5

SIMPLIFIED MULTI-STOREY SHEAR BUILDING MODEL

1. Assumptions in shear building model

The simplest possible multidegree of freedom idealisation is to assume that only the *x* and *y* horizontal sway motions and the *z* vertical motion of each floor need be considered. This kind of idealisation is based on the following assumptions.

I. *Rigid Floors/Diaphramic Action/Rigid Ground*

The floors are totally rigid and incompressible thus transferring the floor displacement to all the columns equally. The floors and the ground provide a fully fixed support to the columns. There are hence no floor rotations. The rigid ground implies that the ground acceleration is applied to all columns equally. The assumptions here will overestimate the stiffness of the building hence under a crude single degree of freedom analysis decrease the fundamental period of vibration of the building. Considering the general trend of the acceleration response spectrum; this decrease fundamental period would tend to result in larger earthquake induced forces. Hence the assumption of floor rigidity etc. can be thought as a safe engineering assumption while not being totally valid.

II. *Lumped Mass at floor level/Massless columns*

All the mass of the floor is lumped at the centre of mass of the floor. If the floors are RC slab constructions then most of the mass is concentrated at the floor levels. The centre of mass of the floor and the centre of area of the floor plan are coincident if the distribution of mass across the floor area is uniform. For dead/self weight loads this is usually true. However for live/imposed load the distribution of loading may not be uniform across the floor area. Thus the centre of mass for the combined dead and live loads may in practice not be at the centre of area. This live load effect is neglected in this analysis.

III. *Symmetric or Regular buildings*

This means that the centre of mass of each floor and the centre of stiffness of each floor are coincident. Symmetric floor plans (symmetric about both *x & y* axis) with symmetric column positioning and size normally result in coincident centres of mass and stiffness. Also the position of this combined centre should be stationary up the height of the building. This assumption restricts changes in the floor plan (*setbacks*) and column positioning up the height of the building. This assumption results in no floor torsional effects.

IV. *Decoupling of Motion in x, y and z directions, linear, small deflections.*

With no torsional effects, small linear deflection theory and all columns having principle axis aligned with global *(x,y,z);* the sway motion in *x* and *y* directions and the *z* vertical motion can be analysed separately.

The structure is thus idealised by a single column with lumped mass at various heights from the floor. The stiffness of the idealised column is the total stiffness of all the columns in the actual structure. The lumped masses are equal the equivalent floor masses. Essentially the total floor load divide by *g*. While only columns have been mentioned as supporting elements for the floors, lift and stairway cores, shear walls, truss-bracing etc. can all be included when calculating the total stiffness of a particular storey. However assumption III must be maintained which is in practice not so simple. Practically speaking any cores, shear wall etc. must by symmetrically placed about the centre of area to validate assumption III. Also it is often quite difficult of estimate stiffnesses of such elements without resorting to some FE analysis. Thus in the rest of this chapter only columns will be considered but if stiffnesses of various other resisting elements is known they can be included in a simple fashion.

Thus the four assumptions do restrict the range of types of buildings that can be analysed. However most buildings are of this type or are nearly in this type hence this simplified shear building, multi-storey analysis is in practice not so restrictive and is a great improvement on the single degree of freedom analysis.

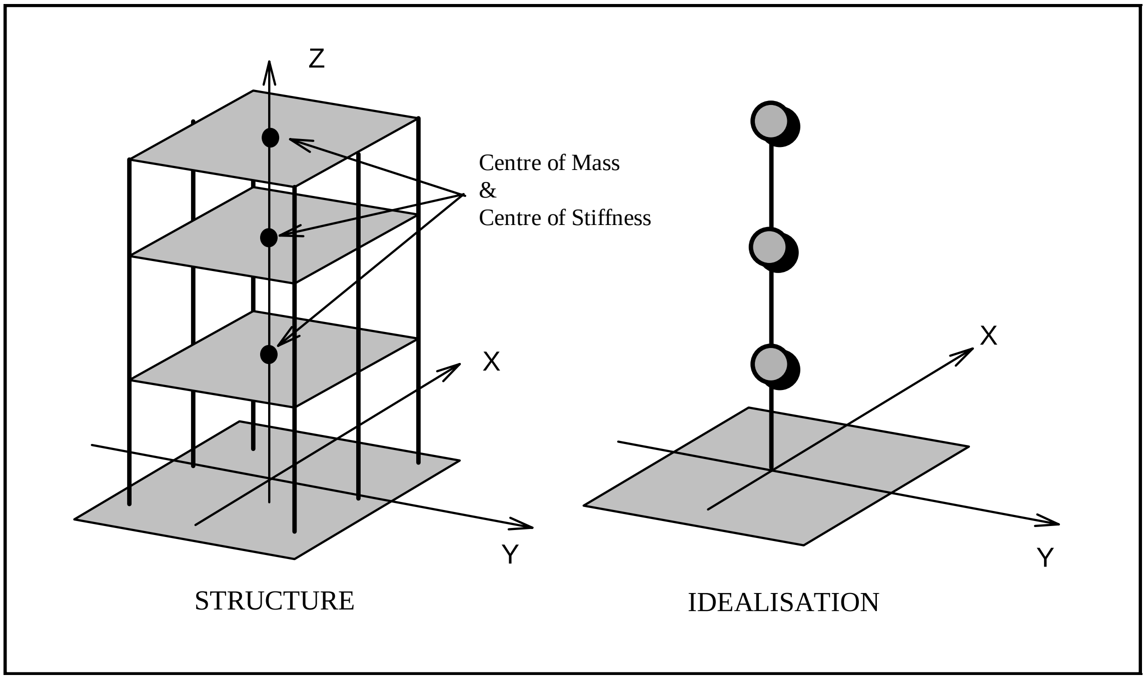


Figure 1: Multi-storey shear building

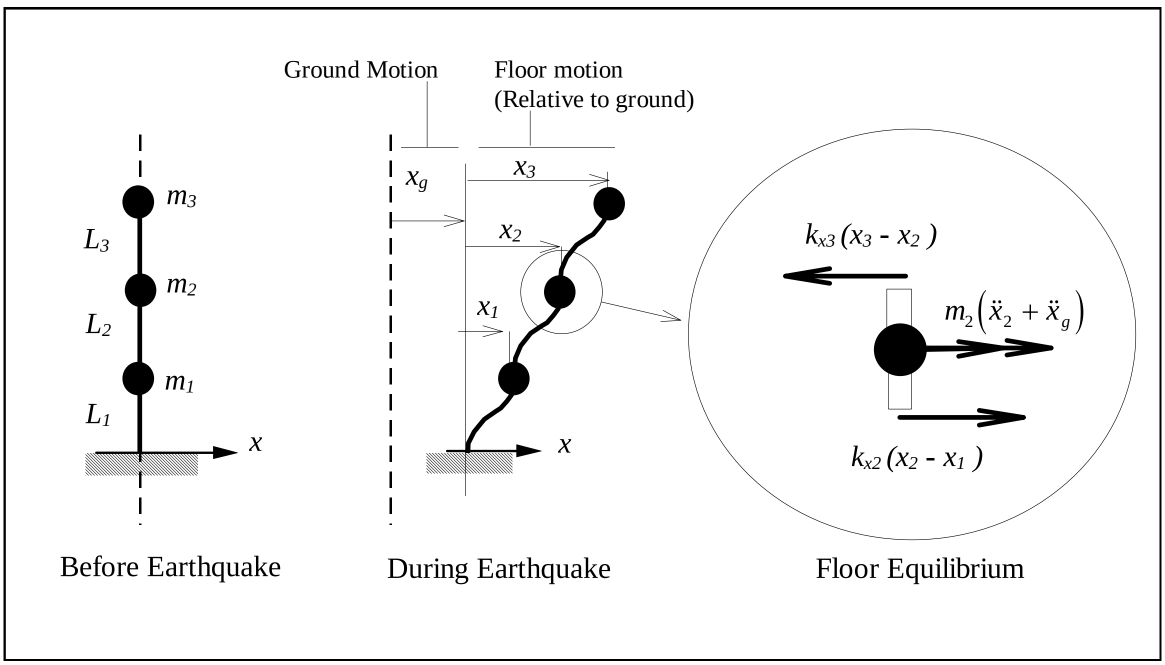


Figure 2: Horizontal motion

2. Horizontal equation of motion (neglecting damping)

Consider Figure 2 of the idealised structure subject to ground motion in the *x* direction. (*y* direction motion is identical: replace *x* with *y* in the following analysis). Consider the floor equilibrium at floor 2 (*this is an typical floor)* The horizontal shear force induced by the bending of the columns above and below (the stiffness forces) must be balanced with the mass horizontal inertia force of the floor. hence

$$\underbrace{m_2\left(\ddot{x}_2+\ddot{x}_g\right)}_{inertia}+\underbrace{k_{x2}\left(x_2-x_1\right)-k_{x3}\left(x_3-x_2\right)}_{Stiffness}=0$$

$$m_2\ddot{x}_2 - k_{x2}x_1 + (k_{x2} + k_{x3})x_2 - k_{x3}x_3 = -m_2\ddot{x}_g$$

It is clear for a general *ith* floor in a *n* floored building

$$m_i\ddot{x}_i - k_{xi}x_{i-1} + (k_{xi} + k_{xi+1})x_i - k_{xi+1}x_{i+1} = -m_i\ddot{x}_g$$

and hence if applied to all floors the matrix system of equations becomes

$$\mathbf{M}\underline{\ddot{x}} + \mathbf{K_X}\,\underline{x} = -\mathbf{M}\underline{1}\ddot{x}_g \qquad (5\text{-}1)$$

The explicit matrix system is for a three-storey building thus

$$\underbrace{\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}}_{\mathbf{M}}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\end{bmatrix} + \underbrace{\begin{bmatrix} k_{x1}+k_{x2} & -k_{x2} & 0 \\ -k_{x2} & k_{x2}+k_{x3} & -k_{x3} \\ 0 & -k_{x3} & k_{x3} \end{bmatrix}}_{\mathbf{K_X}}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = -\mathbf{M}\begin{bmatrix}1\\1\\1\end{bmatrix}\ddot{x}_g$$

and for a four storey building

$$\underbrace{\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix}}_{\mathbf{M}}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\\ \ddot{x}_4\end{bmatrix} + \underbrace{\begin{bmatrix} k_{x1}+k_{x2} & -k_{x2} & 0 & 0 \\ -k_{x2} & k_{x2}+k_{x3} & -k_{x3} & 0 \\ 0 & -k_{x3} & k_{x3}+k_{x4} & -k_{x4} \\ 0 & 0 & -k_{x4} & k_{x4} \end{bmatrix}}_{\mathbf{K_X}}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = -\mathbf{M}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\ddot{x}_g \qquad (5\text{-}2)$$

Note the (roof storey counts as a floor.) Both **M** and $\mathbf{K_X}$ are symmetric matrices while **M** is also diagonal. k_{xi} is the stiffness of the idealised column from floor *i-1* to floor *i* . It is the summation of all the stiffness elements *(here just columns are considered)* between floors *i-1* and *i*. If all columns are identical in cross-section and fully fixed top and bottom[1].

$$k_{xi} = \left(\frac{12EI_{yi}}{L_i^3}\right).m \qquad (5\text{-}3)$$

where *m* is the number of columns between floor *i-1* and floor *i*. L_i is the length of columns. EI_{yi} is the flexural stiffness of a typical column bending in the *x* direction about the *y* axis

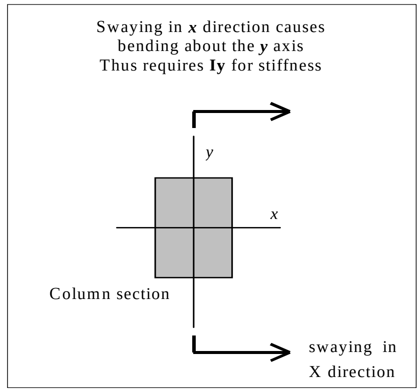


Figure 3: Flexural rigidity

If stiffness elements are not identical then the k_{xi} is the sum of all the stiffness elements between floors *i-1* and *i*. (But remember assumption III means that it is difficult to have elements of different stiffnesses while keeping the centre of stiffness and mass coincident).

3. Vertical equation of motion (neglecting damping)

Consider the following Figure 4 of the idealised structure subject to ground motion in the *z* direction. Consider the floor equilibrium at floor 2 (*this is an typical floor)* The vertical axial force of the columns above and below (the stiffness forces) must be balanced with the mass vertical inertia force of the floor. Hence

$$\underbrace{m_2(\ddot{z}_2 + \ddot{z}_g)}_{inertia} + \underbrace{k_{z2}(z_2 - z_1) - k_{z3}(z_3 - z_2)}_{Stiffness} = 0$$

hence by generalising and applying to all floors the matrix system of equations becomes

$$\mathbf{M}\underline{\ddot{z}} + \mathbf{K_z}\,\underline{z} = -\mathbf{M}\underline{1}\ddot{z}_g \qquad (5\text{-}4)$$

This equation is identical to that for *x* motion Equation (5-1) save that *x's* are replaced by *z's* . The stiffness k_{zi} of the idealised column from floor *i-1* to floor *i* . It is in fact the summation of all the stiffness elements *(here just columns are considered)* between floors *i-1* and *i*. If all columns are identical in cross-section.

$$k_{zi} = \left(\frac{EA_i}{L_i}\right).m \qquad (5\text{-}5)$$

[1] See Appendix E (to include shear deformations)

where *m* is the number of columns between floor *i-1* and floor *i*. L_i is the length of columns. EA_i is the Axial stiffness of a typical column subject to tension/compression.

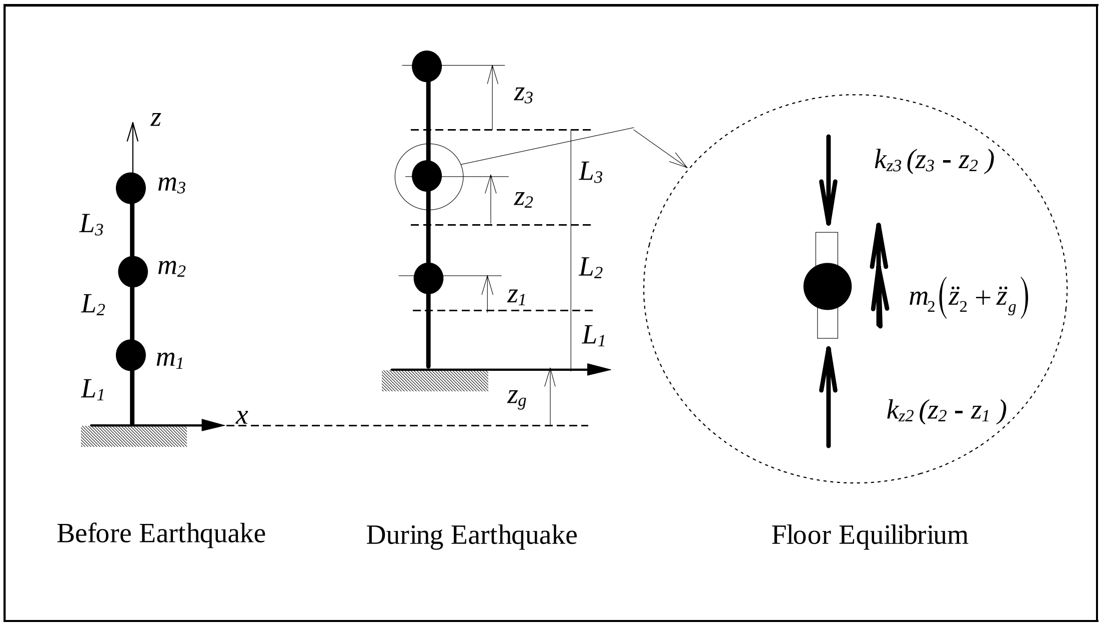
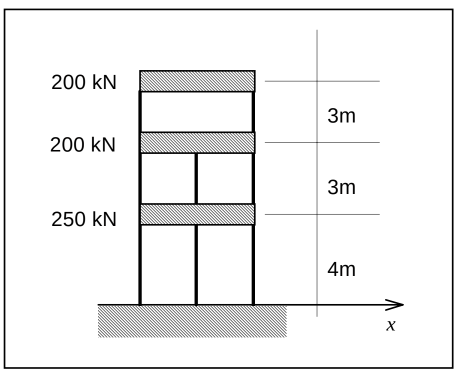


Figure 4: Vertical Motion

4. Example

Derive the Horizontal and vertical stiffness matrix and the mass matrix for the following building frame. Where EI_y is the flexural stiffness (for *x* direction motion) and A is the area of all the columns.



The mass matrix is simply m_1=250/g m_2=200/g etc...

$$M = \begin{bmatrix} 250/g & 0 & 0 \\ 0 & 200/g & 0 \\ 0 & 0 & 200/g \end{bmatrix} \text{kN Sec}^2/\text{m}$$

The horizontal stiffness matrix are derived using equation (5-3)

$$k_{x1} = 3\frac{12EI_y}{4^3}, k_{x2} = 3\frac{12EI_y}{3^3}, k_{x3} = 2\frac{12EI_y}{3^3}$$

$$K_X = \begin{bmatrix} k_{x1}+k_{x2} & -k_{x2} & 0 \\ -k_{x2} & k_{x2}+k_{x3} & -k_{x3} \\ 0 & -k_{x3} & k_{x3} \end{bmatrix}$$

$$= \frac{EI_y}{3}\begin{bmatrix} \frac{91}{16} & -4 & 0 \\ -4 & \frac{20}{3} & -\frac{8}{3} \\ 0 & -\frac{8}{3} & \frac{8}{3} \end{bmatrix} \text{kN/m}$$

For vertical motion, the vertical stiffness matrix is

$$k_{z1} = 3\frac{EA}{4}, k_{z2} = 3\frac{EA}{3}, k_{z3} = 2\frac{EA}{3}$$

$$K_Z = \begin{bmatrix} k_{z1}+k_{z2} & -k_{z2} & 0 \\ -k_{z2} & k_{z2}+k_{z3} & -k_{z3} \\ 0 & -k_{z3} & k_{z3} \end{bmatrix}$$

$$= EA\begin{bmatrix} \frac{7}{4} & -1 & 0 \\ -1 & \frac{5}{3} & -\frac{2}{3} \\ 0 & -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \text{kN/m}$$

5. Free vibration (natural modes of vibration)

When the ground acceleration $\ddot{x}_g$ is zero then the equation (5-1) becomes the free vibration, multi-degree of freedom problem.

$$\mathbf{M}\underline{\ddot{x}} + \mathbf{K}\underline{x} = \underline{0} \qquad (5\text{-}6)$$

Note $\mathbf{K_x}$ shall be denoted as **K** from now on. While the motion in the *x* direction is considered. The analyses of motions in y & z directions are treated identically except that **K** would be $\mathbf{K_Y}$ or $\mathbf{K_z}$. If, as in the case of the single degree of freedom system, the solution to this system of simultaneous second order ordinary differential equations is of the form $\underline{x} = \underline{\phi}\, a \sin(\omega t + \varphi)$. $\underline{\phi}$ is a normalised displacement function describing the relative displacement of each floor mass. It is commonly known as a mode shape. $a \sin(\omega t + \varphi)$ is the time dependent amplitude function which is a sine function of circular frequency ω , phase φ and peak amplitude *a*.

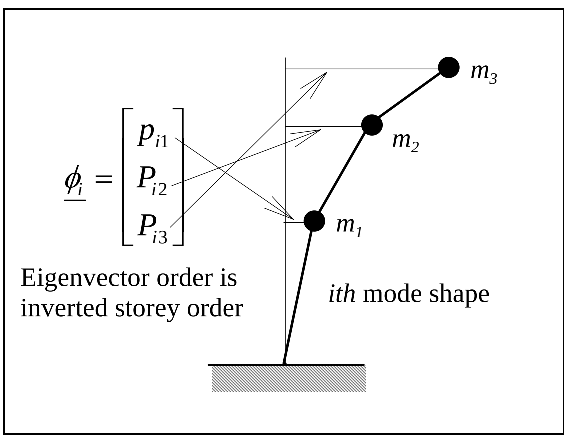


Figure 5: Eigenvector notation

Then $\underline{\ddot{x}} = -\omega^2 a \underline{\phi} \sin(\omega t + \varphi)$ thus substituting into (5-6)

$$(\mathrm{K} - \omega^2 \mathrm{M}) a \underline{\phi} \sin(\omega t + \varphi) = 0$$

For non-trivial solutions, $a \sin(\omega t + \varphi) \neq 0$ thus the resultant equation is a classical Eigenvalue[2] problem thus

$$(\mathrm{K} - \omega^2 \mathrm{M})\underline{\phi} = 0 \quad \text{thus} \quad (D - \lambda\, \mathrm{I})\underline{\phi} = 0 \qquad (5\text{-}7)$$

[2] See Appendix C, Matrix algebra

where **I** is the unity matrix and $\mathbf{M^{-1}}$ is the inverse of the mass matrix. $\mathbf{D{=}M^{-1}K}$ is known as the dynamic matrix **D.** The eigenvector $\underline{\phi}_i$ is the *ith* mode shape and the *ith* eigenvalue λ_i is the square of the circular frequency of that mode shape.

The solution of the eigenproblem means that there are in fact *m* mode shapes with different circular frequencies for an *m* storey shear building that are possible solutions to the equation (5-7). The periods of vibration for each mode shape can be obtained by using the standard relationship $T = 2\pi/\omega$ hence

$$T_i = \frac{2\pi}{\sqrt{\lambda_i}} \qquad (5\text{-}8)$$

Where T_i is the natural period of the *ith* mode of vibration.

6. Example

Find the horizontal modes shapes and natural period of vibration of the following shear building. The flexural stiffness EI_y is 68160 kNm^2 for all columns.

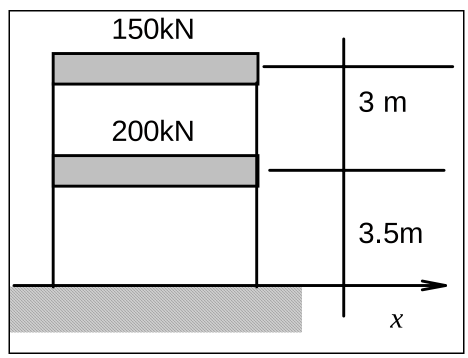


Figure 6: Two storey shear building

The mass matrix is

$$\mathrm{M} = \begin{bmatrix} 200/g & 0 \\ 0 & 150/g \end{bmatrix} = \begin{bmatrix} 20.39 & 0 \\ 0 & 15.29 \end{bmatrix} \ [\mathrm{kNs^2/m}]$$

The stiffness matrix can be derived in the standard way

$$K = \begin{bmatrix} \frac{24EI}{3.5^3} + \frac{24EI}{3^3} & -\frac{24EI}{3^3} \\ -\frac{24EI}{3^3} & \frac{24EI}{3^3} \end{bmatrix} = \begin{bmatrix} 98740 & -60587 \\ -60587 & 60587 \end{bmatrix}$$

[kN/m]

Hence the Dynamic matrix is

$$\mathbf{D} = \begin{bmatrix} 4844.3 & -2972.1 \\ -3963.1 & 3963.1 \end{bmatrix}$$

This is now an eigenvalue problem of the matrix D.

$$\begin{vmatrix} 4484.4-\lambda & -2972.1 \\ -3963.1 & 3963.1-\lambda \end{vmatrix}$$

$$= (4484.1-\lambda)(3963.1-\lambda) - 3963.1\times 2972.1 = 0$$

$$\lambda^2 - 8807\lambda + 0.7142\times 10^6 = 0$$

$$\lambda_1 = 943.4 \ \& \ \lambda_2 = 7863.7$$

$$\underline{\phi}_1 = \begin{bmatrix} 0.762 \\ 1 \end{bmatrix} \quad \underline{\phi}_2 = \begin{bmatrix} -0.984 \\ 1 \end{bmatrix}$$

From equation (5-8) the natural periods of vibration can be calculated. Remember the top element in the eigenvector corresponds to the first floor mass.

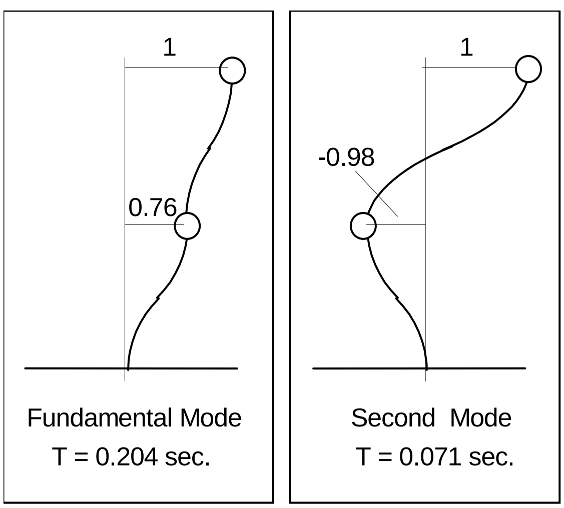


Figure 7: First and second modes of vibration

The eigenvectors are normalised using the unit maximum element norm, common in dynamics. The eigenvector represent the natural mode shape as described above. Notice that the fundamental mode has a period of 0.204 sec which compares very well with the approximation of the period of 0.1N where N is the number of storeys thus 0.2 sec.

7. Superposition of modes of vibration (total solution)

The total solution is given by the following

$$\underline{x} = \sum_{r=1}^{n} \underline{\phi_r} a_r \sin(\omega_r t + \varphi_r) \qquad \text{(5-9)}$$

where a_r is the amplitude for each mode shape and is dependent on the initial conditions of the free vibrations. φ_r is the phase angle of the mode shape and is also dependent on the initial conditions. Basically the each mode shape is a sinusoidal function of amplitude a_r and phase φ_r which are superpositioned (added) together to production the resultant solution.

8. Rayleigh's quotient

From equation (5-7) it follows that

$$\mathrm{K}\underline{\phi}_i = \omega_i^2 \mathrm{M}\underline{\phi}_i$$

And by pre multiplying by $\underline{\phi}_i^{\mathrm{T}}$ and solving for the circular frequency

$$\omega_i^2 = \frac{\underline{\phi}_i^{\mathrm{T}} \mathrm{K} \underline{\phi}_i}{\underline{\phi}_i^{\mathrm{T}} \mathrm{M} \underline{\phi}_i} \qquad \text{(5-10)}$$

This formula is known as Rayleigh's quotient and has be used as a method of approximating the circular frequency of a specific mode shape when the mode shape is only known approximately. However it is far more accurate to solve the complete eigenproblem as already described.

9. Orthogonality of natural modes.

From equation (5-7) it follows $\lambda_1 \mathrm{M}\underline{\phi}_1 = \mathrm{K}\underline{\phi}_1$

By pre-multiplying both sides by $\underline{\phi}_2^{\mathrm{T}}$ thus

$$\lambda_1 \underline{\phi}_2^{\mathrm{T}} \mathrm{M} \underline{\phi}_1 = \underline{\phi}_2^{\mathrm{T}} \mathrm{K} \underline{\phi}_1 \qquad \text{(a)}$$

Note that this multiplication is allowable. Matrix product rule [1.n][n.n][n.1] = [1.1]. Also the matrix product $\underline{\phi}_2^{\mathrm{T}} \mathrm{K} \underline{\phi}_1$ results in a simple scalar number. And if the transpose is taken of both sides of equation (a) we get

$$\lambda_1 \underline{\phi}_1^{\mathrm{T}} \mathrm{M} \underline{\phi}_2 = \underline{\phi}_1^{\mathrm{T}} \mathrm{K} \underline{\phi}_2 \qquad \text{(b)}$$

Remember $(ABC)^T = C^T B^T A^T$. Also M and K are symmetric thus M=M^T and K=K^T.

From equation (5-7) $\lambda_2 \mathrm{M}\underline{\phi}_2 = \mathrm{K}\underline{\phi}_2$

By pre-multiplying both sides by $\underline{\phi}_1^{\mathrm{T}}$ thus

$$\lambda_2 \underline{\phi}_1^{\mathrm{T}} \mathrm{M}\underline{\phi}_2 = \underline{\phi}_1^{\mathrm{T}} \mathrm{K}\underline{\phi}_2 \qquad \text{(c)}$$

By subtracting Equations (c) from (b)

$$(\lambda_1 - \lambda_2)\underline{\phi}_1^{\mathrm{T}} \mathrm{M}\underline{\phi}_2 = 0 \quad \text{hence} \quad \underline{\phi}_1^{\mathrm{T}} \mathrm{M}\underline{\phi}_2 = 0$$

If $\underline{\phi}_1^{\mathrm{T}} \mathrm{M}\underline{\phi}_2 = 0$ then clearly from (b) $\underline{\phi}_1^{\mathrm{T}} \mathrm{K}\underline{\phi}_2 = 0$

Generally $\underline{\phi}_i^{\mathrm{T}} \mathrm{K}\underline{\phi}_j = 0 \;\; \& \; \underline{\phi}_i^{\mathrm{T}} \mathrm{M}\underline{\phi}_j = 0, \;\; i \neq j$ (5-10)

These results in equation (5-10) are known as the orthogonality of the mass and stiffness matrices.

10. Use of initial conditions to find a_r & ψ_r

Equation (5-9) can be written

$$\begin{aligned} \underline{x} &= \sum_{r=1}^{n} \underline{\phi_r} a_r \sin(\omega_r t + \varphi_r) \\ &= \sum_{r=1}^{n} \underline{\phi_r} \{ b_r \sin(\omega_r t) + c_r \cos(\omega_r t) \} \end{aligned} \qquad (5\text{-}11)$$

Given initial condition $\underline{x}$ at $t=0$

$$\underline{x}_{t=0} = \sum_{r=1}^{n} \underline{\phi_r} c_r = c_1 \underline{\phi_1} + c_2 \underline{\phi_2} + \cdots + c_n \underline{\phi_n}$$

To find c_r multiply both sides of the equation by $\underline{\phi}_r^{\mathrm{T}}\mathrm{M}$ and remembering the orthogonality of the mass matrix equation (5-10), thus

$$\begin{aligned} \underline{\phi}_{\mathrm{r}}^{\mathrm{T}} \mathrm{M}\, \underline{x}_{t=0} &= c_1 \underline{\phi}_{\mathrm{r}}^{\mathrm{T}} \mathrm{M}\underline{\phi}_1 + \cdots + c_r \underline{\phi}_{\mathrm{r}}^{\mathrm{T}} \mathrm{M}\underline{\phi}_r + \cdots + c_n \underline{\phi}_{\mathrm{r}}^{\mathrm{T}} \mathrm{M}\underline{\phi}_n \\ &= c_r \underline{\phi}_{\mathrm{r}}^{\mathrm{T}} \mathrm{M}\underline{\phi}_r \end{aligned}$$

hence solving for c_r

$$c_r = \frac{\underline{\phi_r}^{\mathrm{T}} \mathrm{M}\, \underline{x}_{t=0}}{\underline{\phi_r}^{\mathrm{T}} \mathrm{M} \underline{\phi_r}} \qquad (5\text{-}12)$$

Similarly given initial condition $\underline{\dot{x}}$ at $t=0$

$$\underline{\dot{x}}_{t=0} = \sum_{r=1}^{n} \underline{\phi_r} b_r \omega_r$$

Hence by premultiplying by $\underline{\phi}_r^{\mathrm{T}}\mathrm{M}$ and observing the orthogonality condition

$$b_r = \frac{\underline{\phi_r}^{\mathrm{T}} \mathrm{M}\, \underline{\dot{x}}_{t=0}}{\underline{\phi_r}^{\mathrm{T}} \mathrm{M} \underline{\phi_r} \omega_{\mathrm{r}}} \qquad (5\text{-}13)$$

Once b_r & c_r are evaluated a_r & ψ_r can be calculated remembering the simple trigonometrical relationship in equation (2.3-1) where

$$a_r = \sqrt{b_r^{\,2} + c_r^{\,2}} \qquad \varphi_r = \arctan\left(\frac{c_r}{b_r}\right) \qquad (5\text{-}14)$$

11. Diagonalisation of M and K by using eigenvectors

Equation (5-10) the Orthogonality condition is a very useful result in the multidegree of freedom systems analysis. Consider the matrix of eigenvectors Φ where

$$\Phi = \left\lfloor \underline{\phi}_1 \quad \underline{\phi}_2 \quad . \quad . \quad \underline{\phi}_n \right\rfloor$$

Now consider the triple matrix product $\Phi^T \mathrm{K} \Phi$. The result is a square matrix (n.n) where the off diagonal terms are given by $\underline{\phi}_i^{\mathrm{T}} \mathrm{K}\underline{\phi}_j$ which is zero and terms on the main diagonal are given by $\underline{\phi}_i^{\mathrm{T}} \mathrm{K}\underline{\phi}_i$ which is not zero. Hence $\Phi^T \mathrm{K} \Phi$ is a diagonal matrix. By a similar argument $\Phi^T \mathrm{M} \Phi$ is also a diagonal matrix.

12. Modal Analysis, normalised co-ordinates

The evaluation of the modes of free vibration can be turned to some practical use when considering the effects the earthquake ground acceleration $\ddot{x}_g$

$$\mathbf{M}\underline{\ddot{x}} + \mathbf{K}\underline{x} = -\mathbf{M}\hat{\underline{1}}\ddot{x}_g$$

Let $\underline{x} = \Phi \underline{a}$ where $\underline{a}$ is known as the *normalised co-ordinates* and by substituting into matrix equation above and by pre-multiplying by Φ^{T}.

$$\Phi^{\mathrm{T}} \mathrm{M} \Phi \underline{\ddot{a}} + \Phi^{\mathrm{T}} \mathrm{K} \Phi \underline{a} = -\Phi^{\mathrm{T}} \mathrm{M} \hat{\underline{1}} \ddot{x}_g$$

Because of the diagonalisation of the matrices M and K the above system can be re-written in

$$\underbrace{\begin{bmatrix} \underline{\phi}_1^T M \underline{\phi}_1 & 0 & \cdots & 0 \\ 0 & \underline{\phi}_2^T M \underline{\phi}_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \underline{\phi}_n^T M \underline{\phi}_n \end{bmatrix}}_{\Phi^T M \Phi} \underbrace{\begin{bmatrix} \ddot{a}_1 \\ \ddot{a}_2 \\ \vdots \\ \ddot{a}_n \end{bmatrix}}_{\underline{\ddot{a}}} +$$

$$\underbrace{\begin{bmatrix} \underline{\phi}_1^T K \underline{\phi}_1 & 0 & \cdots & 0 \\ 0 & \underline{\phi}_2^T K \underline{\phi}_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \underline{\phi}_n^T K \underline{\phi}_n \end{bmatrix}}_{\Phi^T K \Phi} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{\underline{a}} = -\begin{bmatrix} \underline{\phi}_1^T M \hat{\underline{1}} \\ \underline{\phi}_2^T M \hat{\underline{1}} \\ \vdots \\ \underline{\phi}_n^T M \hat{\underline{1}} \end{bmatrix} \ddot{x}_g$$

$\underline{\phi}_i^T M \hat{\underline{1}}$ is just a number the matrix multiplication is [1.n][n.n][n.1]=[1.1]. Note that because of the diagonalisation of the matrices the matrix equation in effect becomes *uncoupled*. Basically each row of the above matrix equation represents a single differential equations solved without reference to the other equations. The Matrix equation above can be expresses thus

$$\begin{aligned} &\underline{\phi}_1^T M \underline{\phi}_1 \ddot{a}_1 + \underline{\phi}_1^T K \underline{\phi}_1 a_1 = -\underline{\phi}_1^T M \hat{\underline{1}} \ddot{x}_g \\ &\underline{\phi}_2^T M \underline{\phi}_2 \ddot{a}_2 + \underline{\phi}_2^T K \underline{\phi}_2 a_2 = -\underline{\phi}_2^T M \hat{\underline{1}} \ddot{x}_g \\ &\ldots \\ &\underline{\phi}_n^T M \underline{\phi}_n \ddot{a}_n + \underline{\phi}_n^T K \underline{\phi}_n a_n = -\underline{\phi}_n^T M \hat{\underline{1}} \ddot{x}_g \end{aligned} \qquad (5\text{-}15)$$

The solution of each of these separate single degree of freedom systems results in the normalised co-ordinates a_i that are the amplitude of the *ith* modes of vibration. Normalised co-ordinates a_i are functions of time. It is important to point out that a_i is the case of the unforced free vibrations was just a constant and not a function of time.

13. Participation Factors

By dividing each of the above equations (5-15) by the mass (of the ith mode) $M_i = \underline{\phi}_i^T M \underline{\phi}_i$ and remembering Rayleigh quotient equation (5-10) these equations can be expressed in general.

$$\ddot{a}_i + \omega_i^2 a_i = -\Gamma_i \ddot{x}_g \qquad (5\text{-}16)$$

$$\Gamma_i = \frac{L_i}{M_i} = \frac{\underline{\phi}_i^T M \hat{\underline{1}}}{\underline{\phi}_i^T M \underline{\phi}_i} \qquad (5\text{-}17)$$

This is *uncoupled single degree of freedom system* where the ground acceleration is scaled by the factor Γ_i that is known as the participation factor of the ith mode. The participation factor is a scalar number. It turns out that *the normalisation procedure used for the eigenvectors does effect its numerical values*. It is found in practice that the participation factor is largest for the first mode and then continually reduces for the higher modes. Hence generally only the first few modes (perhaps four modes) contribute greatly to the overall structural vibration. Since the participation factor seems to scale the ground acceleration it does influence the effect that a particular mode has on the overall motion.

There are in fact two parameters that effect the contribution of a particular mode to the overall motion of a building. (a) *The participation factor*: modes with smaller participation factors have reduced effect due to reduced ground acceleration exciting this mode. (b) *The period of the mode of vibration:* if the ground acceleration has a frequency component that is near to a modal frequency that mode will be excited resonantly thus increasing its contribution. The acceleration response spectrum is used to determining this effect which shall be discussed later.

14. Quick calculation of participation factors

It can be shown that equation (5-17) is identical to the following given that M is diagonal

$$\Gamma_i = \frac{L_i}{M_i} = \frac{\sum_j m_j P_{ij}}{\sum_j m_j \left(P_{ij}\right)^2} \qquad (5\text{-}18)$$

when $\underline{\phi}_i^T = \begin{bmatrix} P_{i1} & P_{i2} & \cdots & P_{in} \end{bmatrix}$ and the mass matrix is $\mathbf{M} = diag\begin{bmatrix} m_1 & m_2 & \cdots & m_n \end{bmatrix}$.

15. Example

Evaluate the participation factors for the problem in example in section 6.

for Mode 1

$$\Gamma_1 = \frac{0.762 \times 20.39 + 1 \times 15.29}{0.762^2 \times 20.39 + 1^2 \times 15.29} = \frac{30.827}{27.129} = 1.136$$

for Mode 2

$$\Gamma_2 = \frac{-0.984 \times 20.39 + 1 \times 15.29}{(-0.984)^2 \times 20.39 + 1^2 \times 15.29} = \frac{-4.774}{35.03} = 0.136$$

The 2nd mode will most probably have contributed far less than the first mode in the overall structural vibration due to its smaller participation factor.

16. Example

If the eigenvalue problem is solved for the structure in example in section 4 for horizontal x motion with full top storey fixity; the periods of vibration and eigenvectors (using a unit maximum element norm) are

$$T_1 = 0.351s.\ \underline{\phi}_1 = \begin{bmatrix} 0.678 \\ 0.888 \\ 1 \end{bmatrix}$$

$$T_2 = 0.105s.\ \underline{\phi}_2 = \begin{bmatrix} 1 \\ 0.194 \\ -0.85 \end{bmatrix} \quad T_3 = 0.067s.\ \underline{\phi}_3 = \begin{bmatrix} -0.6 \\ 1 \\ -0.48 \end{bmatrix}$$

Find the participation factors for the modes of vibration.

<u>For mode 1</u>

$$\Gamma_1 = \frac{0.678 \times 25.48 + 0.888 \times 20.39 + 1 \times 20.39}{0.678^2 \times 25.48 + 0.888^2 \times 20.39 + 1^2 \times 20.39} = \frac{55.77}{48.18} = 1.157$$

<u>for mode 2</u>

$$\Gamma_2 = \frac{1 \times 25.48 + 0.194 \times 20.39 - 0.85 \times 20.39}{1^2 \times 25.48 + 0.194^2 \times 20.39 + (-0.85)^2 \times 20.39} = \frac{12.1}{40.98} = 0.295$$

<u>for mode 3</u>

$$\Gamma_3 = \frac{-0.6 \times 25.48 + 1 \times 20.39 - 0.48 \times 20.39}{(-0.6)^2 \times 25.48 + 1^2 \times 20.39 + (-0.48)^2 \times 20.39} = \frac{-4.685}{34.26} = -0.137$$

Notice that the participation factors for the higher mode are much smaller indicating that these higher modes tend to contribute much less to the total structure vibrations. The negative sign on a participation factor is not significant, as it will just invert the direction of the ground acceleration.

17. Superposition of modes of vibrations

$\underline{x} = \Phi \underline{a}$ is the total displacement of the storey mass due to the ground acceleration.

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \underline{\phi}_1 & \underline{\phi}_2 & \cdots & \underline{\phi}_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$\underline{x} = a_1\underline{\phi}_1 + a_2\underline{\phi}_2 + \cdots + a_n\underline{\phi}_n = \sum_{i=1}^{n} a_i \underline{\phi}_i$$

$x_1\ x_2$ etc. are the displacement of the floor masses $m_1\ m_2$ etc. Notice that the total displacement is the sum of the mode shape displacements: $a_1\underline{\phi}_1$ and $a_2\underline{\phi}_2$ etc.. Normalised co-ordinate a_i is clearly the amplitude of the *ith* mode of vibration. Thus a_i are sometimes called the modal displacements. a_i is an amplitude which is varying in time. Its exact description for each mode can only be found by solving the differential equation (5-16) for every mode (This would require a numerical procedure such as Newmark's beta method.)

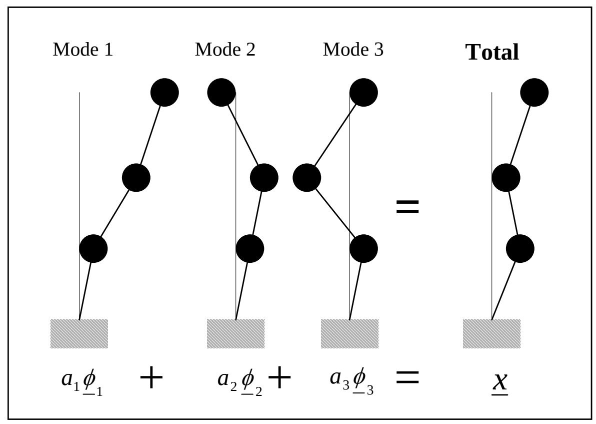


Figure 8: Superposition of modal deformations

Note that $a_i\underline{\phi}_i = a_i\begin{bmatrix} p_{i1} & p_{i2} & \cdots & p_{in} \end{bmatrix}^T$ where $a_i p_{ir}$ is the displacement at floor *r* during the *ith* mode of vibrations.

18. Effective modal mass

Consider the modal single degree of freedom system again

$$\ddot{a}_i + \omega_i^2 a_i = -\frac{L_i}{M_i}\ddot{x}_g \tag{5-19}$$

The solution of this equation could be given by Duhamel's integral, from chapter 1 section 17.

$$a_i(t) = -\frac{L_i}{M_i}\frac{1}{\omega_i}\int_0^t \ddot{x}_g e^{-\gamma\omega_i(t-s)}\sin(\omega_i(t-s))\,ds$$

or $$a_i(t) = \frac{L_i}{M_i}b_i(t)$$

where $b_i(t)$ is the solution to the single degree of freedom system equation (1-18) when the participation factor is one. The Floor displacement are given by

$$\underline{x} = \sum_{i=1}^{n}\underline{\phi}_i a_i = \sum_{i=1}^{n}\underline{\phi}_i \frac{L_i}{M_i}b_i(t)$$

The equivalent static floor force are given by the following,

$$\underline{F} = \mathrm{K}\underline{x} = \sum_{i=1}^{n}\mathrm{K}\underline{\phi}_i\frac{L_i}{M_i}b_i(t) = \sum_{i=1}^{n}\omega_i^2\mathrm{M}\underline{\phi}_i\frac{L_i}{M_i}b_i(t)$$

The sum force at all floors is from the addition of all floor forces

$$V = \underline{1}^T\underline{F} = \sum_{i=1}^{n}\omega_i^2\underline{1}^T\mathrm{M}\underline{\phi}_i\frac{L_i}{M_i}b_i(t) \qquad (5\text{-}20)$$

This sum force is equal to the base shear force when considering horizontal motions and the base vertical reaction when considering vertical motions. Note however that

$$\left(\underline{1}^T\mathrm{M}\underline{\phi}_i\right)^T = \underline{\phi}_i^{\mathrm{T}}\mathrm{M}\underline{1} = L_i$$

Hence equation (5-20) can be simplified

$$V = \sum_{i=1}^{n}\omega_i^2\left[\frac{L_i^2}{M_i}\right]b_i(t) \qquad \mu_i = \left[\frac{L_i^2}{M_i}\right] \qquad (5\text{-}21)$$

Where the term μ_i is known as the *effective modal mass of the ith mode.* Notice that is has dimensions of mass and is not dependent on the procedure used for normalising the eigenvectors. The other noteworthy property of the effective modal mass is that the sum of all the μ_i terms are equal to the total mass of the building. This can be shown as follows. Consider a unit displacement of all floors $\underline{1} = \Phi\underline{a}$ and hence by pre-multiplication by $\underline{\phi}_i^T\mathrm{M}$

$$\underline{\phi}_i^T\mathrm{M}\underline{1} = \underline{\phi}_i^T\mathrm{M}\Phi\underline{a} = M_i a_i \quad \text{hence} \quad a_i = \frac{L_i}{M_i}$$

hence $$\underline{1} = \Phi\begin{bmatrix} L_1/M_1 \\ \vdots \\ L_n/M_n \end{bmatrix}$$

Now the total mass Z of the building is given by

$$\mathrm{Z} = \underline{1}^T\mathrm{M}\underline{1} = \underline{1}^T\mathrm{M}\Phi\begin{bmatrix} L_1/M_1 \\ \vdots \\ L_n/M_n \end{bmatrix}$$

Thus $$\mathrm{Z} = \begin{bmatrix} L_1 & \cdots & L_n \end{bmatrix}\begin{bmatrix} L_1/M_1 \\ \vdots \\ L_n/M_n \end{bmatrix} = \sum_{i=1}^{n}\frac{L_i^2}{M_i} = \sum_{i=1}^{n}\mu_i$$

Most codes of practice suggest that the modes that must be considered should include 90% of the mass of the building. *The summation of all the effective modal mass is equal to the mass of the building!*

19. Example

Evaluate the effective Modal Masses for the problem in section 6.

Total mass of structure $= (200 + 150)/g = 35.67$

<u>for Mode 1</u>

$$\mu_1 = \frac{30.827^2}{27.129} = 35.02\,\mathrm{kNs^2/m} \rightarrow \frac{35.02\times 100}{35.67} = 98.2\%$$

<u>for Mode 2</u>

$$\mu_2 = \frac{(-4.774)^2}{35.03} = 0.65\,\mathrm{kNs^2/m} \rightarrow \frac{0.65\times 100}{35.67} = 1.8\%$$

Note that effect of mode 2 can be neglected according to EC8 since $\mu_1 > 90\%$.

20. Modal analysis including damping

If damping is included into the multidegree of freedom system equation (5-1) becomes

$$\mathbf{M}\ddot{\underline{x}} + \mathbf{C}\dot{\underline{x}} + \mathbf{K}\underline{x} = -\mathbf{M}\hat{\underline{1}}\ddot{x}_g \qquad (5\text{-}22)$$

In the section 12, the equation (5-1) was uncoupled by the normalised co-ordinates to leave n simultaneous 2nd order ODE's. With the introduction of a damping matrix **C** can the matrix system be still *uncoupled?* With the following two orthogonal damping matrices it is still possible to uncouple the matrix system.

20.1 Simple orthogonal damping

In a multidegree of freedom system the damping matrix C is conventionally taken as

$$\mathrm{C} = \alpha\,\mathrm{K} \qquad (5\text{-}23)$$

For practical structures, where light internal damping in the elastic members must be represented approximately, the assumption of C being proportional to K is common and reasonable. The great advantage of a Damping matrix C that is proportional to K is that it can be diagonalised and doesn't hinder the uncoupling of the system of equations. If C was not assumed to be proportional to K then the entire system of differential equations cannot be uncoupled. Hence the matrix differential system equation (5-22) would have to be solved using a timehistory solution using a matrix version of Newmark's procedure.

By expressing equation (5-22) in normalised co-ordinates and by pre-multiplying the equation by Φ^{T} we get

$$\Phi^{\mathrm{T}}\mathrm{M}\Phi\underline{\ddot{a}} + \alpha\,\Phi^{\mathrm{T}}\mathrm{K}\Phi\underline{\dot{a}} + \Phi^{\mathrm{T}}\mathrm{K}\Phi\underline{a} = -\Phi^{\mathrm{T}}\mathrm{M}\underline{\hat{1}}\ddot{x}_g$$

Dividing by the mass of the modes matrix $\Phi^{\mathrm{T}}\mathrm{M}\Phi$

$$\underline{\ddot{a}} + \alpha\,\Omega\underline{\dot{a}} + \Omega\underline{a} = -\Gamma_i\ddot{x}_g$$

where
$$\Omega = \begin{bmatrix} \omega_1^2 & 0 & & 0 \\ 0 & \omega_2^2 & & 0 \\ & & \ddots & \\ 0 & 0 & & \omega_n^2 \end{bmatrix}$$

Thus the *uncoupled single degree of freedom* equations are

$$\ddot{a}_i + \alpha\omega_i^2\dot{a}_i + \omega_i^2 a = -\Gamma_i\ddot{x}_g \qquad (5\text{-}24)$$

To maintain a similarity with the general single degree of freedom system the damping constant are modify by introducing γ the ratio of critical damping

$$\ddot{a}_i + 2\gamma_i\omega_i\dot{a}_i + \omega_i^2 a_i = -\Gamma_i\ddot{x}_g \qquad (5\text{-}25)$$

hence by equating equations (5-24) and (5-25) it is clear that $2\gamma_i\omega_i = \alpha\omega_i^2$. Where γ_i is the ratio of critical damping for the *ith* mode of vibration. Note that if the ratio of critical damping for the first mode γ_1 is known then $\alpha = 2\gamma_1/\omega_1$ thus the ratio of critical damping for modes 2,3 etc. can automatically be found. Thus $\gamma_i = \dfrac{\gamma_1}{\omega_1}\omega_i$ and clearly for the higher modes which have larger circular frequencies the ratio of critical damping is going to be larger. Thus the Higher modes of vibration are damped down quicker because of larger ratios of critical damping. (This is a result of the assumption that $\mathrm{C} = \alpha\,\mathrm{K}$)

20.2 Rayleigh orthogonal damping

$$\mathrm{C} = \alpha\,\mathrm{K} + \beta\,\mathrm{M} \qquad (5\text{-}26)$$

This is in fact a more general form of the previous damping model. By expressing equation (5-22) in normalised co-ordinates and by pre-multiplying the Eqn by Φ^{T} we get

$$\Phi^{\mathrm{T}}\mathrm{M}\Phi\underline{\ddot{a}} + \left(\alpha\,\Phi^{\mathrm{T}}\mathrm{K} + \beta\,\Phi^{\mathrm{T}}\mathrm{M}\right)\Phi\underline{\dot{a}} + \Phi^{\mathrm{T}}\mathrm{K}\Phi\underline{a} = -\Phi^{\mathrm{T}}\mathrm{M}\underline{\hat{1}}\ddot{x}_g$$

hence the uncoupled equations become

$$\ddot{a}_i + \left(\alpha\omega_i^2 + \beta\right)\dot{a}_i + \omega_i^2 a = -L_i\ddot{x}_g \qquad (5\text{-}27)$$

by a similar procedure to section 20.1 equation (5-25) can equated to equation (5-27) and hence if two ratios of critical damping are known the Rayleigh constants α and β can be determined thus.

$$\alpha\,\omega_1{}^2 + \beta = 2\gamma_1\omega_1$$
$$\alpha\,\omega_2{}^2 + \beta = 2\gamma_2\omega_2$$

Thus by solving for α and β

$$\alpha = \frac{2\left(\gamma_1\omega_1 - \gamma_2\omega_2\right)}{\left(\omega_1^2 - \omega_2^2\right)}, \quad \beta = 2\gamma_1\omega_1 - \alpha\omega_1^2 \qquad (5\text{-}28)$$

hence given the circular frequencies and the ratios critical damping of the first two modes it is possible to defined α and β. In can be shown that these equations result in higher modes of vibration having larger ratios of critical damping. While Rayleigh damping model may appear more general than the simpler orthogonal damping it is often very difficult in practice to specify more than one ratio of critical damping. However if the ratio of critical damping in equation (5-28) is set to the same for both mode 1 and 2 then

$$\alpha = \frac{2\gamma}{\left(\omega_1 + \omega_2\right)}, \quad \beta = \frac{2\gamma\omega_1\omega_2}{\left(\omega_1 + \omega_2\right)} \qquad (5\text{-}29)$$

21. Equivalent static actions

The solution of equations (5-25) results in variation in time of the amplitude of the modes of vibration when the building is subject to a general earthquake ground acceleration. This solution would require a computational Duhamel's integral of Newmark's method etc. By adding the displacements at each floor, for each mode (in a similar fashion to that described in section 7) at any instant in time t the displacement $\underline{x}(t)$ of the floors can be calculated. It is now possible to calculate an equivalent set of static forces applied horizontally at each floor that would produce the displacements $\underline{x}(t)$ by the simple force-deflection formula $\underline{F} = \mathrm{K}\underline{x}$. The horizontal force set $\underline{F}$ could then represent the effect of the earthquake and could be used in a static frame analysis to evaluate the internal moments, thrusts and shears (This would generally require a computer frame analysis package). The problem here is to evaluate the worst set of equivalent static forces that could be applied to the structure. For each instant in time t the set of forces $\underline{F}$ will be different. To choose the worst case would thus mean using almost endless number of sets of forces $\underline{F}$ in the static frame analysis. Clearly this is not a practical procedure. In an attempt to overcome this problem the modal components of $\underline{x}(t)$ are considered, remember the total displacement is made up of the superposition of each of the modal displacements.

$$\underline{x} = a_1\underline{\phi}_1 + a_2\underline{\phi}_2 + \cdots + a_n\underline{\phi}_n = \sum_{i=1}^{n} a_i\underline{\phi}_i$$

Thus the equivalent static forces that would produce the ith mode shape $a_i\underline{\phi}_i$ Thus the equivalent static force set for the ith mode is

$$\underline{F}_i = \mathrm{K}a_i\underline{\phi}_i = a_i\omega_i^2\mathbf{M}\underline{\phi}_i$$

The maximum modal displacement a_i can be calculated using the spectral displacement S_{dis}^i to the *ith* mode. Note however while the spectral displacement is defined for a ground motion of $\ddot{x}_g$ each mode is forced by $\Gamma_i\ddot{x}_g$. Thus maximum modal displacement a_i is scaled by the effect of the participation factor hence $\Gamma_i S_{dis}^i$. So the maximum equivalent static action is

$$\left(\underline{F}_i\right)_{\max} = \Gamma_i S_{dis}^i\mathbf{K}\underline{\phi}_i = \Gamma_i\frac{S_{acc}^i}{\omega_i^2}\mathbf{K}\underline{\phi}_i = \Gamma_i S_{acc}^i\mathbf{M}\underline{\phi}_i \quad (5\text{-}30)$$

where S_{acc}^i is the spectral acceleration.

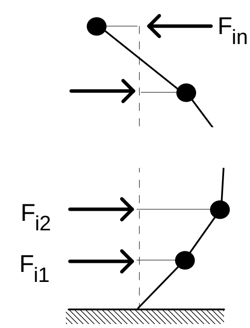


Figure 9: Equivalent static action

And hence by adding up all the contributions of all the modes

$$\underline{F} = \sum_{i=1}^{n}\Gamma_i S_{acc}^i\mathbf{M}\underline{\phi}_i \quad (5\text{-}31)$$

The assumption here is that all the maximum equivalent static actions will occur at the same time t for each mode and hence the simple addition of the effects of each mode is justified. However in reality the modal maxima do not occur at coincident time t. Thus the combined effect of each mode at an instant t is not so straightforward. Certain different modes may tend to act against each other cancelling each other out. The above formulation represents a very conservative (over estimate) of the total equivalent static forces.

22. SRSS combinations of modes

The above problem can be thought of in a probabilistic sense where equation (5-31) represents the worst case scenario of all the modes having coincident maxima. The root of the sum of the squares gives the most probable scenario thus

$$\underline{F}^* = \left\{\underline{F}_1^2 + \cdots + \underline{F}_n^2\right\}^{\frac{1}{2}} = \begin{bmatrix}\sqrt{f_{11}^{\ 2} + \cdots + f_{1n}^{\ 2}} \\ \vdots \\ \sqrt{f_{n1}^{\ 2} + \cdots + f_{nn}^{\ 2}}\end{bmatrix} \quad (5\text{-}32)$$

Thus the maximum equivalent static action for each floor mass for each mode are squared. Note that this is not strictly the square of a column vector that would be undefined but the square of the elements of this column vector. The sum of these squares is added and the resultant is square rooted to produce the compromise solution of the most probable set of equivalent static forces for each floor mass. This is known as the SRSS (*the Square Root of the Sum of the Squares)* combination of modes. This combination is valid as long as the circular frequencies of the modes are not too close to each other. If two modes have almost identical circular frequencies there combined effect would be nearer that given by equation (5-31) for these two modes. Horizontal Forces from equation (5-32) would be used is any subsequent static structural analysis. The foundation (*base*) shear and moments can be calculated by static analysis.

23. Calculating static actions using EC8 design spectrum

The EC8 design response spectrum provides a empirical evaluation of the spectral acceleration S_{acc}^{i}. This code specified design spectrum has great advantages for an engineer because it has been modified to includes ground and site effects, effects of irregularity of the building, effects of nonlinearity when material ductility reduces the magnitude of the response etc. Remember that the design response spectrum has become the vehicle for incorporating the more complex phenomena into the modal design analysis. It is worth noting however that some of the assumptions of modal analysis and the assumptions in the creation of the design response spectra are at odds with each other. The main and most problematic inconsistency is the inclusion of the nonlinearity due to plastic hinging. This would change the nature of the structure stiffness matrix and thus change modal circular frequencies and participation factors. From a mathematical point of view this is not valid however note that as plastic hinging occurs the structure stiffness will reduce. Thus the periods of the modes of vibration will increase (lengthen). Now because the design spectrum acceleration generally reduces for larger periods the equivalent static seismic forces will also reduce. Hence over estimating the structure stiffness tends to be on the safe side as far as determining the seismic actions. Note that this is also the reason why the shear-building model over estimates the seismic actions so is also an approximation on the safe side.

For the EC8 design spectrum the $S_d(T)$ is the design spectrum ordinate (Peak Acceleration/g) hence has to be multiplied by g thus

$$\left(\underline{F}_i\right)_{\max} = \Gamma_i S_{\mathrm{d}}\left(T_i\right) \mathrm{g} \mathrm{M} \underline{\phi}_i \qquad (5\text{-}33)$$

This equation (5-33) is used in conjunction with the SRSS result equation (5-32)

24. Calculation of displacements

The modal analysis is fundamentally a linear analysis but is used with the nonlinear design spectrum. This design spectrum assumes the formation of plastic hinges in the structure and primarily in the beam elements (strong columns - weak beams principle). The formation of plastic hinges has the effect of limiting the stiffness actions in the structure to the value dictated by the moment capacities of the element sections. Thus the actions evaluated by equation (5-32) assume inelastic (nonlinear) behaviour. If these computed actions are used to calculate displacement using the linear equation $\underline{F}^* = \mathrm{K}\underline{x}$ then the displacement $\underline{x}$ will be an under-estimate of the ultimate deflections. They will in effect define the deflections at notional yield-state when the plastic hinges are just forming.

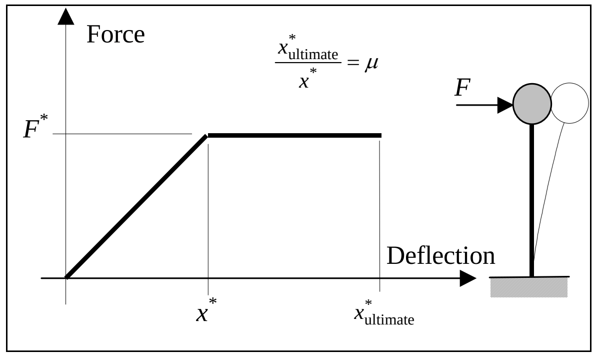


Figure 10: Approximate relationship between ultimate and notional yield-state deflections

Figure 10 is an example of a single column subject to a pushover lateral force *F*. As the force *F* increases a plastic hinge in the column occurs and the stiffness force in the column is limited to F^* while the deflection continues to increase from x^* to $x^*_{ultimate}$. The ultimate deflection is determined by the ductility μ of the column.

24.1 Deflections at notional yield-state

METHOD A (preferred)

By using the force-deflection relationship, the structural deflections at the notional yield-state is given by

$$\underline{x}^* = \mathrm{K}^{-1}\underline{F}^* \qquad (5\text{-}34)$$

The advantage of this equation (5-34) is that the deflections are exactly compatible with the equivalent static actions. The disadvantage, when performing calculations by hand, is that the structure stiffness matrix needs to be inverted. This is clearly a lengthy process for anything more than a [2x2] matrix. Method B uses SRSS modal combinations of deflections to determine the total deflection (at the notional yield-state). The advantage of this method is that it is easier to perform by hand. The disadvantage is the resultant deflections are not exactly compatible with the equivalent static actions.

METHOD B

Equation (5-33) can be rearranged

$$\mathrm{K}\left(\underline{\delta}_i\right)_{\max} = \left(\underline{F}_i\right)_{\max} = \Gamma_i S_{\mathrm{d}}\left(T_i\right)\mathrm{gM}\underline{\phi}_i$$

$$\left(\underline{\delta}_i\right)_{\max} = \Gamma_i S_{\mathrm{d}}\left(T_i\right)\mathrm{gK}^{-1}\frac{1}{\omega^2}\mathrm{K}\underline{\phi}_i$$

$$\left(\underline{\delta}_i\right)_{\max} = \frac{\Gamma_i S_d\left(T_i\right) g T_i^2}{4\pi^2}\underline{\phi}_i \qquad (5\text{-}35)$$

By a similar SRSS combination of modal maxima displacements equation (5-32) can be used thus

$$\underline{x}^* = \left\{\underline{\delta}_1^2 + \cdots + \underline{\delta}_n^2\right\}^{\frac{1}{2}} = \begin{bmatrix} \sqrt{\delta_{11}{}^2 + \cdots + \delta_{1n}{}^2} \\ \vdots \\ \sqrt{\delta_{n1}{}^2 + \cdots + \delta_{nn}{}^2} \end{bmatrix}^{\frac{1}{2}} \qquad (5\text{-}36)$$

Note that this method is very sensitive to the number of significant digits used in describing the periods T_i . It is suggested for reasonable results that at least 4 significant figures are used.

24.2 EC8 Recommendation

EC8 in effect tries to first estimate the deflection at the ultimate Limit State.

$$\underline{d}_s = \mathrm{q}.\underline{d}_e.\gamma_I \qquad (5\text{-}37)$$

where $\underline{d}_s$ is the ultimate deflections (i.e $\underline{x}^*_{\text{ultimate}}$ in Figure 10) q is the behaviour factor (effectively the ductility parameter for the structural system) $\underline{d}_e$ is the peak elastic displacement calculated from the modal analysis (i.e. the deflections $\underline{x}^*$ in section 24.1. And γ_I is the importance factor of the building. Importance class III is for normal offices and residential buildings where γ_I =1.0.

The Serviceability Limit State is expressed in terms of *inter-storey drift deflection*. The inter-storey drift is the difference in deflection for any two adjacent storeys. (i.e deflection at storey (*n*) minus deflection at storey (*n*-1)). For building of importance class III and one where there are brittle non-structural elements that are likely to be adversely effected by large deformation the EC8 prescribes the following

$$d_r \le \frac{h}{250}$$

where *h* is the span (vertical) of the storey. Other cases are described in chapter 3, section 6.

25. Procedure for modal analysis using response spectra

The procedure for calculating the earthquake induced equivalent forces that are applied at each floor mass is as follows. This procedure should be performed in all three directions *(x,y,z)* hence including K_X , K_Y & K_Z .

(a) Calculate Mass M and Stiffness K matrices for the structure.

(b) Solve Eigenvalue problem for dynamic matrix $M^{-1}K$ for each of the three directions

(c) From the eigenvalues calculate all the periods of vibrations of all the modes. Using equation (5-8)

(d) From the eigenvectors calculate the participation factors using equation (5-18) and the effective modal masses using equation (5-21)

(e) Choose damping model. The damping of the first mode $\gamma_i = 0.05$ and hence the ratio for higher mode can be calculated (section 20.1) Or more simply just assume all mode to have the same damping ratio (section 20.2)

(f) Use EC8 Design Spectrum to calculate the spectra accelerations for each mode and hence the maximum modal forces $\left(\underline{F}_i\right)_{\max}$ using equation (5-33). Employ SRSS modal combinations using equation (5-32) to calculate the equivalent static actions.

(g) Use equations (5-34) or equations (5-35) & (5-36) together with EC8 recommendation equations (5-37) to calculate floor displacements. Check inter-storey drift requirement.

(h) Use equivalent static actions for a static structural analysis to calculate member internal actions.

26. Example

Consider the two-storey reinforced concrete structure in Figure 11, to be design to survive a seismic event $a_g = 0.25g$. The foundation soil is deep deposits of medium dense sand (soil class 'B') The behaviour factor is q=3.75. The importance factor for the building is $\gamma_I = 1.0$ (i.e. a domestic structure, importance class III) The ratio of critical damping is $\gamma = 0.04$ (for all modes). All the columns are identical have cross-section of (0.45mx0.45m). E=29E6kNm^2. (a) Produce the mass and horizontal stiffness matrices (assuming shear-building assumptions) (b) Solve the eigenvalue problem and hence calculate the periods and mode shapes. Determine the participation factors and effective modal masses. (c) Using EC8 design spectrum and SRSS modal combinations calculate the equivalent static actions and displacements. (d) Compute the internal storey shears and storey moments.

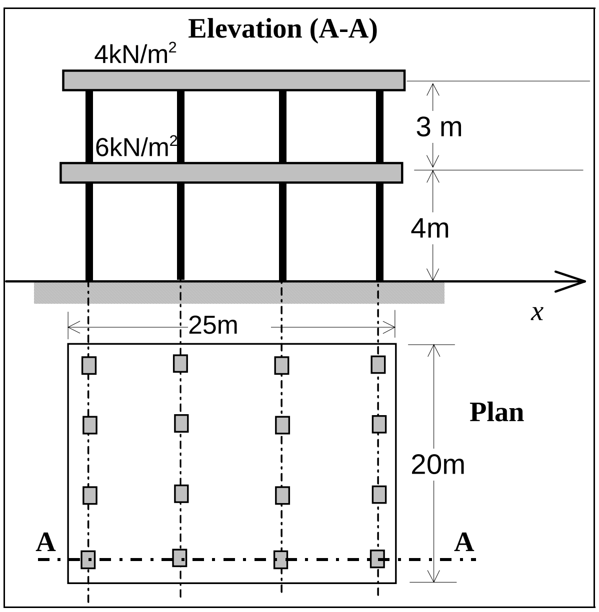


Figure 11: two storey shear building

(a) The Masses of the floors are

$m_1 = 25 \times 20 \times 6 / g = 305.81\text{kNs}^2/\text{m}$

$m_2 = 25 \times 20 \times 4 / g = 203.87\text{kNs}^2/\text{m}$

The stiffnesses (flexural) of the storeys are

$k_1 = 16 \times \left(29E6 \times \left(0.45^4\right)\right) / 4^3 = 2.973E5\text{kN/m}$

$k_2 = 16 \times \left(29E6 \times \left(0.45^4\right)\right) / 3^3 = 7.047E5\text{kN/m}$

hence the mass and stiffness matrices are

$$\mathbf{M} = \begin{bmatrix} 305.81 & 0 \\ 0 & 203.87 \end{bmatrix} \text{kNs}^2/\text{m}$$

$$\mathbf{K} = \begin{bmatrix} 1.002E6 & -7.047E5 \\ -7.047E5 & 7.047E5 \end{bmatrix} \text{kN/m}$$

(b) Dynamic matrix, the solution of eigenproblem on the dynamic matrix **D** is as follows

$$\mathbf{D} = \begin{bmatrix} 3.276E3 & -2.304E3 \\ -3.457E3 & 3.457E3 \end{bmatrix} \text{s}^{-2}$$

$$\underline{\lambda} = \begin{bmatrix} 0.5428E3 \\ 6.190E3 \end{bmatrix} \quad \Phi = \begin{bmatrix} -0.6445 & 0.6203 \\ -0.7646 & -0.7843 \end{bmatrix}$$

The eigenvectors have been normalised using a Euclidean norm

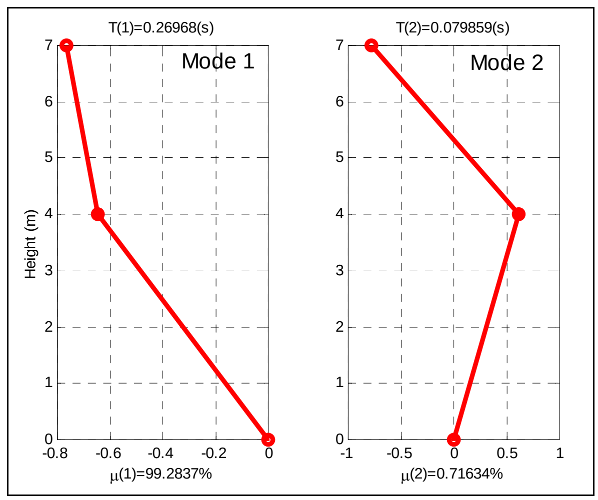


Figure 12: Horizontal Mode shapes

Hence the periods are given by equation (5-8), the participation factors by equation (5-18) and the effective modal masses by using equation (5-21).

$$\underline{T} = \begin{bmatrix} 0.2697 \\ 0.0799 \end{bmatrix} s \quad \underline{\Gamma} = \begin{bmatrix} -1.4336 \\ 0.1226 \end{bmatrix} \quad \underline{\mu} = \begin{bmatrix} 99.28\% \\ 0.72\% \end{bmatrix}$$

(c) The equivalent static actions using equation (5-33) are derived as follows.

$$\begin{bmatrix} S_d(0.2697) \\ S_d(0.0799) \end{bmatrix} = \begin{bmatrix} \dfrac{\alpha S B_0}{q} \\ \alpha S\left(1 + \dfrac{T}{T_B}\left(\dfrac{B_0}{q} - 1\right)\right) \end{bmatrix} = \begin{bmatrix} 0.18 \\ 0.2127 \end{bmatrix}$$

$$(\underline{F}_1)_{\max} = -1.4336\times 0.18gM\begin{bmatrix}-0.6445\\ -0.7646\end{bmatrix} = \begin{bmatrix}498.9\\ 394.6\end{bmatrix}\text{kN}$$

$$(\underline{F}_2)_{\max} = 0.1226\times 0.2127gM\begin{bmatrix}0.6203\\ -0.7843\end{bmatrix} = \begin{bmatrix}48.5\\ -40.9\end{bmatrix}\text{kN}$$

Using SRSS modal combinations, equation (5-32)

$$\underline{F}^* = \begin{bmatrix}\sqrt{498.9^2+48.5^2}\\ \sqrt{394.6^2+(-40.9)^2}\end{bmatrix} = \begin{bmatrix}501.25\\ 396.71\end{bmatrix}\text{kN}$$

The notional yield-state deflections are calculated by equation (5-34)
(Method B)

$$\underline{\delta}_1 = -\frac{1.4336\times 0.18g\times 0.26968^2}{4\pi^2}\begin{bmatrix}-0.6445\\ -0.7646\end{bmatrix} = \begin{bmatrix}3.005E\text{-}3\\ 3.566E\text{-}3\end{bmatrix}$$

$$\underline{\delta}_2 = \frac{0.1226\times 0.2127g\times 0.07986^2}{4\pi^2}\begin{bmatrix}0.6203\\ -0.7843\end{bmatrix} = \begin{bmatrix}0.2566E\text{-}4\\ -0.3244E\text{-}4\end{bmatrix}$$

$$\underline{x}^* = \begin{bmatrix}\sqrt{(3.005E-3)^2+(0.2566E-4)^2}\\ \sqrt{(3.566E-3)^2+(-0.3244E-4)^2}\end{bmatrix} = \begin{bmatrix}3.005E\text{-}3\\ 3.566E\text{-}3\end{bmatrix}\text{m}$$

(Method A)

$$\underline{x}^* = \begin{bmatrix}1.002E6 & -7.047E5\\ -7.047E5 & 7.047E5\end{bmatrix}^{-1}\begin{bmatrix}501.36\\ 396.76\end{bmatrix} = \begin{bmatrix}3.021E\text{-}3\\ 3.584E\text{-}3\end{bmatrix}$$

Method A is the preferred method. Notice that the deflections from method A and B are not completely identical, but they are approximately equal.

EC8 recommendation for deflections

$$\underline{d}_s = q\begin{bmatrix}3.023E\text{-}3\\ 3.584E\text{-}3\end{bmatrix}\gamma_I = \begin{bmatrix}1.1328E\text{-}2\\ 1.344E\text{-}2\end{bmatrix}\text{m}$$

The inter-storey drift is as below, hence the serviceability deflection check is ok.

$$\underline{d}_r = \begin{bmatrix}1.1328E\text{-}2-0\\ 1.344E\text{-}2-1.1328E\text{-}2\end{bmatrix} = \begin{bmatrix}1.1328E\text{-}2\\ 0.2112E\text{-}2\end{bmatrix}\begin{matrix}\le 4/250\\ \le 3/250\end{matrix}$$

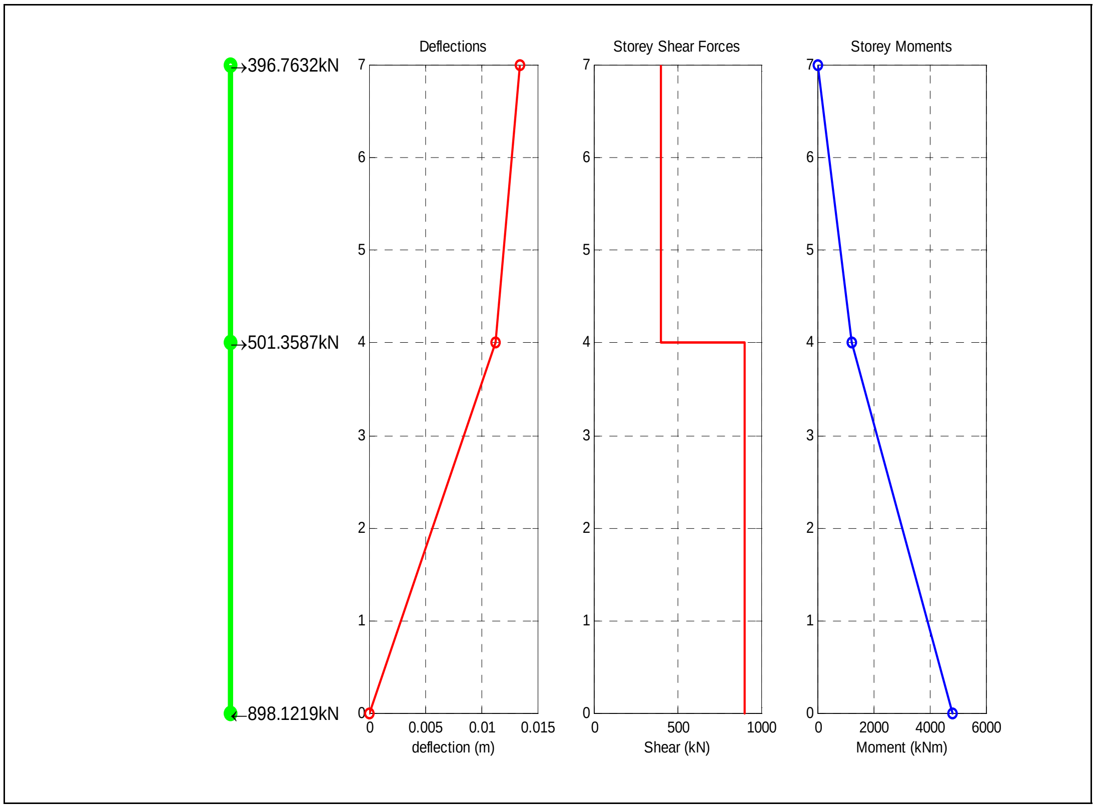


Figure 13: Equivalent static actions, deflections and internal storey actions

(d) The storey shears and moments are calculated by simple statics. For example the Storey Shear V and the storey moment M at the section shown in Figure 14.

$$V = 501.4 + 396.8 = 898.12\text{kN}$$
$$M = 396.76 \times 3 = 1190.3\text{kNm}$$

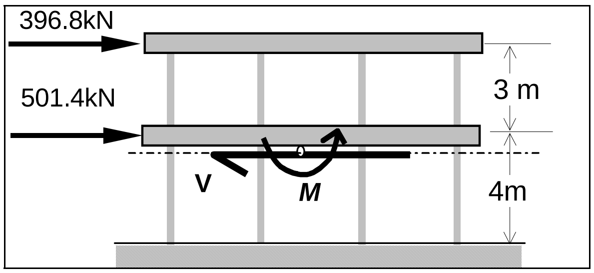


Figure 14: Storey shear and moment

Storey	Storey Shear (kN)	Storey Moment (kNm)
1	898.12	4782.8 to 1190.3
2	396.76	1190.3 to 0

The storey shear at the foundation first storey level is known as *the base shear*. The storey moment at foundation first storey level is known as the *base moment.* These are the ground reactions so are important for the design of foundations. For this example

The base shear force = 898.12kN
The base moment = 4782.8 kNm

Remember all equivalent static actions, storey moments and shears, and deflections can be applied in both positive and negative directions.

27. Relationship between x,y & z motions

Consider the two-storey structure shown in Figure 15. If the ratio of stiffnesses in the *x & y* direction for each storey is given by β i.e.

$$\frac{k_{iy}}{k_{ix}} = \beta \qquad (5\text{-}38)$$

then the stiffness matrices for this structure are

$$\mathbf{K_x} = \begin{bmatrix} k_{1x} + k_{2x} & -k_{2x} \\ -k_{2x} & k_{2x} \end{bmatrix}$$

$$\mathbf{K_y} = \begin{bmatrix} k_{1y} + k_{2y} & -k_{2y} \\ -k_{2y} & k_{2y} \end{bmatrix} = \beta \begin{bmatrix} k_{1x} + k_{2x} & -k_{2x} \\ -k_{2x} & k_{2x} \end{bmatrix} = \beta \mathbf{K_x}$$

hence the Dynamic matrix in the x and y directions are proportional

$$\mathbf{D_x} = \mathbf{M^{-1}K_x}, \quad \mathbf{D_y} = \mathbf{M^{-1}K_y} = \beta\mathbf{M^{-1}K_x} = \beta\mathbf{D_x}$$

now consider the eigenvalue value problem on the dynamic matrix $\mathbf{D_y} = \beta\mathbf{D_x}$

$$(\mathbf{D_y} - \lambda_y I)\underline{\phi} = 0$$
$$(\beta\mathbf{D_x} - \lambda_y I)\underline{\phi} = 0 \qquad (5\text{-}39)$$
$$\beta\left(\mathbf{D_x} - \frac{\lambda_y}{\beta} I\right)\underline{\phi} = 0 \rightarrow (\mathbf{D_x} - \lambda_x I)\underline{\phi} = 0$$

Hence the eigenvalue problem on $\mathbf{D_y}$ is identical to the one on $\mathbf{D_x}$. The eigenvalues of each problem are proportional to each other.

$$\frac{\lambda_y}{\lambda_x} = \beta$$

Hence the periods of the modes in the *x* & *y* directions are also proportional

$$T_{iy} = \frac{2\pi}{\sqrt{\lambda_{iy}}} = \frac{2\pi}{\sqrt{\beta\lambda_{ix}}} = \frac{T_{ix}}{\sqrt{\beta}} \quad \rightarrow \quad \frac{T_{iy}}{T_{ix}} = \frac{1}{\sqrt{\beta}} \qquad (5\text{-}40)$$

Note also from equation (5-39) that because the eigenvectors have only direction but undefined magnitude the normalisation process results in the eigenvectors in *x* & *y* directions being identical. Thus the participation factors and modal mass are also identical since the mass matrix in the in *x* & *y* directions in the same.

$$\Gamma_{ix} = \Gamma_{iy} \ , \ \ \mu_{ix} = \mu_{iy} \qquad (5\text{-}41)$$

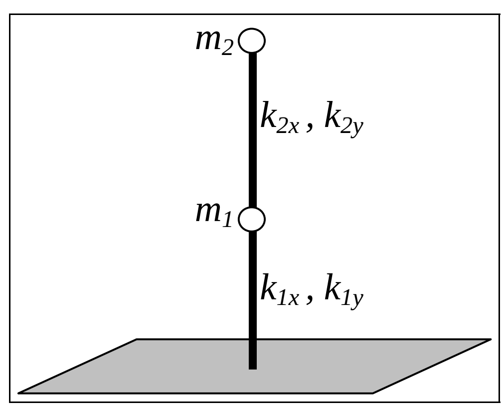


Figure 15: Two-storey shear building

The calculated equivalent static actions and displacements will not be identical as these depend on the design spectrum and periods. This result can be extended to any number of storeys while the equation (5-38) is valid for the structure. Equation (5-38) would normally require a continuation of resisting element through out the entire height of the structure. (i.e. the presence of set-back would normally invalidate this assumption.)

Note also that the above analysis can be extended to the vertical motion.

$$\text{If} \quad \frac{k_{iz}}{k_{ix}} = \beta_z \rightarrow \quad \mathbf{D_Z} = \beta_z \mathbf{D_x} \tag{5-42}$$

then equations (5-40) and (5-41) can be applied to the vertical motions. However equation (5-42) is more difficult, in practice, to achieve if the storey heights vary. This is because the flexural stiffness is inversely proportional to L^3 while the axial stiffness is inversely proportional to L. Hence this result is general only applicable if the storey heights are identical.

28. Example

Perform a horizontal (in the *y* direction) seismic analysis for the structure in example 26. Since the structure is identical is stiffness in the *x* and *y* directions the results for the *x* and *y* are identical. Note that the plan shape is not a factor the analysis of this questions.

29. Example

Perform a vertical seismic analysis for the structure in example 26.

The Axial stiffnesses of the storeys are

$$k_1 = 16 \times \left(29E6 \times \left(0.45^2\right)\right)/4 = 23.49E6\text{kN/m}$$

$$k_2 = 16 \times \left(29E6 \times \left(0.45^2\right)\right)/3 = 31.32E6\text{kN/m}$$

$$\mathbf{K} = \begin{bmatrix} 54.81E6 & -31.32E6 \\ -31.32E6 & 31.32E6 \end{bmatrix} \text{kN/m}$$

Dynamic matrix, the solution of eigenproblem on the dynamic matrix **D** is as follows

$$\mathbf{D} = \begin{bmatrix} 1.7923E5 & -1.0242E5 \\ -1.5362E5 & -1.5362E5 \end{bmatrix} \text{s}^{-2}$$

$$\underline{\lambda} = \begin{bmatrix} 0.4034E5 \\ 2.9251E5 \end{bmatrix} \quad \Phi = \begin{bmatrix} 0.59349 & 0.67063 \\ 0.80484 & -0.74179 \end{bmatrix}$$

Hence the periods are given by equation (5-8), the participation factors by equation (5-18) and the effective modal masses by using equation (5-21).

$$\underline{T} = \begin{bmatrix} 0.0313 \\ 0.0116 \end{bmatrix} s \quad \underline{\Gamma} = \begin{bmatrix} 1.4413 \\ 0.2157 \end{bmatrix} \quad \underline{\mu} = \begin{bmatrix} 97.72\% \\ 2.28\% \end{bmatrix}$$

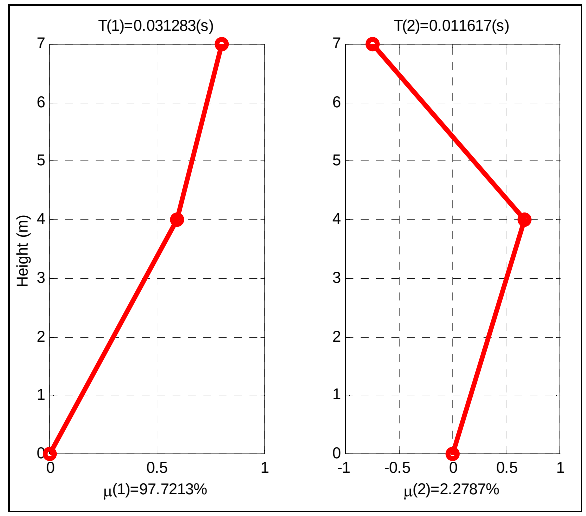


Figure 16: Vertical mode shapes

The equivalent static actions using equation (5-33) are given below. Note that there is a reduction factor when using the EC8 Design Spectrum for the vertical motion, if T<0.15 then the reduction factor is 0.7. Note that EC8 recommend that the behaviour factor q=1.0 for vertical motion. This is because the vertical motion is not thought to induce the same inelastic energy dissipating mechanisms as the horizontal motion.

$$\begin{bmatrix} S_d(0.0313) \\ S_d(0.0116) \end{bmatrix} = \begin{bmatrix} 0.7 \times \alpha S\left(1 + \dfrac{T}{T_B}\left(\dfrac{B_0}{1} - 1\right)\right) \\ 0.7 \times \alpha S\left(1 + \dfrac{T}{T_B}\left(\dfrac{B_0}{1} - 1\right)\right) \end{bmatrix} = \begin{bmatrix} 0.2371 \\ 0.198 \end{bmatrix}$$

$$\left(\underline{F}_1\right)_{\max} = 1.4413 \times 0.2371 g\text{M} \begin{bmatrix} 0.59349 \\ 0.80484 \end{bmatrix} = \begin{bmatrix} 608.44 \\ 550.07 \end{bmatrix} \text{kN}$$

$$\left(\underline{F}_2\right)_{\max} = 0.2157 \times 0.198 g\text{M} \begin{bmatrix} 0.67063 \\ -0.74179 \end{bmatrix} = \begin{bmatrix} 85.92 \\ -63.36 \end{bmatrix} \text{kN}$$

Using SRSS modal combinations, equation (5-32)

$$\underline{F}^* = \begin{bmatrix} \sqrt{608.44^2 + 85.92^2} \\ \sqrt{550.07^2 + (-63.36)^2} \end{bmatrix} = \begin{bmatrix} 614.4 \\ 553.6 \end{bmatrix} \text{kN}$$

The deflections are calculated using Method A, equation (5-34). Since the behaviour factor for vertical motion q=1.0 and the importance factor $\gamma_I = 1.0$ the notional yield-state deflections equation (5-34) and the ultimate deflections (5-37) produce the same results.

$$\underline{x}^* = \underline{d}_s = \begin{bmatrix} 54.81E6 & -31.32E6 \\ -31.32E6 & 31.32E6 \end{bmatrix}^{-1} \begin{bmatrix} 614.4 \\ 553.6 \end{bmatrix}$$
$$= \begin{bmatrix} 4.9721E\text{-}5 \\ 6.7396E\text{-}5 \end{bmatrix} \text{m}$$

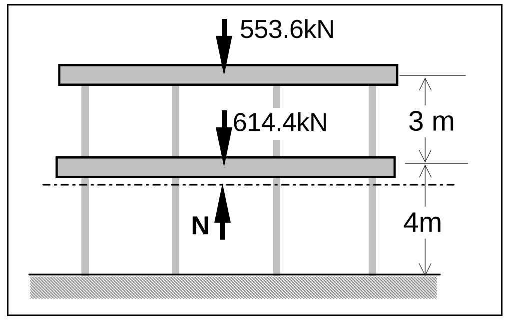


Figure 17: Storey axial forces

The storey axial forces are calculated by simple statics. For example the Storey axial force N at the section shown in Figure 17 is

$$N = 614.4 + 553.6 = 1168\text{kN}$$

Storey	Storey Axial forces (kN)
1	1168
2	553.6

The storey axial force at the first storey is equal to the *base vertical reaction.*

Base vertical reaction = 1168 kN

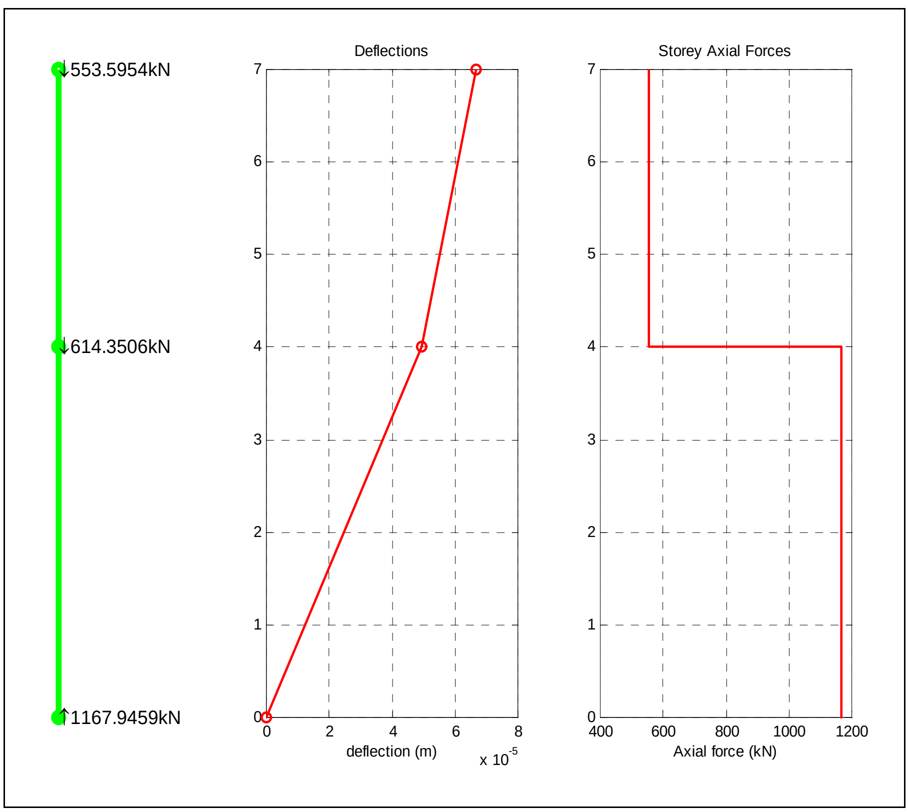


Figure 18: Vertical equivalent static actions, displacements and Storey axial forces

30. Calculation of element actions from global actions and displacements[3]

The global actions, i.e. the equivalent static actions, are applied at the centre of mass of the floors. Due to the symmetry of shear building model there is no torsion (about the z axis) thus all resisting elements on a particular storey are subject to the same deflections horizontal and vertical deflections. From these storey deflections it is possible to calculate the element shear forces, moments and axial forces. Due to the fixity of the ends of the column elements there are no rotations at the end.

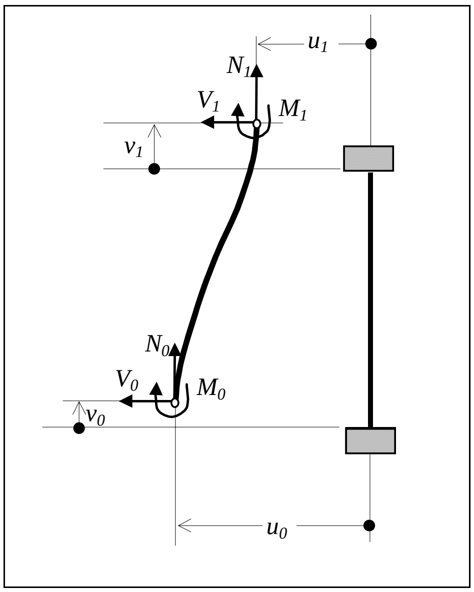


Figure 19: Element actions and deformations

The shear forces are constant along the column element. While the moments vary linearly with a zero moment at the centre span of the column. The axial forces are also constant along the column element. Equation (5-43) is derived from classical static analysis of a beam-column element that is fully fixed at both ends.

$$V = k_{xi}(u_1 - u_0),\ M = \frac{VL}{2},\ N = k_{zi}(v_1 - v_0) \qquad (5\text{-}43)$$

where k_{xi} is the element flexural stiffness and k_{zi} is the element axial stiffness.

[3] The displacements calculated previously are notionally at a yield state. Thus there is an ambiguity here since the actions are supposedly meant to be the ultimate actions.

31. Example

Calculate the actions for a typical column element in example in section 26. Using equation (5-42)

<u>A typical first storey column</u>

$$k_{xi} = \frac{12(EI_y)_i}{L_i^3} = 1.8581E4\ ,\ k_{zi} = \frac{(EA)_i}{L_i} = 1.4681E6$$

$$V = 1.8581E4(3.021E\text{-}3 - 0.0) = 56.13\text{kN}$$

$$M = 56.13 \times 4/2 = 112.3\text{kNm}$$

$$N = 1.4681E6(4.9721E\text{-}5 - 0.0) = 72.99\text{kN}$$

<u>A typical second storey column</u>

$$k_{xi} = \frac{12(EI_y)_i}{L_i^3} = 0.4404E5\ ,\ k_{zi} = \frac{(EA)_i}{L_i} = 1.9575E6$$

$$V = 0.4404E5(3.584E\text{ - }3 - 3.021E\text{ - }3) = 24.8\text{kN}$$

$$M = 24.8 \times 3/2 = 37.2\text{kNm}$$

$$N = 1.9575E6(6.7396E\text{-}5 - 4.9721E\text{-}5) = 34.6\text{kN}$$

Note that the sum of all element shear forces in the first storey foundation level is equal to the base shear force. (i.e. $56.13 \times 16 = 898.1\text{kN}$) The sum of all the element axial forces in the first storey at the foundation level is equal to the base vertical reaction. (i.e. $72.99 \times 16 = 1167.8\text{kN}$). However the sum of all the element moments in the first storey is not equal to base moment. This is because the base moment also includes the moment effects of the axial, vertical reactions induced by the horizontal equivalent static actions.

32. EC8 accidental torsional motion approximation recommendation

Due to an adverse distribution of live load (variable action) the centre of mass and centre of stiffness may not coincide. The effect of torsional motion of structures that are symmetric in stiffness and mass at all floors can be modelled by increasing the actions obtained from the horizontal simplified analysis by the factor ζ where

$$\varsigma = 1 + 0.6\left(\frac{x}{L}\right) \qquad (5\text{-}43)$$

Where x is the distance from the element under consideration to the centre of the building measured perpendicularly to the direction of the seismic action considered. L is the distance between the two outermost lateral resisting elements measured perpendicularly to the direction of the seismic action considered.

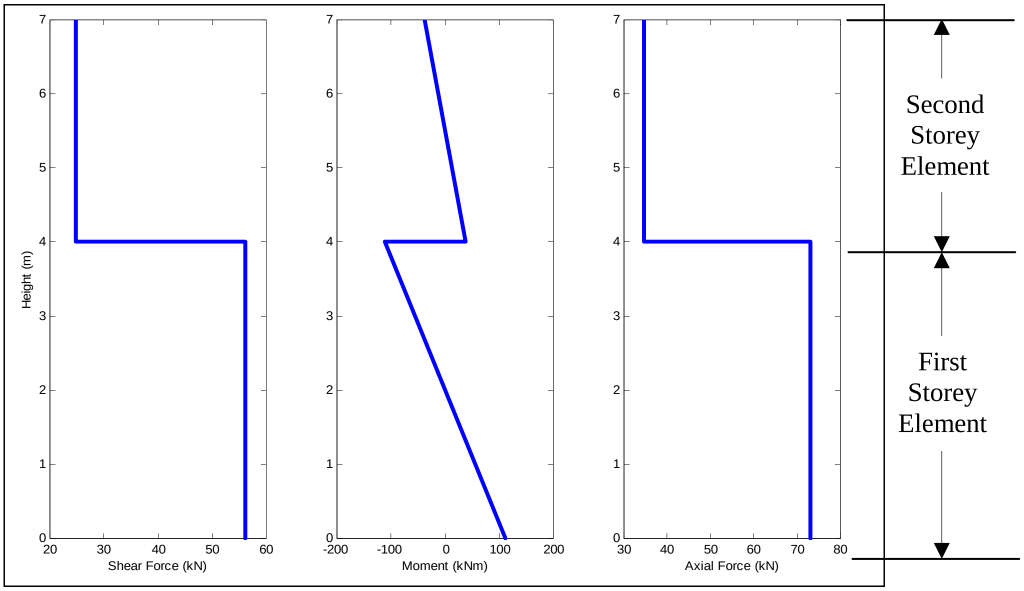


Figure 20: Element shear force and bending moment diagrams

33. Example

Consider the plan shown in Figure 21. It is the continuation of example 26. The torsion of the building is expected about the centre of mass of the building floor. Elements further away from the centre of mass are subject to increased actions due to the effects of torsion.

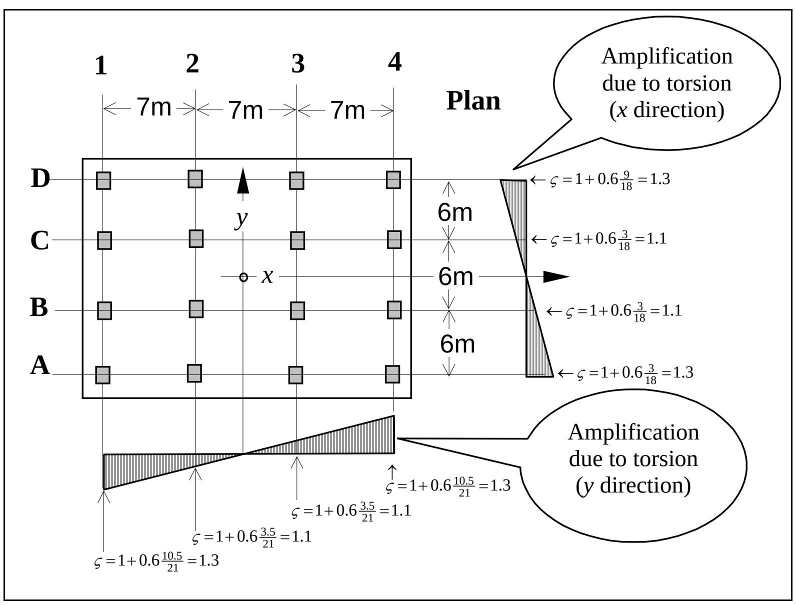


Figure 21: Simplified method of building torsion analysis

Consider element A1 in x direction

The amplification factor $\varsigma = 1 + 0.6\frac{9}{18} = 1.3$ in the x direction is given by the following. (L=18, building dimension in y direction.) This means than all seismic actions in the *x* direction are amplified by a value of 1.3. e.g. This element on the first storey has an internal shear of 56.13kN and an maximum internal moment of 112.3kNm. (see **Figure 20**) Thus EC8 recommends that these action should be increased by 1.3

$$\begin{aligned} V &= 56.13 \times 1.3 = 72.97\text{kN} \\ M &= 112.3 \times 1.3 = 146\text{kNm} \end{aligned} \quad \text{(element A1 in x direction)}$$

Consider element B1 in x direction

The amplification factor $\varsigma = 1 + 0.6\frac{3}{18} = 1.1$ in the x direction is given by the following. (L=18, building dimension in y direction.) This means than all seismic actions in the *x* direction are amplified by a value of 1.1. e.g. This element on the first storey has an internal shear of 56.13kN and an maximum internal moment of 112.3kNm. (see **Figure 20**) Thus EC8 recommends that these action should be increased by 1.1

$$\begin{aligned} V &= 56.13 \times 1.1 = 61.74\text{kN} \\ M &= 112.3 \times 1.1 = 123.5\text{kNm} \end{aligned}$$ (element B1 in x direction)