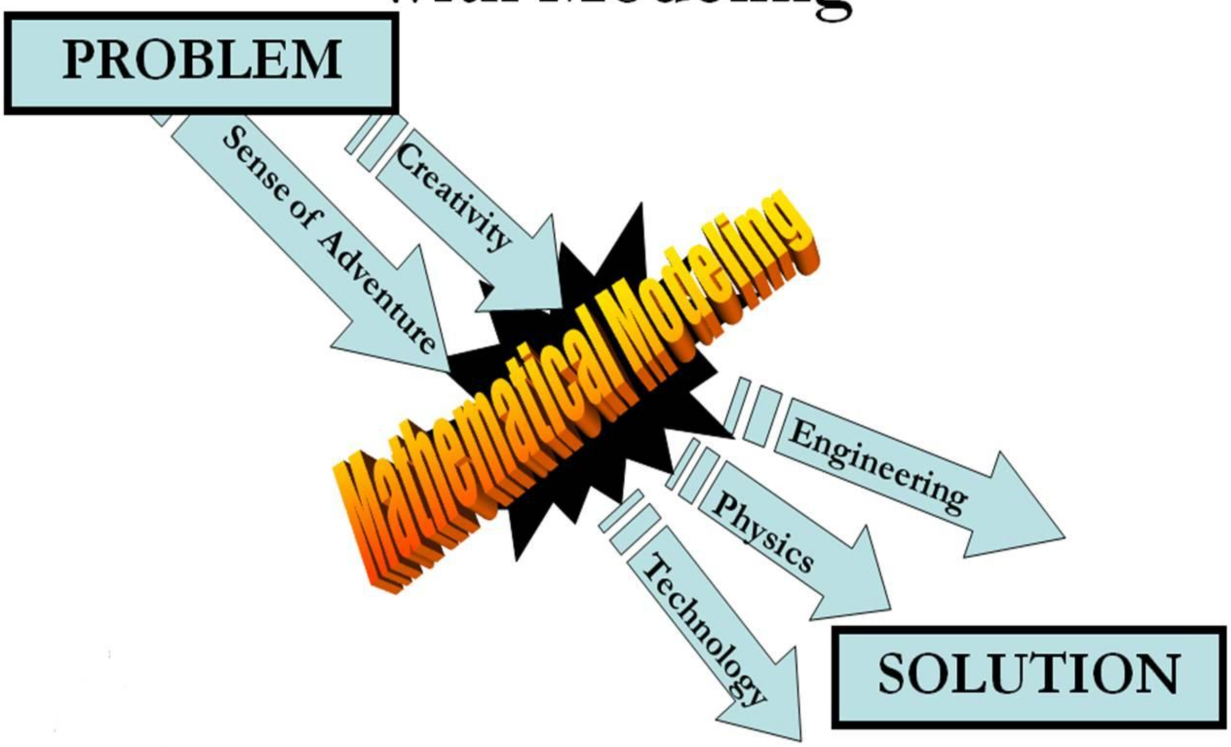
Teaching Differential Equations with Modeling



JMM Minicourse #11
January 9 & 11, 2013

Preface

Mathematical modeling has been an important part of our differential equations courses for many years and we appreciate the opportunity to share our passion and ideas with you. We hope that you find the projects in this booklet entertaining, thought provoking, and most of all, useful in your differential equations course. Feel free to use these projects as they are or to modify them to better reflect the interests and abilities of your students.

The projects in this booklet in many ways reflect the interests and backgrounds of the authors. Who but a former Airborne officer teaching mathematics at West Point would dare to write a differential equations project about parachuting? All of us see mathematics in general, and differential equations in particular, as a way to describe the changing world around us. And we use these inspirations to construct demonstrations and projects for our students in the hopes that one day they will share the excitement and joy of modeling a natural (or sometimes supernatural) phenomenon and using mathematics to understand it in a deeper and more fulfilling manner.

Most of these projects are organized in a similar fashion. We begin with an introduction to the problem, outline or construct the mathematical model, and then lead the student through a sequence of steps to solve and analyze the model. In some cases, the focus of the project is more on the the development of the model equations. In these projects we generally outline how differential equations are used to model phenomena of this type and then guide the student through the modeling development process.

In past presentations, we have each fielded numerous questions about the practical classroom aspects of incorporating projects into a differential equations class. Below are just a few.

How do you find the time during the semester to do these projects? Incorporating projects into a differential equations class does take class time. One should budget at least one class period per project so that students can get started working on the projects under the guidance of the instructor. If all goes well here, they should be in great shape to independently complete the projects within a week.

Are projects done individually or in groups? They certainly can be done individually but for practical and pedagogical reasons we find that group work works best here. The practical side is obvious, working in groups means fewer papers to grade. Pedagogically, using group work here encourages students to develop interpersonal skills that are not always a big part of mathematics classes but are an essential component of most work in both academia and the corporate world. Moreover, having students work in groups and encouraging them to think beyond the confines of the written projects allows the students to bring some creativity to the project as they think about how to extend and build on the basic framework of the project.

How many projects do you do in a semester? The scale of these modeling projects can be greatly varied. Some of these projects can be scaled down to the size of a large homework problem, while others would be the equivalent of writing a term paper in a class. So, the number of projects that one uses deends greatly on the size of the projects being used. If you are going to assign students to work on these projects in groups, one way to balance individual mathematical work and collaborative modeling work is to assign as many projects as you do exams.

What do you expect the final product to look like? It depends on the class somewhat, but we generally expect a typed report and explicitly tell the students that in addition to proper mathematical reasoning, they will be graded on these things:

- complete sentences, good grammar, etc.,
- figures that are labeled, captioned, and easily understood,
- citations if appropriate to the project, and
- a structure and organization consistent with good project writing.

Do students present the projects in class? This depends on the focus of the class. In a differential equations class, students generally don't present their results to the rest of the class. When we teach Mathematical Modeling (a course beyond differential equations at our institutions) then we do require presentations and formal write-ups.

How do you grade all this? It certainly is different than grading calculus problems! A good rubric helps a lot. It is worthwhile to put together a rubric before you start grading and share it with the students so that they have an idea about how they will be graded. A sample rubric is included in Appendix B.

Incorporating mathematical modeling projects into a differential equations course is not always easy. We have been doing this for quite some time and have not only built a large repertoire of projects, but a collection of methods and ideas that have been honed via both successes and failures. We encourage you to use the projects and ideas of this minicourse the next time you teach differential equations.

Michael Huber, Muhlenberg College
Dan Flath, Macalester College
Thomas LoFaro, Gustavus Adolphus College
Ami Radunskaya, Pomona College
Darryl Yong, Harvey Mudd College

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A Summary of Projects in this Booklet

Below is a summary of the projects in this book. For each project we have listed the author(s), a summary of the mathematical topics covered, and the technology requirements of the project.

A listed topic of "Modeling" means that the students are expected to derive differential equations describing some phenomenon.

There are three **Technology** possibilities. "Pencil" indicates that the entire project can be done by hand. However, calculators may be required to assist in some calculations. The "*ODE Toolkit*" designation indicates that numerical methods will be needed to plot solution curves or phase portraits. We recommend ODE Toolkit (available from `http://odetoolkit.hmc.edu`) but there are other packages available to do this work. "CAS" stands for Computer Algebra System such as *Maple* or *Mathematica*. These projects contain some calculations that we believe requires the power these software packages provides.

Project 1: **Modeling Airborne Situations** (*Michael Huber*)
Topics: Modeling, Phase Plane, Equilibrium Points
Technology: *ODE Toolkit*

Project 2: **Aircraft Flight Strategies** (*Michael Huber*)
Topics: Separation of variables
Technology: Pencil

Project 3: **Modeling Deflection in a Rigid Beam**(*Michael Huber*)
Topics: Boundary value Problem, Laplace Transforms
Technology: CAS

Project 4: **A Bungee Jumping Problem** (*Michael Huber*)
Topics: Modeling, second-order equations
Technology: Pencil & *ODE Toolkit*

Project 5: **Another Bungee Jumping Problem** (*Kevin Cooper & Tom LoFaro*)
Topics: Second-order equations
Technology: Pencil & *ODE Toolkit*

Project 6: **The Fifth Labor of Hercules** (*Michael Huber*)
Topics: Modeling, mixing problems
Technology: Pencil

Project 7: **A Partially Insulated Rod** (*Tom LoFaro*)
Topics: Partial Differential Equations
Technology: Pencil & CAS

Project 8: **Murder at the Mayfair** (*Tom LoFaro*)
Topics: Newton's Law of Cooling, Laplace Transforms
Technology: Pencil & CAS

Project 9: **Gnomeo and Juliet** (*Michael Huber*)
Topics: Modeling, linear systems, eigenvalues & eigenvectors
Technology: Pencil

Project 10: **Hercules Meets Laplace in the Classroom** (*Michael Huber*)
Topics: Modeling, Laplace Transforms
Technology: Pencil

Project 11: **Laplace's Equation in Spherical Coordinates** (*Tom LoFaro*)
Topics: Partial Differential Equations, Legendre's Equation
Technology: Pencil

Project 12: **Modeling Malaria in Central America** (*Michael Huber*)
Topics: Systems, SIR models
Technology: *ODE Toolkit*

Project 13: **Who Shot Mr. Burns?** (*Michael Huber*)
Topics: Modeling, Newton's law of cooling
Technology: Pencil

Project 14: **Introduction to Predator/Prey Problems** (*Ami Radunskaya*)
Topics: phase portraits, nullclines
Technology: Pencil & *ODE Toolkit*

Project 15: **Resonance** (*Tom LoFaro*)
Topics: Fourier series, method of undetermined coefficients
Technology: Pencil

Project 16: **A Vibrating Chain** (*Tom LoFaro*)
Topics: Boundary-value problems, Bessel's equation and Bessel functions
Technology: Pencil & CAS

Project 17: **A Vibrating Square Membrane** (*Tom LoFaro*)
Topics: PDE, Fourier series
Technology: Pencil

Project 18: **Immortal Differential Equations** (*Michael Huber*)
Topics: Modeling, predator-prey models, phase plane analysis
Technology: Pencil & *ODE Toolkit*

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Project 1

Modeling Airborne Situations

Stand Up, Hook Up, Shuffle to the Door

A is for Airborne.
I is for In the Sky.
R is for Ranger.
B is for Born to Run.
O is for On the Go.
R is for Rock and Roll.
N is for Never Quit.
E is for Every Day.

C-130 rolling down the strip;
Airborne troopers gonna take a little trip.
Stand up, hook up, shuffle to the door,
Jump on out and count to four.

If my main doesn't open wide,
I've got another one by my side.
If that one should fail me, too,
Look out below, I'm comin' through.

Purpose

In this project, you will study various scenarios involving falling objects. In most cases, the falling object is a human with a parachute (a paratrooper); however, in some cases the object is not human. What forces act upon the falling body? How can we predict time of landing after the paratrooper has exited an aircraft? How does the drag force change after a parachute opens? We all have read about the experiments that Galileo conducted at the Leaning Tower of Pisa. Upon leaving the tower, visitors see a memorial plaque dedicated to Galileo.

After we spend time with falling bodies in general, we will examine the case of D.B. Cooper, a fugitive

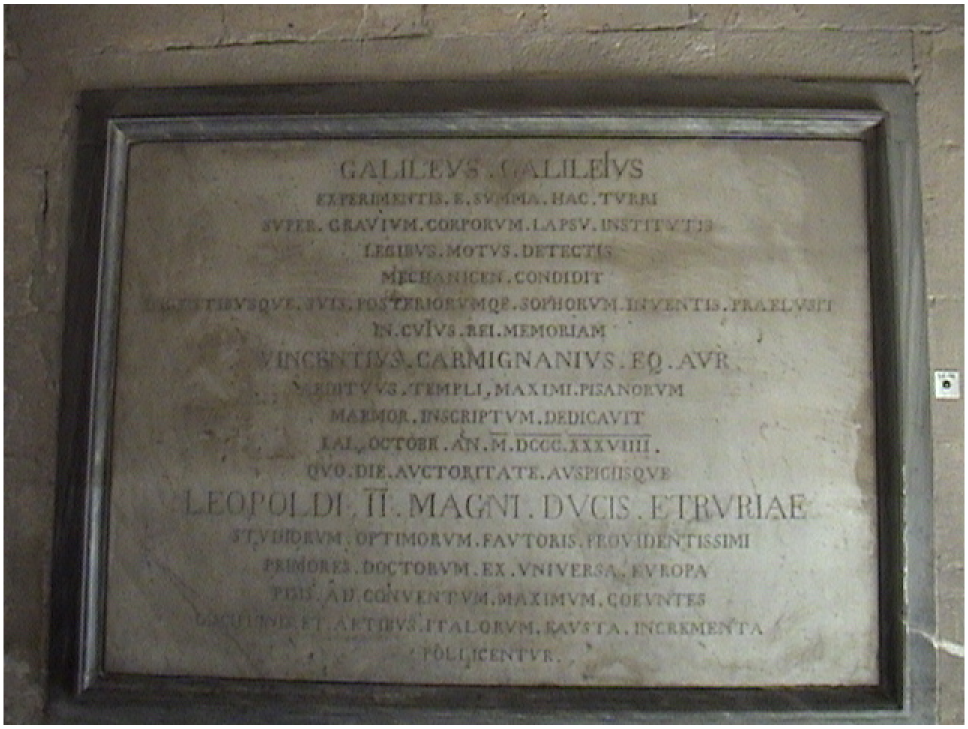

Figure 1.1: Honoring Galileo at Pisa

who hijacked a plane, traded passengers for a ransom, and then jumped from the aircraft while in flight. Did he survive? If he did, where is he now?

Armed with your expert skills in solving differential equations, you will have the opportunity to do just that – solve problems from actual scenarios. You will summarize your findings in a report and you will also prepare an in-class presentation of your solutions. Hoo-ah!

Background – Try Jumping From 30,000 Meters!

Ever hear of Joe Kittinger? In airborne lore, it is said that Superman wears Joe Kittinger pajamas. During the late 1950s, Captain Joseph Kittinger of the United States Air Force was assigned to the Aerospace Medical Research Laboratories at Wright-Patterson Air Force Base in Dayton, Ohio. For Project Excelsior, as part of research into high altitude bailout, he made a series of three high altitude parachute jumps wearing a pressurized suit, from a helium balloon with an open gondola.

The first, from 76,400 feet (23,290 meters) in November 1959 was a near tragedy when an equipment malfunction caused him to lose consciousness, but the automatic parachute saved him (he went into a flat spin at a rotational velocity of 120 rpm; the g-force at his extremities was calculated to be over 22 times that of gravity, setting another record). Three weeks later he jumped again from 74,700 feet (22,770 m).

On August 16, 1960 he made the final jump from a balloon dubbed the Excelsior III. An hour and a half after lifting off from the surface of the earth, he rose to an altitude of 102,800 feet (nearly 19.5 miles or approximately 31,330 meters). At that altitude, Captain Kittinger had 99.2 percent of the earth's atmosphere beneath him. He was, for all practical purposes, in space. YouTube video from a BBC broadcast (Google it and watch the video). Towing a small drogue chute for stabilization, he fell for 4 minutes and 36 seconds, reaching a maximum speed of 614 mph (988 km/h). Only when Kittinger reached the much thicker atmosphere at 14,000 feet (4,270 m) did he open his parachute . He had exceeded the speed of sound in his free-fall descent. Pressurization for his right glove malfunctioned during the ascent, and his right hand swelled to twice its normal size. He set records for highest balloon ascent, highest parachute jump, longest drogue-fall (4 min), and fastest speed by a human through the atmosphere. He finally safely landed in the New Mexico desert after a 13 minute, 45 second descent.

Paratroopers in the United States Army are trained to jump from a height of 450 feet under combat conditions. This is about 1/230th of what Captain Kittinger experienced.

The Glider Problem

The Air Force considered taking a glider up to altitude to view Joe Kittinger's descent. Do you think that would be a good idea? Why or why not? A typical glider problem is based on the system of equations

$$\frac{dv}{dt} = -\sin(\theta) - D\,v^2 \tag{1.1}$$

$$\frac{d\theta}{dt} = \frac{v^2 - \cos(\theta)}{v}, \tag{1.2}$$

where $v > 0$. The parameter D is a coefficient for the drag force on the glider. Also, note that the equations are periodic in θ. Actually these equations are not the true equations of motion but convenient "rescalings" of those equations.

Requirement 1:
(a) Identify the different types of motions of the glider both in the no-drag case ($D = 0$) and when drag is present ($D > 0$). What do the variables represent? Does the system have equilibrium points? What is the physical interpretation of the equilibria, both in the $D = 0$ and $D > 0$ cases.

(b) Select three different values of D between 0 and 4. What are the possible behaviors? Are the solutions periodic? What eventually happens to the flight of the glider?

Green Light ... Go!

One Saturday morning in the spring, you decide to go to the county airfield and learn how to skydive. After an hours training in the classroom and a few minutes practicing landing in a sand pit, you board a small plane, take off, and exit from the aircraft at a specific height, x_0, above ground level (AGL) and fall toward the Earth under the influence of gravity (disregard horizontal motion). You are jumping solo (none of that tandem nonsense for you!). Assume the force due to air resistance is proportional to your velocity, with a different constant of proportionality when the chute is closed (free-fall) and open (final descent). Given the condition that determines when the chute is deployed, how long is it before you reach the ground? And don't worry – you will hit the ground, because gravity is not just a good idea, it's the law!

Requirement 1

Model this as a second-order differential equation. The motion is covered by Newton's Second Law of Motion. Balance the forces in your initial value problem. For x_0, select the last three digits of a group member's social security number, as long as it is greater than 500. If the number is less than 500, subtract it from 1000 (for example, if your number is 996, then $x_0 = 996$ m. If your number is 265, then $x_0 = 735$ m). Select one of your group and use that member's mass. Assume that the ratio of the drag coefficient to the jumper's mass is initially $\frac{1}{5}$. After the chute deploys, assume that the ratio of the drag coefficient to the jumper's mass is $\frac{7}{5}$.

Requirement 2: Here are three different situations that you will consider.

(1) Life is good. Everything (static line, deployment bag, etc.) works as it should, and your parachute deploys successfully four seconds after exiting the aircraft (assume it opens instantaneously). You then descend under control of an open parachute. You land in a cornfield amidst whoops and hollers from your roommate, who was too chicken to join you.

(2) Oh, no! Your main parachute does not open. However, you are trained for this and after four more seconds, you successfully deploy your reserve parachute, which opens immediately, and then you descend, controlling the reserve. Whew!

(3) Your instructor miscalculated! The drag force is really proportional to the velocity squared, even after your chute deploys, with the ratio of the drag coefficient to your mass equal $\frac{1}{2}$.

Address requirements (a) and (b) for each situation above. Then address requirements (c) and (d).

(a) What are your assumptions bearing on the model? What information is given? Develop a model and include initial conditions. How long until you reach the ground? Provide a plot of height AGL versus time.

(b) What are the terminal velocities of the different stages of the jump? What is the velocity when the chute is opened? Upon reaching the ground? When the chute deploys, there will be an almost instantaneous rate of change in the acceleration (or deceleration) of the jumper. The opening shock of the parachute will cause a jerk in the jumper's descent. Provide a plot of velocity versus time and acceleration versus time from the time you exit the helicopter until you reach the ground.

(c) In the original situation, what is the latest time that the parachute can be opened while keeping the impact velocity below 100 m/s? How much time do you have before you hit the ground if your parachute doesn't open?

(d) Suppose we model this second-order equation as a system of first-order equations (hint, hint). What type of equilibrium point is expected? What is the equilibrium point? Is it stable or unstable? Find the eigenvalues and eigenvectors of the system and give the particular solution in vector form for each stage. Provide a direction field plot and show the trajectory pertaining to your initial conditions.

Whatever Happened to D.B. Cooper?

On November 24, 1971, an unknown subject, also known as Dan Cooper, purchased a one-way ticket on Northwest Orient Airlines Flight 305. The flight was carrying 36 passengers and crew. The flight originated in Portland, Oregon with the final destination of Seattle, Washington. The plane was hijacked just prior to its arrival in Seattle. In Seattle, the hijacker allowed the passengers and two stewardesses to depart the plane. In exchange for the safe release of these people, Northwest Orient Airlines paid the hijacker $200,000. The plane departed Seattle for Reno, Nevada. It is believed the hijacker parachuted from the plane during this flight. Authorities and personnel from Fort Lewis, Washington searched for Mr. Cooper but he was never found. In 1980, an 8-year-old boy found $5,800 on the bank of the Columbia River. This is the only money ever recovered from the ransom.

Requirement 3: Suppose with all of the trouble of the wind shear, Cooper has trouble holding onto the briefcase of money and he drops it when he is approximately 500 meters above the ground. The briefcase is designed to withstand an impact of 100 meters per second. Will it explode upon hitting the ground? Recall that the acceleration due to gravity is 9.8 meters per second squared.

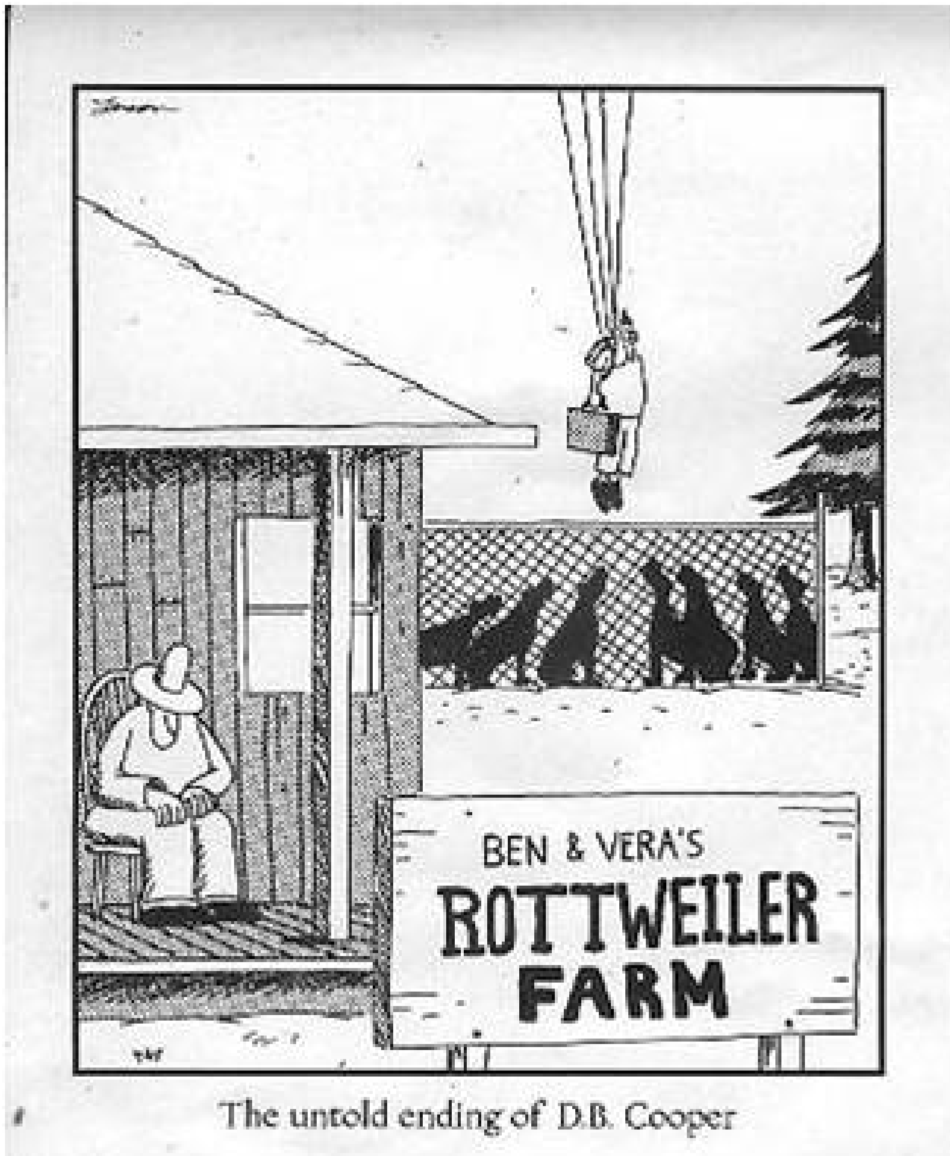

Figure 1.2: What Really Happened to D.B. Cooper

Project 2

Aircraft Flight Strategies

Purpose

The purpose of this project is to develop a sense of how single variable calculus and differential equations can assist in analyzing the governing equations to yield important information about flight operations. This Interdisciplinary Lively Applications Project was used in “Single Variable Calculus and Introduction to Differential Equations,” a freshman mathematics course at the United States Military Academy, and it was developed by Michael Jaye and Joseph Myers in the Department of Mathematical Sciences, together with David Arterburn and Kip Nygren in the Department of Civil and Mechanical Engineering, and it has been modified since by Michael Huber in the Department of Mathematics and Computer Science at Muhlenberg College.

Introduction

Three important considerations in every flight operation are the altitude (possibly variable) at which to travel, the velocity (possibly variable) at which to travel, and the amount of lift that we choose to generate (at the expense of fuel consumption again possibly variable) during the flight. It turns out that when planning a flight operation, one cannot just choose any desired value for each of these three quantities; they are dependent upon one another. We can relate these three quantities through a set of equations known as the Breguet Range Equations. These equations are derived in Handout A. Deriving these equations shows that once we decide to choose constant values for any two of altitude, lift coefficient, and velocity, the third is automatically determined. Thus there are three basic independent flight strategies: constant altitude/constant lift coefficient, constant velocity/constant altitude, and constant velocity/constant lift coefficient. Exercise 1 asks you to analyze how the third quantity must vary under each of these flight strategies.

Commercial flight operations are generally conducted at constant velocity/constant lift coefficient in order to save fuel. In military operations, however, there are often other considerations that override cost efficiency, and thus dictate the choice of a different flight strategy. Surveillance/reconnaissance flights generally dictate flying at constant velocity/constant altitude in order to best gather required intelligence. Phased air operations are sometimes better coordinated when restricted to constant velocity. When several sorties are in the air at the same time, especially both outbound and inbound, safe airspace management often dictates flights at constant specified altitudes. You will more closely analyze which flight strategy may be most appropriate for which military mission. Thus unlike most commercial operations, the military planner must be prepared to operate under any of several different flight strategies.

Scenario: A-10 Close Air Support

You are the pilot of an A-10 Thunderbolt, Close Air Support (CAS) aircraft. Among the many things for which you are responsible, some of the particular aspects are to determine within what radius your plane can safely service CAS targets, how long it can ”loiter” in a target area, and when it must return for refueling.

Now, an interesting aspect of your job is that, at times, some of the instruments malfunction. This forces you to double-check your instruments' accuracy through other means, or to rely on these other means to plan your plane's flight. In this project you are going to answer several questions about the flight of your craft based primarily on your plane's fuel consumption. (Your fuel gauge is known to be working).

Strategy 1: Flying at Constant Velocity/Constant Lift Coefficient

The Range Differential Equation:

You can answer questions regarding how far the plane can travel by relating the distance traveled by the plane to the weight of fuel that it consumes. Assume that you fly at constant velocity and with a constant coefficient of lift (thus, you increase altitude over time as your plane gets progressively lighter). From our knowledge of fluid dynamics, we have the following relationship (this and all following relationships are derived in Handout A):

$$\frac{dx}{dW} = -\frac{V\ C_L}{c\ C_D\ W}, \tag{2.1}$$

where x is the distance traveled, W is the weight of the aircraft, V is the velocity, c is the coefficient of fuel consumption ($c = 0.3700$ pounds of fuel per hour per pound of thrust), and the ratio C_L/C_D is 3.839 for the constant lift coefficient. This differential equation models the distance x as a function of the weight, and the domain of the weight is given by $W_{\text{start}} \leq W \leq W_{\text{finish}}$.

Requirement 1

You take off weighing 40,434 pounds (this force includes fuel, armament, and ordnance) and travel at $V = 347.5$ miles per hour. When you arrive at the target, your weight is 36,434 pounds. Using a numerical technique, estimate the distance you have traveled. Does the answer depend on the increment size? What appears to be the limit of the distance traveled as the number of increments (partitions) increases without bound?

Requirement 2

Evaluate the definite integral to find the exact distance traveled.

Requirement 3

Solve the differential equation for the distance traveled, using $x(W_{\text{start}}) = 0$, and plot the solution as W decreases. Compare $x(W_{\text{finish}})$ to your solution in Requirement 2.

Requirement 4

Calculate the "Endurance Equation," which is the time the aircraft will loiter in the target area.

Strategy 2: Flying at Constant Velocity/Constant Altitude

For tactical reasons, you are required to return home at constant velocity and constant altitude. You must, therefore, decrease your lift as your plane lightens by decreasing your lift coefficient. It turns out, after some work, that we can derive the relationship

$$\frac{dx}{dW} = -\frac{V}{c\ \bar{q}\ S\ C_{D0}\ (1 + aW^2)}, \tag{2.2}$$

where $a = 2.33 \times 10^{-11}$, $C_{D0} = 0.03700$, $\bar{q} = 541.894$, $S = 506.0$ square feet (the surface area of the wing), and c = 0.3700 pounds of fuel per hour per pound of thrust.

Requirement 5

First, calculate the general distance traveled in miles at constant velocity/constant altitude. Second, suppose you have expended all of your ordnance, and your mission is complete. However, you find yourself 478.0 miles from the airfield. You will return to the field at a constant velocity of $V = 460.4$ miles per hour and at a constant altitude. Can you make it home on 4500 pounds of fuel? If so, how much fuel do you have remaining when you arrive at the airfield? If not, how much additional fuel do you need? You aircraft weighs 24,959 pounds when empty of both fuel and ordnance.

Strategy 3: Flying at Constant Altitude/Constant Lift Coefficient

We have discussed two flight strategies, namely flight at constant velocity/constant lift coefficient, and flight at constant velocity/constant altitude. A third strategy is constant altitude/constant lift coefficient. Now, constant lift coefficient will require you to slow down over time as your plane lightens (otherwise your plane will climb). It turns out for this strategy that we can derive the relationship

$$\frac{dx}{dW} = -\frac{\sqrt{2\,C_L}}{c\,C_D\,\sqrt{\rho\,S\,W}}, \tag{2.3}$$

where ρ is the air density.

Requirement 6

Find the distance traveled in miles if $\rho = 0.002377$ slugs per cubic feet and $\frac{\sqrt{C_L}}{C_D} = 9.997$. Assume S and c have the same values as above. Use a full weight of 40,434 pounds.

Exercise

Repeat Requirements 1 through 6 for the F-15E Eagle, using the aircraft data found in Handout B.

References

[1] J. D. Anderson. 1989. *Introduction to Flight*. New York: McGraw-Hill.

[2] C. B. Millikan. 1941. *Aerodynamics of the Airplane*. New York: Wiley.

[3] C. D. Perkins and R. E. Zhage. 1949. *Airplane Peformance, Stability, and Control*. New York: Wiley.

Handout A: Derivation of the Breguet Range and Endurance Equations

Mathematical Model: Lift (L) = Weight of the aircraft (W) (by Newtons second law, assuming no or negligible vertical acceleration).

Thrust (T) = Drag on the aircraft (D) (by Newtons second law, assuming no or negligible horizontal acceleration).

Velocity (V) = $\frac{dx}{dt}$ (where x is the position of the plane at time t).

Weight Loss:

$$-\frac{dW}{dt} = c\ T.$$

The loss of weight, all due to fuel consumption, is directly proportional to the thrust produced; c is the specific fuel consumption in units of pounds of fuel per hour per pounds of thrust.

Definitions: Coefficient of lift:

$$C_L = \frac{L}{\bar{q}\ S}.$$

Coefficient of drag:

$$C_D = \frac{D}{\bar{q}\ S}.$$

The following relationship is developed:

$$C_D = C_{D0} + K(C_L)^2,$$

where $\bar{q} = \frac{1}{2}\ \rho\ V^2$, ρ is the air density, S is the wing area, and C_{D0} and K are constants.

Derived Relationships:

$$\frac{L}{D} = \frac{C_L}{C_D}.$$

$$T = D = W\ \frac{D}{L} = W\ \frac{C_L}{C_D}.$$

The velocity can be shown to be:

$$V = \sqrt{\frac{2\ W}{\rho\ S\ C_L}}.$$

Range Equation for Constant Altitude (ρ constant) and constant C_L: The loss of weight is given by

$$-\frac{dW}{dt} = -\frac{dW}{\frac{dx}{V}},$$

or

$$-\frac{dW}{dt} = \frac{c\ T}{V},$$

and

$$\frac{dx}{dW} = -\frac{V}{c\ T}.$$

By substituting for V, we obtain:

$$\frac{dx}{dW} = -\frac{1}{c}\sqrt{\frac{2\ W}{\rho\ S\ C_L}}\ \frac{C_L}{C_D}\ \frac{1}{W} = -\frac{1}{c\ C_D}\sqrt{\frac{2\ C_L}{\rho\ S\ W}}.$$

Range Equation for Constant Velocity and Constant C_L:

$$\frac{dx}{dW} = -\frac{V\ C_L}{c\ C_D\ W}.$$

Range Equation for Constant Velocity and Constant Altitude:

$$\frac{dx}{dW} = -\frac{V}{c\ T} = -\frac{V}{c\ D}.$$

Substituting for Drag, where

$$D = \bar{q}\ S\ C_D = \bar{q}\ S\ (C_{D0} + K(C_L)^2) \text{ and } C_l = \frac{W}{\bar{q}\ S},$$

yields

$$\frac{dx}{dW} = -\frac{V}{c\ (\bar{q}\ S\ C_{D0} + \frac{KW^2)}{\bar{q}\ S}} = -\frac{V}{c\ \bar{q}\ S\ C_{D0}\ (1 + a\ W^2)},$$

where

$$a = \frac{K}{\bar{q}^2\ S^2\ C_{D0}}.$$

Endurance Equation for a Jet Aircraft at Constant C_L:

$$\frac{dt}{dW} = -\frac{1}{c\ T} = -\frac{C_L}{c\ C_D\ W}.$$

Handout B: Aircraft Data for the F-15E Eagle

Fuel consumption = 0.9 pounds per hour per pound $\frac{C_L}{C_D} = 6.193$

Take Off Weight = 62,323 pounds

Arrival Weight = 58,323 pounds

Flight Velocity = 347.5 miles per hour

Aircraft Weight (no fuel, with ordnance) = 49,200 pounds

$a = 5.866\ E^{11}$ per square pound

$C_{D0} = 0.026$

$\bar{q} = 518.503$ square pounds per square feet

$S = 608$ square feet

Aircraft Empty Weight = 31,700 pounds

Distance = 325 miles

$\frac{\sqrt{C_L}}{C_D} = 13.928$

Project 3

Modeling Deflection in a Rigid Beam

Purpose

The purpose of this project is to develop an understanding of modeling the deflection in a rigid beam subject to uniform loading. Such beams are being used in the construction of the extension of the student union building at Muhlenberg College. The solution for the general case is derived, and then beam-specific parameters are applied to determine actual deflections. The simplified problem is then adjusted to add a more appropriate representation of the beam's curvature. As an additional requirement, a maximum deflection is offered and the reader must determine the appropriate beam from a table of W12-type steel beams. Finally, we investigate the deflection of a beam supported by an elastic foundation and Laplace transforms are employed to determine the deflection.

Figure 3.1: Installing the Student Beam in Muhlenberg's Seegers Union extension

Background

In the Summer of 2009, construction began on the extension of the student union building at Muhlenberg College. The new structure would be a 130-by-100 feet addition to an existing building. The frame for the building consisted of several W12-type steel beams, including a 14-foot long W12x14 beam which was painted red and then autographed in silver ink by students and members of the campus community (see Figure 3.1).

Classical beam theory is a simplification of the linear theory of elasticity, which provides a means to determine the deflection and load-bearing properties of beams. Many mechanics textbooks credit Galileo Galilei's attempt to determine the load carrying capacity of a transversely-loaded beam as the beginning of classic beam theory. Recent developments hypothesize that Leonardo da Vinci made a fundamental contribution to beam theory 100 years before Galileo. A notebook discovered in 1967 (called the *Codex Madrid I*) contains ideas from da Vinci. The normal stress and moment-curvature formulas for slender elastic beams were developed by Leonhard Euler and Daniel Bernoulli in the mid-1700s, after Jacob Bernoulli had made significant discoveries in the field. These formulas are commonly referred to as the Euler-Bernoulli beam theory. The latter Bernoulli assumed correctly that strain on a beam is proportional to the distance from the neutral surface, and he named this constant of proportionality the curvature. It has been argued that the *Codex Madrid I* shows that da Vinci established all of the essential features of the strain distribution in a beam while pondering the deformation of springs. However, Leonardo da Vinci obviously did not know calculus, so he would not have been able to formulate or solve differential equations, but his input into the field of beam theory is considered significant nonetheless (see [6] for more information). Any fundamental mechanical engineering text (such as [7]) will have a chapter devoted to the Euler-Bernoulli beam theory.

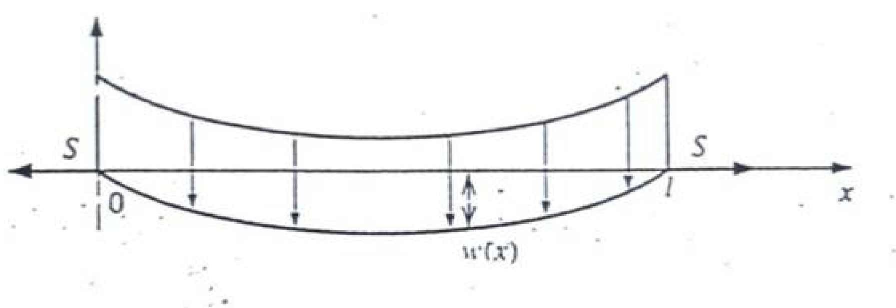

Figure 3.2: Typical Beam Deflection, from [2]

The Theory

A common problem in construction concerns the deflection of a rigid beam which is subject to uniform loading. In many cases, the ends of the beam are supported so that they undergo no deflection (see Figure 3.2. Using this assumption, this creates a boundary value problem which can be modeled by an ordinary differential equation. Let's further assume that the beam whose length is L is homogeneous and has uniform cross sections along its entire length. From the theory of elasticity, the bending moment $M(x)$ at any point x along the beam is related to the load per unit length $p(x)$ by the second-order equation

$$\frac{d^2M}{dx^2} = p(x). \tag{3.1}$$

This bending moment is also proportional to the curvature κ of the deflection curve by the equation

$$M(x) = EI\kappa, \tag{3.2}$$

where the product EI is often called the *flexural rigidity* of the beam. E is Young's modulus (a measure of the elastic property of the material of the beam), and I is the moment of inertia of a cross section of the beam.

The curvature of the beam can be determined from the equation

$$\kappa = \frac{y''}{\left[1 + (y')^2\right]^{\frac{3}{2}}},$$

where $y(x)$ is the deflection of the beam at some point x along its length. For beams with small deflection ($y(x)$ near 0), the slope $y'(x)$ is also near zero and can be neglected. This allows us to assume that $\kappa \approx y''$, so that Equation 3.2 can be written as $M(x) = EIy''$, and Equation 3.1 becomes

$$\frac{d^2M}{dx^2} = EI\,\frac{d^2(y'')}{dx^2} = EI\,\frac{d^4y}{dx^4}.$$

Therefore,

$$EIy^{(\mathrm{iv})} = p(x). \tag{3.3}$$

The Requirements

There are six requirements to this project.

Requirement 1: Assume the beam is fixed at both ends with no vertical deflection, giving the following boundary conditions:

$$y(0) = 0, \quad y'(0) = 0, \quad y(L) = 0, \quad y'(L) = 0.$$

Solve Equation 3.3 for the deflection, $y(x)$, if $p(x) = p_0$, a constant load. Determine the position on the beam where the maximum deflection occurs. What is $y(x)$ at this point?

Requirement 2: We can also model the deflection of the beam using a second-order differential equation:

$$\frac{d^2y}{dx^2} = \frac{S}{EI}y + \frac{p_0x}{2EI}(x - L), \tag{3.4}$$

for $0 < x < L$, where $y = y(x)$ is the deflection of a distance x (measured in inches) from the left end of the beam, and L and S represent the length of the beam in inches and the stress at the end points, respectively. The moment of inertia which we are concerned with is about the neutral axis perpendicular to the web at the center of the beam. Associated with Equation 3.4 are two boundary conditions: as in Requirement 1, we assume that no deflection occurs at the ends of the beam, so $y(0) = y(L) = 0$.

When the steel beam is of uniform thickness, the product EI (the flexural rigidity) is constant, and we can easily determine the exact solution of Equation 3.4. In many real-world applications, however, this thickness is not uniform, so the moment of inertia I is a function of the distance x and numerical approximation techniques are required to determine a solution.

For the structure, we will use a W18x35 type steel I-beam with the following characteristics: the length L is 30 feet, the nominal weight of the beam is 35 pounds per foot, the intensity of the load p_0 is 100 pounds per foot, the modulus of elasticity E is 2.9×10^7 pounds per square inch, and the stress at the ends S is 1000 pounds. The central moment of inertia I from Figure 3.4 is 510 inches^4. A cross-section of the I-beam is shown in Figure 3.3.

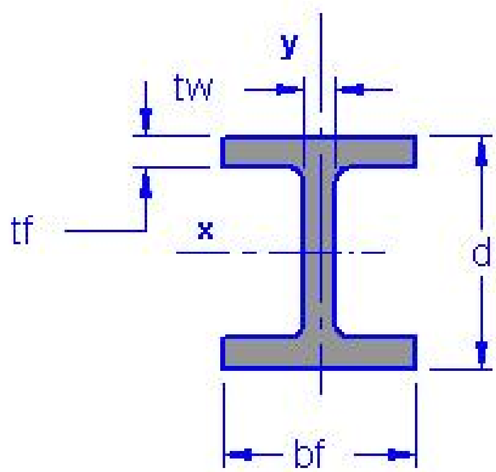

Figure 3.3: Cross-section of I-beam, from [4]

Determine the deflection $y(x)$ of the beam and graph the deflection as a function of the beam length. Assume that there is uniform loading on the fixed beam. Using the parameter values listed, compare your solution to that from Requirement 1. Is there a difference in the location and magnitude of the maximum deflection?

Requirement 3: Using a more appropriate representation of the beam's curvature, the differential equation can be written as

$$\left[1 + (y')^2\right]^{-\frac{2}{3}} y'' = \frac{S}{EI} y + \frac{px}{2EI}(x - L), \tag{3.5}$$

for $0 < x < L$. Approximate the beam's deflection at 6-inch intervals and compare the results with those from Requirement 1.

Requirement 4: According to [4], the maximum deflection of a simply supported beam with a uniform load is given by

$$y = \frac{5\, p_0\, L^3}{384\, EI}. \tag{3.6}$$

The code assumes that this deflection should occur at the center of the beam. Compare this value with those obtained in Requirements 1 and 2. In addition, consider the following: In construction, the rule of thumb is "the lighter, the cheaper, and the cheaper, the better." Select the lightest beam that will perform to the required standard. If 6 inches of deflection is acceptable, which beam from Figure 3.4 would you recommend? How would you calculate the maximum deflection? What would be the maximum deflection of your new, lighter beam?

Requirement 5: Let's consider the entire beam instead of the point of maximum deflection. According to [4], the deflection at any point on a structural steel beam supported on both ends under uniform loading is

$$y(x) = \frac{p_0 x(L-x)}{24EIL} \left[L^2 + x(L-x)\right] \tag{3.7}$$

Determine the deflection at 6-inch intervals and compare to Requirement 4. Do they provide equivalent measures?

Requirement 6: When a structural beam is supported by an elastic foundation, the differential equation which models the deflection is:

$$EIy^{(\mathrm{iv})} + ky = p(x), \tag{3.8}$$

where k is the foundation's modulus. This means that the restoring force of the foundation is given by $-ky$ and it acts in a direction opposite to that of the load, $p(x)$. For algebraic convenience, Equation 3.8 can be written as

$$y^{(\mathrm{iv})} + 4a^4 y = \frac{p(x)}{EI}, \tag{3.9}$$

where $a = (k/4EI)^{1/4}$. For the sake of this requirement, let's model Muhlenberg's autographed red beam with $L = 98$ inches and $a = 1$. Use Laplace transforms to find the deflection $y(x)$ of the beam that is supported on an elastic foundation when the beam is embedded at both ends and $p(x)$ is a concentrated load $p_0 = 1000$ applied at $x = L/2$. Assume that the flexural rigidity $EI = 15 \times 10^9$.

Wide flange beams

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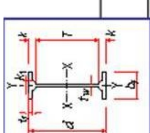

Designation	Area A	Depth d		Web Thickness t_w		$\frac{t_w}{2}$	Flange Width b_f		Flange Thickness t_f		Distance T	Distance k	Distance k_1
	in.^{2}	in.	in.	in.	in.	in.	in.	in.	in.	in.	in.	in.	in.
W12x136	39.9	13.41	13 3/8	0.790	13/16	7/16	12.400	12 3/8	1.250	1 1/4	9 1/2	1 15/16	1
W12x120	35.3	13.12	13 1/8	0.710	11/16	3/8	12.320	12 3/8	1.105	1 1/8	9 1/2	1 13/16	1
W12x106	31.2	12.89	12 7/8	0.610	5/8	5/16	12.220	12 1/4	0.990	1	9 1/2	1 11/16	15/16
W12x96	28.2	12.71	12 3/4	0.550	9/16	5/16	12.160	12 1/8	0.900	7/8	9 1/2	1 5/8	7/8
W12x87	25.6	12.53	12 1/2	0.515	1/2	1/4	12.125	12 1/8	0.810	13/16	9 1/2	1 1/2	7/8
W12x79	23.2	12.38	12 3/8	0.470	1/2	1/4	12.080	12 1/8	0.735	3/4	9 1/2	1 7/16	7/8
W12x72	21.1	12.25	12 1/4	0.430	7/16	1/4	12.040	12	0.670	11/16	9 1/2	1 3/8	7/8
W12x65	19.1	12.12	12 1/8	0.390	3/8	3/16	12.000	12	0.605	5/8	9 1/2	1 5/16	13/16
W12x58	17.0	12.19	12 1/4	0.360	3/8	3/16	10.010	10	0.640	5/8	9 1/2	1 3/8	13/16
W12x53	15.6	12.06	12	0.345	3/8	3/16	9.995	10	0.575	9/16	9 1/2	1 1/4	13/16
W12x50	14.7	12.19	12 1/4	0.370	3/8	3/16	8.080	8 1/8	0.640	5/8	9 1/2	1 3/8	13/16
W12x45	13.2	12.06	12	0.335	5/16	3/16	8.045	8	0.575	9/16	9 1/2	1 1/4	13/16
W12x40	11.8	11.94	12	0.295	5/16	3/16	8.005	8	0.515	1/2	9 1/2	1 1/4	3/4
W12x35	10.3	12.50	12 1/2	0.300	5/16	3/16	6.560	6 1/2	0.520	1/2	10 1/2	1	9/16
W12x30	8.79	12.34	12 3/8	0.260	1/4	1/8	6.520	6 1/2	0.440	7/16	10 1/2	15/16	1/2
W12x26	7.65	12.22	12 1/4	0.230	1/4	1/8	6.490	6 1/2	0.380	3/8	10 1/2	7/8	1/2
W12x22	6.48	12.31	12 1/4	0.260	1/4	1/8	4.030	4	0.425	7/16	10 1/2	7/8	1/2
W12x19	5.57	12.16	12 1/8	0.235	1/4	1/8	4.005	4	0.350	3/8	10 1/2	13/16	1/2
W12x16	4.71	11.99	12	0.220	1/4	1/8	3.990	4	0.265	1/4	10 1/2	3/4	1/2
W12x14	4.16	11.91	11 7/8	0.200	3/16	1/8	3.970	4	0.225	1/4	10 1/2	11/16	1/2

Figure 3.4: Dimensions of I-Beams, from [5]

Wide flange beams

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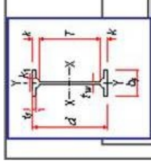

Desig-nation	nominal Wt. per ft	Compact Section Criteria			X_1	$X_2 \times 10^6$	Elastic Properties						Plastic Modu
							Axis X-X			Axix Y-Y				
		$\frac{bf}{2tf}$	$\frac{h}{tw}$	Fy'''			I	S	r	I	S	r	Zx	
				ksi	ksi	(1/ksi)^2	in.^{4}	in.^{3}	in.	in.^{4}	in.^{3}	in.	in.^{3}	i
W12x136	136	5.0	12.3	-	5850	119	1240	186	5.58	398	64.2	3.16	214	9
W12x120	120	5.6	13.7	-	5240	184	1070	163	5.51	345	56.0	3.13	186	8
W12x106	106	6.2	15.9	-	4660	285	933	145	5.47	301	49.3	3.11	164	7
W12x96	96	6.8	17.7	-	4250	405	833	131	5.44	270	44.4	3.09	147	6
W12x87	87	7.5	18.9	-	3880	586	740	118	5.38	241	39.7	3.07	132	6
W12x79	79	8.2	20.7	-	3530	839	662	107	5.34	216	35.8	3.05	119	5
W12x72	72	9.0	22.6	-	3230	1180	597	97.4	5.31	195	32.4	3.04	108	4
W12x65	65	9.9	24.9	-	2940	1720	533	87.9	5.28	174	29.1	3.02	96.8	4
W12x58	58	7.8	27.0	-	3070	1470	475	78.0	5.28	107	21.4	2.51	86.4	3
W12x53	53	8.7	28.1	-	2820	2100	425	70.6	5.23	95.8	19.2	2.48	77.9	2
W12x50	50	6.3	26.2	-	3170	1410	394	64.7	5.18	56.3	13.9	1.96	72.4	2
W12x45	45	7.0	29.0	-	2870	2070	350	58.1	5.15	50.0	12.4	1.94	64.7	1
W12x40	40	7.8	32.9	59	2580	3110	310	51.9	5.13	44.1	11.0	1.93	57.5	1
W12x35	35	6.3	36.2	49	2420	4340	285	45.6	5.25	24.5	7.47	1.54	51.2	1
W12x30	30	7.4	41.8	37	2090	7950	238	38.6	5.21	20.3	6.24	1.52	43.1	9
W12x26	26	8.5	47.2	29	1820	13900	204	33.4	5.17	17.3	5.34	1.51	37.2	8
W12x22	22	4.7	41.8	37	2160	8640	156	25.4	4.91	4.66	2.31	0.847	29.3	3
W12x19	19	5.7	46.2	30	1880	15600	130	21.3	4.82	3.76	1.88	0.822	24.7	2
W12x16	16	7.5	49.4	26	1610	32000	103	17.1	4.67	2.82	1.41	0.773	20.1	2
W12x14	14	8.8	54.3	22	1450	49300	88.6	14.9	4.62	2.36	1.19	0.753	17.4	1

Figure 3.5: Properties of I-Beams, from [5]

References

[1] Figure 3.1 courtesy of David Rabold, Muhlenberg College Plant Operations.

[2] R. L. Burden and J. D. Faires [1993]. *Numerical Analysis*. Fifth Edition. Boston: PWS Publishing Company.

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[4] `http://www.engineersedge.com`. Accessed February 24, 2010.

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Project 4

A Bungee Jumping Problem

Purpose

The purpose of this project is to develop a sense of oscillatory behavior, given a simplified model of a bungee jumper. This project was used in “Single Variable Calculus and Introduction to Differential Equations,” a freshman mathematics course at the United States Military Academy, developed by members of the Department of Mathematical Sciences. It has been revised and used by Michael Huber at Muhlenberg College. [**Note:** The bold-faced, underlined numbers can be changed for use with mulitple sections or groups of students.]

Introduction

In bridge jumping, a participant attaches one end of a bungee cord to himself, attaches the other end to a bridge railing and then drops off of the bridge. In this problem, the jumper will be dropping off of the Royal Gorge Bridge, a suspension bridge that is 1053 feet above the floor of the Royal Gorge in Colorado. The jumper will use a **200**-foot-long bungee cord. It would be nice if the jumper had a safe jump, meaning that the jumper will not crash into the floor of the gorge or run into the bridge on the rebound, so you’re going to do some analysis of the fall.

The Problem

To begin with, the jumper weighs **160** pounds, and the jumper will free-fall until the bungee cord begins to exert a force that acts to restore the cord to its natural position. In order to determine the spring constant of the bungee cord, you found that a mass weighing **4** pounds stretches the cord **8** feet. This spring force, hopefully, will help to slow the descent sufficiently so that the jumper does not hit the floor of the gorge. Throughout this problem, we will assume that down is the positive direction.

Requirement 1

Before the bungee cord begins to retard the fall of the jumper, the only forces that act on the jumper are his weight and the force due to wind resistance.

a. If the force due to wind resistance is **0.9** times velocity of the jumper, then use Newton's Second Law to write a differential equation that models the fall of the jumper.

b. Solve this differential equation and find:

   (1) a function that describes the jumper's velocity (as a function of time), and

   (2) a function that describes the jumper's position (as a function of time).

c. What is the velocity of the jumper after he has fallen **200** feet?

d. What is the terminal velocity of the jumper, if any?

Requirement 2

After a bit more research, you've found that the force due to wind resistance is not linear, as assumed above. Apparently, the force due to wind resistance is more closely modeled by **0.9** v + **0.0009** v^2.

a. Write a new differential equation governing the velocity of the jumper (prior to the bungee cord coming into effect).

b. You should notice that the new equation is no longer an easy one to solve. Nonetheless, you believe that you can determine the velocity of the jumper after 4 seconds by using a numerical solution technique. Choose a suitable h value, and estimate the velocity of the jumper after 4 seconds.

c. Find the terminal velocity, if any.

d. How do your results compare with those found under the first assumption?

e. What is the velocity of the jumper after he has fallen **200** feet? (Hint: what is the relationship between velocity and position?)

Requirement 3

For the last part of our analysis, we'll consider what happens after the bungee cord comes into play. When the jumper reaches the point **200** feet below the bridge, the bungee cord begins to stretch. As we know, it takes some force to stretch the bungee cord, and the cord exerts an equal and opposite force upward. Hooke's law tells us that the force of the bungee cord will be directly proportional to the amount that it is stretched ($F = ks$, where F is the force exerted by the bungee cord, k is a constant of proportionality – also known as the spring constant) – and s is the distance that the cord is stretched beyond its normal length). To further retard the speed of the jumper, assume that there is a drag force due to air resistance. We'll assume that this force is as in requirement 1 (that is, assume that the force due to air resistance is equal to the **0.9** times the velocity of the jumper).

a. Write a differential equation that models the jumper's motion after the bungee cord begins to stretch. What are the conditions that will allow you to solve the differential equation?

b. Solve the differential equation and find:

   (1) a function that describes the jumper's velocity (as a function of time), and

   (2) a function that describes the jumper's position (as a function of time).

c. Does the jumper crash into the floor of the gorge? If not, then what is the distance from the floor of the gorge when the jumper begins to move upward?

d. How close to the bridge does the jumper come on the first rebound? Explain the assumptions that you make in solving this portion of the problem.

e. How far below the bridge will the jumper be once the bouncing stops?

Requirement 4

What assumptions have you made in your model? What effect will these assumptions have on your results? How could you refine your model?

Requirement 5 (Difficult)

How long should the bungee cord be if the jumper wants to barely touch the water on his jump? Make an attempt at finding the length of the cord that would make the jumper's velocity zero at the time when the jumper reaches a point 1053 feet below the bridge. You might wish to find reasonable bounds on the length and discuss how you might solve this problem.

Project 5

Another Bungee Jumping Problem

Suppose that you have no sense[1]. Suppose that you are standing on a bridge above the Malad River canyon. Suppose that you plan to jump off that bridge. You have no suicide wish. Instead, you plan to attach a bungee cord to your feet, to dive gracefully into the void, and to be pulled back gently by the cord before you hit the river that is 53 meters below. You have brought several different cords with which to affix your feet, including several standard bungee cords, a climbing rope, and a steel cable. You need to choose the stiffness and length of the cord so as to avoid the unpleasantness associated with an unexpected water landing. You are undaunted by this task, because you know math!

Each of the cords you have brought will be tied off so as to be 30 meters long when hanging from the bridge. Call the position at the bottom of the cord 0, and measure the position of your feet below that "natural length" as $x(t)$, where x increases as you go down and is a function of time t. Then at the time you jump, $x(0) = -30$, while if your 2 meter frame hits the water head first, then at that time $x(t) = 53 - 30 - 2$.

You know that the acceleration due to gravity is a constant, called g, so that the force pulling downwards on your body is mg. You know that when you leap from the bridge, air resistance will increase proportionally to your speed, providing a force in the opposite direction to your motion of about av, where a is a constant and v is your velocity. Finally, you know that Hooke's law describing the action of springs says that the bungee cord will eventually exert a force on you proportional to its distance past its natural length. Thus, you know that the force of the cord pulling you back from destruction may be expressed as

$$b(x) = \begin{cases} 0, & x \leq 0 \\ -kx, & x > 0 \end{cases}$$

The number k is called the *spring constant*, and is where the stiffness of the cord you use influences the equation. For example, if you used the steel cable, then k would be very large, giving a tremendous stopping force very suddenly as you passed the natural length of the cable. This could lead to discomfort, injury, or even a Darwin award. You want to choose the cord with a k value large enough to stop you above or just touching the water, but not too suddenly. Consequently, you are interested in finding the distance you fall below the natural length of the cord as a function of the spring constant. To do that, you must solve the differential equation that we have derived in words above: the force

[1] This work is adapted from Tom Lofaro's Physics 230 projects.

mx'' on your body is given by

$$mx'' = mg + b(x) - ax'.$$

Here m is your mass, 72 kg, and x' is the rate of change of your position below the equilibrium with respect to time; i.e. your velocity. The constant a for air resistance depends on a number of things, including whether you wear your skin-tight pink spandex or your skater shorts and XXL T-shirt, but you know that the value today is 2.8.

This is a nonlinear differential equation, unlike any we have seen before. However, inside this nonlinear equation are two linear equations, struggling to get out. When $x < 0$, the equation is $mx'' = mg - ax'$, while after you pass the natural length of the cord it is $mx'' = mg - kx - ax'$. We will solve these separately, and then piece the solutions together when $x(t) = 0$.

Requirement 1. Solve the equation $mx'' + ax' = mg$ for $x(t)$, given that you step off the bridge — no jumping, no diving! Stepping off means $x(0) = -30; x'(0) = 0$. You may use $m = 72$, $a = 2.8$, and $g = 9.8$.

Requirement 2. Use the solution from Requirement 1 to compute the length of time you freefall (the time it takes to go the natural length of the cord: 30 meters).

Requirement 3. Compute the derivative of the solution you found in Exercise 1 and evaluate it at the time you found in Requirement 2. You have found your downward speed when you pass the point where the cord starts to pull.

Requirement 1 has given you an expression for your position t seconds after you step off the bridge, before the bungee cord starts to pull you back. Notice that it does not depend on the value for k. When you pass the natural length of the bungee cord, it does start to pull back, so the differential equation changes. Let t_1 denote the time you computed in Requirement 2, and let v_1 denote the speed you calculated in Requirement 3.

Requirement 4. Solve the initial value problem

$$\begin{aligned} mx'' + ax' + kx &= mg; \\ x(t_1) &= 0; \\ x'(t_1) &= v_1. \end{aligned}$$

For now you may use the value $k = 200$, but eventually you will need to replace that with the actual values for the cords you brought. The solution $x(t)$ represents your position below the natural length of the cord after it starts to pull back. Now we have an expression for our position as the cord is pulling on us. All we have to do is to find out the time t_2 when we stop going down. When we stop going down, our velocity is zero, i.e. $x'(t_2) = 0$.

Requirement 5. Compute the derivative of the expression you found in Requirement 4 and solve for the value of t where it is zero. This time is t_2. Be careful that the time you compute is greater than t_1 — there are several times when your motion stops at the top and bottom of your bounces! After you find t_2, substitute it back into the solution you found in Requirement 4 to find your lowest position.

Requirement 6. You have brought a soft bungee cord with $k = 102$, a stiffer cord with $k = 196$, and a climbing rope for which $k = 284$. Which, if any, of these may you use safely under the conditions given?

As you can see, knowing a little bit of math is a dangerous thing. We remind you that the assumption that the drag due to air resistance is linear applies only for low speeds. By the time you swoop past the natural length of the cord, that approximation is only wishful thinking, so your actual mileage may vary. Moreover, springs behave nonlinearly in large oscillations, so Hooke's law is only an approximation. Do not trust your life to an approximation made by a man who has been dead for two hundred years. Leave bungee jumping to the professionals.

Still Curious?

Exercise 1. You have a bungee cord for which you have not determined the spring constant. To do so, you suspend a mass of 5 kg from the end of the thirty meter cord, causing the cord to stretch 0.35 m. What is the k value for this cord? You may neglect the mass of the cord itself.

Exercise 2. If your friend uses the bungee cord with $k = 196$, what happens? Your friend has a mass of 102 kg.

Exercise 3. If your heavy friend wants to jump anyway, how short should you make the cord so that he does not get wet?

Project 6

The Fifth Labor of Hercules

Purpose[1]

The mythology surrounding Hercules has been a part of human culture for over two and a half thousand years. In the ancient Greek mythology, Eurystheus assigns various labors to Hercules, who has to perform them in order to cleanse his soul. This article treats one of the more famous labors, the fifth labor: The Augean Stables. The labor is provided verbatim from Apollodorus and then I describe three tasks needed to accomplish the labor, followed by their solutions. The solutions are in a separate section, in case the reader wishes to solve them before reading the solutions I provide. Each of the three tasks may be used as problems in the classroom to enhance learning.

Introduction

Hercules (or, *Herakles*, as he was known to Greeks) is a hero well-known for his strength and ingenuity. He "shared the characteristics of, on one hand, a hero (both cultic and epic), and on the other hand, a god" [6]. He is arguably the greatest and most famous of the classical Greek heroes. The mythology surrounding Hercules has been a part of human culture for over two and a half thousand years. Many students get a glimpse of Hercules long before they come to high school or college. Whether it is from a civilizations history course, a literature course, or watching a Disney movie, most of us know who Hercules is and we sometimes wonder if such a hero really existed. A walk through a famous museum, such as the Louvre in Paris or the Metropolitan Museum of Art in New York City will provide an abundance of vases, sculptures, and paintings of Hercules performing one of his famous labors. Did you ever wonder how he accomplished them? Can we model the labors in the mathematics classroom?

There are at least four authors from classical times who wrote in significant detail about Hercules and his labors: Apollodorus, Diodorus, Hyginus, and Euripides . Each of these four authors tells the tale of Hercules and his twelve labors, but not all are the same. I chose the Apollodorus version [1], as his treatise, *The Library*, is deemed by many as the authority. In each of the legends, Hercules is assigned various labors to accomplish. This paper deals with the fifth labor.

[1] This work is adapted from *Mythematics: Solving the Twelve Labors of Hercules* (Princeton University Press, 2009), by Michael Huber.

From Apollodorus [1]:

The fifth labour he laid on him was to carry out the dung of the cattle of Augeas in a single day. Now Augeas was king of Elis; some say that he was a son of the Sun, others that he was a son of Poseidon, and others that he was a son of Phorbas; and he had many herds of cattle. Hercules accosted him, and without revealing the command of Eurystheus, said that he would carry out the dung in one day, if Augeas would give him the tithe of the cattle. Augeas was incredulous, but promised. Having taken Augeas's son Phyleus to witness, Hercules made a breach in the foundations of the cattle-yard, and then, diverting the courses of the Alpheus and Peneus, which flowed near each other, he turned them into the yard, having first made an outlet for the water through another opening.

The Problems

Augeus reigned in the district of Elis in the northwestern Peleponnesus [7]. His father had given him many herds of cattle, but after years of neglect in cleaning up after the cattle, the stables were over a meter deep in dung. Our hero has three tasks to complete, which are extracted from the Apollodorus reading. We start with a simple algebraic puzzle to determine how many herds of cattle Augeas owned (**The Herds of Augeas** problem). Next, suppose that Hercules had not "made breaches in the foundation of the cattle-yard." An view of Olympia in Elis, where Augeas ruled, shows a valley which had an elevated plateau. The walls of the stables that contain the cattle sit at the edge of this plateau, and they measured 750 meters long, 400 meters wide, and 2 meters high. The two rivers flow on either side of the stables, toward the edge of the plateau. Hercules must determine the hydrostatic pressure that the water will exert on the walls of the stable (the **Hydrostatic Pressure on the Stable Walls** problem). Finally, as Hercules does indeed divert the Alpheus and Peneus Rivers as Apollodorus suggests, he creates a mixing problem of water and cattle dung. He must determine know how long it will take to rid the stables of the dreaded cattle dung (the **Cleaning the Stables with Torricelli** problem).

The Herds of Augeas

Requirement 1

The following task is taken verbatim from *The Greek Anthology* [4]:

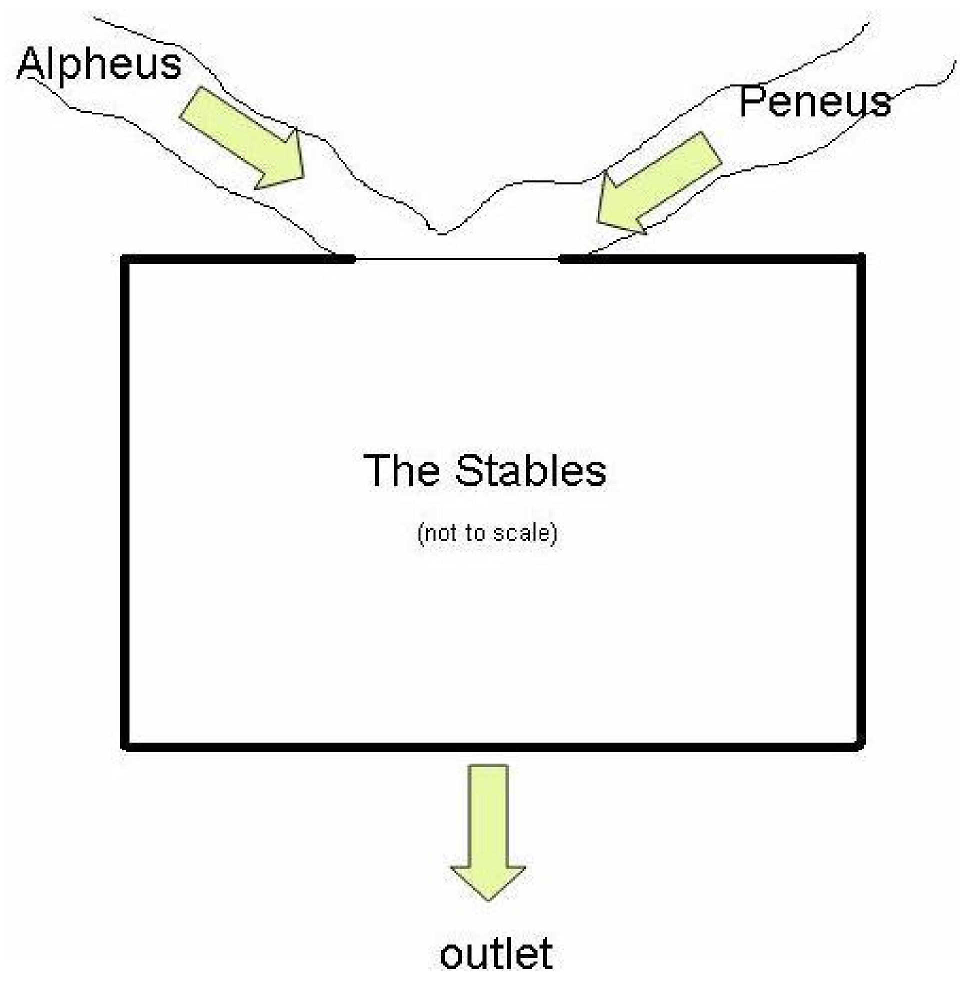
> Hercules the mighty was questioning Augeas, seeking to learn the number of his herds, and Augeas replied: "About the streams of Alpheius, my friend, are half of them; the eighth part pasture around the hill of Cronos, the twelfth part far away by the precinct of Taraxippus; the twentieth part feed in holy Elis, and I left the thirtieth part in Arcadia; but here you see the remaining fifty herds."

How many herds did Augeas have?

Hydrostatic Pressure on the Stable Walls

Requirement 2

In the cattle-yard at Elis, where only a part of the total cattle is situated, Augeas had built a rectangular stable that was 750 meters long, 400 meters wide, and 2 meters high. Calculate the volume of water needed to fill the stables to a height of two meters. What would be the pressure at the base of the stables compared to the pressure a half meter off the ground? Further, what is the total force on each end of the rectangular stables (where the stables are 400 meters wide)? The density of water is 1000 kilograms per cubic meter.



Cleaning the Stables With Torricelli

Requirement 3

Let's assume that Hercules must fill the stables with water to the full height of two meters to ensure that all of the dung is dissolved and then washed away. Model the situation with the two rivers (the

Alpheus and the Peneus – see Figure 6) flowing into the valley. Suppose that the flow of each river into the stables is at a rate of 30 and 25 cubic meters per second, respectively. Suppose also that Hercules digs a hole at the opposite end that is a two-meter-diameter circle underneath the end of the stables. The amount of dung in the yard is initially 25,000 kilograms. Note: an average cow produces 14 to 18 kg of dung per day [2], but we'll assume that all of the cattle are not always inside the stables. Further assume that the stables fill up with water before Hercules opens the hole. Determine the total time to cleanse the stables, meaning that no dung or water is left inside the cattle-yard. Can the stables be cleaned in one day?

Project 7

A Partially Insulated Rod

In this project[1] we will investigate a partially insulated rod. Consider a bar of length 1 that is held at a constant temperature of 0 at one end. The other end is partially insulated and we assume that the heat flux through this end of the bar is proportional to the temperature at that end.

Requirement 1. Explain why this situation can be modelled using the heat equation

$$u_t = u_{xx} \tag{7.1}$$

with boundary conditions

$$\begin{aligned} u(0,t) &= 0 \\ u_x(1,t) + hu(1,t) &= 0 \end{aligned}$$

where $h > 0$ with an initial condition of $u(x,0) = f(x)$. In particular, discuss the form of the boundary condition at $x = 1$. (**Note:** For simplicity, I've let the thermal conductivity $c = 1$.)

Requirement 2. Assume that $u(x,t) = F(x)G(t)$ and use the technique of separation of variables to show that F and G satisfy the ordinary differential equations

$$F'' + \lambda F = 0 \tag{7.2}$$

$$G' + \lambda G = 0. \tag{7.3}$$

Requirement 3. Use the boundary conditions to show that

$$\begin{aligned} F(0) &= 0 \\ F'(1) + hF(1) &= 0. \end{aligned}$$

Requirement 4. Use equation (2) with the boundary conditions above to show that the constant

[1] This work is adapted from Tom Lofaro's Physics 230 projects.

$\lambda > 0$. In other words, show that if $\lambda \leq 0$ then $F(x) = 0$.

Requirement 5. Solve equation (2) with $\lambda = k^2$ and use the first boundary condition to show that $F(x) = A\sin(kx)$ where A is a constant.

Requirement 6. Use the second boundary condition to show that the constant k is determined by the equation

$$\tan k = -\frac{k}{h}. \tag{7.4}$$

Requirement 7. The values of k that satisty equation 7.4 cannot be found analytically but can be found graphically or numerically. Plot $y = -k/h$ and $y = \tan k$ on the same set of axes and describe how you could use this to determine the correct k values. Use $h = 1$ to approximate the first four k values. (**Note:** The *Maple* command `fsolve(eqn, x=a..b)` finds a solution to the equation `eqn` in the interval $[a, b]$.)

Compute

$$\int_0^1 \sin(k_i x)\sin(k_j x)dx$$

for $k \neq j$. What does this calculation suggest about the set of functions $\{\sin(k_n x)\}$?

Requirement 8. Solve equation (3) and use this to find an expression for the solution $u(x,t)$.

Requirement 9. Finally it is time to use the initial conditions to determine the constants in the above series. Explain why when $h = 1$ each constant c_n is given by

$$c_n = \frac{\int_0^1 f(x)\sin(k_n x)dx}{\int_0^1 \sin^2(k_n x)dx}.$$

Requirement 10. Let $f(x) = 1$ and $h = 1$. Compute $c_1, \ldots c_4$ and write out the approximate solution using the first 4 series terms.
Can you describe what each term of this solution looks like?

Project 8

Murder at the Mayfair

Dawn at the Mayfair Diner[1]. The amber glow of streetlights mixed with the violent red flash of police cruisers begins to fade with the rising of a furnace orange sun. Detective Daphne Marlow exits the diner holding a steaming cup of hot joe in one hand and a summary of the crime scene evidence in the other. Taking a seat on the bumper of her tan LTD, Detective Marlow begins to review the evidence.

At 5:30AM the body of one Joe D. Wood was found in the walk-in refrigerator in the diner's basement. At 6:00AM the coroner arrived and determined that the core body temperature of the corpse was 85 degrees Fahrenheit. Thirty minutes later the coroner again measured the core body temperature. This time the reading was 84 degrees Fahrenheit. The thermostat inside the refrigerator reads 50 degrees Fahrenheit.

Daphne takes out a fading yellow legal pad and ketchup stained calculator from the front seat of her cruiser and begins to compute. She knows that Newton's Law of Cooling says that the rate at which an object cools is proportional to the difference between the temperature T of the body at time t and the temperature T_m of the environment surrounding the body. She jots down the equation

$$\frac{dT}{dt} = k(T - T_m), \qquad t > 0, \tag{8.1}$$

where k is a constant of proportionality, T and T_m are measured in degrees Fahrenheit, and t is time measured in hours. Because Daphne wants to investigate the past using positive values of time, she decides to correspond $t = 0$ with 6:00 A.M., and so, for example, $t = 4$ is 2:00 A.M. After a few scratches on her yellow pad, Daphne realizes that with this time convention the constant k in (8.1) will turn out to be *positive*. She jots a reminder to herself that 6:30 A.M. is now $t = -1/2$.

As the the cool and quiet dawn gives way to the steamy midsummer morning, Daphne begins to sweat and wonders aloud "But what if the corpse was moved into the fridge in a feeble attempt to hide the body? How does this change my estimate?" She re-enters the restaurant and finds the grease streaked thermostat above the empty cash register. It reads 70 degrees Fahrenheit.

"But when was the body moved?" Daphne asks. She decides to leave this question unanswered for now

[1] This work is adapted from Tom Lofaro's Physics 230 projects.

simply letting h denote the number of hours the body has been in the refrigerator prior to 6:00AM . For example, if $h = 6$ then the body was moved at midnight.

Daphne flips a page on her legal pad and begins calculating. As the rapidly cooling coffee begins to do its work she realizes that the way to model the environmental temperature change caused by the move is with the unit step function $\mathcal{U}(t)$. She writes

$$T_m(t) = 50 + 20\,\mathcal{U}(t - h) \tag{8.2}$$

and below it the differential equation

$$\frac{dT}{dt} = k(T - T_m(t)). \tag{8.3}$$

Daphne's mustard stained polyester shirt begins to drip sweat under the blaze of a midmorning sun. Drained from the heat and the mental exercise, she fires up her cruiser and motors to Boodle's Cafe for another cup of java and a heaping plate of scrapple and fried eggs. She settles into the faux leather booth. The intense air-conditioning conspires with her sweat soaked shirt to raise goose flesh on her rapidly cooling skin. The intense chill serves as a gruesome reminder of the tragedy that occurred earlier at the Mayfair.

While Daphne waits for her breakfast, she retrieves her legal pad and quickly reviews his calculations. She then carefully constructs a table that relates refrigeration time h to time of death.

Shoving away the empty platter, Daphne picks up her cell phone to check in with her partner Marie. "Any suspects?" Daphne asks.

"Yeah," she replies, "we got three of 'em. The first is the late Mr. Wood's ex-wife, a dancer by the name of Twinkles." She was seen in the Mayfair between 5 and 6PM in a shouting match with Wood."

"When did she leave?"

"A witness says she left in a hurry a little after six. The second suspect is a South Philly bookie that goes by the name of Slim. Slim was in around 10 last night having a whispered conversation with Joe. Nobody overheard the conversation but witnesses say there was a lot hand gesturing, like Slim was upset or something."

"Did anyone see him leave?"

"Yeah. He left quietly around 11. The third suspect is the cook."

"The cook?"

"Yep, the cook. Goes by the name of Shorty. The cashier says he heard Joe and Shorty arguing over the proper way to present a plate of veal scaloppine. She said that Shorty took an unusually long break at 10:30 P.M. He took off in a huff when the restaurant closed at 2:00 AM. Guess that explains why the place was such a mess."

"Great work partner. I think I know who to bring in for questioning."

Related Problems

Guidelines Your report should be written in the style of a police report. Include graphics and tables where appropriate. Remember that the report needs to be understood by Diff's police captain so these items should be explained carefully. Include your computations in an appendix.

Requirement 1. Solve equation (8.1) which models the scenario where Joe Wood is killed in the refrigerator. Use this solution to estimate the time of death (recall that normal living body temperature is 98.6 degrees Fahrenheit.)

Requirement 2. Solve the differential equation (8.3) using Laplace transforms. Your solution $T(t)$ will depend on both t and h (Use the value of k found in Requirement 1).

Requirement 3. Complete Daphne's table. In particular explain why large values of h give the same time of death.

h	time body moved	time of death
12	6:00PM	
11		
10		
9		
8		
7		
6		
5		
4		
3		
2		

Requirement 4. Who does Daphne want to question and why?

Requirement 5. Still Curious? The process of temperature change in a dead body is known as *algor mortis* (*rigor mortis* is the process of body stiffening) and although it is not perfectly described by Newton's Law of Cooling, this topic is covered in most forensic medicine texts. In reality, the cooling of a dead body is determined by more than just Newton's Law. In particular, chemical processes in the body continue for several hours after death. These chemical processes generate heat and thus a near constant body temperature may be maintained during this time before the exponential decay due to Newton's Law of Cooling begins.

A linear equation, known as the *Glaister equation*, is sometimes used to give a preliminary estimate of the time t since death. The Glaister equation is

$$t = \frac{98.4 - T_0}{1.5} \tag{8.4}$$

where T_0 is measured body temperature (98.4°F is used here for normal living body temperature instead of 98.6°F). Although we do not have all of the tools to derive this equation exactly (the 1.5 degrees per hour was determined experimentally), we can derive a similar equation via linear approximation.

Use equation (8.1) with an initial condition of $T(0) = T_0$ to compute the equation of the tangent line to the solution through the point $(0, T_0)$. *Do not use the values of T_m or k found in Requirement 1.* Simply leave these as parameters. Next let $T = 98.4$ and solve for t to get

$$t = \frac{98.4 - T_0}{k(T_0 - T_m)}. \tag{8.5}$$

Project 9

Gnomeo and Juliet

Figure 9.1: Gnomeo and Juliet

The Movie

Gnomeo and Juliet is an animated movie which was directed by Kelly Asbury and released in 2011 by Touchstone Pictures/Walt Disney Pictures. It features music by Elton John, Nelly Furtado, Kiki Dee, and Lady Gaga. It is a version of the greatest love story ever told, starring — garden gnomes! According to the official movie website, the audience is invited to

> step into the world of garden gnomes....Caught up in a feud between neighbors, Gnomeo and Juliet must overcome as many obstacles as their namesakes. But with flamboyant pink flamingos and epic lawnmower races, can this young couple find lasting happiness?

The Differential Equations of Shakespeare

There have been a few different treatments of modeling relationships with differential equations (see the References). Let's model a few of the cases here. First and foremost, we define $g(t)$ as Gnomeo's love for Juliet at time t, and $j(t)$ as Juliet's love for Gnomeo at time t. Positive values of each variable will indicate love, while negative values would indicate *the opposite* (can gnomes really hate one another?).

In a first scenario, assume that Gnomeo is in love with Juliet, but she tends to be a fickle lover. As Gnomeo becomes more enamored with her, Juliet grows weary and tries to run away. When Gnomeo lays off the charm, Juliet suddenly wants him more than ever. Gnomeo, a blue gnome, follows Juliet's lead. If she shows more interest, so does he; if Juliet, a red gnome, feigns disinterest, so does Gnomeo.

Requirement 1. Develop a system of equations to model this scenario.

Requirement 2. Do the two gnomes ever achieve simultaneous love for each other? Draw a phase portrait of this scenario.

Figure 9.2: Unknown Feelings?

Scenario Two: Gnomeo asserts that his love for Juliet decreases in proportion to her love for him, while Juliet tells her confidante (a water-squirting frog on the movie) that her love for Gnomeo grows in proportion to his love for her. Assume that the constant of proportionality for the rate of change of Gnomeo's love is 0.2, while the constant of proportionality for Juliet's love change is 0.8. Further, assume that Gnomeo's love starts out strong ($g(0) = 2$) while Juliet is indifferent.

Requirement 3. Develop a system of equations to model this scenario.

Requirement 4. Draw a phase portrait of this scenario. When is Gnomeo's love for Juliet the strongest?

Scenario Three: Suppose that each gnome's changing love depends on both his or her own love but also on the perceived love from the other gnome. Mathematically, we develop the following equations:

$$\begin{aligned} \frac{dg}{dt} &= \alpha g + \beta j \\ \frac{dj}{dt} &= \eta g + \xi j, \end{aligned} \tag{9.1}$$

where α, β, η, and ξ are parameters that can have either sign.

Requirement 5. Taking just one equation (i.e., $\frac{dg}{dt} = \alpha g + \beta j$), suggest names for each of the four possible sets of parameters (romantic styles). For example, if both α and β are positive, does that mean that Gnomeo is overly eager (his love grows when Juliet favors him and when his own affections are positive).

Requirement 6. Assume that $\alpha = \xi < 0$ (this could be a measure of cautiousness) and $\beta = \eta > 0$ (this could be a measure of responsiveness). What happens to the two gnomes?

*Elton*values and *Elton*vectors

Figure 9.3: *Elton*values

Continue with Scenario Three and the outcomes of Requirement 6. Write the equations in matrix form.

Requirement 7. What type of fixed point emerges? What is the trace? The determinant? In deference to the music from the movie, let's call the eigenvalues and eigenvectors of the system *Elton*values and *Elton*vectors. Determine the *Elton*values and corresponding*Elton*vectors. Draw the phase portrait with *Elton*vectors for each type of fixed point. In the long run, do the trajectories settle down to a pattern?

A Few More Scenarios

In each of the following scenarios, write the equations and support your analysis with phase portraits.

(1) Suppose Gnomeo and Juliet react to each other, but not to themselves. Develop the equations. What happens?

(2) Do opposites attract? What happens if the two equations are $\frac{dg}{dt} = \alpha g + \beta j$ and $\frac{dj}{dt} = -\beta g - \alpha j$?

(3) Suppose Juliet's feelings for Gnomeo are constant, but Gnomeo's rate of change for his love dependes on both his and Juliet's feelings. Does Gnomeo end up loving Juliet or hating her?

Figure 9.4: Gnomeo and Juliet

References

[1] All figures copies from `http://www.gnomeoandjuliet.com`, accessed Oct–Nov 2011.

[2] S. Strogatz. 1994. *Nonlinear Dynamics and Chaos*. New York: Addison-Wesley Publishing Company.

[3] J. M. McDill and B. Felsager. *The College Mathematics Journal*. Vol. 25, No. 5, November 1994: pages 448 - 452.

Project 10

Hercules Meets Laplace in the Classroom

Purpose[1]

The mythology surrounding Hercules has been a part of human culture for over two and a half thousand years. Laplace transforms, on the other hand, have been used in solving differential equations for about one-tenth the time. This article treats The Eleventh Labor: The Apples of the Hesperides, in which Hercules must wrestle and defeat the giant Antaeus in order to pass through Libya. Each time Hercules throws Antaeus to the ground, the giant recoups energy from the earth. Hercules then lifts him off the ground and Antaeus' strength decreases. This interdisciplinary problem is a nice application of a model which can use Laplace transforms to determine a solution of how Hercules can defeat the giant. It has been used in the differential equations classroom to enhance learning in a liberal arts environment.

Introduction

The following passage is taken from Apollodorus' *The Library* [1], from the English translation by Sir James George Frazer:

When the [ten] labours had been performed in eight years and a month, Eurystheus ordered Hercules, as an eleventh labour, to fetch golden apples from the Hesperides, for he did not acknowledge the labour of the cattle of Augeas nor that of the hydra. These apples were not, as some have said, in Libya, but on Atlas among the Hyperboreans. They were presented by Earth to Zeus after his marriage with Hera, and guarded by an immortal dragon with a hundred heads, offspring of Typhon and Echidna, which spoke with many and divers sorts of voices. With it the Hesperides also were on guard, to wit, Aegle, Erythia, Hesperia, and Arethusa. So journeying he came to the river Echedorus. And Cycnus, son of Ares and Pyrene, challenged him to single combat. Ares championed the cause of Cycnus and marshalled the combat, but a thunderbolt was hurled between the two and parted the combatants. And going on foot through Illyria and hastening to the river Eridanus he came to the nymphs, the daughters of Zeus and Themis. They revealed Nereus to him, and Hercules seized him while he slept, and though the god turned himself into all kinds of shapes, the hero bound him and did not release him till he

[1] This work is adapted from *Mythematics: Solving the Twelve Labors of Hercules* (Princeton University Press, 2009), by Michael Huber.

Figure 10.1: Hercules Wrestling Antaeus

had learned from him where were the apples and the Hesperides. Being informed, he traversed Libya. That country was then ruled by Antaeus, son of Poseidon, who used to kill strangers by forcing them to wrestle. Being forced to wrestle with him, Hercules hugged him, lifted him aloft, broke and killed him; for when he touched earth so it was that he waxed stronger, wherefore some said that he was a son of Earth.

Background

Apollodorus is often referenced as "Apollodorus of Athens," and it is believed that he was born around 180 B.C. According to the *The Oxford Classical Dictionary* [3], Apollodorus lived in Alexandria, but he left there in approximately 146 B.C., moving to Athens, where he spent the rest of his life. He had varied interests and was considered to be a great scholar. His *Library* was a study of the ancient Greek heroic mythology. The oldest discovered copy of this book dates to the first or second century A.D. His account of the Herculean myth is commonly accepted by most as the authority.

The Hesperides were known as the "nymphs of the West," and were often compared in mythology to the Three Graces. The apples grew on a magical tree which had golden leaves and a golden bark. These apples supposedly gave eternal life to whoever ate them. Zeus, the king of the gods, had given the tree to Hera as a wedding present. She planted the tree in a garden at the base of Mount Atlas (which gave rise to the Atlantic Ocean, which surrounded the world near Mount Atlas). The Hesperides, daughters of Atlas, liked to pilfer from the tree, so Hera placed a ferocious serpent with one hundred heads named Ladon to guard her precious tree. To successfully accomplish this labor, Hercules must grapple with the giant Antaeus in order to pass through Libya. Each time Hercules throws Antaeus to the ground, the giant recoups energy from the earth. Hercules then lifts him off the ground and Antaeus' strength decreases.

The Requirement

Suppose that Hercules and Antaeus wrestle for five minutes before Antaeus is thrown to the ground. In the next minute, his strength is instantly recharged. The ground provides a one-minute pulse of energy to Antaeus (similar to a step or Heaviside function with an amplitude of one), and he again wrestles Hercules for five minutes, and his strength diminishes as before. This periodic forcing function continues throughout the battle. In the absence of the forcing function, the rate of change of Antaeus' strength would decline at a rate proportional to   of his original strength. Model Antaeus' strength and determine if Hercules can defeat him as long as he is on land (at least 20 minutes).

If Hercules cannot defeat Antaeus under these conditions, what recommendations do you have?

Provide plots of the forcing function and the final solutions of strength versus time.

References

[1] Apollodorus. 1939. *Apollodorus, The Library*, with an English Translation by Sir J. G. Frazer, in 2 Volumes. Cambridge, Massachusetts: Harvard University Press. Includes Frazer's notes.

[2] R. Borelli and C. Coleman. 2004. *Differential Equations: A Modeling Perspective*. 2d Edition. New York: John Wiley & Sons, Inc.

[3] M. Huber. 2009. *Mythematics: Solving the 12 Labors of Hercules*. Princeton, New Jersey: Princeton University Press.

[4] S. Hornblower and A. Spawforth, editors. 1996. *The Oxford Classical Dictionary*. New York: Oxford University Press.

Project 11

Laplace's Equation in Spherical Coordinates

Introduction[1]

Recall that Laplace's equation $\nabla^2 u = 0$ arises in a variety of applications including steady-state heat flow, fluid flow, electrostatic potential, and gravitational potential. In this project we will be solving Laplace's equation in spherical coordinates. For the sake of definiteness let's assume that we have a sphere of radius 1 that has a given temperature distribution and we want to compute the temperature at any point in space.

Recall that given a point P with cartesian coordinates (x, y, z), spherical coordinates are defined by

$$x = r\cos\theta\sin\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\phi$$

where $r^2 = x^2 + y^2 + z^2$ is the distance from the origin, θ is the angle made with the positive x-axis and the point $(x, y, 0)$ and ϕ is the angle made with the positive z-axis. Laplaces equation in spherical coordinates is

$$\nabla^2 u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}\right] = 0. \tag{11.1}$$

a. Suppose on the sphere $r = 1$ centered at the origin the temperature $u(1, \theta, \phi) = f(\phi)$ (in other words the temperature is independent of θ or equivalently the temperature is constant along lines of latitude.) Explain why Laplace's equation can be simplified to

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) = 0. \tag{11.2}$$

b. In addition to the "boundary condition" on the unit sphere given above we also require two additional conditions

$$\lim_{r\to\infty} u(r, \phi) = 0 \tag{11.3}$$
$$u(0, \phi) = 0. \tag{11.4}$$

Explain why these assumptions are physically relevant.

[1]This work is adapted from Tom Lofaro's Physics 230 projects.

c. Use separation of variable $u(r,\phi) = G(r)H(\phi)$ to reduce equation (11.2) to the pair of ordinary differential equations

$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dH}{d\phi}\right) + kH = 0 \tag{11.5}$$
$$\frac{1}{G}\frac{d}{dr}\left(r^2\frac{dG}{dr}\right) = k \tag{11.6}$$

where k is an arbitrary constant.

d. Let's solve equation (6) first. Let $k = n(n+1)$ and show that (6) becomes

$$r^2G'' + 2rG' - n(n+1)G = 0.$$

Show that two linearly independent solutions of this differential equation are $G_n(r) = r^n$ and $G_n^*(r) = r^{-n-1}$.

e. Now let's solve the other equation. Let $\cos\phi = w$ and use this to show that

$$\frac{d}{d\phi} = -\sin\phi\frac{d}{dw}.$$

Use this and the fact that $k = n(n+1)$ to rewrite (5) as

$$(1-w^2)\frac{d^2H_n}{dw^2} - 2w\frac{dH_n}{dw} + n(n+1)H_n = 0.$$

f. Note that this is Legendre's Equation. Explain why the solution to equation (5) is thus $H(\phi) = P_n(\cos\phi)$.

g. Explain why each of the functions $u_n(r,\phi) = A_n r^n P_n(\cos\phi)$ is a solution to equation (11.2) *inside* the unit sphere while each of the functions $u_n^*(r,\phi) = B_n r^{-n-1}P_n(\cos\phi)$ are solutions *outside* the unit sphere.

h. Let's restrict our attention to solutions inside the sphere only. By the principal of superposition

$$u(r,\phi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\phi)$$

is the general solution to the partial differential equation in this region. Find a formula for each of the coefficients A_n. Justify your answer.

Project 12

Modeling Malaria in Central America

Purpose

Since 1995, military clinicians have reported an average of 42 cases of malaria per year in United States soldiers, with the majority of these cases acquired while serving in the Republic of Korea. Although this incidence rate and the epidemiological pattern have been relatively stable over the past decade, outbreaks associated with an increase in the number of military troops deployed to malarial areas have occurred and may continue to account for an increase in malaria cases imported into the United States. Malaria is endemic to more than 100 countries and territories worldwide and is predominantly found in the tropic and subtropic regions. More than 90% of malaria cases occur on the African continent, with the remainder concentrated in parts of the Pacific, Latin America, and Asia.

Most U.S. soldiers currently deployed in war zones are in Afghanistan or Iraq where malaria transmission is seasonal and varies geographically. While *Plasmodium vivax* historically accounts for 80% to 90% of indigenous cases in Afghanistan and 95% of cases in Iraq, with *Plasmodium falciparum* causing the majority of the remaining cases, these numbers are likely to be inaccurate due to unreliable reporting in recent years from these war-torn areas.

The U.S. Army directs soldiers operating in these areas to consume antimalaria chemoprophylaxis and use personal protective measures, to include minimizing exposed skin through proper wear of the uniform and use of bed nets, impregnating uniforms and bed nets with permethrin, and frequently applying topical insect repellent (33% diethyltoluamide [DEET]) to exposed skin. Although bed nets are an integral component of this directive, front-line soldiers, like those described here, may be afforded only limited protection through this measure because nighttime patrols and vigilance during dusk and dawn (when mosquitoes are prevalent) often preclude their intended use.

Deployment to Central America exposes the soldiers to risk of infection.

Introduction

The United States Army has deployed soldiers to the Central American country of Honduras for several years, beginning in the late 1980s, to support Joint Task Force-Bravo operations in the region.

JTF-B headquarters is located at Soto Cano (formerly known as Palmerola) Air Base, Honduras, just outside the capital of Tegucigalpa. In January 1989, the 37th Engineer Battalion (Combat)(Airborne Corps) deployed as part of a task force in support of Operation *Ahuas Tara*. The unit's three-month mission was to construct a 5200-foot flight landing strip (an airfield), complete with a taxiway and parking apron, to allow C-130 aircraft to take off and land in the region. The engineer battalion's area of operations was in southwest Honduras, near the Pacific Ocean, between El Salvador and Nicaragua; its base camp was just outside the town of San Lorenzo, which is east of San Miguel de la Frontera. In addition to the airfield construction project, the engineers also conducted road construction and repair operations in the surrounding countryside. The battalion task force accomplished humanitarian projects as well, which included building an orphanage, upgrading a local school, and drilling wells to provide water for local inhabitants. Military missions involved training with the Honduran armed forces, participating in joint field operations with a host engineer battalion and conducting several airborne training operations with the Honduran Special Forces Battalion. Many American soldiers earned the Honduran Parachutist Badge after making five such jumps with the *Paracadista*.

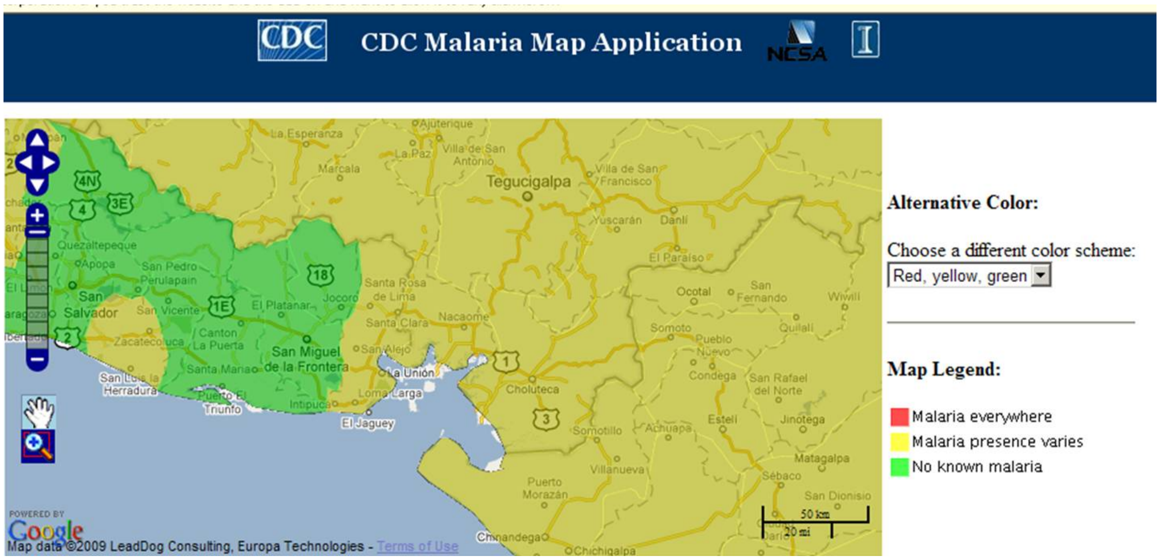


Figure 12.1: Malaria in Honduras from [1]

As you might imagine, the soldiers of the 37th traveled extensively throughout southwestern and central Honduras. Malaria is known to be present throughout the country at altitudes below 1000 meters ($< 3,281$ ft), which included the 37th's Area of Operations (see Figure 12.1). This meant that preventive measures were taken to protect American soldiers from malaria infection. Figure 12.2 shows the area of operations of the battalion. This exposed the soldiers to the local population, animals, and, unfortunately, diseases.

Although malaria is endemic to more than 100 countries and territories worldwide and is predominantly found in the tropic and subtropic regions (more than 90% of malaria cases occur on the African continent), there is a significant risk associated with parts of the Pacific, Latin America, and Asia. There were 247 million cases of malaria world-wide in 2006, causing nearly one million deaths. Approximately half of the world's population is at risk of malaria.

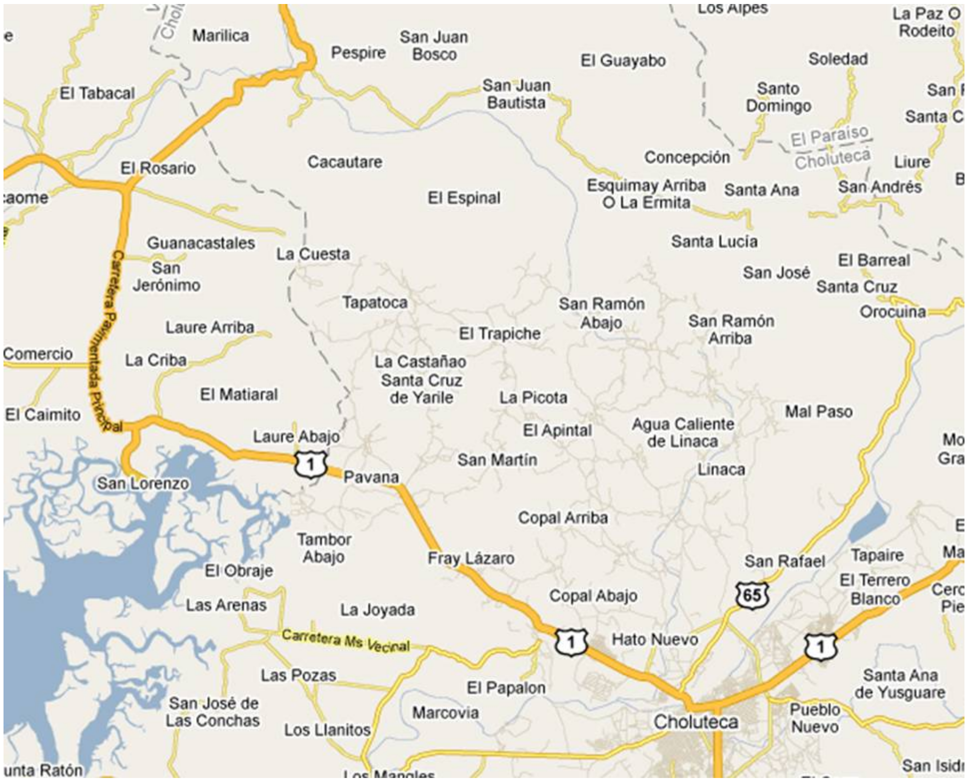


Figure 12.2: Area of Operations in Honduras from [2]

The United States Army directs soldiers operating in these areas to consume antimalaria chemoprophylaxis and use personal protective measures, to include minimizing exposed skin through proper wear of the uniform and use of bed nets, impregnating uniforms and bed nets with permethrin, and frequently applying topical insect repellent (33% diethyltoluamide [DEET]) to exposed skin. Although bed nets are an integral component of this directive, front-line soldiers, like those described in this study, may be afforded only limited protection through this measure because nighttime patrols and vigilance during dusk and dawn (when mosquitoes are prevalent) often preclude their intended use.

Antimalaria Chemoprophylaxis

Malaria symptoms may include fever, chills, sweats, headache, body aches, nausea and vomiting, and fatigue . Malaria symptoms will occur at least 7 to 9 days after being bitten by an infected mosquito. Fever in the first week of travel in a malaria-risk area is unlikely to be malaria; however, you should see a doctor right away if you develop a fever during your trip. Malaria may cause anemia and jaundice. Malaria infections with *plasmodium falciparum*, if not promptly treated, may cause kidney failure, coma, and death. Despite using the protective measures outlined above, travelers may still develop malaria up to a year after returning from a malarious area.

The first problem we study is one of pharmacokinetics. There are a few anti-malarial drugs currently considered for use for preventing malaria in Honduras: Atovaquone/proguanil, chloroquine, doxycycline, or mefloquine. In order to understand the kinetics of any drug dosing regimen, we need to know a few things, such as the amount of the initial dosage, the drug's rate of absorption in the body, the volume of distribution, and the rate of elimination. To simplify this problem, we will assume that there is one rate of metabolism, so that we only consider one rate of elimination (each soldier's rate of absorption and rate of elimination is the same). Some pharmacists believe that malaria can be adequately prevented if three times the dosage is achieved in the bloodstream in a relatively short period and five times the dosage amount is achieved in a longer period. In 1989, soldiers of the 37th Engineer Battalion took one tablet of chloroquine (Aralen or its generic equivalent) once each week, usually after the Friday breakfast meal. The dosage was 300 mg per tablet. Soldiers began taking one tablet per week two weeks before deploying to the region.

Requirements

1. As a first requirement, suppose the body's system breaks down the chemicals in chloroquine at a rate that corresponds to a half-life of four weeks. If the soldiers take the anti-malarial drug once per week, when are they getting the required dosage to prevent malaria and maintain an immunity? How much of the drug is in the body after several weeks (i.e., does $a(t)$ reach an equilibrium value)? Plot the amount of Aralen in the bloodstream if the regimen is continued for a half-year deployment.

2. What does the body's absorption rate x need to be to take a 300 mg tablet once each week and build up to 1200 mg in the bloodstream?

How Many Mosquitoes Are There?

By some estimates, in heavily infected regions, the number of mosquitoes that can be sustained by the environment is equal to one hundred million times the number of humans. For every 10 people, there are one billion mosquitoes. Not all of them are capable of transmitting the disease, but one billion is a very large number.

Suppose the population $p(t)$ of mosquitoes in Central America can be modeled with the logistics equation:

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{M}\right), \tag{12.1}$$

where r is the net growth rate per unit population and M is the carrying capacity of the environment. Honduras covers an area of just over 112,000 square kilometers, with an estimated population of almost eight million people. Assuming a density of 60 people per square kilometer in the southwest region of the country, that would translate to a carrying capacity of mosquitoes of 6,000,000,000 mosquitoes.

Requirements

1. Consider an initial mosquito population near San Lorenzo that numbers four billion. Plot and discuss the number of mosquitoes that will be present as time increases if the net growth rate per unit population is 0.1, 0.01, and 0.001.

2. What effect would lowering the carrying capacity of the environment to below the initial amount have on the population? Plot examples for both $M > p_0$ and $M < p_0$.

3. When is the rate of change of the mosquito population the greatest?

A Compartment Model for Susceptibility and Infection of Humans

One of the most important interactions being studied regarding malaria is whether immunity can be a consequence of infection. Several studies are taking place with the goal of inducing artificial immunity through the use of vaccines. These studies have led to the further investigation of the natural dynamics of immunity to the disease. According to Anderson [4], both the maintenance of immunity and the degree of immunity depend on reinfections. The standard characterization of the epidemiology of malaria is what is termed an "age-prevalence" curve, which shows the proportion of each age group whose blood have the parasites present. Models are needed to study whether the effects of malaria on the human host depend on the infection's intensity, rather than whether or not the infection is present.

Malaria is caused by the multiplication of parasitic protozoa of the family *Plasmodiidae* within the blood cells of other tissues of the host. Anderson writes:

> Infection of a human host begins with the bite of a female mosquito and the injection of sporozoite stages into the bloodstream. These stages of the parasite are carried to the liver where they develop in the parenchymal cells. After an incubation period of several days, these exoerythrocytic stages grow, divide and release merozoites back into the bloodstream. The merozoites penetrate red blood cells, where they grow and subdivide to produce more merozoites that rupture host cells and invade other red blood cells....A portion of the merozoites develop into sexual stages, the gametocytes. Only gametocytes are infective to the mosquito. When a vector mosquito bites a human and ingests male and female gametocytes, these are freed from the blood cell, the female gamete is fertilized, and develops into an oocyst on the wall of the mosquito's gut. After 10 days or so (the actual development time is temperature-dependent), immature sporozoites migrate from the ruptured oocyst to the mosquito's salivary glands, mature to infectivity, and the cycle is ready to repeat itself.

Let's study a compartment model which just treats the infection among humans. The case where everyone is born susceptible, becomes infected, and then recovers to become permanently immune has been often called the SIR model. Let $x(t)$ represent the susceptible class of the population, $y(t)$ the infected class, and $z(t)$ the recovered class. If the infection rate is h and the constant rate of recovery is r, then we can describe the dynamics of the disease by the following differential equations:

$$\begin{aligned} \frac{dx}{dt} &= -hx \\ \frac{dy}{dt} &= hx - ry \\ \frac{dz}{dt} &= ry. \end{aligned} \tag{12.2}$$

By scaling, we can assume that $x(0) = 1$ (the entire population), and that $x(t) + y(t) + z(t) = 1$ as well, for all times t.

Perhaps a better model can be expressed as an SIRS model. Susceptible members of the population are repeatedly infected, recover, become *temporarily* immune, and then become susceptible again. Introduce γ as the average rate of movement out of the immune state. In other words, it is the inverse of the average time spent with immunity. The coupled differential equations become:

$$\begin{aligned} \frac{dx}{dt} &= \gamma z - hx \\ \frac{dy}{dt} &= hx - ry \\ \frac{dz}{dt} &= ry - \gamma z. \end{aligned} \tag{12.3}$$

Requirements

1. What is the solution to the first set of equations? Typical values for h are 2 year^{-1}; allow r to vary between 0.07 and 0.03 year^{-1}. Plot the solutions for each class.

2. Use Equations 12.3 to plot solutions to each class. Let $r = 0.5$ year^{-1} and $\gamma = 0.5$ year^{-1}. The parameter h can vary between 0.2 and 5 year^{-1}. Further, an important assumption is that the parameters r and γ are independent of the parameter h. Explain why. As h increases, what happens to the immunity of the population?

Controlling the Mosquito Population

Let's examine a small part of the base camp; in particular, we will model the number of mosquitoes that existed in the area of the battalion tent city. Imagine that we could track the number of mosquitoes. Assume that the mosquito population in the area surrounding the tents grows at a rate of 5% per day. They are drawn to the living quarters by many factors. To combat the mosquitoes, the operations section of the battalion tasks a platoon to spray insect repellent throughout the area. This causes a 1000-mosquito drop in the population over the next 24 hours after spraying. The platoon sprays the area on the 3rd, 6th, 9th, 12th, etc., days after the operations section begins tracking mosquitoes.

Requirements

1. If there are initially 5000 mosquitoes, what happens to the population over time? Approximately how many mosquitoes will there be after two weeks? Plot both the effects of the spraying (the forcing function) and the overall mosquito population.

2. Is this strategy effective or should it be adjusted?

Modeling the Interaction of Humans and Mosquitos

The modeling of the spread of malaria more accurately relies on two populations — the number of parasite-laden humans and the number of parasite-laden female mosquitos — and the interaction among man, mosquito, and *plasmodium*. The above-mentioned SIRS model, while informative on the number of infected humans, does not look at the mosquito population at all. The Ross-MacDonald equations assume that the transmission of *plasmodium*, both from mosquito to man and from man to mosquito, will depend jointly on the number of susceptible and infected population of each species. With regard to a model, that involves the product of the two variables.

Let N designate the number of parasite-laden humans and M the number of parasite-laden female mosquitos. Let $X(t)$ be the number of infected humans at time t and $Y(t)$ be the number of *plasmodium*-bearing (infected) mosquitos. The transfer from mosquito to human ($\frac{dX}{dt}$) is proportional to the difference of the interaction between the number of *plasmodium*-bearing mosquitos ($Y(t)$) and the number of malaria-free humans ($N - X(t)$) and the number of infected humans. Mathematically, we write this as

$$\frac{dX}{dt} = A\ B\ Y(N - X) - rX, \tag{12.4}$$

where A, B, and r are all positive constants. A is the mosquito bite rate, B is the number of new human infections, and r is the rate of recovery of infected humans. The product AB is the per mosquito rate of newly infected humans.

The transfer from human to mosquito ($\frac{dY}{dt}$) is proportional to the difference of the interaction between the number of infected humans ($X(t)$) and the number of *plasmodium*-free mosquitos ($M - Y(t)$) and the number of infected mosquitos. Mathematically, we write this as

$$\frac{dY}{dt} = A\ C\ X(M - Y) - mY, \tag{12.5}$$

where A, C, and m are all positive constants. C is the number of new mosquito infections, and m is the mosquito death rate. The product AC is the per human production rate of newly infected mosquitos.

What are typical values for the equation parameters? First, we will non-dimensionalize the equations. Let $x = X/N$, the proportion of infected humans. Let $y = Y/M$, the proportion of infected mosquitoes. Define $a = ACN$ to be the biting rate on a human per mosquito. Typical values for a are 0.2 to 0.5 per day. Define $b = B/C$ to be the infected mosquito to human transmission efficiency. Typical values for

b are 0.50. Finally, let $q = M/N$ be the ratio of mosquitoes to humans. We will let this value be 2. Typical values for r fall between 0.01 and 0.05 per day. Typical values for m fall between 0.05 and 0.5 per day. After substitution, the dimensionless version of the Ross-MacDonald equations become

$$\begin{aligned} \frac{dx}{dt} &= a\,b\,q\,y(1-x) - rx, \\ \frac{dy}{dt} &= a\,x(1-y) - my, \end{aligned}$$

Assumptions for the Ross-MacDonald Model

Several assumptions need to be made to create an initial simple model of the situation. We will assume that the population totals, M and N, remain constant. Malaria-free humans are susceptible to the disease. Infected humans who recover become immediately susceptible to the disease (unfortunately).

We also assume that mosquitos feed only on humans. When other animals are "part of the menu," a three- or more species model would be required. Each host would have a different set of epidemiological parameters. Our current model will remain a simplification, relying only on two species.

We cannot assume that the population will be quarantined, but as an initial assumption, the population size does not change.

Requirements

1. A major question about any epidemic is: Once it has begun, can it be controlled? What factor in the model determines how bad the disease spreads? Then, once you have determined this factor, what constraints on it will inform us as to whether the disease can be controlled?

2. Create a phase portrait of the mosquito-human interaction. Where on this slope field are the isoclines? If $x(0) = 1$ and $y(0) = 2$, what happens in the long term?

3. What are the limitations on the model? Discuss the changes to your model as the parameters a, r, and m are varied over time.

References

[1] Figure 12.1 found online at `http://www.cdc.gov/malaria/map`. Accessed 31 January 2010.

[2] Figure 12.2 found online at `http://maps.google.com`. Accessed 21 January 2010.

[3] Merriam-Webster Online Dictionary, found online at *http://www.m-w.com/dictionary/*. Accessed 26 April 2006.

[4] R. M. Anderson (1982). *The Population Dynamics of Infectious Diseases: Theory and Applications*. Chapman and Hall: London.

[5] M. J. Keeling and P. Rohani (2009?). *Modeling Infectious Diseases in Humans and Animals*. Princeton University Press: Princeton, NJ.

[6] Personal conversation with Dr. (Captain) Paul Arguin, Chief, Domestic Response Unit of the Malaria Branch, Division of Parasitic Diseases, Center for Disease Control and Prevention, January 2010.

[7] Personal conversation with Dr. Marten Edwards, Department of Biology, Muhlenberg College, January 2010.

[8] urlhttp://bill.srnr.arizona.edu/classes/182/Malaria/RossEqs.html. Accessed 29 January 2010.

[9] `http://www.who.int/mediacentre/factsheets/fs094/en/`. Accessed 1 February 2010.

[10] Personal conversation with Dr. Chrys Cronin, Department of Biology, Muhlenberg College, September 2010. Dr. Cronin provided information about the drug Aralen after a discussion with Mr. Vince Astolf, a pharmacist.

Project 13

Who Shot Mr. Burns?

Purpose

This project is a classic whodunnit, requiring you to solve a mystery with differential equations.

The Show

"Who Shot Mr. Burns?" is the only two-part episode of *The Simpsons* to date. Part One is the twenty-fifth and final episode of the sixth season and originally aired on the Fox network on May 21, 1995[1]. Part Two is the season premiere of the seventh season and originally aired on September 17, 1995. Springfield Elementary School strikes oil, but Mr. Burns steals it and at the same time brings misery to many of Springfield's citizens. The first part has a cliffhanger ending where Mr. Burns is shot by an unidentified assailant. In the second part, Springfield's police try to find the culprit, with their main suspects being Waylon Smithers and Homer Simpson. The animated episodes intended to mimic the controversy that had resulted when the character J. R. Ewing was shot on the series *Dallas*, known by most as "Who shot J.R.?"

In the first part, Principal Skinner walks into school and discovers that a class gerbil has died. Skinner orders Groundskeeper Willie to bury it and as he is digging the grave, he strikes oil, suddenly making Springfield Elementary rich. Skinner and Superintendent Chalmers lavishly think of ways to spend the money, taking many student requests, including Lisa's suggestion of hiring Tito Puente as a music teacher.

At the Springfield Nuclear Power Plant, Homer is disturbed that Mr. Burns cannot remember his name, even after working for him for ten years. Burns finds out about the school's oil and immediately decides that he must have it. He establishes a slant drilling operation to take it and the Springfield Elementary oil pump fails as Burns had tapped into the oil first. Mr. Burns's drilling operation causes harm and distress to many Springfield citizens: Moe's Tavern is closed and Moe and Barney are enraged; the Springfield Retirement Castle collapses and Grampa has no home; Bart's treehouse is destroyed and Santa's Little Helper is injured; and because the school has lost a lot of

[1]Note: Figure 13.1 is from a screen capture at *http://www.youtube.com* many years ago.

Figure 13.1: Mr. Burns is shot

money, Tito Puente and Groundskeeper Willie are fired.

Burns encounters his assailant. Burns then reveals to Smithers his grandest scheme: the construction of a giant, movable disk that will permanently block out the sun in Springfield, forcing the residents to continuously use the electricity from his nuclear power plant. A horrified Smithers finally stands up to Burns, insisting that he has gone too far - but Burns just fires him in response. Homer later sneaks into Burns's office and spray paints his name on the wall, hoping to help Burns remember it. Burns catches him in the act, but still does not recognize him, and in a rage Homer attacks him. Homer is hauled away by security, vowing revenge on Burns.

At a city hall meeting, the Simpson family, along with many other citizens, angrily come to Mayor Quimby about their problems with Mr. Burns. Suddenly, Burns himself appears, and he reveals that he has armed himself with a gun after his encounter with Homer in the office. He then proceeds to activate the sun-blocking device, stating that nobody can stop him. Laughing evilly, Burns leaves the city hall.

The camera shows him walking into an alley, obscuring him from view. Burns can be heard saying, "Oh it's you, what are you so happy about? I see. I think you'd better drop it," and can be heard struggling with someone before a gunshot rings out. He then stumbles out into the open and collapses on the town's sundial. The townspeople find his body and since Burns has angered so many people recently, no one can guess who the culprit is. Chief Wiggum says that he will find out.

Now, in Part Two, Mr. Burns is hospitalized in a coma, and the Springfield police are working to find the shooter. Their primary suspect is Waylon Smithers, who, after waking up in his apartment with a hangover, vaguely remembers shooting someone the night before. Smithers is arrested until Sideshow

Mel realizes that Smithers must have been home watching Pardon My Zinger at the time of the shooting. It turns out that Smithers had actually shot Jasper's wooden leg; following this, Smithers is released.

With one of the prime suspects cleared, the police eliminate other suspects, including Tito Puente, Principal Skinner, Groundskeeper Willie, and Moe. While checking the suit Burns was wearing, Wiggum finds an eyelash which matches Simpson DNA. At the same time, Burns wakes up from his coma and cries, "Homer Simpson!" The police raid the Simpson home and find a gun under the seat of their car, loaded with bullets that match the one fired and covered with Homer's fingerprints. Homer is arrested for attempted murder, but escapes from a paddywagon when it overturns. Smithers offers a reward for his capture.

At the hospital it is revealed that "Homer Simpson" are the only words that Burns can speak. Lisa returns to the scene of the crime to investigate and finally figures out the identity of Burns' true assailant. At the same time, Homer arrives at the hospital to silence Burns, who keeps saying his name. A police bulletin reports Homer's location, and Lisa, the police, and citizens of Springfield all race to the hospital. Upon entering Burns's room, everyone finds an enraged Homer shaking Mr. Burns vigorously. The shaking returns Burns's ability to speak normally, and he reveals the true assailant: Maggie Simpson.

Burns reveals what really happened on the night he was shot: after leaving the town meeting, he came across Maggie with a lollipop in the Simpsons' car. Burns decided to try stealing candy from a baby again but Maggie's strength proved comparable to his own due to his frailness, and there was a struggle for the lollipop. As he finally yanked it away, his gun slipped from its holster into Maggie's hands and she pulled the trigger. The gun and lollipop both then fell beneath the car seat; Homer would later unknowingly leave fingerprints on the gun while feeling around under the seat.

Burns demands that Maggie be arrested for the crime, but he is dismissed by everyone. Marge adds that the shooting must have been an accident. However, a final shot of Maggie's shifting eyes could suggest otherwise...

A Twist to the Ending

There are many possibilities which allow this to be used in the classroom. One of the more challenging problems is to create a scenario where Mr. Burns actually dies from the gunshot wound (rest assured, however, that he is resurrected in a later episode). His body is found on the town's sundial at 11:00 PM by the townspeople. The coroner comes and at 11:30 PM takes the temperature of Mr. Burns' body, determining it to be 90 degrees. Fifteen minutes later, just before taking the body to the morgue, he takes the body temperature again, and it has fallen two degrees. Further, the weatherman strangely remarks that the night air has been at a constant temperature for hours – 60 degrees. Police Chief Wiggum has five suspects, each of whom gives an alibi. Smithers was watching Pardon My Zinger from 10:30 to 11:00 PM. Lisa was practicing her saxophone at the school for Tito Puente and Marge from 10:15 to 10:40 PM. Homer arrived at Moe's Tavern looking shaken at 10:25 and started making a long list of things he'd like to do to Mr. Burns with Moe from 10:30 PM to 10:45 PM, but then he suddenly left and was unaccounted for. Bart was walking Maggie in her stroller in the town park, but, upset over the loss of the treehouse, Bart left her alone from 10:35 to 10:50 to spray paint some grafitti on the side of city hall (this was captured on the police camera). Bart claims he returned to the stroller to find Maggie fast asleep. Lisa indeed returns to the scene of the crime to investigate and finally figures out the identity of Burns' true assailant. Who was it? State all assumptions!

Project 14

Introduction to Predator / Prey Problems

This is a REAL LIFE STORY[1]

In 1926, the Transylvanian biologist Jean-Luc Dracula completed a statistical study of vampires and their prey (mortal humans) in an isolated mountain region. He tabulated the fraction of vampires in the population of the isolated mountain region over a span of 10 years. Here is his data:

1914	1915	1916	1917	1918	1919	1920	1921	1922	1923
11%	15%	16%	16%	27%	36%	21%	22%	21%	12%

Notice that during the war years — from 1914 to 1918, both vampires and normal mortals left to fight the common enemy. There were fewer prey AND fewer predators, but the data shows that there was a smaller percentage of predators. Jean-Luc was puzzled, so he turned to his uncle (Volterra), who lived over the mountains in Italy, for help. Volterra obligingly came up with the following model:

$$x' = -ax + bxy \tag{14.1}$$

$$y' = cy - dxy \tag{14.2}$$

where a, b, c, and d are positive constants.

Requirement 1. Justify the terms in the model, draw nullclines, and try to sketch in a few orbits.

Requirement 2. Write $\frac{dy/dt}{dx/dt} = \frac{dy}{dx}$ and solve the resulting separable ODE.

Requirement 3. If you have the technology, graph a few of the resulting orbits. Are you convinced that the orbits are cycles? Well, they are. (See Section 2.6, Problem 11 in *Differential Equations: A Modeling Perspective*, 2nd ed, by Borrelli and Coleman, for one method of proof.)

[1]This work is adapted from Ami Radunskaya's Math 102 projects.

Law of Averages

Let $x(t)$ and $y(t)$ be solutions with period T. Define the average values of x and y over one cycle as:

$$\bar{x} = \frac{1}{T}\int_0^T x(t)dt \quad \bar{y} = \frac{1}{T}\int_0^T y(t)dt$$

Requirement 4. Use the differential equations of the model, Equations 14.1 and 14.2 to write an equation for x(t) in terms of y and y0 only.

Requirement 5. Integrate this equation over one cycle, then find the average value of $x(t)$ over a cycle. Do the same for $y(t)$. Does your answer make sense?

In terms of our mountain town, we assume yearly cycles due to the change in the length of the day (*or rather ... night*), and the seasonal availability of garlic. How does this assumption change the model? Solve the new model, using a computer if necessary, and describe how the solutions differ from the solutions of the original model.

Harvesting

Some people dislike vampires, and, as a result, they indiscriminately bomb our isolated mountain region. This method of population control is sometimes known as "harvesting." Volterra came up with the following model for constant effort harvesting, where a fixed percentage of each type of population is harvested:

$$\begin{aligned} x' &= -ax + bxy - H_1 x \\ y' &= cy - dxy - H_2 y \end{aligned}$$

where H_1 and H_2 are both positive constants.

Requirement 6. Where is the new equilibrium?

Requirement 7. What happens as H_1 and H_2 increase??

Requirement 8. Who does the bombing hurt the most? (You can assume that $H_1 = H_2$.)

Requirement 9. What is the fraction of vampires (their average value over the year) in the total (in the total average over the year)?

Requirement 10. How does this explain the data?

Project 15

Resonance

In this project[1] we will investigate the phenomenon known as resonance.

Let
$$g(t) = \sin \omega t$$
and
$$f(t) = \begin{cases} -1 & \text{if } -\frac{\pi}{\omega} < x < 0 \\ 1 & \text{if } 0 < x < \frac{\pi}{\omega} \end{cases}$$

Requirement 1. Express the period P of $f(t)$ and $g(t)$ in terms of ω.

Requirement 2. Show that the Fourier series representation is
$$f(t) = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \sin((2k-1)\omega t).$$

Requirement 3. Now consider the differential equation $x'' + x = g(t)$ where $g(t)$ is the function defined above. The general solution is $x(t) = x_h(t) + x_p(t)$ where x_h is the solution to $x_h'' + x_h = 0$ and x_p is called a *particular solution.*

- What is $x_h(t)$ and what is its period?
- Find the particular solution to this differential equation by assuming
$$x_p(t) = A \sin(\omega t)$$
and then finding an expression for A by differentiating x_p and substituting these expressions into the differential equation.
- Using the answers from above, what is $x(t)$?
- Under what conditions on ω is A undefined? Also express this conditions in terms of the period P of the forcing term. What are the physical implications of this?

[1] This work is adapted from Tom Lofaro's Physics 230 projects.

Requirement 4. Now consider the differential equation $x'' + x = f(t)$ where $f(t)$ is the function defined above.

- Find the particular solution to this differential equation by assuming

  $$x_p(t) = \sum_{n=1}^{\infty} \alpha_k \sin((2k-1)\omega t).$$

  and then finding an expression for each α_k.

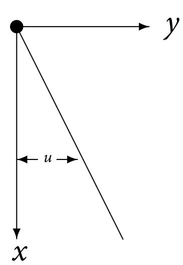
- Under what conditions on ω does there exist a k such that α_k is undefined? Also express this conditions in terms of the period P of the forcing term. What are the physical implications of this?

Project 16

A Vibrating Chain

Introduction[1]

A flexible cable, chain, or rope of length L and density (mass per unit length) ρ is fixed at the upper end ($x = 0$) and allowed to make *small* vibrations (small angles α in the horizontal displacement $u(x,t)$ where t is time) in a vertical plane.



The partial differential equation modeling the motion of this chain is

$$u_{tt} = g\left[-u_x + (L - x)\, u_{xx}\right] \tag{16.1}$$

where the subscripts denote derivatives (i.e. u_{xx} is the second derivative of u with respect to x).

Requirement 1. Assume that $u(x,t) = y(x)\cos(\omega t + \delta)$ and derive the differential equation

$$(L - x)\frac{d^2 y}{dx^2} - \frac{dy}{dx} + \lambda^2 y = 0 \tag{16.2}$$

where $\lambda^2 = \omega^2/g$ and $' = d/dx$.

Requirement 2. The next step is to change variables to convert (16.2) to a differential equation we recognize. Let $L - x = z$ and $s = 2\lambda z^{1/2}$. The next few problems will lead you to the new differential equation for y as a function of s. (**Hint:** Remember that using the chain rule is exactly like dimensional analysis.)

[1] This work is adapted from Tom Lofaro's Physics 230 projects.

- Use the chain rule to show that $\frac{dy}{dx} = -\lambda z^{-1/2}\frac{dy}{ds} = -2\lambda^2 s^{-1}\frac{dy}{ds}$.

- Use the chain rule to show that $\frac{d}{dx} = -\lambda z^{-1/2}\frac{d}{ds} = -2\lambda^2 s^{-1}\frac{d}{ds}$.

- Use the results above to show that

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 4\lambda^4 s^{-1}\left(-s^{-2}\frac{dy}{ds} + s^{-1}\frac{d^2y}{ds^2}\right).$$

Requirement 3. Verify that this is Bessel's equation of order 0 and thus one solution is $J_0(s)$. Using the expressions for s and z in problem 2, show that a solution to differential equation (16.2) is

$$y(x) = J_0\left(2\omega\sqrt{(L-x)/g}\right).$$

Requirement 4. Recall that we assumed that $u(x,t) = y(x)\cos(\omega t + \delta)$. Use this to find an expression for $u(x,t)$.

Requirement 5. Recall that the top of the chain is fixed so that $u(0,t) = 0$ for all t. Use this boundary condition to show that the only possible frequencies are $\omega_k = \frac{z_k\sqrt{g}}{2\sqrt{L}}$ where z_k is the kth zero of J_0 (i.e. $J_0(z_k) = 0$ for all k). The corresponding solutions $u_k(x,t)$ are called the *normal modes*.

Extra Credit: Can you describe what the first normal mode looks like? What about the second? Note that $z_0 = 2.40483\ldots$ and $z_1 = 5.52008\ldots$. Feel free to assume that $L = 1$ and $\delta = 0$ here and graph and/or animate $u(x,t)$.

Project 17

A Vibrating Square Membrane

Introduction[1]

In this project you will consider a model of a vibrating square membrane (a square drum). The partial differential equation modeling the vertical displacement of this membrane is

$$u_{tt} = c^2 \left(u_{xx} + u_{yy}\right). \tag{17.1}$$

We will assume that the drum is a square having length and width equal to π. The edges of the drum are fixed at a reference height of zero and therefore $u(x,0,t) = u(x,\pi,t) = u(0,y,t) = u(\pi,y,t) = 0$.

Requirement 1. Assume that $u(x,t) = F(x)G(y)H(t)$ and derive the differential equations

$$F''G + FG'' + K_1^2 FG = 0 \tag{17.2}$$
$$\ddot{H} + c^2 K_1^2 H = 0 \tag{17.3}$$

Requirement 2. Show that the first equation above can be further reduced to the pair of ordinary differential equations

$$F'' + K_2^2 F = 0 \tag{17.4}$$
$$G'' + (K_1^2 - K_2^2)G = 0 \tag{17.5}$$

Requirement 3. Solve (17.4) subject to the boundary conditions above. In particular, explain why $K_2 = m$ where $m = 1, 2, \ldots$ and $F_m(x) = \sin(mx)$.

Requirement 4. Solve (17.5) subject to the boundary conditions above. In particular, explain

[1] This work is adapted from Tom Lofaro's Physics 230 projects.

why $K_1^2 - K_2^2 = n^2$ where $n = 1, 2, \ldots$ and $G_{mn}(y) = \sin(ny)$.

Requirement 5. Solve (17.3). In particular, explain why

$$H_{mn}(t) = A_{mn}\cos(\lambda_{mn}t) + B_{mn}\sin(\lambda_{mn}t)$$

where $\lambda_{mn} = c\sqrt{m^2+n^2}$.

Requirement 6. What is u_{mn}? Describe the vibrational modes of u_{11}, u_{12}, u_{21}, and u_{22}.

Using the principle of superposition, we know that the sum of solutions is again a solution. Therefore the general solution to this differential equation is a double summation

$$u = \sum_{m=1}\sum_{n=1}\left[A_{mn}\cos\left(\lambda_{mn}t\right) + B_{mn}\sin\left(\lambda_{mn}t\right)\right]\sin\left(mx\right)\sin\left(ny\right). \tag{17.6}$$

This is called a double Fourier series. The goal of the next few steps is to find a formula for the coefficients. We will assume initial conditions of $u(x, y, 0) = f(x, y)$. and $u_t(x, y, 0) = 0$.

Requirement 7. Show that the derivative condition implies that $B_{mn} = 0$ for all m and n.

Requirement 8. Let

$$\alpha_m = \sum_{n=1} A_{mn}\sin(ny). \tag{17.7}$$

Write a Fourier series for $f(x, y)$ having coefficients α_m. Use the Fourier coefficient formula to express α_m as an integral.

Requirement 9. Use equation (17.7) to express A_{mn} as a double integral.

Project 18

Immortal Differential Equations

Purpose[1]

This project is a combination and variation of the Lotka-Volterra model for predator-prey interaction and the SIR (Susceptible-Infected-Removed) model for the spread of infectious diseases. By incorporating the undead (zombies and vampires, etc.) into the models, we can relate the concepts from previous known models using popular (albeit unrealistic) examples.

Introduction

A popular genre in television and the Hollywood screen involves how immortal creatures, such as vampires and zombies, have interacted with humans. From television shows (such as *Dark Shadows*, *Buffy the Vampire Slayer*, and *True Blood*) to motion pictures (*Night of the Living Dead*, the *Twilight Saga* series, *Zombieland*, and the *Underworld* movies), to video games and websites, students have become fascinated with vampire/werewolf/zombie and human interaction.

Vampires and the Predator-Prey Model

Vampires are creatures that subsist by feeding on living beings, generally by drinking their blood. Stories of vampires have been in popular culture for almost 200 years, when John Polidori's *The Vampyre* appeared in print in 1819. It featured the vampire Lord Ruthven and introduced the vampire as we now know it. Bram Stoker's novel *Dracula* appeared in 1897. *Dracula* drew on earlier mythologies of werewolves and similar legendary demons. A distinctive vampire genre was spawned, still popular in the 21st century, with books, films, video games, and television shows. It seems as if we are obsessed with stories of vampires. There are many characteristics of vampires[2]. These include:

- vampires belong to the group known as the "undead,"
- they use of fangs to feed by draining blood from the victim,

[1] This work is adapted from "Differential Equations and the Undead," a presentation by Daniel M. Look, St. Lawrence University.

[2] [Note: This is a good exercise for students. Part of the modeling process is to make valid assumptions.]

- they are immortal, and
- they can be created only from other vampires (usually by draining victims completely of blood and then having victims feed on the vampire's blood).

Try to name at least two or three more characteristics.

How can a vampire be destroyed? A vampire can be killed by being beheading, by sprinkling it with holy water (with or without a cross), or by driving a wooden stake through its heart. Most accounts name daylight as fatal to vampires as well, although there are exceptions (*Blade*, for example, can walk around in daylight).

Can we effectively model the human-vampire interaction with a Predator-Prey Model? First, develop some valid assumptions. Vampires are removed from the environment when slain by a human. In the absence of humans, vampires could survive (in *Underworld Evolution*, Marcus feeds on horses), although there is a belief that they prefer human blood. New vampires enter the environment by being sired or created (as outlined above). The human population has a carrying capacity. Interaction of humans with vampires causes one of three outcomes: the vampire does not feed, the vampire does feed and a human is killed, or the vampire feeds and the human is turned into a vampire.

Define $h(t)$ to be the number of humans in the environment at time t. Let $v(t)$ to be the number of vampires in the environment at time t. The carrying capacity is denoted as M, and the natural growth rate for humans (taking birth/death and immigration/emigration into account) is r. In the absence of any human-vampire interaction, the rate of change of humans follows a logistics-type relationship:

$$\frac{dh}{dt} = rh\left(1 - \frac{h}{M}\right). \tag{18.1}$$

The vampires' rate of change (without interaction) depends on the rate at which vampires are added to the environment (including humans who are turned into vampires) and on the rate at which vampires are slain. Mathematically,

$$\frac{dv}{dt} = mv - sv = (m - s)v, \tag{18.2}$$

where m is the vampire migration rate into the environment and s is the slaying rate. If we account for interaction between humans and vampires, then the human population is decreased by a ratio of human/vampire attacks to human/vampire encounters (let's call it a). The vampire population is increased by the product of a and the ratio of sirings to attacks (let's call it b). Not all encounters lead to attacks, and not all attacks lead to the creation of new vampires.

Requirement 1. Derive the predator-prey differential equations.

Requirement 2. What are the equilibria? What are the conditions (i.e.., the values of the parameters), that will lead to coexistence of the two populations?

Requirement 3. What is the effect of a large (or small) human carrying capacity for the vampire population? In order for the vampire population to increase, what does the carrying capacity need to be?

Determining Parameters

Let's have some fun and replace the parameters with some numerical values. Suppose the human population growth is approximately 10% annually. Assume also that the vampires migrate to the environment with an annual migration rate of 10%. A vampire feeds every three days (on average), and in that time, the vampire encounters 100 potential victims in the course of an evening. So, 1 out of 300 encounters involves feeding. The human carrying capacity is 105,000 people. A vampire sires another vampire toughly once every other year. With the aggressive actions of individuals like Buffy, approximately one-third of the vampire population is slain annually. [3]

Requirement 4. Determine the equilibria. Do they make sense? What type of fixed point does the system present? Plot the long-term populations.

Requirement 5. A typical vampire coven (according to *Buffy*) has 4 to 6 members, and we suspect there are 3 to 5 covens in our community. How does this fit into your determinations in Requirement 4?

[3] [Note: These are based on actual *Buffy the Vampire Slayer* episodes and the town of Sunnyvale. Students can argue for other values, or you can create different values for each group in the class. For r, s, and m, we need to convert from annual discrete values to continuous parameters. r = ln(annual human population growth rate), m = ln(annual vampire population migration rate), and s = ln(annual vampire attack-to-encounter rate). With our values above, $r = m \approx 0.0953$, $s \approx 0.594$, $a \approx 0.00333$, $b \approx 0.00417$, and $M = 105,000$.]

Zombies!

A zombie is a reanimated corpse that feeds on living human flesh. Yikes! Zombies have been appearing in popular media for decades, from movies (*Night of the Living Dead*, *Zombieland*, and) to video games (*Red Dead Redemption Undead Nightmare*, as an example) to even the *Harry Potter* books. In *Harry Potter and the Half-Blood Prince*, Albus Dumbledore explains that

> Inferi are corpses, dead bodies that have been bewitched to do a Dark wizard's bidding. Inferi have not been seen for a long time, however, not since Voldemort was last powerful.... He killed enough people to make an army of them, of course.

An inferius is a dead body, reanimated by a Dark Wizard, similar to a zombie.[4] Inferi have no free will, and cannot think for themselves; their purpose is merely to serve as puppets of the Dark Wizard who reanimated them.

"Humans vs. Zombies (HvZ)" is a game of moderated tag played at schools, camps, neighborhoods, military bases, and conventions across the world. Human players must remain vigilant and defend themselves (usually with socks and dart blasters) to avoid being tagged by a growing zombie horde. For more information, visit `http://humansvszombies.org`. Perhaps you can incorporate a game of HvZ into the project....

Just as with vampires, there are many characteristics of zombies. These include:

- zombies belong to the "undead,"
- they are slow-moving,
- they are decomposing, shambling creatures,
- they have no intelligence – they are instinctive and not usually controlled (inferi are exceptions), and
- they eat living flesh (so zombies do not feed on other zombies).

Zombies can only be killed through dismemberment/decapitation and cremation. Zombies are sired by other zombies (if a zombie bites and kills a human, that human usually becomes a zombie, and it usually occurs rapidly). The first zombies in a specific community have been "created" by forms of radiation, plagues, and viruses (often from some scientific experiment gone wrong). As "zombification" spreads like a virus, a modified SIR model seems appropriate (let's call it the HZR model – Human-Zombie-Removed).

What assumptions can we make?

- All humans begin in the Human class and are susceptible.
- Human/zombie interaction can be won by either the human or the zombie.

[4] According to `http://harrypotter.wikia.com/wiki/Inferius.`

- If a human loses a fight with a zombie, the human moves to the Zombie class.
- If a zombie loses a fight with a human, the zombie moves to the Removed class.
- Zombies do not fight other zombies.

List a few more, based upon your "experiences." For instance, do we need to explicitly state that humans cannot go directly from the Human class to the Removed class?

The HZR Model

We will need to define parameters for each differential equation.[5] The rate of change of the Human class ($h(t)$) depends on the growth and decay rates of the class, and on the interaction with zombies. Let's define β as the natural growth rate of humans (counting immigration and births), δ as the natural decay rate of humans (counting emigration and deaths), and γ as the ratio of Human/Zombie interactions that result in a human moving from the Human to the Zombie class.

The rate of change of the Zombie class ($z(t)$) depends on the interaction with zombies, whether the human becomes a zombie (affected by γ) or the zombie is destroyed. Let's define α as the ratio of Human/Zombie interactions that result in a zombie moving from the Zombie to the Removed class; i.e., a zombie is destroyed.

Finally, the rate of change of the Removed class ($r(t)$) depends on the decay rate of the Human class and on the interaction of susceptible humans with zombies. As humans die, the Removed class grows.

In addition, the population is assumed to be closed (the number of humans plus zombies plus removed is fixed, say N). Is this realistic? Well, if we consider a short-term scenario, then the answer is Yes. Initially, both β and δ are zero. How long do zombies "survive" without feeding?

Requirement 1. Derive the HZR equations. Are they realistic? What are the equilibria? When, if ever, can humans and zombies coexist?

Requirement 2. Suppose $N = 2500$ (the zombie virus is created in the science building of a liberal arts college and the campus is quarantined), $\gamma = 0.0095$ and $\alpha = 0.005$. Plot the long-term behavior for each class.

Requirement 3. Let's add some complexity to the environment. Assume that $\beta = 0.02$ and $\delta = 0.005$. Now, what are the equilibria? Plot the long-term behavior for each class.

Requirement 4. Suppose scientists have created a technique/serum to bring zombies back from the undead into the Human class, with all zombie effects gone. Obviously, this reversal would have to occur within τ minutes of a human being attacked (and losing the attack) by a zombie. Let ρ be a probability that the human-to-zombie transition is reversed. Develop new equations. How does the

[5] Assume all parameters are positive.

equilibrium value change for zombies? Given previous values and $\rho = 0.35$, what are the long-term values for each class?

Variations to the Undead Models – the *Underworld* Factor

We have studied a predator-prey model with vampires versus humans, and a virus problem with zombies versus humans. What happens when a second predator is introduced in the environment? In the *Underworld* movies, vampires battle lycans (werewolves) as competing species. Each is a predator against the human prey.

There are now interactions of humans with vampires, humans with werewolves, and vampires with werewolves. The weapons used by vampires and werewolves against each other are pretty sophisticated (in the *Underworld* stories), so they are much more of a threat to each other than the humans are. List assumptions for each predator. For example, the feeding rates for vampires on humans is quite different than for werewolves on humans. Werewolves tend to feed only at the full moon (once every 28 days). Are the interaction parameters similar?

Requirement 1. Derive the differential equations. When, if ever, can humans, vampires, and werewolves coexist?

This competing species problem allows students to develop the parameters. Assumptions are key. For instance, does every successful werewolf attack on a human result in the death of the human and creation of another werewolf? How successful are humans in killing vampires and werewolves, or do the predators kill each other at a higher rate than the humans kill them? have some fun with this.

Now to the Real World

Think about how you modeled the Human-Zombie-Removed classes. Is this an effective way to model an infectious disease? Currently, there are Susceptible-Infected-Removed (SIR) models used in cases such as modeling influenza outbreaks, SARS, and even swine or bird flu. How could you adjust such models to include diseases which allow persons to move from the Removed class to the Susceptible class again. Can you provide any example (two or three) of such diseases? What parameters would you need to define?

Requirement 1. Develop a generic model for two different types of infectious diseases, complete with definitions of parameters, classes, and make several valid assumptions.

Appendix A

Teaching Differential Equations With Modeling and Visualization

Introduction[1]

In this article, I explain the history of using Interdisciplinary Lively Applications Projects (ILAPs) in an ordinary differential equations course. Students are seeking to learn methods to "solve real world problems," and incorporating ILAPs into the syllabus has been an effective way to apply solutions methods to situations that students may encounter in other disciplines. Feedback has been positive and will be shared . Examples of ILAPs currently used will be referenced.

Confident and Competent Problem Solvers

One of my goals each semester is to develop students into confident and competent problem solvers. The competence comes from practice, solving problem after problem. The confidence also comes with experience, from knowing what to do when facing a new problem. Many textbooks give students an equation and ask them to solve it. In the past, if the textbook gave students the harmonic oscillator equation, complete with values for the mass, damping coefficient and spring constant, students could put in the values and then use the appropriate technique to solve for the displacement or velocity of the mass. However, when confronted with a word problem and asked to develop the model's differential equation, students struggled. Students were comfortable with their "Plug and Chug" method of solving, as long as they knew what and where to plug. That got me thinking: What if we just gave them the forces acting on the problem, to include any non-homogeneous driving function, and asked them to predict long-term behavior?

A few years ago, as an experiment suggested by my colleague Don Small at West Point, I gave a group of students a differential equations problem in the form of a long word problem and asked them to set up the model (define variables, state what is given, make a few valid assumptions, write down what they were asked to determine, and explain the technique they would use to solve it). The initial value problem involved modeling the alligator population in the Okefenokee Swamp, and it

[1]This work is adapted from *Teaching Differential Equations With Modeling and Visualization*, by Mike Huber, published online at `http://www.codee.org/ref/CJ10-0157`.

involved harvesting which differed depending on which year it was (a piecewise Heaviside forcing function). How many gators would be in the swamp after 6, 7, or 8 years? Further, what would the harvesting need to be to wipe out the gator population? I told them NOT to solve it, unless they had time. We had already covered using Laplace transforms to solve these types of problems, but I wanted to see if they could establish the problem to be solved. Afterwards, I asked the students to explain their problem-solving process, to include any assumptions they had to make. The real value of this exercise came from the explanations. Some students wrote about their anxieties about having a "vague" problem, while others wrote about their happiness in being able to set up a "real-world" problem.

The Problem-Solving / Modeling Process

Students must transform a problem from a real world setting into a math model, solve that model, and then interpret the results in the real world scenario. With computer algebra systems, the Solution Process leg of the process has become less important than the other two legs. Instead of giving the students the model or algorithm and asking them to simply substitute numbers into the equation to find an answer, I began to ask students to develop the equation or the heuristics of the model. I also found that students were more receptive to learning solution methods when confronted with applications-oriented scenarios. Model creation depends heavily on defining variables with appropriate units, stating what is given, making valid assumptions, and figuring out what it is we are trying to find. The interpretation leg requires students to discuss the solution back in the context of the situation. The answer to the gator problem is not 50. Students are expected to discuss the effects of harvesting and how the population changes when harvesting rates are adjusted. That takes the solution back into the "real world"

Creating the model and interpreting the solution are just as important—maybe more important—than simply finding a solution. Today's computer algebra systems and numerical solvers (*Mathematica, Maple, ODETOOLKIT,* etc.) can solve most traditional differential equations, take derivatives, determine integrals, plot slope fields. Knowing what to enter as the equation to be solved becomes the critical task. Describing a situation where the rate of change of a body's temperature is proportional to a difference of the temperature of the body and the surrounding environment requires students to understand each term in the ODE; simply telling them to use Newton's Law of Cooling does not. Describing interaction terms in a competing species problem forces students to determine which might be a prey and which might be a predator, based on the situation. Making valid assumptions to simplify the model is stressed in the classroom. Asking, "So what?" or "Is my answer reasonable?" leads students into the interpretation stage. Writing is incorporated into the process by requiring students to explain their solutions in the context of the problem, with appropriate units. Does the solution pass the common sense test? The "visualization" aspect of a computer algebra system becomes a force multiplier in student learning. Plot the solution of the ODE. What happens as time increases? Does it make sense? Does the phase portrait or direction field of the system match our intuition (complex eigenvalues give rise to a spiral)? Can we predict the behavior based upon the direction field (without finding the particular solution)? Can a student hand-draw a portion of the slope field near an equilibrium point? Far away from an equilibrium point? Then can they compare their drawing to that from the computer and claim success?

Usings ILAPs

My lessons are now geared to solving groups of applied problems. The methods needed to approach applications are still very important, but I try to sequence theory with solving problems. Eigenvalue/eigenvector and matrix algebra skills are introduced as students try to solve systems of first-order ODEs. Visualization is incorporated into every solution, either by plotting a solution, direction field, or forcing function. Does the solution satisfy the ODE's forcing function? Plot them both and compare. Each block of material culminates in a word problem assessment (see the ILAPs below). A possible drawback is that not every type of problem can be assessed in a one-hour exam. However, this drawback existed with the previous mid-terms, before I began this modeling approach. In addition, students are now writing about their mathematics. The discussion section of the problem is not simply an answer that is double-underlined. It is the student's attempt at answering an applied problem and explaining the results.

As an example, I found an old exam from several years ago in our department, in which we asked, "Find the general solution to the second-order equation: $\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0$."

We may have asked to determine the position at some time t, but students were given the model. They expected to be given the equation to be solved, and they really didn't question how it was derived. Now, we have placed a little more emphasis on how to put together the model. How do we sum the change of position in a first-order equation? What happens when we sum the forces acting on a body? Now, for instance, I like to give the following type of problem:

Figure A.1: The Leaning Tower of Pisa (photo by author)

> A ball is dropped (no initial velocity) from the side of the Leaning Tower of Pisa (approximately 35 meters above the ground – see Figure A.1) and bounces repeatedly; each time, the rebound is a bit lower. The following forces apply:
>
> - the ball has a mass of 3 kilograms and acceleration is due to gravity alone;
> - the ball has a resistive force that is equal to two-and-one-half times its velocity, always reducing the speed of the ball;
> - the ball exhibits a Hooke's Law-type force that can be modeled with a spring constant of 12 kilograms per second squared;
> - there is no external forcing function applied.
>
> Model the motion of the ball as a system of equations. Write the problem in matrix form. What are the eigenvalues and eigenvectors of the coefficient matrix? What is the expected behavior of the ball? Plot the particular solution. How many times does the ball bounce above 2.5 meters during a rebound?

Several objectives are assessed with this problem. Students show that they can transform a higher-order ODE into a system of first-order ODEs. Sometimes I don't explicitly ask for the eigenvalues, but by determining them, students can predict motion based upon the signs of the eigenvalues' real components. Most students can get the model of a damped harmonic oscillator and solve for the particular solution. However, not all will correctly plot the motion of a ball, as the solution crosses the t-axis (going from positive to negative). The more astute students realize they need to plot the absolute value of the ball's position, in order to get a bouncing effect (see Figure A.2). In evaluating student writing about the solution, I look to see if students question the fact that the motion of their ball goes through the earth, as in the left side of Figure A.2, or if it bounces, as in the right side.

This is a nice problem, but it can be viewed as contrived for a classroom setting. Don Small suggested that we develop connections with other departments at West Point, in order to expose students to problems that they may have to solve in their engineering or science or humanities classes.

When I was teaching at the United States Military Academy, we created several Interdisciplinary Lively Applications Projects (ILAPs). In the 1990s, our department at West Point led a consortium of 12 schools located throughout the country in an NSF-sponsored program hoping to improve the educational culture through the enhancement of interdisciplinary cooperation and coordination. The program was entitled *Project INTERMATH*, and its primary activity was to develop ILAPs. In 1997, the Mathematics Association of America published a book of ILAPs, edited by Dr. David C. Arney, department head at West Point at the time. In the preface, now-retired General Arney wrote, "From the student's perspective, ILAPs provide applications which motivate the need to develop mathematical concepts and skills, provide interest in future subjects that become accessible through further study and mastery of mathematics, and enable a broader, more interdisciplinary outlook at an earlier stage of development."

I was fortunate to have helped in some ILAP development, and I still try to develop them today. There is a cost. I have to find a colleague in another department who is willing to devote some time to assist with the context of the scenario. I have recently used ILAPs with ties to mechanical engineering, environmental engineering, physical education, and even mythology. I have submitted

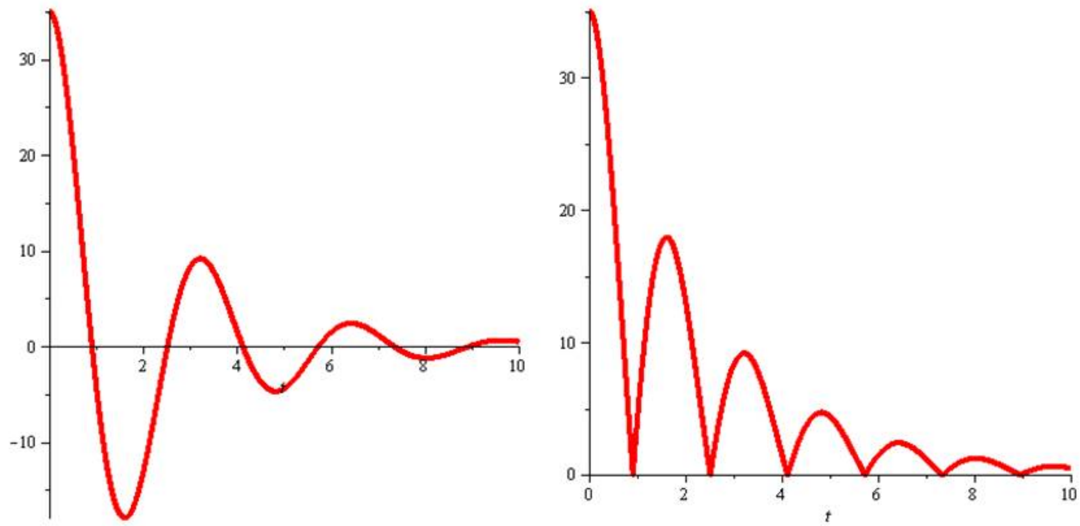


Figure A.2: On the left, solution to damped harmonic oscillator; on the right, absolute value of solution

some of these ILAPs to the CODEE web site.

Some ILAPs are more sophisticated than others. For example, "The Phenomena of Mechanical Resonance" is a nice introduction to resonance with a linear second-order equation, while "Aircraft Flight Strategies" requires a basic understanding of lift, altitude, velocity, etc., and has nonlinear equations. Recently, I have been researching the Labors of Hercules and have developed ILAPs with connections to Greek Mythology. The Fifth Labor of Hercules required the hero to clean the stables of Augeas. This is a unique problem using Torricelli's law. The Eleventh Labor of Hercules involved wrestling a giant who gained energy from contact with the earth. The forcing function is a Heaviside term (a series of step functions), and we use Laplace transforms in developing the solutions. There are ILAPs for mixing problems. There was an episode on The Simpsons ("Who Shot Mr. Burns?") that I have used involving the connection of mathematics and pop culture. Students have to solve a Newton's Law of Cooling problem to find the culprit. I have used ILAPs centered around bungee jumping and modeling oxygen in the bloodstream after exercise.

There is also a wealth of possibilities in COMAP journal issues. I am working to write up a few of them for use in my course. Many of the COMAP articles are very detailed, so I will condense them a bit, hoping to keep the interdisciplinary flavor for the students while extracting necessary information to develop the models.

Students and Feedback

What prerequisites have my students had? At Muhlenberg, the student population in my ODEs class is usually mixed when considering the instruments in their mathematical toolkit. I usually have 20 to 25 students each semester. Often I have sophomore mathematics majors who have just completed Integral Calculus (Calculus II is the only prerequisite for the course), and this is their first mathematics elective. In the same class I have juniors and seniors who take ODEs as their last elective to fulfill a major or minor in mathematics. Their abilities are much more advanced, but in other areas of mathematics (many have taken our proofs course, linear algebra, abstract algebra, and other electives). Many of the upper-classmen are physics majors, minoring in mathematics. Usually, the differential equations course is the first (or maybe second, after linear algebra) course which is heavily applications-oriented. However, as it is an elective, some mathematics majors will graduate from Muhlenberg without having had a course in differential equations.

When I assign ILAP projects, the students work in 2- , 3- , or 4-person teams. I usually allow up to two weeks for students to work the problems, and I require a formal write-up with all aspects of the modeling process. Format for the write-ups are provided in my syllabus. About a week into the projects, I hold informal 10-minute in-progress reviews (IPRs) with the student teams. This requires them to have done some work (instead of waiting until the last minute), and the teams brief me on their progress, addressing any questions they might have. The IPRs keep the groups focused and I find that the groups are trying to accomplish as much as they can before the IPRs. When the projects are turned in for a grade, the student groups are selected at random to brief a portion of the requirements and solutions. This allows for more groups to brief, reduces unnecessary repetition, and gets more students involved. The discussions we have are very valuable, to both the groups and to me. I find that they try to explain their thinking to me, before I have graded the write-ups), and the groups benefit from each other. I also have a portion of my final examination devoted to the ILAP, to ensure students were participating and understood the learning objectives.

Student feedback has been very positive. My experience has shown that students are more apt to really learn the mathematics when they can apply that math to solve problems that they might see again in a science or engineering course. Course-end surveys have revealed that once students sense that the lessons and assessments focus on problems which they might encounter in the real world, a metamorphosis occurs. The students feel that classes are more interesting. They also believe that their confidence levels rise when using mathematics to solve problems which surround them in life, because they have a feel as to whether the answer makes sense. They are beginning to understand that their reasoning skills are as important as their analytic skills. If they are confident in their model, and they obtain an answer that just does not make sense, they can receive partial credit for explaining what their intuition tells them should be the answer. For instance, “When I solved for the position of the mass in the damped harmonic oscillator problem, the solution to my equation had a positive exponent, which indicated exponential growth. Since there is damping, the displacement should decrease with time. I must have an algebra mistake somewhere, but I ran out of time.” When our department chairman conducts his exit interviews with graduating mathematics majors, most mention how valuable the applications-oriented courses have been to them, even naming the differential equations course.

Nationally, I see more sessions at the annual mathematics meetings on teaching/assessment using modeling or problem-solving every year. Several other professors across the country are getting their students to visualize and model word problems, and others are having great success in getting students to write about their mathematics. Textbooks are popping up with “modeling” in the title. The technology keeps getting better and more user-friendly; visualization of solutions is critical in learning to predict long-term behavior. Modeling population growth or decay, interaction between competing species, the spread of disease in a community, or even solving problems with piecewise continuous forcing functions are now problems that students think, "Oh, I've heard of that," and "I think I can solve it." Competence and confidence are achievable goals.

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Appendix B

Sample Grading Rubric

On frequent question that comes up with discussing modeling projects or mathematical writing is, how do I grade it? Rubrics are very helpful because they give instructors a way of being more systematic in their assessments of student work. And, when a rubric is distributed to students before their assignments are due, is also helps students understand what is expected of them.

On the next page is a sample rubric that could be used to assess students' write-ups of their modeling projects. If students are not creating their own mathematical models, some of the first two categories may not be applicable.

Description of the model	The description of the mathematical model is well-motivated, convincing, clearly conveyed, and tailored to the intended audience of the paper.	The mathematical model is described well, but is missing one or two things that the audience would need to know to fully understand the problem.	The mathematical model is hastily described or is missing significant details.	The mathematical model is ill defined.
Problem formulation	The mathematical model is accurately formulated; all assumptions are clearly stated and warranted.	Some unwarranted assumptions were made in the mathematical model.	There are some flaws in the formulation of the mathematical model.	The mathematical model is either largely incorrect or inappropriate for the given problem.
Quality of mathematical work	The analysis techniques chosen are well-suited to the problem. The calculations performed are thorough, complete and correct.	The analysis techniques work well for the problem, but there may be other techniques that are superior. The calculations are mostly complete and correct.	The analysis techniques chosen are not well implemented or inappropate. Some calculations are incorrect.	There are serious mathematical errors that lead to incorrect results.
Insightfulness and depth of understanding	Numerous insightful and thoughtful conclusions from data are presented. It is clear that the author designed the model and calculations well to reach these conclusions and that the author learn many thing through the model.	Some insightful conclusions are reached through the model and calculations. There are a few obvious things the author could have done to draw more conclusions from the model.	The author makes a cursory attempt to draw conclusions from the data. The author demonstrates a minimal advancement of technical ability through the model.	The write-up gives no insights on the original problem or shows no signs of learning on the part of the student.
Grammar and mechanics	Excellent	Few errors	Several errors	Poorly written