characteristics of the graph (or function) distinguish it from and rule out four of the five possibilities really quickly once you know the properties. We need to start with some definitions for this stuff to make sense.

The pictures that get drawn from the equations are called phase portraits. Phase portraits, just like a phase lines from section 2.9-ish area, describe what is happening at any given point. Applying what you know about the phase lines to a phase portrait may be useful. Do you remember from eons ago that a phase line describes stability at a critical point? This is a similar concept, but now we are applying the concept to a graph as a whole, not just a single point. So, to be a stable graph, all of the arrows you plot on the graph must be going towards a single point. It is unstable otherwise.

All of the words that create the phrase you use to label a graph mean something. Knowing the practical definition of each word can make a big difference in understanding. So what does it mean to be a node or nodal? If you look up the definition of “node” on dictionary.com, they provides one definition that is quite nice: a centering point of component parts. A node or being nodal is a type of solution where each and every part of the solution (i.e. every curve that is known to exist for the function) goes toward a single point and eventually will meet at that point (usually the origin). This includes the half-lines as well because the half-lines are just degenerate cases of the curves. What do I mean when I say degenerate? Being a degenerate solution is a very specific case of the solution. Yes, you can argue then that every solution is degenerate but you would be arguing a lost cause. Why would you be arguing a lost cause though? You would be arguing a lost cause because degenerate solutions can be seen without graphing a thing. They are the absolute simplest solution to the system.

Now, what do I mean when I say half-lines? A half-line is the thing that every other solution will come towards at some time. This means that every solution, at some point, will want to look like one or both of the half-lines[1]. Now, the next term you need to know is spiral. A spiral is a type of solution that looks exactly like what you think it does. It will look something like Figure 7.1. A source is a class of solutions that, as time goes forward, tells everything to come away from the origin. Hopefully I am not stretching you mind to much if I tell you going backwards in time in a source makes the solutions get closer to the origin. A sink is a class of solution that (as time goes forward) gets closer to the origin; as time goes backwards, the solution gets farther away from the origin. The final word you need to know is saddle. A saddle is a type of solution where all bets are off. Some parts of the solution will force it toward the origin and some parts of the solution with force it away from the origin. In the end, neither group will win so the solution gets "confused" at a point and gives up.

Figure 7.1: A Generic Spiral (Don't look at this too long!)



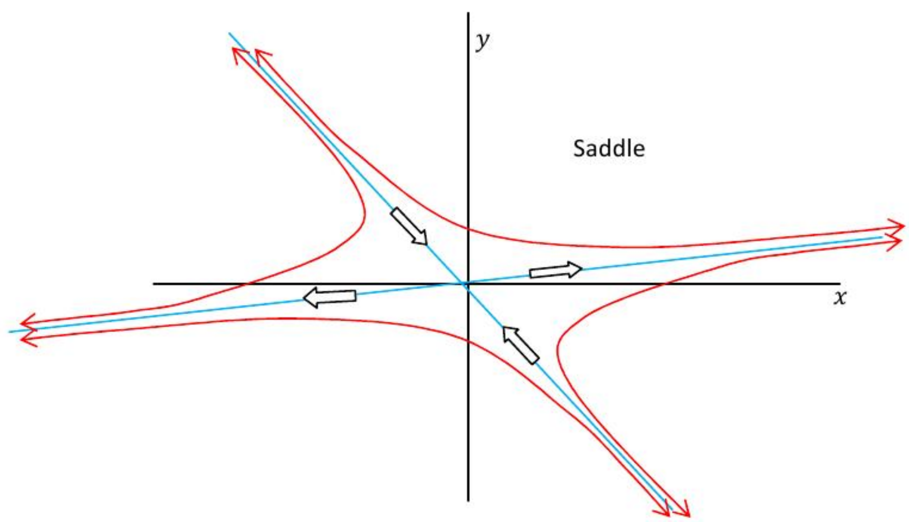
I just gave you a lot of definitions and you are probably thinking "What in the world did he just say?" That's okay, here are some pictures to help explain what I was saying. I will start with what a saddle looks like, then I will do nodal sink, nodal source, spiral sink, and spiral source in that order. Be patient in the explanation. I am going to wave my hands a bit and then explain why certain things are true in my example. The blue lines are the half-lines and the red lines are solutions. It is important to note that I have only graphed a limited number of solutions in the following five phase portraits. You need to know that there are infinitely many solutions in all five categories depending on the initial conditions you choose.

[1]The peer pressure from these two half-lines must be pretty big because every solution wants to be like them. Maybe the half-lines are just violent and beat every other solution into submission.

7.1.1 Saddle Solutions

Figure 7.2 is what a saddle will look like. Notice that a saddle is unstable because all of the arrows are not pointing towards a single point. Another important thing to notice is how I didnt label the solution lines with arrows, I only labeled the half-lines. In a saddle, the solutions are following directions from the half lines. The only reason to put arrows on the solution lines to the saddle is to know which way time is going. That should be fairly easy to understand though. If you remember that a source pulls you away from the origin and sink pushes you towards the origin, you will know where to put the arrows.

Figure 7.2: Saddle Solution

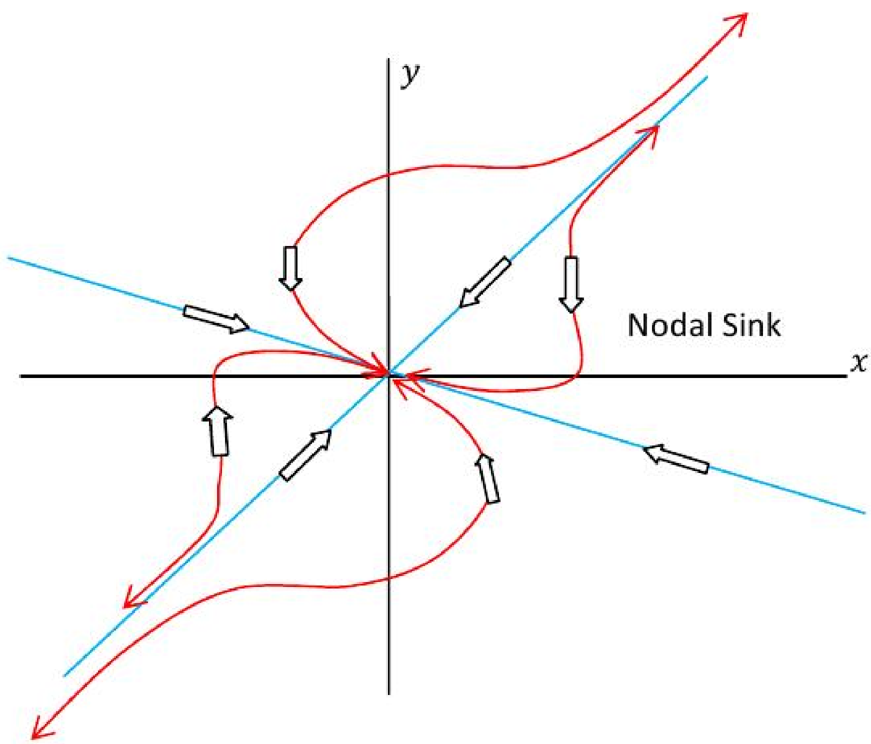


So this is what it looks like, but how do you know that a set of equations forms a saddle? A saddle is formed from two real eigenvalues. One eigenvalue will be positive and the other will be negative. Now, lets assume you dont know where to find the eigenvalues in the equation and you dont know how to find them given a matrix. It is still possible to tell given the general solution to a system of equations. You need to graph the half-lines and check which way the arrows are pointing. I will do an example later in real detail to explain how this will be done. Once you know how to do this for one category, this process will work to graph any of the other four categories.

7.1.2 Nodal Sink

Now for the nodal sink. A nodal sink looks something like Figure 7.4. A nodal sink is stable because all of the arrows are pointing towards a single point. Recall that "nodal" means everything comes to one point. Look at the graph. Do you notice how everything is meeting up at the origin (a single point)? Do you remember that "sink" means head towards a single point (again, the origin)? Everything in this picture is meeting at a point and is moving towards a single point. That is the condition for a nodal sink: everything must meet at a single point and head toward the point. Now, how could you tell from a set of eigenvalues that you have a nodal sink? The eigenvalues will be real numbers and both eigenvalues will need to be negative.

Figure 7.3: Nodal Sink Solution

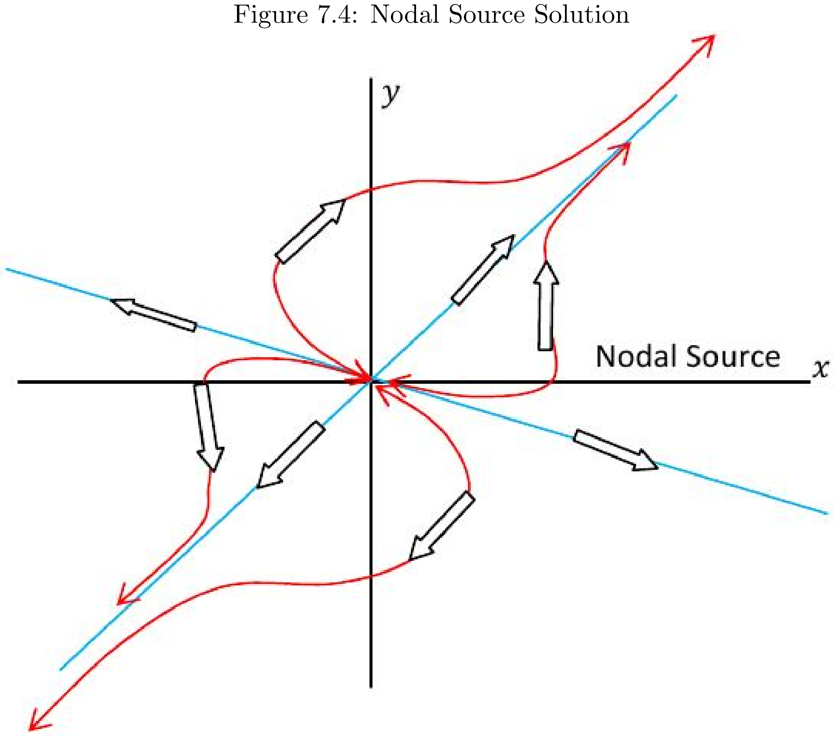


How could you tell you have a nodal sink from an equation assuming you didnt know how to find the eigenvalues in the equation and you weren't given a matrix to start with? Again, plot the half-lines with arrows and check whether the arrows are pointing toward the origin or away from the origin. If they are pointing toward the origin, you have a nodal sink.

7.1.3 Nodal Source

A nodal source is the same concept as a sink, only the arrows are reversed. This means that a nodal source, using the process of elimination, is unstable. “Nodal” means “meet at a single point (the origin)” and “source” means “head away from that point.” The condition for a “nodal source” is that all of the lines need to meet at a single point and they need to head away from the point.

Figure 7.4: Nodal Source Solution

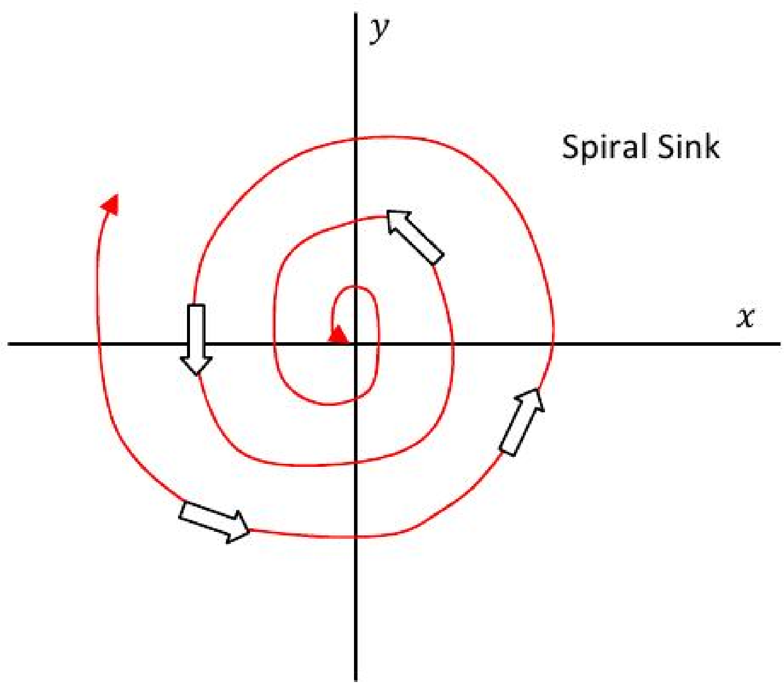


How could you tell from only a set of eigenvalues that you have a nodal source? Both eigenvalues will be real numbers and both eigenvalues will be positive. Now lets assume you dont know where to find the eigenvalues in the equation and you are not given an initial matrix to work from, how would you know you have a nodal source? Plot the half-lines and then check which directions the arrows are pointing. If all arrows are pointing away from the origin, then you have a nodal source.

7.1.4 Spiral Sink

Spirals are a bit more involved, so be especially patient with yourself (and us) in this explanation. Spirals are an attempt at being a conic section[2]. If you think about it, you will eventually come to the conclusion that a spiral looks something (in a really weird sense) like a circle. A circle (when defined parametrically in terms of sines and cosines) just repeats itself like a broken record if you consider all possible values to put in. Well, a spiral (in a sense) is periodic as well. In a spiral, you will repeatedly cross both the x and y axes. The difference between a spiral and a circle is that a circle has a scale factor of one (i.e. nothing is growing or shrinking the value that can be obtained, so you stay on the same path) where as a spiral has a scale factor that is a function that is non-zero and not equal to one. It is the scale factor that is moving the point off the path on every period. For example, if I scale a parametric equation by $\{e^t | t \in \Re\}$, as t is really large, the function is obviously different then if t is small, or negative.

Figure 7.5: Spiral Sink Solution



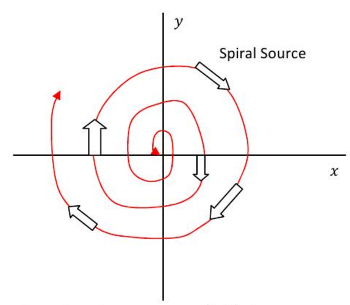
[2]For those of you that dont remember, a conic section is a hyperbola, parabola, or ellipse. Yes, a circle is a conic section, but a circle is just a very special (or degenerate case) of an ellipse.

Keeping that in mind, we are ready to talk about spirals. Spirals look like what you normally think of when you hear the word spiral. I want to apply the vocabulary again, so remember that "sink" means "to go toward the origin (or a single point)" so, a "spiral sink" is something that spirals toward the origin. But how do you know this from the eigenvalues? I dont want to go through all of the work explaining that to you in writing because I dont think I can do a decent job, but both of the eigenvalues need to be complex and the real part of the eigenvalue needs to be negative if you have a spiral sink. What do I mean when I say the real part? If you recall, every complex number is written in the form of $a \pm bi$ (a is the real part and b is the complex part). If a is negative, you have a sink. Let me finish my description of a spiral source and then we will talk about how to tell this from the equations.

7.1.5 Spiral Source

Finally, a spiral source is a spiral that spirals away from the origin. You can tell this from the eigenvalues. If your eigenvalues are complex and the real part of the eigenvalues are positive (i.e. a is positive), then you have a spiral source. How do you know from the equations that you have a spiral source or spiral sink assuming you cant find the eigenvalues in the equation and you dont have an initial matrix to work from? For arguments sake, lets assume you are given the general solution. Look to see if your eigenvectors are sines and cosines. If this is true, then you have a spiral. That much is guaranteed. To determine whether it is a source or a sink, you absolutely need an initial set of equations. Assuming you are given two initial equations, pick a point that lies very, very close to or on the spiral (on the spiral is better). Put that point into the initial set of equations. Because a spiral is parametrically defined, you are going to get a vector out as your answer. Plot that vector from the point you chose. Treat the point you choose as the origin (even if it isnt) so the vector will graph fairly easily and quickly. Depending on which way the vector points in relation to the spiral tells you whether you have a sink or source. If the vector points in the direction of the spiral that heads toward the real origin, then you have a sink. If the vector points in the direction of the spiral that heads away from the real origin, then you have a source. Hopefully that made sense.

Figure 7.6: Spiral Source Solution



Chapter 8

Series Solutions to Differential Equations

If you recall from the chapter on the Laplace Transform, I used the concept of summations to motivate the idea of a transform. Now we are going to use sequences and series more as they were intended. The concept of solving differential equations using series is probably quite foreign, and I agree, it is really weird. After you grasp the concept, the idea behind it is really ingenious and cool, but the actual implementation can be quit a pain. Basically, the idea isn't too bad, but if you aren't amazingly comfortable with sequences and series, doing the problems can be quite a challenge.

8.1 Ummm...What Are They?

Series solutions to differential equations isn't so different from the idea of Undetermined Coefficients for solving second-order differential equations. Basically, we decide to "guess" a solution as an infinite sequence and try to come up with coefficients that make it work. First, let me try to motivate how this works with series. Suppose we have a differential equation,

$$y' + yx = 0.$$

This isn't too bad to solve using conventional methods we've already talked about (think variable separation here), but just humor me here.[1] Since we know the answer will be some function $y = f(x)$, we simply "guess" a solution in the *form* of a power series:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

[1] No, I don't actually find this funny. :)

Now the idea is that we take the derivative of $y(x)$ (it's actually not too bad):

$$y'(x) = \sum_{n=0}^{\infty} a_n n x^{n-1}.$$

and plug the whole thing into our differential equation:

$$y' + yx = 0 \to \sum_{n=0}^{\infty} a_n n x^{n-1} + x\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Then it's simply a matter of solving for various unknown coefficients (like all the a_n terms). Okay, that's not too bad, right? Basically just guess a series solution, plug it in and make the coefficients work out. However, the hardest part is solving for coefficients because this often involves being quite clever with moving around an infinite set of terms and grouping things into patterns. But if you get the hang of it, you'll probably have more power than you know what to do with.

Now why are we able to guess a power series and make it work? How do we know that a power series will be the solution? The answer is, we don't. If you recall back in the good ol' days of calculus, we discussed how functions can be *represented* by a power series. There are various conditions for this to work though, such as the radius of convergence for the series, etc. But for the most part, many of our usual, friendly functions such as e^x, $\sin(x)$, and $\frac{1}{1-x}$ have power series expansions which can be found using a Taylor series. The concept here is that although we don't know if the solution to a particular differential equation can be written in terms of elementary functions, or if we have to find some series expansion for it, but we make the fundamental *assumption* that the the solution to a differential equation can at least be *written* in the form of a power series. Then we have a go at it and see if it works.

This concept allows us to solve many previously unsolvable problems. Specifically, many non-linear differential equations do not yield closed-form solutions We will hopefully be able to write down a power series solution that, if all the infinite terms are taken into account, will provide the exact solution. But in practice, we cannot compute an infinite number of terms, so we will usually truncate the series at some term and use it as an approximation. In fact, this is how Newton would calculate many integrals—write them out in a power series and compute each term by hand until he got the accuracy he wanted.

8.2 Solving Differential Equations Using Series Solutions

In our class, the idea of series solutions was somewhat a footnote near the end of the course. Basically, "These exist. Now let's take the final exam." So...this is obviously a very cursory overview. First we'd recommend you review sequences and series from your calculus book, and then jump into these problems. Again, the idea is simple, implementation, not so much. So, let's do a *very* simple example so you hopefully don't get too lost in all the math. Let's try and solve the following differential equation:

$$y' = y.$$

The beauty in this is that we already know the answer. Of course, $y = Ce^x$.[2] It's basically the most simple differential equation in existence. But let's try and solve this using a power series method. Let's guess that the solution will be,

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

To be able to plug it in the equation, we need the derivative of the power series:

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}.$$

Now, we will plug in both y and y' into the equation $y' = y$:

$$\sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n.$$

Now we have this messy looking equation, and we want to solve for a_n. That is, we want to find a general expression for a_n that will satisfy this equation. Don't panic yet. This is the hardest part, but it just takes some time. To start, let's begin by expanding some terms on both sides of the equation.

$$0 + a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Now here's where the "Undetermined Coefficients" comes in. We are going to relate *like terms* on both sides of the equation. That is, we are going to say that the coefficients of x^3 on the LHS are equal to the coefficients of x^3 on the RHS. It only makes sense, right? So, equating all the coefficients for like terms (even constant terms with no x).

$$\begin{aligned} a_1 &= a_0 \\ 2a_2 &= a_1 \\ 3a_3 &= a_2 \\ 4a_4 &= a_3 \\ 5a_5 &= a_4 \\ &\vdots \end{aligned}$$

[2] The constant C is the initial condition and will be important in a minute.

If you look closely, we can actually relate all the coefficients back to a_0 by a series of back solving. Basically we want to rewrite these equations all in terms of one common coefficient, and in ascending order, $a_0, a_1, a_2, \cdots$, until we can figure out what the pattern is and write a_n.

$$\begin{aligned}
a_1 &= a_0 \\
a_2 &= \frac{a_1}{2} = \frac{a_0}{2} \\
a_3 &= \frac{a_2}{3} = \frac{a_0}{2 \cdot 3} \\
a_4 &= \frac{a_3}{4} = \frac{a_0}{2 \cdot 3 \cdot 4} \\
a_5 &= \frac{a_4}{5} = \frac{a_0}{2 \cdot 3 \cdot 4 \cdot 5} \\
&\vdots
\end{aligned}$$

Are you seeing the pattern? For each value of n, we're just dividing a_0 by $n!$. So here's our "answer."

$$a_n = \frac{a_0}{n!}$$

Now we can't forget what we're doing. Remember we are finding the coefficients for our power series which we guessed as a solution. So let's plug our new coefficient expression into it:

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n.$$

If you have any recollection of the subject of Taylor series from calculus, you may remember that the power series representation of e^x is

$$y = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

And notice that's exactly what we got!! Well, ignoring the a_0 term. But it turns out that a_0 is simply the initial condition. Since it doesn't depend on n, we can pull it out of the power series giving us $y = a_0 e^x$.

There. How cool is that? We can guess that a solution can be written as a power series, take derivatives, plug it in, solve for coefficients, and bamo! we have a solution! In this case, we already knew the solution, and we showed that the power series was equivalent with that solution.

Appendix A

Practice Exams

A.1 Exam Information

The following appendices contain several sample tests and exams[1]. To clarify where these came from, here is a rundown:

- **Appendix A.2 - Sample Test 1** The practice test provided by our instructors before the first exam.
- **Appendix A.3 - Sample Test 2** The practice test provided by our instructors before the second exam.
- **Appendix A.4 - Test 1 With Answers** Actual exam administered in Fall 2009.
- **Appendix A.5 - Test 2 With Answers** Actual exam administered in Fall 2009.
- **Appendix A.6 - Test 3 With Answers** In our class we had a comprehensive final with perhaps a third of it covering material since the second exam. We were not allowed to take back or make copies of our final exams like we were with the previous ones, so this is the substitute we created ourselves. However, this Test 3 With Answers is not a comprehensive final. Instead it is meant to represent a true third exam (before the final). The questions on this exam are probably more difficult than what would be found in a real exam situation but should be useful for studying the material covered in chapters 7 - 11.

We would encourage you to not study for the sake of the exams, but instead to study for the sake of learning the material. Putting in the extra effort to read these notes, doing extra practice problems, and making every effort to

[1]Sponsored by that annoying high school teacher who always gave you "surprises" on the exams.

actually understand the material instead of just copying the solutions manual or mimicking the problem solving steps your instructor shows you, will absolutely ensure your success in this class. If you merely study these practice exams and are only in it for the grade, regardless of your performance on the exams, you will come away from this class with only a mediocre understanding of differential equations. Although, these sample tests are provided as tools to guide your studying so you can do well on the exams, just keep in mind that what you walk away with from this course is what you put into it.[2]
We would recommend the following study plan in case you care.

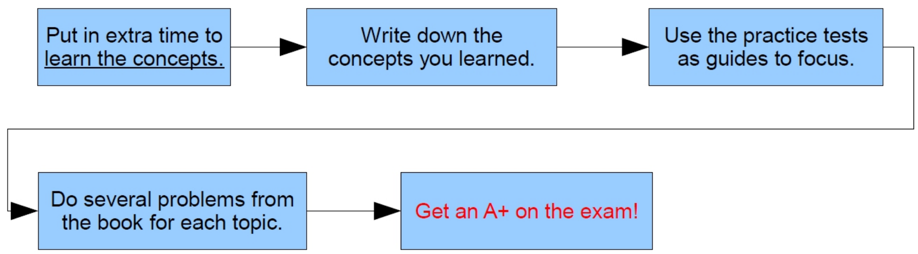


Figure A.1: How to Ace the Exams

The first two steps are the most important. It is essential that you take the time to really understand the concepts. Then it is also important that you write down (in your own words) what those concepts mean like we did with these notes. If you don't want to write anything down, find someone who will patiently listen to you explain the concepts. You can verify that you really know the concepts by outwardly explaining them somehow. It is one thing to understand the concepts in your head, but entirely another to convey that information to someone else (which is essentially what an exam situation is: can you apply the concepts and explain your steps/results to the grader?). Even if you don't remember all the nitty-gritty details of solving a particular kind of problem, you should be able to figure it in an exam out just by understanding the concepts. Again these practice tests will be useful for you to see which topics you might need to focus on or what kind of problems may be on the exam. Finally, it is prudent to practice the application of the concepts by doing as many problems for each topic as you can. At this point, you have undoubtedly stopped reading this section long ago and moved on to something else, so I won't waste any more time here.

[2]Lecturing provided the by the "Brown-Nosed Student Handbook"....and your mom.

A.2 Sample Test 1

WSU Math 315 Sample Test 1 (Fall, 2009)

1. For the differential equation,

$$y' = y^5$$

   (a) Is the function $y = 0$ a solution?
   (b) Are there other solutions with $y(0) = 0$?

2. Solve the following first-order differential equation:

$$\frac{dy}{dt} = 2t(1 + 5y),\ y(0) = 1.$$

3. Find the general solution for the following differential equation:

$$\frac{dy}{dt} - y = 5t$$

4. For the differential equation,

$$(y^2 - x^2)dx + (2xy - \sin(y))dy = 0$$

   (a) Verify that the equation is exact.
   (b) Find the general solution

5. For the following equation:

$$\frac{dy}{dt} = -y^2 + 4$$

   (a) Find all equilibrium solutions.
   (b) For each equilibrium solution, determine it as asymptotically stable or unstable.

6. Find a *fundamental set of solutions* for the following equation:

$$y'' + 2y' + y = 0$$

7. Find the *general solution* for the following differential equation:

$$y'' + y' + 2y = 0$$

8. For the following differential equation, find the general solution and Wronskian determinant.

$$y'' + 2y' = 0$$

9. Find the solution for the following problem:

$$y'' + 3y' + 2y = 0,\ y(0) = -1,\ y'(0) = 1$$

A.3 Sample Test 2

WSU Math 315 Sample Test 2 (Fall 2009)

1. An object with mass $0.2kg$ stretches a spring $0.49m$. Suppose there is no damping force nor external force acting on the object. Suppose the object is set in motion from its equilibrium position with a downward velocity of $0.1m/s$.
   (a) Find the spring constant and the natural frequency of the motion.
   (b) Formulate an initial-value problem that describes the motion of the mass.
2. For the motion of the object in Problem 1,
   (a) Find the solution.
   (b) Find the amplitude of the harmonic motion.
3. Find the general solution for the following equation:
$$y'' + 3y' = 5e^{-t}$$
4. Find a particular solution for the following equation:
$$y'' - 4y = 3e^{2t}$$
5. Find the Laplace transform for
$$f(t) = (t+2)e^{-3t}$$
6. Let $g(t)$ be defined as follows:
$$g(t) = \begin{cases} 2t: & 0 \le t < 3 \\ 5\ : & t \ge 3 \end{cases}$$
   Find the Laplace transform for $g(t)$.
7. Find the Laplace transform for $3H_3(t) + 4\delta_2(t)e^{2t}$.
8. Find the inverse Laplace transform for
$$F(s) = \frac{1}{(s^2+1)(s^2+3)}.$$
9. Find the inverse Laplace transform for
$$F(s) = \frac{e^{-3s}}{s^2+2s+5}.$$
10. Use Laplace transform to solve the following equation:
$$y'' = \delta_2(t),\ y(0) = 0,\ y'(0) = 0.$$

A.4 Test 1 With Answers

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ID: 10877217

Name: Travis Mallett

Differential Equations, Math 315 , Section 1
September 28, 2009
NO CALCULATORS!

1. Find the solution of the following first order initial value problem using the method of integrating factors:

$y' = y + e^{-t}$, $y(0) = 1$.

$$y' - y = e^{-t}$$

$$u = e^{\int -1\,dx} = e^{-t}$$

$$y = e^{t} \int e^{-t} \cdot e^{-t}\, dt$$

$$= e^{t} \int e^{-2t}\, dt$$

$$= e^{t}\left[-\frac{1}{2} e^{-2t} + C\right]$$

$$= -\frac{1}{2} e^{-2t} \cdot e^{t} + Ce^{t}$$

$$y = -\frac{1}{2} e^{-t} + Ce^{t}$$

$y(0) = 1$

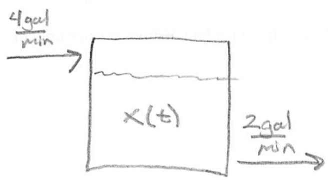
$$1 = \frac{-1}{2} + C$$

$$C = \frac{3}{2}$$

$$\boxed{y = \frac{-1}{2} e^{-t} + \frac{3}{2} e^{t}}$$

✓ $\overline{10}$

2. A tank contains 100 gallons of pure water. Water containing a salt concentration of 2 grams per gallon flows into the tank at a rate of 4 gallons per minute. The well-mixed salt solution in the tank is pumped out at the rate of 2 gallons per minute. What is the initial value problem that describes the amount of salt in the tank at any time t until the ~~time~~ tank overflows? DO NOT SOLVE THIS INITIAL VALUE PROBLEM.



$V = 100 + 2t \qquad x(0) = 0$

$$\frac{dx}{dt} = \left(\text{rate salt in}\right) - \left(\text{rate salt out}\right)$$

$$= \frac{4\,\text{gal}}{\text{min}} \cdot \frac{2\,\text{gram}}{\text{gal}} - \frac{2\,\text{gal}}{\text{min}} \cdot \frac{x}{100+2t}$$

$$= \frac{8g}{\text{min}} - \frac{2x}{100+2t}$$

$$\boxed{\frac{dx}{dt} = 8 - \frac{2x}{100+2t}, \quad x(0) = 0}$$

/10

3. Find the general solution, in implicit form, of the differential equation:

$$x' = \frac{2tx}{1+x}.$$

$$\frac{dx}{dt} = \frac{2tx}{1+x}$$

$$\frac{x+1}{x}\,dx = 2t\,dt$$

$$\int\left(1+\frac{1}{x}\right)dx = \int 2t\,dt$$

$$\boxed{x + \ln|x| = t^2 + C}$$

10

4. Find the general (implicit) solution of the following differential equation:

$$\overset{P}{(2x+y)}dx + \overset{Q}{(x-6y)}dy = 0$$

① Verify exact equation:

$P_y = Q_x$

$P_y = 1$ and $Q_y = 1$ ✓

② Find F from P

$F_x = P = 2x + y$

$F = \int (2x+y)dx + \phi(y)$

$= x^2 + yx + \phi(y)$

③ Find F_y from newly found F

$F_y = x + \phi'(y) = Q = x - 6y$

$\phi'(y) = -6y$

$\phi(y) = -3y^2$

④ Plug $\phi(y)$ into F

$$\boxed{C = x^2 + yx - 3y^2}$$ ✓

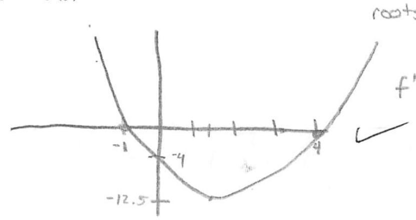
10

$y^2 - 4y + y - 4 = y^2 - 3y - 4$

5. The differential equation $y' = (y+1)(y-4)$ has the form $y' = f(y)$ with $f(y) = (y+1)(y-4)$.

(a) Sketch a graph of $f(y)$

4



roots to $f(y) = 0$ are $-1, 4$

$f'(y) = 0 = 2y - 3$

$y = \frac{3}{2}$

$f(\frac{3}{2}) = (\frac{5}{2})(\frac{3}{2} - \frac{8}{2})$

$= (\frac{5}{2})(-\frac{5}{2})$

$= -\frac{25}{2} = -12.5$

(b) Use the graph to develop a phase line for the differential equation.

4

On the intervals: $(-\infty, -1)$ $f(y) > 0$

$(-1, 4)$ $f(y) < 0$

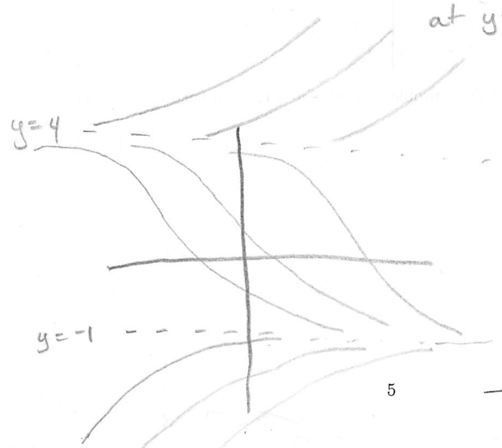
$(4, \infty)$ $f(y) > 0$

(c) Classify each equilibrium point as either unstable or asymptotically stable.

4

The equilibrium points: at $y = -1$ is Stable

at $y = 4$ is unstable



← unstable

← stable

/12

6. (a) Find the general solution in the form $y(x) = c_1y_1(x)+c_2y_2(x)$ of the initial value problem $y'' + y' - 12y = 0$.

try $\rightarrow y = e^{rt}$

$r^2 + r - 12 = 0$

$r = \frac{-1 \pm \sqrt{1-(4)(1)(-12)}}{2} = \frac{-1 \pm \sqrt{49}}{2} = \frac{-1}{2} \pm \frac{7}{2}$

$r_1 = \frac{6}{2} = 3, \quad r_2 = \frac{-8}{2} = -4$

$\boxed{y = C_1 e^{3t} + C_2 e^{-4t}}$ ✓

(b) Calculate the Wronskian of y_1 and y_2 from part (a).

$y_1 = e^{3t} \quad y_1' = 3e^{3t}$

$y_2 = e^{-4t} \quad y_2' = -4e^{-4t}$

$W = \begin{vmatrix} e^{3t} & e^{-4t} \\ 3e^{3t} & -4e^{-4t} \end{vmatrix} = -4e^{-4t}e^{3t} - 3e^{3t}e^{-4t} = -4e^{t} - 3e^{t}$ ✓

$= \boxed{-7e^{t}} \neq 0$

(c) Find the solution of the initial value problem for $y'' + y' - 12y = 0$ with $y(0) = 1$ $y'(0) = 0$.

$y = C_1 e^{3t} + C_2 e^{-4t} \quad y(0) = 1$

$1 = C_1 + C_2$

$y' = 3C_1 e^{3t} - 4C_2 e^{-4t} \quad y'(0) = 0$

$0 = 3C_1 - 4C_2$

$C_1 + C_2 = 1$

$3C_1 - 4C_2 = 0$

$\frac{3}{4}C_1 - C_2 = 0$

$C_1 + C_2 = 1$

$\frac{7}{4}C_1 = 1$

$C_1 = \frac{4}{7} \rightarrow C_2 = \frac{3}{7}$

$\boxed{y = \frac{4}{7}e^{3t} + \frac{3}{7}e^{-4t}}$ ✓

6

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A.5 Test 2 With Answers

100

+2 Travis Mallett
10877217

Differential Equations, Math 315
Midterm 2, November 9, 2009

14

1. A object with mass 0.5 kg stretches a spring 0.1 m (meters).

(a) Find the spring constant k. Assume that the acceleration due to gravity is $g = 9.8m/(sec)^2$.

$$K = \frac{mg}{x(0)} = \frac{.5 \times 9.8}{.1} = \frac{4.9}{.1} = \boxed{49}$$ ✓

$$\frac{4.9}{.1} = 4.9 \times 10$$

(b) The object moves in a viscous medium that imparts a viscous force of 8N (Newtons) when the speed of the object is 4 m/sec. Find the damping constant μ.

$$N = \frac{8N}{4m/s} = \frac{2s}{NM} = \boxed{2}$$ ✓

(c) Formulate the initial value problem that describes the motion of the object. Assume that there is no external force acting on the object and that the object is set in motion from its equilibrium position with an initial downward velocity of 2 m/sec.

DO NOT SOLVE THIS INITIAL VALUE PROBLEM!

Of the form: $my'' + Ny' + ky = 0$ (Change to x's)

$$= \boxed{.5x'' + 2x' + 49x = 0,\ x(0) = .1,\ x'(0) = -2}$$ ✓

1

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2. Find a particular solution of the differential equation

$y'' - y' - 2y = \sin t.$

① Homogeneous Version: $y'' - y' - 2y = 0$

$r^2 - r - 2 = 0$

$(r+1)(r-2) = 0 \rightarrow r = -1, \; r = 2$

$y_1 = e^{-t}, \quad y_2 = e^{2t}, \quad g(t) = \sin(t)$

② Variation of Parameters

$$V_1 = \int \frac{-e^{2t}\sin(t)}{e^{-t}2e^{2t} + e^{-t}e^{2t}}\,dt = \int \frac{-e^{2t}\sin(t)}{3e^{-t}\cancel{e^{2t}}}\,dt$$

Too hard to integrate right now.

$\boxed{\text{Undetermined Coefficients:}}$

$y = a\sin(t) + b\cos(t)$

$y' = a\cos(t) - b\sin(t)$

$y'' = -a\sin(t) - b\cos(t)$

$-a\sin(t) - \cancel{b\cos(t)} - \cancel{a\cos(t)}$
$+ \cancel{b\sin(t)} - 2a\sin(t) - 2b\cos(t)$
$= \sin(t)$

$\sin(t)(-a + b - 2a) + \cos(t)(-b - a - 2b) = \sin(t)$

$\sin(t)(b - 3a) + \cos(t)(-a - 3b) = \sin(t)$

$b - 3a = 1$
$-a - 3b = 0 \rightarrow a = -3b \rightarrow b + 9b = 1 \rightarrow b = \frac{1}{10} \rightarrow a = \frac{-3}{10}$

$$\boxed{y_p = \frac{-3}{10}\sin(t) + \frac{1}{10}\cos(t)}$$

2 ✓

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3. Find the general solution to the differential equation

$$y'' - 5y' + 6y = 2e^t.$$

① Homogeneous Version: $y'' - 5y' + 6y = 0$

$r^2 - 5r + 6 = 0$

$(r-3)(r-2) = 0 \rightarrow r = 3, \; r = 2$

$y_1 = e^{3t}, \quad y_2 = e^{2t}$

$y_h = C_1 e^{3t} + C_2 e^{2t}$

② Particular Solution

Try $y = ae^t \rightarrow y' = ae^t \rightarrow y'' = ae^t$

$\rightarrow ae^t - 5ae^t + 6ae^t = 2e^t$

$e^t(a - 5a + 6a) = 2e^t$

$2ae^t = 2e^t \rightarrow a = 1$ CHECK THIS: $e^t - 5e^t + 6e^t = 2e^t$

$y_p = e^t$ $e^t(1 - 5 + 6) = 2e^t$ ✓

③ General Solution:

$y_g = y_h + y_p = \boxed{C_1 e^{3t} + C_2 e^{2t} + e^t}$ ✓

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4. Show that the Laplace transform of $f(t) = e^{at}$ is $F(s) = \dfrac{1}{s-a}$ for $s > a$ using the definition;

$$F(s) = \int_0^\infty e^{-st} f(t)dt$$

✓

$$\int_0^\infty e^{-st} \cdot e^{at}\, dt = \int_0^\infty e^{at-st}\, dt = \int_0^\infty e^{t(a-s)}\, dt$$

$$= \frac{1}{a-s} e^{t(a-s)} \Big|_0^\infty \qquad \cancel{\text{If } s>a, \text{ then: } -\frac{1}{s-a} e^{-t(s-a)} \Big|_0^\infty}$$

$$= -\frac{1}{s-a} e^{-t(s-a)} \Big|_0^\infty = 0 - \frac{-1}{s-a} = \boxed{\frac{1}{s-a},\ s>a} \therefore$$

This is only true if $s > a$

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5. Find the Laplace transform for $2H_4(t) + \delta_2(t)e^{3t}$.

For this problem you may use Theorem 6.3 or Proposition 2.12.

Theorem 6.3: Suppose $p \geq 0$ and let ϕ be any function that is continuous at $t = p$. Then

$$\int_0^\infty \delta_p(t)\phi(t)dt = \phi(p).$$

Proposition 2.12: Suppose f is a piecewise continuous function of exponential order. Let $F(s)$ be the Laplace transform of f, and let c be any constant. Then

$\mathcal{L}\{e^{ct}f(t)\}(s) = F(s-c)$

Provides a shift

$$\mathcal{L}\left(2H_4(t) + \delta_2(t)e^{3t}\right) = \frac{2e^{-4s}}{s} + \underbrace{\mathcal{L}(\delta_2(t))}_{\text{shifted by } 3}$$

$$= \frac{2e^{-4s}}{s} + \underbrace{e^{-2s}}_{\text{shifted by } 3} = \boxed{\frac{2e^{-4s}}{s} + e^{-2(s-3)}}$$

14

6. Find the inverse Laplace transform of

$$F(s) = \frac{2e^{-2s}}{s^2 - 4}$$

$\mathcal{L}(H(t-c)f(t-c)) = e^{-cs}F(s)$

$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{2}{s^2-4}\right) = \frac{2}{(s-2)(s+2)} = f(t)$

$= \frac{A}{s-2} + \frac{B}{s+2} \rightarrow A(s+2) + B(s-2) = 2$

$s = -2 \rightarrow -4B = 2 \Rightarrow B = -\frac{1}{2}$

$s = 2 \rightarrow 4A = 2 \rightarrow A = \frac{1}{2}$

$= \mathcal{L}^{-1}\left(\frac{1}{2}\frac{1}{s-2} - \frac{1}{2}\frac{1}{s+2}\right)$

$= \frac{1}{2}e^{2t} - \frac{1}{2}e^{-2t}$

Now shift this by 2:

$\rightarrow \frac{1}{2}e^{2(t-2)} - \frac{1}{2}e^{-2(t-2)}$

$$\boxed{H(t-2)\left(\frac{1}{2}e^{2(t-2)} - \frac{1}{2}e^{-2(t-2)}\right)}$$

✓

14

7. Use the Laplace transform to solve the initial value problem

$y'' - 2y' - 3y = 0, \qquad y(0) = 1,\ y'(0) = 0.$

$$\underbrace{s^2 \mathcal{L}(y) - s\,\cancel{y(0)}^{\,1} - \cancel{y'(0)}^{\,0}}_{\mathcal{L}(y')} - \underbrace{\left(2s\,\mathcal{L}(y) - 2\,\cancel{y(0)}^{\,1}\right)}_{\mathcal{L}(y)} - 3\,\mathcal{L}(y) = 0$$

$$s^2\mathcal{L}(y) - s - 2s\,\mathcal{L}(y) + 2 - 3\,\mathcal{L}(y) = 0$$

$$\mathcal{L}(y)\left(s^2 - 2s - 3\right) - s + 2 = 0$$

$$\mathcal{L}(y) = \frac{s-2}{s^2-2s-3} = \frac{s-2}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$$

$$A(s+1) + B(s-3) = s-2$$

$$s = -1 \rightarrow -4B = -3 \rightarrow B = \frac{3}{4}$$

$$s = 3 \rightarrow 4A = 1 \rightarrow A = \frac{1}{4}$$

$$\rightarrow = \frac{1}{4}\frac{1}{s-3} + \frac{3}{4}\frac{1}{s+1}$$

$$y = \mathcal{L}^{-1}\left(\frac{1}{4}\frac{1}{s-3}\right) + \mathcal{L}^{-1}\left(\frac{3}{4}\frac{1}{s+1}\right)$$

$$= \boxed{\frac{1}{4}e^{3t} + \frac{3}{4}e^{-t}}$$

✓

14

A.6 Test 3 With Answers

Practice Exam 3
Differential Equations Fall 2009
Problems and Solutions

1. Is the following system linear? Is the system autonomous? Is the system homogeneous or inhomogeneous? If possible, write the systems of equations in vector form. Assume the independent variable is t.

$$\begin{aligned} x_1' &= -x_1 + tx_2 - t^2x_3 \\ x_2' &= -2x_1 + 2tx_2 \\ x_3' &= -t^2x_2 + \sin(x_1) + x_3\sin(t) \end{aligned}$$

SOLVED:
No, the system is not linear because $\sin(x_1)$ is in the x_3 equation.
The system is <u>autonomous</u>.
The system is <u>homogeneous</u>.
It is <u>not possible</u> to write the system in matrix vector form because $sin(x_1)$ cannot be separated.

2. Find the characteristic polynomial and eigenvalues for the matrix.

$$\begin{bmatrix} -2 & 3 \\ 0 & 5 \end{bmatrix}$$

SOLVED:
Using this formula for the characteristic polynomial,

$$\lambda^2 - (a+d)\lambda + (ad - cb) = 0,$$

plug in the numbers from the matrix:

$$\lambda^2 - (3)\lambda + (-10 - 0) = 0.$$

Simplify: $\lambda^2 - 3\lambda - 10 = 0 \rightarrow$ **Characteristic Polynomial**
Factor: $(\lambda - 5)(\lambda + 2) = 0$
Solve: $\lambda = 5$ and $\lambda = -2 \rightarrow$ **Eigenvalues**

3. Find the radius of convergence of the series.

$$\sum_{n=0}^{\infty} \frac{n!x^n}{(2n)!}$$

SOLVED:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| &= \lim_{n\to\infty}\left|\frac{n!x^n}{(2n)!}\cdot\left(\frac{[2(n+1)]!}{(n+1)!x^{n+1}}\right)\right| \\
&= \lim_{n\to\infty}\left|\frac{n!x^n}{(2n)!}\cdot\left(\frac{(2n+2)(2n+1)(2n)!}{(n+1)n!x^n x}\right)\right| \\
&= \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)}{(n+1)x}\right| \\
&= \left|\frac{1}{x}\right| \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)}{(n+1)}\right| = \infty
\end{aligned}$$

Thus the radius of convergence is $R = \infty$.

4. Initially, the first tank contains 100 gallons of pure water, the second contains 80 gallons of pure water, and the third contains 60 gallons of pure water. Salt solution containing 2 pounds of salt per gallon of water magically appears in the first tank at a rate of 5 gallons per minute. Salt solution from the first tank drains into the second tank at a rate of 5 gallons per minute. Finally, salt solution drains from the second tank into the third tank at a rate of 5 gallons per minute. Setup the system of equations with initial conditions that models the amount of salt in each tank over time. Place your solution in matrix vector form. Is the system homogeneous or inhomogeneous?

SOLVED:
Because it is pure water, $\bar{x}(0) = \langle 0, 0, 0\rangle$ represents the initial condition where the concentration of salt is zero when we start (i.e. pure water).

<u>Tank 1 Information:</u>
The rate of salt in, $x_1 = \frac{2lb}{gal}\left(\frac{5gal}{min}\right) = \frac{10lb}{min}$.
Rate of salt out, $x_1 = \frac{5gal}{min}\left(\frac{x_1 lb}{100gal}\right) = \frac{x_1 lb}{20min}$.
This leads to the differential equation: $\frac{dx_1}{dt} = \text{rate in} - \text{rate out} = 10 - \frac{x_1}{20}$.

<u>Tank 2 Information:</u>
Rate of salt in is $x_2 = \frac{x_1}{20}$ because it is coming from Tank 1's output.
Rate of salt out is $x_2 = \frac{5gal}{min}\left(\frac{x_2 lb}{80gal}\right) = \frac{x_2 lb}{16min}$.

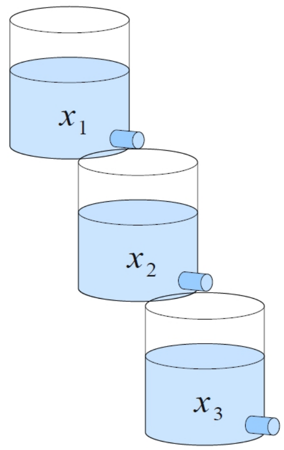

Figure A.2: Scary, Multi-dimensional Mixing Problem

This leads to the differential equation, $\frac{dx_2}{dt} = \text{rate in} - \text{rate out} = \frac{x_1}{20} - \frac{x_2}{16}$.

Tank 3 Information:
Rate of salt in is $x_3 = \frac{x_2}{16}$ because it is coming from Tank 2's output.
Rate of salt out is $x_3 = \frac{5gal}{min}\left(\frac{x_3 lb}{60gal}\right) = \frac{x_3 lb}{12min}$.
This leads to the differential equation, $\frac{dx_3}{dt} = \text{rate in} - \text{rate out} = \frac{x_2}{16} - \frac{x_3}{12}$.

System of Equations:
Finally we can write our system of equations in matrix vector form as requested.

$$\bar{x}' = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} -1/20 & 0 & 0 \\ 1/20 & -1/16 & 0 \\ 0 & 1/16 & -1/12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

Subject to the initial condition,

$$\bar{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This system is inhomogeneous because $\bar{f}(t) \neq \bar{0}$.

5. Show that the given functions are solutions to the system. Then verify that any linear combination is also a solution to the system.

$$x_1' = -x_1 + 3x_2$$
$$x_2' = 2x_2$$

$$\alpha(t) = \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} \text{ and } \beta(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$$

SOLVED:
We will need the derivatives of the solutions:

$$\alpha'(t) = \begin{pmatrix} -e^{-t} \\ 0 \end{pmatrix} \text{ and } \beta'(t) = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix}$$

Now just plug in the values.

Checking the α **vector:** $-e^{-t} = -(e^{t}) + 3(0)$ and $0 = 2(0)$. Both of these check out so the α vector is a solution.

Next we check the β **vector:** $2e^{2t} = -(e^{2t}) + 3(e^{2t})$ and $2e^{2t} = 2(e^{2t})$. Both of these check out so the β vector is a solution.

Now for the linear combination. The word *any* implies that there exist scalars c_1 and c_2 not zero such that $\gamma = c_1\binom{-e^{-t}}{0} + c_2\binom{2e^{2t}}{2e^{2t}}$ is also a solution.

So lets check γ: $-c_1e^{-t} + 2c_2e^{2t} = -(c_1e^{-t}) + 3(0c_2) - (2c_2e^{2t}) + 3(c_2e^{2t})$ and $0c_1 + 2c_2e^{2t} = 2(0) + 2(c_2e^{2t})$.

Since this works, **any linear combination of α and β is also a solution.**

6. Find the general solution of the system $y' = Ay$. Then find the solution of the initial value problem for the system with the given initial condition.

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \text{ given } \bar{y}(0) = [3, 2]^T$$

SOLVED:
The characteristic polynomial is: $\lambda^2 - 5\lambda + 6 = 0$.

Factor: $(\lambda - 2)(\lambda - 3) = 0$.

Solving yields the following eigenvalues: $\lambda = 2$ and $\lambda = 3$.

- If $\lambda = 2$:
$$A - \lambda I = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \to \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$$
This leads to the system of equations:
$$\begin{aligned} x_1 &= 2x_2 \\ x_2 &= x_2 \end{aligned}$$
which produces the eigenvector, $\binom{2}{1}$.
- If $\lambda = 3$:
$$A - \lambda I = \begin{bmatrix} -2 & -2 \\ -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$
This leads to the system of equations:
$$\begin{aligned} x_1 &= x_2 \\ x_2 &= x_2 \end{aligned}$$
which produces the eigenvector, $\binom{1}{1}$.

Finally we can write our **general solution:**

$$\bar{y}_{gen} = c_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Now we can apply our initial condition to obtain the system of equations,

$$\begin{aligned} 3 &= 2c_1 + c_2 \\ 2 &= c_1 + c_2 \end{aligned}$$

Solving this gives, $c_1 = 1$ and $c_2 = 1$.

Thus we can write our **particular solution:**

$$\bar{y}_p = e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

7. Find a fundamental set of real solutions of the system $y' = Ay$ with the given A.

$$A = \begin{bmatrix} -1 & -2 \\ 4 & 3 \end{bmatrix}$$

SOLVED:
The characteristic polynomial is found using this formula: $\lambda^2 - (a+d)\lambda + (ad - cb) = 0$.

Plug in the numbers from the matrix: $\lambda^2 - 2\lambda + 5 = 0$.

Solve, $\lambda = \frac{2 \pm \sqrt{4-4(1)(5)}}{2} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$.

If $\lambda 1 + 2i$:

$$A - \lambda I = \begin{bmatrix} -2-2i & -22 \\ 4 & 2-2i \end{bmatrix} \to \begin{bmatrix} 1+i & 1 \\ 2 & 1-i \end{bmatrix} \to \begin{pmatrix} 1 \\ -1-i \end{pmatrix}$$

$$\begin{aligned}
\bar{z}(t) &= e^{(1+2i)t} \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] \\
&= e^t(\cos(2t) + i\sin(2t)) \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] \\
&= e^t \left[\cos(2t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\sin(2t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\cos(2t) \begin{pmatrix} 0 \\ -1 \end{pmatrix} - \sin(2t) \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] \\
&= e^t \begin{pmatrix} \cos(2t) \\ \sin(2t) - \cos(2t) \end{pmatrix} + ie^t \begin{pmatrix} \sin(2t) \\ -\sin(2t) - \cos(2t) \end{pmatrix}
\end{aligned}$$

The real and imaginary parts form a **fundamental set of solutions:**

$$\bar{y_1} = \Im(z) = e^t \begin{pmatrix} \sin(2t) \\ -\sin(2t) - \cos(2t) \end{pmatrix}$$

$$\bar{y_2} = \Re(z) = e^t \begin{pmatrix} \cos(2t) \\ \sin(2t) - \cos(2t) \end{pmatrix}$$

8. Classify the equilibrium point of the system $y' = Ay$. Using the (T, D) plane, sketch the phase portrait.

$$A = \begin{bmatrix} -4 & 10 \\ -2 & 4 \end{bmatrix}$$

SOLVED:
The trace T of the matrix A is $T = a_{11} + a_{22} = 0$. The determinant of the matrix A is $\det(A) = a_{11}a_{22} - a_{12}a_{21} = 4$. Now we can find the eigenvalues for the matrix A. We know $\lambda_1, \lambda_2 = \frac{T \pm \sqrt{T^2 - 4D}}{2}$. So $\lambda_1 = 2i$ and λ_2.

Since we have complex eigenvalues, we get a spiral sink, spiral source, or a center. Since $T = 0$, we get a center. This is shown in the following phase portrait:

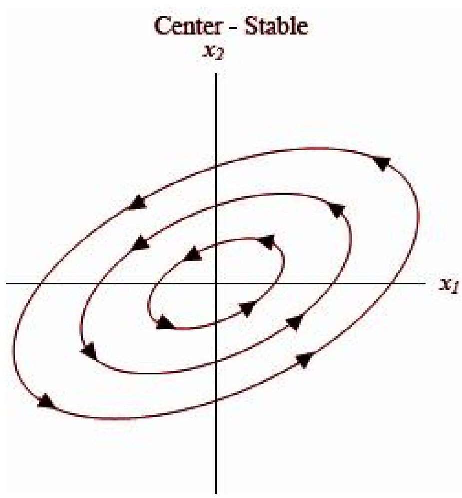

Figure A.3: A Random Phase Portrait

9. Find the solution to the initial value problem.

   $y^{(4)} + 16y = -8y'' + 0t$ Given $y(0) = 0,\ y'(0) = -1,\ y''(0) = 2,\ y'''(0) = 0$

   SOLVED:
   There are a couple of ways to go about this. The Laplace transform and undetermined coefficients will both get you to the same place. As to which is better, it is probably a toss up. This solution will use the Laplace transform.

   The initial equation $y^{(4)} + 16y = -8y'' + 0t$ implies $y^{(4)} + 8y' + 16y = 0$. This new thing is what we are going to apply the transform to:

$$\begin{aligned}
\mathscr{L}\left(y^{(4)} + 8y' + 16y\right) &= \mathscr{L}\left(y^{(4)}\right) + 8\mathscr{L}(y'') + 16\mathscr{L}(y) \\
&= \left[s^4Y - s^3y(0) - s^2y'(0) - s^1y''(0) - s^0y'''(0)\right] \\
&\quad + 8\left[s^2Y - s^1y(0) - s^0y'(0)\right] + 16Y \\
&= s^4Y - (0)s^3 - (-1)s^2 - (2)s^1 - (0)s^0 \\
&\quad + 8\left[s^2Y - (0)s^1 - (-1)s^0\right] + 16Y \\
&= s^4Y + s^2 - 2s^1 + 8s^2Y + 8 + 16Y = 0
\end{aligned}$$

   Now solve for Y and obtain (after a bit of algebra) $Y = \frac{2s-s^2}{(s^2+4)^2}$. Now partial fraction decomposition on this yields,

$$Y = \frac{2s - s^2}{(s^2+4)^2} = \frac{As + B}{s^2 + 4} + \frac{Cs + D}{(s^2+4)^2}.$$

   I will choose $s = -1, 0, 1,$ and 2 as values to solve for the unknown constants. You will get the following equations.

$$0A + \frac{B}{4} + 0C + \frac{D}{4} = 0,\ \frac{A}{5} + \frac{B}{5} + \frac{C}{25} + \frac{D}{25} = \frac{1}{25},\ \frac{A}{4} + \frac{B}{8} + \frac{C}{32} + \frac{D}{64} = 0$$
$$\text{and} -\frac{A}{5} + \frac{B}{5} - \frac{C}{25} + \frac{D}{25} = -\frac{3}{25}$$

   Putting this into a matrix and reducing gives, $A = -\frac{3}{8}$, $B = -\frac{1}{4}$, $C = \frac{31}{8}$, and $D = \frac{1}{4}$. These coefficients now go into Y giving,

$$Y = \frac{2s - s^2}{(s^2+4)^2} = -\frac{3s}{8(s^2+2^2)} - \frac{1}{4(s^2+2^2)} + \frac{31s}{8(s^2+2^2)} + \frac{1}{4(s^2+s^2)^2}.$$

   Finally we can take the inverse Laplace Transform:

$$\begin{aligned}
y(t) &= \mathscr{L}^{-1}\left(-\frac{3s}{8(s^2+2^2)} - \frac{1}{4(s^2+2^2)} + \frac{31s}{8(s^2+2^2)} + \frac{1}{4(s^2+s^2)^2}\right) \\
&= -\frac{3}{8}\mathscr{L}^{-1}\left(\frac{s}{s^2+2^2}\right) - \frac{1}{4}\mathscr{L}^{-1}\left(\frac{1}{s^2+2^2}\right) + \frac{31}{8}\mathscr{L}^{-1}\left(\frac{s}{(s^2+2^2)^2}\right) + \frac{1}{4}\mathscr{L}^{-1}\left(\frac{1}{4(s^2+2^2)^2})\right) \\
&= -\frac{\mathbf{3}\cos(\mathbf{2t})}{\mathbf{8}} - \frac{\sin(\mathbf{2t})}{\mathbf{8}} + \frac{\mathbf{31}\sin(\mathbf{2t})}{\mathbf{32}}
\end{aligned}$$

10. Find the power series for the function $f(x)$.

$$f(x) = e^x - e^{-x}$$

SOLVED:
You can work the series out yourself using the Taylor polynomial coefficients, or you just recognize (and remember) the e^x power series. You know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. So,

$$f(x) = e^x - e^{-x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

Expand the series to give:

$$\begin{aligned}
f(x) &= \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \frac{(-x)^0}{0!} + \frac{(-x)^1}{1!} + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} \\
&= 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots 1 + \frac{(-x)^1}{1!} + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} \\
&= 2 + 2\frac{x^2}{2!} + 2\frac{x^4}{4!} + \cdots \\
&= 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!}\right) \\
&= \mathbf{2\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}}.
\end{aligned}$$

Appendix B

Resources

Because of the deficiencies of our differential equations textbook, we relied heavily on other resources to learn the material. We didn't just pull all the information in these notes out of thin air now did we? Following are a few of the resources we found extremely useful.

B.1 Video Lectures

There are two very good video sets covering differential equations that can be found freely on the web. The first is the MIT lectures by Dr. Arthur Mattuck. This is the complete course of lectures from the MIT classroom available freely in video format. Professor Mattuck is *excellent* at explaining the concepts and making tough material understandable. Sometimes the lectures went a little beyond the scope of our class at WSU by diving a bit deeper into the theory behind why some of this stuff works the way it does. We cannot extend enough thanks to MIT for providing these lectures to students like us through their OpenCourseWare program.

MIT OpenCourseWare 18.03 Differential Equations
As taught in: Spring 2010
http://ocw.mit.edu/

We did not find this next set of videos until after we had taken the class and written these notes. But after reviewing them they seem to cover the material quite thoroughly and in an easy-to-learn format. Mr. Khan is very informal in his teaching but conveys the information effectively. Also check out his impressive array of other courses that he has recorded videos for, you can't miss them!

Khan Academy Differential Equations
`http://www.khanacademy.org/`

B.2 Books

Even though you've probably already purchased a very expensive differential equations text book, you should consider spending a few extra dollars and obtain this great little book. Although this book is not very verbose in its explanations (that's what these notes are for), it is an awesome reference guide and contains many problems for each section, all fully worked out. We highly recommend this book to aid you in your studies. The other thing is it's so cheap it's not worth complaining about...just get one!

Ordinary Differential Equations
"An Elementary Textbook for Students of Mathematics, Engineering, and the Sciences"
by Morris Tenenbaum and Harry Pollard
Dover Publishing
ISBN: 0-486-64940-7

Because the Polking book was so useless, I sold mine back right after the end of the semester. After some searching, I came across this textbook which, although not containing the most thorough coverage of all the topics, seems to provide good explanations in almost an informal manner. Well, not *nearly* as casual as this set of notes. So if you are interested in an additional reference, this book is a recommendation.

A Modern Introduction to Differential Equations
by Henry J. Ricardo
Academic Press
ISBN: 978-0-12-374746-4

B.3 Online Resources

Appendix C

Table of Laplace Transforms