The Ordinary Differential Equations Project

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Preface

This text is intended for a one-semester undergraduate course in ordinary differential equations.

There are additional exercises or computer projects at the ends of many of the chapters. The computer projects often require a basic knowledge of programming. All of these exercises and projects are more substantial in nature and allow the exploration of new results and theory.

Sage (sagemath.org) is a free, open source, software system for advanced mathematics, which is ideal for assisting with a study of ordinary differential equations. Sage can be used either on your own computer, a local server, or on SageMathCloud (https://cloud.sagemath.com). The Sage code has been tested for accuracy with the most recent version available at this time: Sage Version 7.3 (released 2016–08–5).

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1

A First Look at Differential Equations

1.1 Modeling with Differential Equations

Calculus tells us that the derivative of a function measures how the function changes. An equation relating a function to one or more of its derivatives is called a ***differential equation***. The subject of differential equations is one of the most interesting and useful areas of mathematics. We can describe many interesting natural phenomena that involve change using differential equations. In addition, the theory of the subject has broad and important implications.

1.1.1 Exponential Growth

We begin our study of ordinary differential equations by modeling some real world phenomena. For a particular situation that we might wish to investigate, our first task is to write an equation (or equations) that best describes the phenomenon. Suppose that we wish to study how a population $P(t)$ grows with time t. We might make the assumption that a constant fraction of population is having offspring at any particular time. If we also assume that the population has a constant death rate, the change in the population ΔP during a small time interval Δt will be

$$\Delta P \approx k_{\text{birth}} P(t)\Delta t - k_{\text{death}} P(t)\Delta t,$$

where k_{birth} is the fraction of the population having offspring during the interval and k_{death} is the fraction of the population that dies during the interval. Equivalently, we can write

$$\frac{\Delta P}{\Delta t} \approx kP(t),$$

where $k = k_{\text{birth}} - k_{\text{death}}$. Since the derivative of P is

$$\frac{dP}{dt} = \lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t},$$

the rate of change of the population is proportional to the size of the population, or

$$\frac{dP}{dt} = kP. \tag{1.1}$$

The equation

$$\frac{dP}{dt} = kP$$

is one of the simplest differential equations that we will consider. The equation tells us that a population grows in proportion to its current size. It is not too difficult to see that $P(t) = Ce^{kt}$ is a ***solution*** to this equation, where C is an arbitrary constant. Indeed, if we differentiate $P(t)$, we obtain

$$\frac{d}{dt}P(t) = kCe^{kt} = kP(t).$$

In addition, if we know the value of $P(t)$, say when $t = 0$, we can also determine the value of C. For example, if the population at the time $t = 0$ is $P(0) = P_0$, then

$$P_0 = P(0) = Ce^{k\cdot 0} = C$$

or $P(t) = P_0e^{kt}$. The differential equation

$$\begin{aligned} P'(t) &= kP(t), \\ P(0) &= P_0 \end{aligned}$$

is an example of an ***initial value problem***, and we say that $P(0) = P_0$ is an ***initial condition***. Since the solution to equation (1.1) is $P(t) = Ce^{kt}$, and we say that the population grows ***exponentially***.

As an example, suppose that $P(t)$ is a population of a colony of bacteria at time t, whose initial population is 1000 at $t = 0$, where time is measured in hours. Then

$$1000 = P(0) = Ce^0 = C,$$

and our solution becomes $P(t) = 1000e^{kt}$. If the population grows at three percent per hour, then

$$1030 = P(1) = 1000e^k,$$

after one hour. Consequently,

$$k = \ln 1.03 \approx 0.0296$$

and the solution to our initial value problem is

$$P(t) = 1000e^{0.0296t}.$$

Of course, it is important to realize that this is only a model. If t is small, our model might be reasonably accurate. However, if we let t be very large, our colony of bacteria could very well exceed the mass of the earth.

The growth rate of a population need not be positive. For example, Japan has experienced negative growth in recent years.[1] The equation $dP/dt = kP$ can also be used to model such phenomena such as radioactive decay and compound interest—topics which we will explore later.

To summarize, we say that the expression $x(t) = Ce^{kt}$ is a ***general solution*** of the equation $x' = kx$, and $x(t) = x_0e^{kt}$ is a ***particular solution*** to the differential equation. The general solution to our equation $x(t) = Ce^{kt}$ graphs as an infinite family of curves, which we will call ***integral curves*** or ***solution curves*** (Figure 1.1).

[1] See http://data.worldbank.org/indicator/SP.POP.GROW.

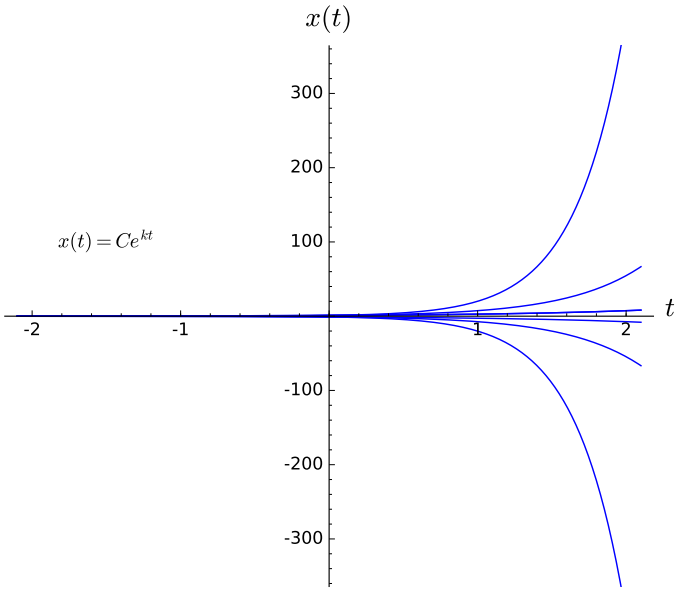

Figure 1.1: Integral curves

1.1.2 Logistic Growth

Not all populations grow exponentially; otherwise, a bacteria culture in a petri dish would grow unbounded and soon be much larger than the size of the laboratory. To see what happens if there are limiting factors to population growth, let us consider the population of fish in a children's trout pond. The number of trout will be limited by the available resources, food supply, and spawning habitat. A small population of fish might grow exponentially if the pond is large and food is abundant, but the growth rate will decline as the population increases and the availability of resources declines. We can use the ***logistic equation*** to model population growth in a resource limited environment.[2]

To see how the logistic model works, let us try to adjust our model of exponential growth to account for the limited resources of the pond. We will make the following assumptions.

- If the population of trout is small and there are abundant resources, the rate of growth will be approximately exponential,

$$\frac{dP}{dt} \approx kP.$$

- If N is the maximum population of trout that the pond can support, then any population larger than N will decrease. In other words,

$$\frac{dP}{dt} < 0$$

for $P > N$. We say that N is the ***carrying capacity*** for the population.

Our assumptions suggest that we might try an equation of the form

$$\frac{dP}{dt} = kf(P)P,$$

[2]The logistic model was first used by the Belgian mathematician and physician Pierre François Verhulst in 1836 to predict the populations of Belgium and France.

where $f(P)$ is a function of P that is close to 1 if the population is small, but negative if the population is greater than N. The simplest function satisfying these conditions is

$$f(P) = \left(1 - \frac{P}{N}\right).$$

Thus, the ***logistic population model*** is given by the differential equation

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P.$$

Example 1.2. Suppose we have a pond will support 1000 fish, and the initial population is 100 fish. In order to determine the number of fish in the lake at any time t, we must solve the initial value problem

$$\begin{aligned}\frac{dP}{dt} &= k\left(1 - \frac{P}{1000}\right)P\\ P(0) &= 100.\end{aligned}$$

It is easy to verify that

$$P(t) = \frac{1000}{9e^{-kt} + 1}$$

is the solution to our initial value problem.[3]

If we differentiate P, we will obtain the righthand side of the differential equation,

$$\begin{aligned}\frac{dP}{dt} &= \frac{d}{dt}\left(\frac{1000}{9e^{-kt} + 1}\right)\\ &= 1000k\frac{9e^{-kt}}{(9e^{-kt} + 1)^2}\\ &= k\frac{9e^{-kt}}{(9e^{-kt} + 1)} \cdot \frac{1000}{9e^{-kt} + 1}\\ &= k\frac{(9e^{-kt} + 1) - 1}{(9e^{-kt} + 1)} \cdot \frac{1000}{9e^{-kt} + 1}\\ &= k\left(1 - \frac{1000}{1000(9e^{-kt} + 1)}\right)\frac{1000}{9e^{-kt} + 1}\\ &= k\left(1 - \frac{P}{1000}\right)P,\end{aligned}$$

and certainly $P(0) = 100$. In addition, if we know that the population is 200 fish after one year, then

$$200 = P(1) = \frac{1000}{9e^{-k} + 1}$$

and we can determine that

$$k = \ln\left(\frac{9}{4}\right) \approx 0.8109.$$

Consequently, the solution to our intial-value problem is

$$P(t) = \frac{1000}{9e^{-0.8109t} + 1}.$$

The graph of our solution certainly fits the situation that we are modeling (Figure 1.3).

Of course, we have not allowed for the possibility that children will be catching fish in our pond. We will examine how fishing affects our model in Section 1.7.

[3] We will learn how to solve initial-value problems such as the one described here in Section 1.2. For the time being, we will be satisfied with being able to verify the fact that we have a solution.

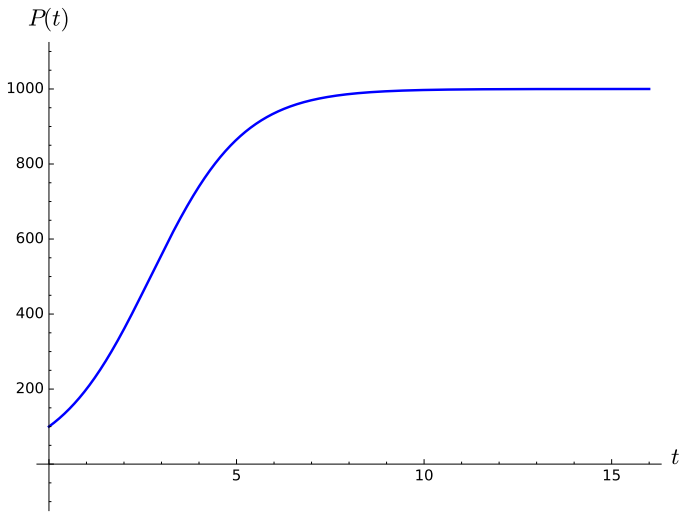

Figure 1.3: Logistic growth

1.1.3 A Spring-Mass Model

Sometimes it is necessary to consider the second derivative when constructing a mathematical model. Suppose that we have a mass lying on a flat, frictionless surface and that this mass is attached to one end of a spring with the other end of the spring attached to a wall. We denote the spring displacement by x. If $x > 0$, then the spring is stretched. If $x < 0$, the spring is compressed. If $x = 0$, then the spring is in a state of equilibrium (Figure 1.4). If we pull on the mass, then the mass will oscillate back and forth across the table.

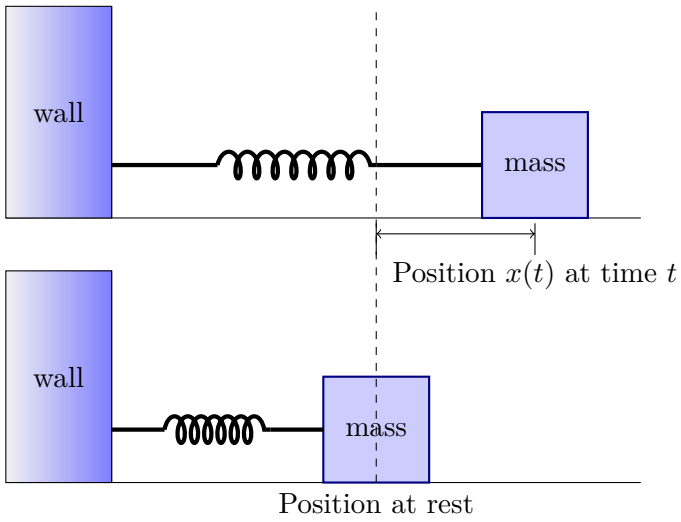

Figure 1.4: A spring-mass system

We can construct a differential equation that models our oscillating mass. First, we must consider the restorative force on the spring. We will make the assumption that this force depends on the displacement of the spring, $F(x)$. Using Taylor's Theorem from calculus, we can expand F to obtain

$$\begin{aligned} F(x) &= F(0) + F'(0)x + \frac{1}{2}F''(0)x^2 + \cdots \\ &= -kx + \frac{1}{2}F''(0)x^2 + \cdots , \end{aligned}$$

where $F'(0) = -k$ and $F(0) = 0$. If the displacement is not too large, then x^n will be small for $n \geq 2$, and we can ignore higher ordered terms. Thus, we can consider the restorative force on the spring to be proportional to displacement of the spring from its equilibrium length,

$$F = -kx.$$

This equation is known as is ***Hooke's Law***. We can test this law experimentally, and it is reasonably accurate if the displacement of the spring, x, is not too large.

By Newton's second law of motion, the force on the mass must be

$$F = ma = m\frac{d^2x}{dt^2} = mx''.$$

Setting the two forces equal, we have a ***second order*** differential equation,

$$mx'' = -kx,$$

which describes our oscillating mass. The spring-mass system is an example of a ***harmonic oscillator***.

Suppose that we have a spring-mass system where $m = 1$ and $k = 1$. If the initial velocity of the spring is one unit per second and the initial position is at the equilibrium point, then we have the following initial value problem,

$$\begin{aligned} x'' + x &= 0 \\ x(0) &= 0 \\ x'(0) &= 1. \end{aligned}$$

Since $x''(t) = -x(t)$ for both the sine and cosine functions, we might guess that a general solution of our differential equation has the form

$$x(t) = A\cos t + B\sin t.$$

Noting that

$$x'(t) = -A\sin t + B\cos t,$$

and using our initial conditions, we can determine that $A = 0$ and $B = 1$ or

$$x(t) = \sin t.$$

The graph of the position of the mass as a function of time is given in Figure 1.5.

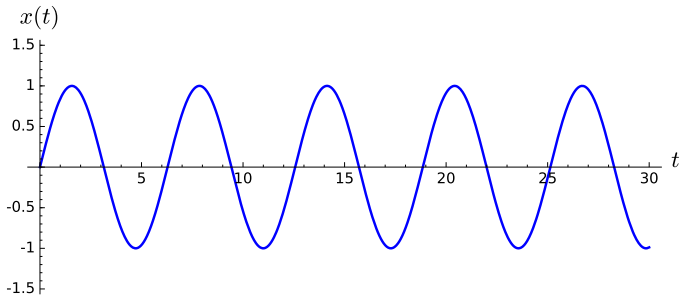

Figure 1.5: A undamped spring-mass system

Now let us add a damping force to our system. For example, we might add a dashpot, a mechanical device that resists motion, to our system. Think of a dashpot as the small

cylinder that keeps your screen door from slamming shut. The simplest assumption would be to take the damping force of the dashpot to be proportional to the velocity of the mass, $x'(t)$. Thus, we have will have an additional force,

$$F = -bx'$$

acting on our mass, where $b > 0$. Our new equation for the spring-mass system is

$$mx'' = -bx' - kx.$$

or

$$mx'' + bx' + kx = 0,$$

where m, b, and k are all positive constants. If we let $m = 1$, $b = 3$, and $k = 2$, then we would have an oscillating mass governed by the equation

$$x'' + 3x' + 2x = 0.$$

We will learn how to solve equations of the form $ax'' + bx' + cx = 0$ in Chapter 4, but let us assume that the solution is of the form $x(t) = e^{rt}$ for now. Then

$$\begin{aligned} x'' + 3x' + 2x &= r^2e^{rt} + 3re^{rt} + 2e^{rt} \\ &= e^{rt}(r^2 + 3r + 2) \\ &= e^{rt}(r + 2)(r + 1) \\ &= 0. \end{aligned}$$

Since e^{rt} is never zero, it must be the case that $r = -2$ or $r = -1$ if $x(t) = e^{rt}$ is to be a solution to our equation. Thus, we might guess that

$$x(t) = Ae^{-t} + Be^{-2t}$$

is a general solution to our equation. If the initial velocity of our mass is one unit per second and the initial position is zero, then we have the initial value problem

$$\begin{aligned} x'' + 3x' + 2x &= 0 \\ x(0) &= 0 \\ x'(0) &= 1. \end{aligned}$$

Using the fact that $x'(t) = -Ae^{-t} - 2Be^{-2t}$, our initial conditions give us the following system of linear equations,

$$\begin{aligned} A + B &= 0 \\ -A - 2B &= 1. \end{aligned}$$

Thus, $A = 1$ and $B = -1$, and our spring-mass system is modeled by the function

$$x(t) = e^{-t} - e^{-2t}.$$

Notice that the additional damping negates any oscillation in the system. In this case, we say that the system is ***over-damped*** (Figure 1.6).

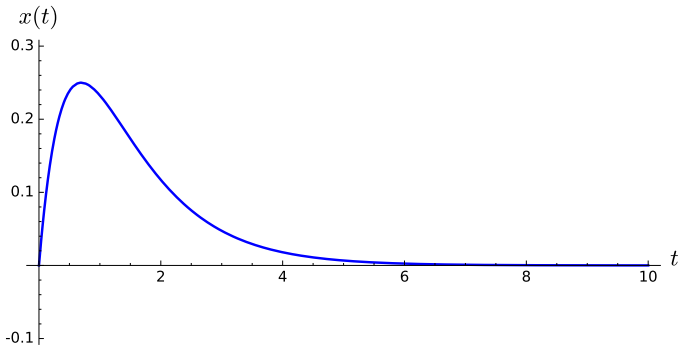

Figure 1.6: An over-damped spring-mass system

Of course, if we have a very strong spring and only add a small amount of damping to our spring-mass system, the mass would continue to oscillate, but the oscillations would become progressively smaller. In other words, our system would be under-damped. For example, our spring-mass system might be described by the initial value problem

$$\begin{aligned} x'' + 2x' + 50x &= 0 \\ x(0) &= 0 \\ x'(0) &= 1. \end{aligned}$$

It is easy to verify that

$$x(t) = \frac{1}{7}e^{-t}\sin 7t$$

is a solution to the initial value problem (Figure 1.7).

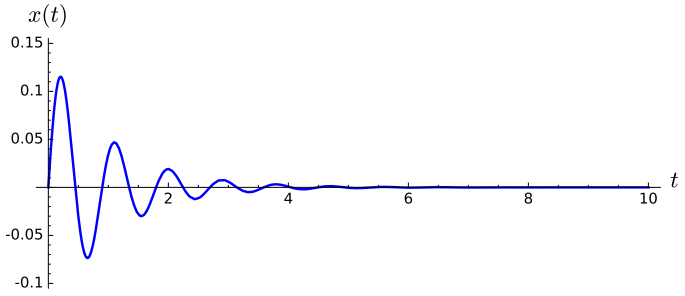

Figure 1.7: An under-damped spring-mass system

We will explore harmonic oscillators and second-order differential equations more fully in Chapter 4.

1.1.4 A Predator-Prey System

Some situations require more than one differential equation to model a particular situation. We might use a system of differential equations to model two interacting species, say where one species preys on the other.[4] For example, we can model how the population of Canadian lynx (*lynx canadienis*) interacts with a the population of snowshoe hare (*lepus americanis*) (see https://www.youtube.com/watch?v=ZWucOrSOdCs).

We have good historical data for the populations of the lynx and snowshoe hare from the Hudson Bay Company. This large Canadian retail company, which owns and operates

[4]The predator-prey model was discovered independently by Lotka (1925) and Volterra (1926).

a large number of retail stores in North America and Europe, including Galeria Kaufhof, Hudson's Bay, Home Outfitters, Lord & Taylor, and Saks Fifth Avenue, was originally founded in 1670 as a fur trading company. The Hudson Bay Company kept accurate records on the number of lynx pelts that were bought from trappers from 1821 to 1940. The company noticed that the number of pelts varied from year to year and that the number of lynx pelts reached a peak about every ten years [11]. The ten year cycle for lynx can be best understood using a system of differential equations.

The primary prey for the Canadian lynx is the snowshoe hare. We will denote the population of hares by $H(t)$ and the population of lynx by $L(t)$, where t is the time measured in years. We will make the following assumptions for our predator-prey model.

- If no lynx are present, we will assume that the hares reproduce at a rate proportional to their population and are not affected by overcrowding. That is, the hare population will grow exponentially,
$$\frac{dH}{dt} = aH.$$
- Since the lynx prey on the hares, we can argue that the rate at which the hares are consumed by the lynx is proportional to the rate at which the hares and lynx interact. Thus, the equation that predicts the rate of change of the hare population becomes
$$\frac{dH}{dt} = aH - bHL.$$
We are thinking of HL as the number of possible interactions between the lynx and the hare populations.
- If there is no food, the lynx population will decline at a rate proportional to itself,
$$\frac{dL}{dt} = -cL.$$
- The lynx receive benefit from the hare population. The rate at which lynx are born is proportional to the number of hares that are eaten, and this is proportional to the rate at which the hares and lynx interact. Consequently, the growth rate of the lynx population can be described by
$$\frac{dL}{dt} = -cL + dHL.$$

We now have a ***system*** of differential equations that describe how the two populations interact,

$$\begin{aligned} \frac{dH}{dt} &= aH - bHL, \\ \frac{dL}{dt} &= -cL + dHL. \end{aligned}$$

We will learn how to analyze and solve systems of differential equations in subsequent chapters; however, we will give a graphical solution in Figure 1.8 to the system

$$\begin{aligned} \frac{dH}{dt} &= 0.4H - 0.01HL, \\ \frac{dL}{dt} &= -0.3L + 0.005HL. \end{aligned}$$

Notice that the predator population, L, begins to grow and reaches a peak after the prey population, H reaches its peak. As the prey population declines, the predator population also declines. Once the predator population is smaller, the prey population has a chance to recover that the cycle begins again.[5]

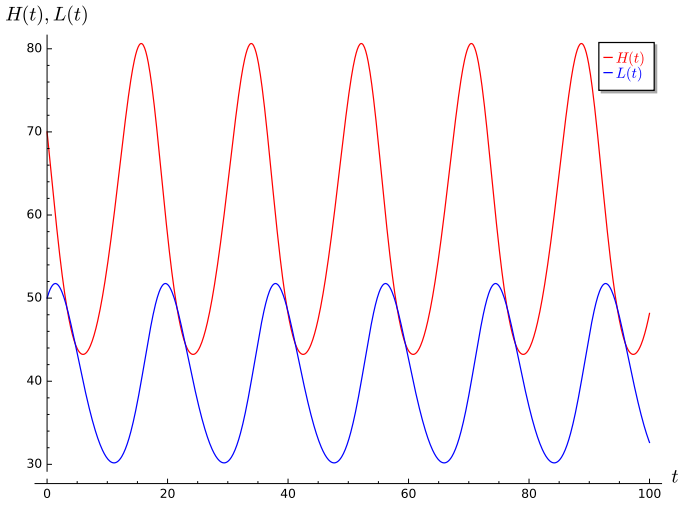

Figure 1.8: The predator-prey relationship between the lynx and the snowshoe hare

1.1.5 Modeling the HIV-1 Virus

The interaction of the HIV-1 virus with the body's immune system can be modeled by a system of differential equations similar to a predator-prey system. After an individual is infected with the HIV-1 virus, the amount of the virus in the bloodstream rises dramatically and the person will often suffer from flu-like symptoms. However, these symptoms will disappear after a period of weeks or months as the body begins to manufacture antibodies against the virus. Tests have been developed to determine the presence of HIV-1 antibodies. If a individual has such antibodies, then they are said to be HIV-1 positive. Once infected with the HIV-1 virus, it can be years before an HIV-positive patient exhibits the full symptoms of AIDS. The body's immune system fights the HIV-1 virus with white blood cells. The CD4-positive T-helper cells, a specific type of white blood cell, is especially important since it helps other cells fight the virus. However, the HIV-1 virus can destroy CD4-positive T-helper cells.

We can develop a system of differential equations to better understand the dynamics of the HIV-1 virus [20]. Let $V = V(t)$ be the population of the HIV-1 virus at time t. We will assume that the virus concentration is governed by the following differential equation,

$$\frac{dV}{dt} = P - cV.$$

The first term, P is some function of t that determines the rate at which new viral particles are created. The term $-cV$ is the death rate for the virions. If someone discovers a drug that blocks the creation of new HIV-1 virions, then P would be zero and the virions would clear the body at the following rate,

$$\frac{dV}{dt} = -cV,$$

and $V(t) = V_0 e^{-ct}$, where V_0 is the initial viral population.

[5] An excellent account of the actual lynx and snowshoe hare data and model can be found in [5].

Now let us consider a model for the concentration of $T = T(t)$ of (uninfected) CD4-positive T-helper cells,

$$\frac{dT}{dt} = s + pT\left(1 - \frac{T}{T_{\max}}\right) - d_T T.$$

The constant s represents the rate at which T cells are created from sources in the body, such as the thymus. New CD4-positive T-helper cells can also be created from the proliferation of existing CD4-positive T-helper cells, and the second term in the equation represents the logistic growth of the T cells, where p is the maximum proliferation rate and $T_{\max}$ is the T cell population density where proliferation ceases. Finally, d_T is the death rate of the T cells.

Like the influenza virus, the HIV-1 virus is an RNA virus. An RNA virus cannot reproduce on its own and must use the DNA from a host cell. To do this, the virus attaches itself to a CD4-positive T-helper cell and injects its RNA into the cell. This way the virus can use the T cell's DNA to replicate itself using a process called reverse transcription, where a DNA copy of the virus's RNA is made. New virus particles are created, and the T cell eventually bursts releasing the virions into the body. If we let T^* be the concentration of infected T cells, we can model this process with the following system of equations,

$$\begin{aligned}\frac{dT^*}{dt} &= kTV - \delta T^* \\ \frac{dV}{dt} &= N\delta T^* - cV,\end{aligned}$$

where δ is the rate of loss of the virus producing T cells and N is the number of virions produced per infected T cell during its lifetime. The term kTV tells us the rate at which the HIV-1 virus infects T cells. This is the same idea as modeling how predators interact with prey in a predator-prey model. Thus, our complete model becomes

$$\begin{aligned}\frac{dT}{dt} &= s + pT\left(1 - \frac{T}{T_{\max}}\right) - d_T T - kTV \\ \frac{dT^*}{dt} &= kTV - \delta T^* \\ \frac{dV}{dt} &= N\delta T^* - cV.\end{aligned}$$

One class of drugs that HIV infected patients receive are reverse transcriptase (RT) inhibitors. RT inhibitors block the action of reverse transcription and prevent the virus from duplicating. If one could find the perfect RT inhibitor, then $k = 0$ and our system becomes

$$\begin{aligned}\frac{dT}{dt} &= s + pT\left(1 - \frac{T}{T_{\max}}\right) - d_T T \\ \frac{dT^*}{dt} &= -\delta T^* \\ \frac{dV}{dt} &= N\delta T^* - cV.\end{aligned}$$

Unfortunately, no one has discovered a perfect RT inhibitor, so we will need to modify the system to account for the effectiveness of the RT inhibitor. We can accomplish this by adding an effectiveness factor, $1 - \eta$, to the kVT term. Our system now becomes

$$\frac{dT}{dt} = s + pT\left(1 - \frac{T}{T_{\max}}\right) - d_T T - k(1-\eta)TV$$

$$\frac{dT^*}{dt} = k(1-\eta)TV - \delta T^*$$
$$\frac{dV}{dt} = N\delta T^* - cV.$$

If $\eta = 1$, then the RT inhibitor is completely effective. On the other hand, if $\eta = 0$, then the RT inhibitor is completely ineffective. We now have a model for how the HIV-1 virus interacts we the immune system. Researchers can use data to estimate the parameters and see exactly what types of solutions are possible.

1.1.6 SageOpen Source Software

Technology can prove very useful when studying differential equations. Software packages such Maple, Mathematica, and Matlab each have their advantages and disadvantages. We will use *Sage*, a readily available open source computer algebra system, as our choice of software. Sage can be run on an individual computer or over the Internet on a server. You can even access *Sage* from your smart phone. For our purposes, *Sage* cells are embedded into the textbook, so there is nothing to install on your computer. Simply, evaluate the cell. You can even change the preloaded commands in the cell if you wish. For example, let us evaluate the derivative of $f(x) = x^2 \cos x$.

```
x = var('x')
y = x^2*cos(x)
diff(y, x)
```

```
-x^2*sin(x) + 2*x*cos(x)
```

We can use Sage to plot functions. For example, we can plot the function $f(x) = x^2 \cos x$ as well as its derivative on the same graph.

```
x = var('x')
y = x^2*cos(x)
yprime = diff(y, x)
p = plot(y, (x, -2, 2), color ='blue', legend_label='$f$',
    legend_color='blue')
p += plot(yprime, (x, -2, 2), color='red', legend_label='$df/dx$',
    legend_color='red')
p
```

We can use *Sage* to solve differential equations. Suppose that we wish to solve the initial value problem

$$\frac{dP}{dt} = kP$$
$$P(0) = 1000.$$

We can use the following commands.

```
k,t = var('k, t') #declare variables k and t
P = function('P')(t) #declare P to be a function of t
de = diff(P, t) == k*P  #differential equation
#solve specifying initial conditions
#and independent variable solution
solution = desolve(de, P, ivar=t, ics=[0, 1000])
solution
```

```
1000*e^(k*t)
```

We will provide abundant examples of how to use *Sage* to solve and analyze differential equations throughout the book, and we encourage the reader to experiment by altering the Sage commands inside the individual Sage cells. If you make a mistake, you can simply reload the webpage and start again.

The reader will find plenty of resources to learn how to use *Sage*. A good place to start is http://www.sagemath.org/help.html or [1]. Although we will be using *Sage* as the technology of choice, this book can be read independently of *Sage*. Finally, we would like to emphasize once again that the reader who chooses not to use some sort of technology will be at a disadvantage.

1.1.7 Some Questions for Thought

In this section we have provided a general notion of what a differential equation is as well as several modeling situations where differential equations are useful; however, we have left many questions unanswered. First, we need a more rigorous definition of a differential equation. What is the proper way to define a system of differential equations? Does a differential equation or a system of differential equations always have a solution? Are solutions to differential equations unique? If a unique solution to a differential equation exists, can we find it? If it is not possible to find a precise solution algebraically, can we estimate the solution numerically? If neither is possible, can we still say anything useful about the solution? Of course, other questions will come to mind as we continue our study of differential equations.

1.1.8 Important Lessons

- A ***differential equation*** is an equation relating a function to one or more of its derivatives.
- An ***initial value problem*** is a differential equation
$$\frac{dx}{dt} = f(t, x),$$
where the ***initial condition***, $x(t_0) = x_0$, is specified.
- A population that is not affected by overcrowding can be modeled by the differential equation $P' = kP$ and is said to grow ***exponentially***.
- A population that must compete for limited resources can be modeled by the ***logistic equation***,
$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P,$$
where N is the carrying capacity of the population.
- Some phenomenon, such as the relationship between a population of predators and a population of prey, are best modeled by systems of differential equations.
- The three principle steps in modeling any phenomenon with differential equations are:
  - Discovering the differential equation or equations that best describe a specified physical situation.
  - Finding—either exactly or approximately—the appropriate solution of the equation or equations.
  - Interpreting the solution in terms of the phenomenon.

1.1.9 Exercises

1. Verify that $y(t) = 3e^{-2t}$ is a solution to the differential equation $y' = -2y$

Hint. $y' = -6e^{-2t} = -2(3e^{-2t}) = -2y$

2. Verify that $y(t) = Ce^{-t}$ is a solution to the differential equation $y' + y = 0$. Determine C if $y(0) = 3$.

Hint. We can determine that $C = 3$. Since $y' + y = -3e^{-t} + 3e^{-t} = 0$, $y(t) = 3e^{-t}$ is a solution to the initial value problem.

3.

(a) Verify that $y(t) = 2/(1 + Ce^{-2t})$ is a solution to the differential equation $y' = y(2 - y)$.

(b) Sketch solution curves for $C = 1, 2, \ldots, 5$.

(c) Verify that $y = 0$ is a solution to the differential equation in part (a). Can you find a value for C such that

$$y(t) = \frac{2}{1 + Ce^{-2t}} = 0?$$

4. Write a differential equation that models a population P, where P changes at a rate proportional to the square root of P.

Hint. $dP/dt = k\sqrt{P}$

5. We can modify the logistic growth model to account for a threshold population. That is, if the population of a species drops below a certain level, the species is endanger of extinction. Consider the case of the black rhinoceros, once the most numerous of all rhinoceros species and now critically endangered. The black rhino, native to eastern and southern Africa, was estimated to have a population of about 100,000 around 1900. Because of hunting, habitat changes, competing species, and most of all illegal poaching, the number of black rhinos today is estimated to be below 3000. If the wild population becomes too low, the animals may not be able to find suitible mates and the black rhino will become extinct. There must be a minimum population for the species to continue. Suppose the this minimum or threshold population for the black rhino is 1000 animals and that remaining habitant in Africa will support no more that 20,000 rhinos.

(a) For what values of P is the rhino population increasing? What can be said about the value of dP/dt for these values of P?

(b) For what values of P is the rhino population decreasing? What can be said about the value of dP/dt for these values of P?

(c) For what values of P is the rhino population in equilibrium? What can be said about the value of dP/dt for these values of P?

(d) Find a differential equation that models the population of rhinos at time t.

Hint. (a) The population is increasing if $dP/dt > 0$ and $1000 < P < 20000$.

6. Given the equation $x' + px = q(t)$, where p is a constant and $q(t)$ is a continuous function defined on an interval I, show that

$$x(t) = Ce^{-at} + e^{-at} \int_{t_0}^{t} e^{as} b(s)\, ds$$

is a solution of this equation, where c is any constant and $t_0 \in I$.

Hint. Rewriting the differential equation as $x' + px - q(t) = 0$ and using the fact that

$$x'(t) = -aCe^{-at} - ae^{-at}\int_{t_0}^{t} e^{as}b(s)\,ds + b(t),$$

we see that

$$\begin{aligned} x'(t) + ax(t) - b(t) &= -aCe^{-at} - ae^{-at}\int_{t_0}^{t} e^{as}b(s)\,ds + b(t) \\ &+ \; aCe^{-at} + ae^{-at}\int_{t_0}^{t} e^{as}b(s)\,ds - b(t) \\ &= 0. \end{aligned}$$

7. (Radiocarbon Dating) Carbon 14 is a radioactive isotope of carbon, the most common isotope of carbon being carbon 12. Carbon 14 is created when cosmic ray bombardment changes nitrogen 14 to carbon 14 in the upper atmosphere. The resulting carbon 14 combines with atmospheric oxygen to form radioactive carbon dioxide, which is incorporated into plants by photosynthesis. Animals acquire carbon 14 by eating plants. When an animal or plant dies, it ceases to take on carbon 14, and the amount of isotope in the organism begins to decay into the more common carbon 12. Carbon 14 has a very long half-life, about 5730 years. That is, given a sample of carbon 14, it will take 5730 years for half of the sample will decay to carbon 12. The long half-life is what makes carbon 14 dating is very useful in dating objects from antiquity.

(a) Consider at sample of material that contains $A(t)$ atoms of carbon 14 at time t. During each unit of time a constant fraction of the radioactive atoms will spontaneously decay into another element or a different isotope of the same element. Thus, the sample behaves like a population with a constant death rate and a zero birth rate. Make use of the model of exponential growth to construct a differential equation that models radioactive decay for carbon 14.

(b) Solve the equation that you proposed in (a) to find an explicit formula for $A(t)$.

(c) The Chauvet-Pont-d'Arc Cave in the Ardèche department of southern France contains some of the best preserved cave paintings in the world. Carbon samples from torch marks and from the paintings themselves, as well as from animal bones and charcoal found on the cave floor, have been used to estimate the age of the cave paintings. If a particular sample taken from the Cauvet Cave contains 2% of the expected cabon 14, what is the approximate age of the sample?

8. Consider the following predator-prey systems of differential equations

$$\begin{aligned} \frac{dx}{dt} &= -\frac{x}{2} + \frac{xy}{2+y}, \\ \frac{dy}{dt} &= y(1-y) - \frac{xy}{2+y}. \end{aligned}$$

(a) Which equation models the prey population and which equation models the predator population?

(b) How does the prey population grow if there are no predators present?

(c) What happens if there are a lot of prey present?

Hint. Think about the limit of the interaction term as the number of prey becomes very large.

1.1.10 An Introduction to Sage

Write an introduction to Sage.

1.2 Separable Differential Equations

We will define a ***differential equation*** of ***order*** n to be an equation that can be put in the form

$$F(t, x, x', x'', \ldots, x^{(n)}) = 0,$$

where F is a function of $n+2$ variables. A ***solution*** to this equation on an interval $I = (a, b)$ is a function $u = u(t)$ such that the first n derivatives of u are defined on I, and

$$F(t, u, u', u'', \ldots, u^{(n)}) = 0.$$

We will concentrate on first-order differential equations in this chapter. That is, we will consider equations of the form

$$\frac{dx}{dt} = f(t, x).$$

1.2.1 Separable Differential Equations

For some first-order differential equations such as

$$\frac{dx}{dt} = kx$$

the solution is given by an explicit formula. In this particular case, the solution is $x(t) = Ce^{kt}$. In general, we cannot generally find such a formula for an arbitrary first-order differential equation. We can, however, solve a differential equation $y' = f(x, y)$ if we can write the equation in the form

$$f(x) + g(y)\frac{dy}{dx} = 0.$$

Such equations are called ***separable***. We can solve separable equations by integrating the first term with respect to x and the second term with respect to y.

Example 1.9. Suppose that we wish to solve the initial value problem

$$\frac{dy}{dx} = xy$$
$$y(0) = 1.$$

We can rewrite this equation in the form

$$\frac{1}{y}\frac{dy}{dx} = x$$

or in the alternate form

$$\frac{1}{y}\,dy = x\,dx.$$

Integrating both sides of the equation, we have

$$\ln|y| = \frac{1}{2}x^2 + C,$$

where C is an arbitrary constant. Using the initial condition, $y(0) = 1$ to find C, we see that

$$0 = \ln|1| = 0 + C.$$

Thus, the solution to our initial value problem can be given implicitly by $\ln y = x^2/2$. In this example, we can actually write down an explicit solution that is defined everywhere,

$$y = e^{x^2/2}.$$

The *Sage* commands for solving our initial value problem are below.

```
x = var('x') #declare x as a variable
y = function('y')(x) #declare y as a function of x
de = diff(y, x) == x*y #write the differential equation
solution = desolve(de, y, ics=[0, 1]) #find the solution
solution
```

```
e^(1/2*x^2)
```

If we ask *Sage* to solve a differential equation that is impossible to solve analytically, the computation will return an error. *Sage* will not solve the initial value problem

$$\frac{dy}{dx} = \sin(xy)$$
$$y(0) = 1.$$

```
x = var('x')
y = function('y')(x)
de = diff(y, x) == sin(x*y)
solution = desolve(de, y, ics=[0, 1])
solution
```

Example 1.10. Consider the initial-value problem

$$\frac{dy}{dt} = \frac{t}{y - t^2 y}$$
$$y(0) = 4.$$

First, we separate the variables of the equations and write

$$y\,dy = \frac{t}{1 - t^2}\,dt.$$

Integrating both sides of the equation, we have

$$\frac{1}{2}y^2 = -\frac{1}{2}\ln|1 - t^2| + C \qquad \text{or} \qquad y^2 = -\ln|1 - t^2| + C.$$

Using the initial condition, $y(0) = 4$, we can determine the value of C,

$$y^2 = 16 - \ln|1 - t^2| \qquad \text{or} \qquad y = \sqrt{16 - \ln|1 - t^2|}.$$

Notice that the solution does not make sense for all values of t. In fact, the solution is only defined on the interval $-1 < t < 1$, if we require that our solution be continuous. Let us see what *Sage* has to say.

```
t = var('t')
y = function('y')(t)
de = diff(y, t) == t/(y - t*2*y)
solution = desolve(de, y, ics=[0, 4])
show(solution)
```

```
-1/2*y(t)^2 == -1/4*I*pi + 1/2*t + 1/4*log(2*t - 1) - 8
```

Sage does return a solution even if it looks a bit different than the one that we arrived at above. Notice that we have an imaginary term in our solution, where $i^2 = -1$. We will examine the role of complex numbers and how useful they are in the study of ordinary differential equations in a later chapter, but for the moment complex numbers will just muddy the situation.

Example 1.11. The initial value problem in Example 1.2 is a good example of a separable differential equation,

$$\begin{aligned} \frac{dP}{dt} &= k\left(1 - \frac{P}{1000}\right)P \\ P(0) &= 100. \end{aligned}$$

Using partial fractions, we can rewrite this equation as

$$\frac{1000}{P(1000-P)}\,dP = \left(\frac{1}{P} + \frac{1}{1000-P}\right)\,dP = k\,dt. \tag{1.2}$$

Integrating both sides of (1.2), we obtain

$$\ln|P| - \ln|1000 - P| = \ln\left|\frac{P}{1000-P}\right| = kt + C.$$

Taking the exponential of both sides yields

$$\frac{P}{1000-P} = e^{kt+C} = e^{kt}e^{C}.$$

Since C is an arbitrary constant, we know that e^C is an arbitrary postive constant, which we will also call C. So we can rewrite this last equation as

$$P = Ce^{kt}(1000 - P) = 1000Ce^{kt} - Ce^{kt}P.$$

Solving for P yields

$$P = \frac{1000Ce^{kt}}{Ce^{kt}+1} = \frac{1000}{Ce^{-kt}+1}.$$

Using our initial condition $P(0) = 100$, we can determine that $C = 9$.

1.2.2 Newton's Law of Cooling

Separable equations arise in a wide range of application problems. The time of death of a murder victim is an important question on many popular movies and television programs. How does a forensic scientist or a medical examiner determine the time of death? Human beings have a temperature of 98.6°F. If the surrounding temperature is cooler, then the body will cool down after death. Eventually, the temperature of the body will match the temperature of the environment. We should not expect the body to cool at a constant rate

either. Think of how a hot cup of coffee or tea cools. The liquid will cool quite quickly during the first few minutes but will remain relatively warm for quite a long period.

The answer to our forensic question can be found by using ***Newton's law of cooling***, which tells us that the rate of change of the temperature of a object is proportional to the difference between the temperature of the object and the temperature of the surrounding medium. Newton's law of cooling can be easily stated as a differential equation,

$$\frac{dT}{dt} = k(T - T_m),$$

where T is the temperature of the object, T_m is the temperature of the surrounding medium, and k is the proportionality constant.

Suppose that the temperature of the surrounding environment is 70°F, and we know from experience that a body under these conditions cools off approximately 2°F during the first hour after death. In order to determine a formula for the time of death, we must solve the initial value problem

$$\begin{aligned} \frac{dT}{dt} &= k(T - 70) \\ T(0) &= 98.6, \end{aligned}$$

where $T(1) = 96.6$. If we rewrite the equation

$$\frac{dT}{dt} = k(T - 70)$$

as

$$\frac{1}{T - 70}\frac{dT}{dt} = k,$$

we see that this equation is separable. Integrating both sides of the last equation, we obtain $\ln|T - 70| = kt + C$. Since we are assuming that $T > 70$, we can write $T - 70$ instead of $|T - 70|$. Thus, we have

$$\ln(T - 70) = kt + C \qquad \text{or} \qquad T - 70 = e^{kt+C} = e^{kt}e^{C}.$$

Letting $D = e^C$, the solution becomes

$$T(t) = De^{kt} + 70.$$

The initial condition, $T(0) = 98.6$, tells us that $D = 28.6$. Thus,

$$T(t) = 28.6e^{kt} + 70.$$

Since

$$96.6 = T(1) = 28.6e^{k\cdot 1} + 70,$$

we can determine the constant k to be

$$k = \ln\left(\frac{26.6}{28.6}\right) \approx -0.0725.$$

and

$$T(t) = 28.6e^{-0.0725t} + 70.$$

The graph of the T seems appropriate to our model (Figure 1.12).

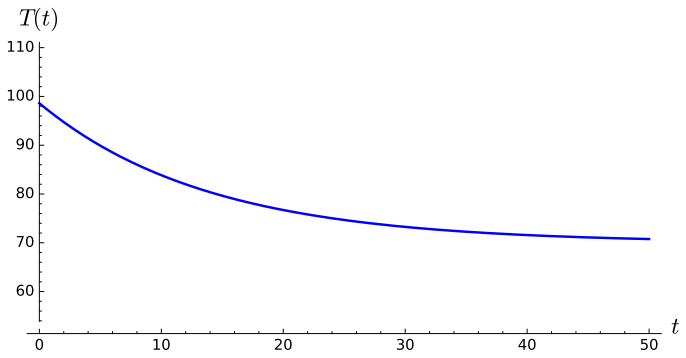

Figure 1.12: Newton's law of cooling

Let us solve our differential equation using *Sage*.

```
k,t = var('k,t')
T = function('T')(t)
de = diff(T,t) == k*(T - 70)
solution = desolve(de, T, ivar=t)
solution
```

```
(c + 70*e^(-k*t))*e^(k*t)
```

We can use *Sage* to plot our solution.

```
t = var('t')
T(t) = 28.6 * exp(-0.0725 * t) + 70
plot(T, (t, 0, 24))
```

1.2.3 Mixing Problems

There is a large class of problems in modeling known as ***mixing problems***. These problems refer to situations where two are more substances are mixed together in a container or containers. For example, we might wish to model how chemicals are mixed together in a refinery, how pollutants are mixed together in a pond or a lake, how ingredients are mixed together when brewing beer, or even how various greenhouse gases mix together across different layers of the atmosphere.

Suppose that we have a large tank containing 1000 gallons of water and that water containing 0.5 pounds of salt per gallon flows into the tank at a rate of 10 gallons per minute. If the tank is also draining at a rate of 10 gallons per minute, the water level in the tank will remain constant. We will assume that the water in the tank is constantly stirred so that the mixture of salt and water is uniform in the tank.

We can model the amount of salt in the tank using differential equations. If $x(t)$ is the amount of salt in the tank at time t, then the rate at which the salt is changing in the tank is the difference between the rate at which salt is flowing into the tank and the rate at which it is leaving the tank, or

$$\frac{dx}{dt} = \text{rate in} - \text{rate out}. \tag{1.3}$$

Of course, the salt flows into the tank at the rate of $10 \cdot 0.5 = 5$ pounds of salt per minute. However, the rate at which the salt leaves the tank depends on $x(t)$, the amount salt in the tank at time t. At time t, there is $x(t)/1000$ pounds of salt in one gallon. Therefore, salt

flows out of the tank at a rate of $10x(t)/1000 = x(t)/100$ pounds per minute. Equation (1.3) now becomes

$$\begin{aligned} \frac{dx}{dt} &= 5 - \frac{x}{100} \\ x(0) &= 0. \end{aligned}$$

This equation is separable,

$$\frac{dx}{500 - x} = \frac{dt}{100}.$$

Integrating both sides of the equation, we have

$$-\ln|500 - x| = \frac{t}{100} + k$$

or

$$\ln|500 - x| = -\frac{t}{100} - k.$$

Consequently,

$$500 - x = Ce^{-0.01t},$$

where $C = e^{-0.01k}$. From our intial condition, we can quickly determine that $C = 500$ and

$$x(t) = 500 - 500e^{-0.01t}$$

models the amount of salt in the tank at time t.

1.2.4 A Retirement Model

Differential equations have many applications in economics and finance. For example, Dr. J., a college professor, wisely started saving for his retirement as soon as he entered the workforce, and he now has \$500,000 in a retirement account earning an interest of 5% compounded continuously. The initial value problem,

$$\begin{aligned} \frac{dP}{dt} &= 0.05P \\ P(0) &= 500 \end{aligned}$$

provides a nice model of Dr. J.'s investment, where $P(t)$ is the amount in thousands of dollars in the fund at time t. The solution to our initial value problem is

$$P(t) = 500e^{0.05t}.$$

If Dr. J. plans to retire in 10 years, he can expect a nest egg of $P(10) \approx 824.360635350064$ or about \$824,360.

Of course, Dr. J. still plans to make contributions to his retirement fund during his next ten years of employment. His annual contribution will be \$5,000, which his employer will generously match. If we assume that these contributions will spread out evenly over the course of the year, we can incorporate this information into our original initial value problem,

$$\begin{aligned} \frac{dP}{dt} &= 0.05P + 10 \\ P(0) &= 500 \end{aligned}$$

This differential equation is separable, so we have

$$\int \frac{dP}{0.05P + 10} = \int dt.$$

Integrating both sides of this equation, we have

$$20 \ln |0.05P + 10| = t + k,$$

where k is an arbitrary constant. Since $0.05P + 10 > 0$, we have

$$20 \ln(0.05P + 10) = t + k.$$

This last equation is equivalent to

$$0.05P + 10 = e^{0.05(t+k)} = e^{0.05t} e^{0.05k}.$$

If we let $C = e^{0.05k}$ and solve for P, we obtain

$$P = 20Ce^{0.05t} - 200.$$

Using our initial condition,

$$500 = P(0) = 20Ce^{0.05 \cdot 0} - 200 = 20C - 200,$$

we have $C = 35$. Thus, the solution that we seek is

$$P(t) = 700e^{0.05t} - 200.$$

Dr. J.'s nest egg is now $P(10) \approx 954.104889490090$ or about \$954,105.

Once Dr. J. retires, he will need to begin withdrawing money from his account. He estimates that he will need to withdraw \$60,000 a year for living expenses if he wishes to travel and enjoy his golden years. Of course, whatever remains in his account at any given time will still collect interest. We describe J.'s retirement situation with the initial value problem,

$$\frac{dP}{dt} = \begin{cases} 0.05P + 10, & t \leq 10 \\ 0.05P - 60, & t > 10 \end{cases}$$
$$P(0) = 5.$$

If $P = 954$, then

$$\frac{dP}{dt} = 0.05P - 60 \approx -12.3 < 0.$$

Hence, the rate of withdrawal exceed the rate at which Dr. J.'s account is earning interest. Eventually, Dr. J.'s retirement fund will be disappear. This may pose a problem, if Dr. J. plans to retire early and live a long life.

Again, the differential equation $dP/dt = 0.05P - 60$ is separable, and we have

$$\int \frac{dP}{0.05P - 60} = \int dt.$$

Intergrating both sides of this equation yields

$$20 \ln |0.05P - 60| = t + k.$$

Since $0.05P - 60 < 0$,

$$|0.05P - 60| = 60 - 0.05P,$$

and

$$20\ln(60 - 0.05P) = t + k.$$

Consequently,

$$60 - 0.05P = e^{0.05(t+k)/20} = Ce^{0.05t},$$

or

$$P = 1200 - Ce^{0.05t},$$

where $C = e^{0.05k}$. Now, we can apply our initial condition $P(10) = 954$ to determine that $C \approx 149.21$. Therefore,

$$P = 1200 - 149.21e^{0.05t}$$

describes how much money Dr. J. after he retires ($t \geq 10$).

If Dr. J. want to know how long his retirement fund will last, he must solve the equation

$$1200 - 149.21e^{0.05t} = 0.$$

In this case,

$$t = 20\ln\left(\frac{1200}{149.21}\right) \approx 41.7.$$

This means that if Dr. J. retires in 10 years at the age of 65, he can expect his retirement to last into his mid 90s.

1.2.5 Some Theory

We now give a theoretical basis for solving first-order separable differential equations. A differential equation $y' = F(x, y)$ is called ***separable*** if it can be written in the form

$$f(x) + g(y)\frac{dy}{dx} = 0. \tag{1.4}$$

We now will prove that such an equation can solved by integrating the first term with respect to x and the second term with respect to y. If

$$h_1(s) = \int f(s)\,ds$$
$$h_2(s) = \int g(s)\,ds,$$

then we can rewrite equation (1.4) as

$$h_1'(x) + h_2'(y)\frac{dy}{dx} = 0.$$

Applying the chain rule to the second term, we obtain

$$h_2'(y)\frac{dy}{dx} = \frac{d}{dx}[h_2(y)].$$

Hence, equation (1.4) now becomes

$$\frac{d}{dx}(h_1(x) + h_2(y)) = 0.$$

Integrating, we obtain

$$h_1(x) + h_2(y) = C,$$

where C is any arbitrary constant.

Now suppose that $y(x_0) = y_0$ is an initial condition for

$$f(x) + g(y)\frac{dy}{dx} = 0.$$

Then $h_1(x_0) + h_2(y_0) = C$. By the Fundamental Theorem of Calculus

$$h_1(x) - h_1(x_0) = \int_{x_0}^{x} f(s)\, ds,$$
$$h_2(y) - h_2(y_0) = \int_{y_0}^{y} g(s)\, ds.$$

Consequently, we can replace equation (1.4) with the ***integral equation***

$$\int_{x_0}^{x} f(s)\, ds + \int_{y_0}^{y} g(s)\, ds = 0.$$

In other words, we simply need to integrate each term to solve the differential equation.

1.2.6 What Can Go Wrong

Example 1.13. It is not always possible to explicitly solve a separable differential equation. Consider the equation

$$\frac{dy}{dx} = \frac{y(1+y^2)}{y^2+y+1}.$$

This equation can be rewritten in the form

$$\left(\frac{1}{1+y^2} + \frac{1}{y}\right) dy = dx$$

Integrating both sides of the equation, we have

$$\arctan y + \ln|y| = x + C.$$

However, we have no method of solving this last equation explicitly for y.

Example 1.14. Another difficulty arises if we consider the equation

$$y' = te^{-y^2}.$$

This equation is separable since we can rewrite it in the form

$$e^{y^2}\, dy = t\, dt.$$

Although the Fundamental Theorem of Calculus guarantees that every continuous function has an antiderivative, we cannot find an antiderivative for the function e^{y^2} in terms of elementary functions. Thus, we are forced to write our solution as

$$\int_0^y e^{s^2}\, ds = \frac{1}{2}t^2 + C.$$

Example 1.15. Even if we have a separable differential equation, we are not guaranteed a unique solution. Consider the initial value problem $y' = y^{1/3}$ with $y(0) = 0$ and $t \geq 0$. Separating the variables,

$$y^{-1/3}\,dy = dt.$$

Thus,

$$\frac{3}{2}y^{2/3} = t + C$$

or

$$y = \left(\frac{2}{3}(t + C)\right)^{3/2}.$$

If $C = 0$, the initial condition is satisfied and

$$y = \left(\frac{2}{3}t\right)^{3/2}$$

is a solution for $t \geq 0$. However, we can find at least two additional solutions for $t \geq 0$:

$$y = -\left(\frac{2}{3}t\right)^{3/2},$$
$$y \equiv 0.$$

In Section 1.5 we will learn sufficient conditions for a first-order initial value problem to have a unique solution.

Example 1.16. Suppose that $y' = y^2$ with $y(0) = 1$. Separating the variables,

$$\frac{1}{y^2}\,dy = dt,$$

we see that

$$y = -\frac{1}{t + C}$$

or

$$y = \frac{1}{1 - t}.$$

Therefore, a continuous solution also exists on $(-\infty, 1)$ if $y(0) = -1$. In the case that $y(0) = -1$, the solution is

$$y = -\frac{1}{t + 1},$$

and a continuous solution exists on $(-1, \infty)$.

1.2.7 Important Lessons

- A differential equation of ***order*** n is an equation that can be put in the form

  $$F(t, x, x', x'', \ldots, x^{(n)}) = 0,$$

  where F is a function of $n + 2$ variables. A ***solution*** to the equation on an interval $I = (a, b)$ is a function $u = u(t)$ such that the first n derivatives of u are defined on I, and

  $$F(t, u, u', u'', \ldots, u^{(n)}) = 0.$$

- A ***first-order differential equation*** is an equation that can be written in the form

$$\frac{dx}{dt} = f(t, x).$$

- A differential equation is separable if it can be written in the form

$$\frac{dy}{dx} = M(x)N(y).$$

In this case we can rewrite the equation in the form

$$f(x) + g(y)\frac{dy}{dx} = 0$$

or

$$g(y)\,dy = f(x)\,dx$$

and solve by integrating both sides.

1.2.8 Exercises

1. Solve each of the following differential equations.

(a) $\dfrac{dy}{dx} = 8 - 2x$

(b) $\dfrac{dy}{dx} = \dfrac{x+1}{y}$

(c) $\dfrac{dx}{dt} = x^2t^2$, $x(0) = 1$

(d) $y' = \dfrac{t^2y - y}{y+1}$, $y(0) = -1$

Hint. (a) $y = -x^2 + 8x + C$; (c) $x = 3/(3 - t^3)$.

2. Solve each of the following equations in Exercise 1.2.8.1 using Sage.

Hint.

(a)

```
x = var('x')
y = function('y')(x)
de = diff(y, x) == 8 - 2*x
desolve(de, y)
```

```
-x^2 + _C + 8*x
```

3. Solve the initial value problem

$$\frac{dy}{dt} = t^2y^2 + t^2;$$
$$y(0) = 1.$$

What is the domain of your solution?

4. *Homogeneous Equations.* A first-order differential equation, $y' = f(x, y)$, is *homogeneous* if $f(x, y) = f(tx, ty)$.

(a) Show that the equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

is homogeneous.

(b) Let $y = xv$ and show that the equation in part (a) can be written as

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2vx^2}.$$

Use the fact that this new equation is separable to solve for y.

(c) Show that any homogeneous equation $y' = f(x, y)$ can be transformed into a separable differential equation by making the substitution $y = vx$.

(d) A function f is said to homogeneous of degree n if $f(tx, ty) = t^n f(x, y)$ for $n = 1, 2, \ldots$. Show that differential equation

$$P(x, y)dx + Q(x, y)dy = 0,$$

where P and Q are both homogeneous of degree n, can be transformed into a separable differential equation using the substitution $y = vx$.

(e) Solve the differential equation

$$x^2y' = 2y^2 - x^2.$$

5. Mr. Ratchett, an elderly American, was found murdered in his train compartment on the Orient Express at 7 A.M. When his body was discovered, the famous detective Hercule Poirot noted that Ratchett had a body temperature of 28 degress. The body had cooled to a temperture of 27 degrees one hour later. If the normal temperature of a human being is 37 degrees and the air temperature in the train is 22 degrees, estimate the time of Ratchett's death using Newton's Law of Cooling.

1.2.9 Sage—Quick Start Guide to Solving Ordinary Differential Equations

Sage has powerful algorithms for finding exact and numerical solutions of differential equations. In addition, we can plot solutions and direction fields. Although some differential equations have an exact solution and can be solved using analytic techniques with calculus, many differential equations can only be solved nusing numerical techniques. This should not be too surprising if we consider how we solve polynomials. It is quite easy to find the roots of any equation of the form $ax^2 + bx + c = 0$ by either factoring or using the quadratic equation, but solving an equation such as

$$5x^7 - 6x^4 + 3x^3 - 23x^2 + 3x - 17 = 0$$

is a much more difficult problem. Unlike the situation for quadratic equations, there does not exist a general formula for solving seventh degree equations. We can even encounter difficulties when using numerical algorithms such as the Newton-Raphson algorithm.

In general, *Sage* needs three things to solve a differential equation:

- An abstract function
- A differential equation
- A *Sage* command to solve the equation.

Suppose we wish to solve the equation

$$\frac{dy}{dx} = x + y.$$

We can use the following sequence of *Sage* commands.

```
y = function('y')(x)
de = diff(y,x) == x + y
h = desolve(de, y)
h
```

```
-((x + 1)*e^(-x) - _C)*e^x
```

The first command defines the abstract function. The second describes the actual differential equation. Finally, we use the Sage command `desolve` to find the actual solution. Try replacing the `h` command with `show(h)` or `show(expand(h))`.

We can also specify an initial condition for our differential equation, say $y(0) = 1$.

```
y = function('y')(x)
de = diff(y,x) == x + y
h = desolve(de, y, ics=[0,1])
h
```

```
h
```

There are many other commands and packages to solve ordinary differential equations using Sage. For more information, see http://www.sagemath.org/doc/reference/calculus/sage/calculus/desolvers.html. An empty Sage cell is below for practice and exploration.

```

```

1.3 Geometric and Quantitative Analysis

If we view the differential equation $y' = f(t, y)$ as a formula for the slope of a tangent line to a solution curve, we can approximate the graph of a solution curve. For example, if we consider the equation $y' = t + y$, then a solution curve will have a slope of 2 at the point $(1, 1)$. We can use this information to obtain a geometric description of the solutions to the equation.

1.3.1 RC Circuits

Suppose that we wish to analyze how an electric current flows through a circuit. An ***RC circuit*** is a very simple circuit might contain a voltage source, a capacitor, and a resistor (Figure 1.17). A battery or generator is an example of a voltage source. The glowing red heating element in a toaster or an electric stove is an example of something that provides resistance in a circuit. A capacitor stores an electrical charge and can be made by separating two metal plates with an insulating material. Capacitors are used to power the electronic flashes for cameras. Current, $I(t)$, is the rate at which a charge flows through this circuit and is measured in amperes or amps (A). We assign a direction to the current. A current flowing in the opposite direction will be given negative values.

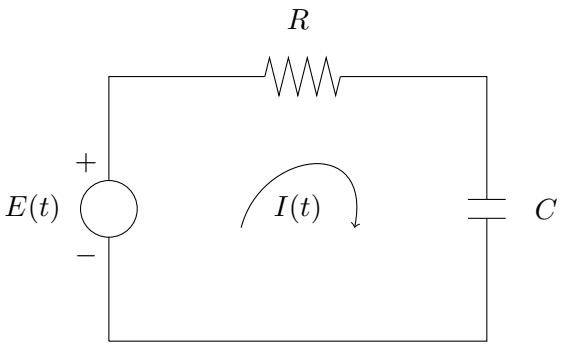

Figure 1.17: An RC circuit

The source voltage, $E(t)$, is measured in volts (V). ***Kirchhoff's Second Law*** tells us that the impressed voltage in a closed circuit is equal to the sum of the voltage drops in the rest of the circuit. Thus, we need only compute the voltage drop across the resistor, E_R, and the voltage drop across the capacitor, E_C. According to Kirchhoff's Law, this is

$$E_R + E_C = E.$$

Resistance, R, to the current is measured in ohms (Ω). ***Ohm's Law*** tells us that the voltage drop across a resistor is given by

$$E_R = IR. \tag{1.5}$$

Finally, capacitance, C, is measured in farads (F). ***Coulomb's Law*** tells us how current flows across a capacitor,

$$I = C\frac{dE_C}{dt}. \tag{1.6}$$

Thus, if we combine the equations (1.5) and (1.6), our equation $E_R + E_C = E$ becomes

$$RC\frac{dE_C}{dt} + E_C = E(t). \tag{1.7}$$

We will now investigate how our circuit reacts under different voltage sources. For example, we might have a zero voltage source (the capacitor could still hold a charge). We could also have a constant nonzero source of voltage such as a battery or a fluctuating source of voltage such as a generator. We might even have a series of pulses of voltage where the current is periodically turned on and off. We would like to be able to understand the solutions to the equation (1.7) for different voltage sources $E(t)$. If we view the differential equation (1.7) as an expression for computing how fast current is flowing across the capacitor, we can analyze our circuit from a geometric point of view and can actually say a great deal about circuits without solving a differential equation.

1.3.2 Direction Fields

Any differential equation

$$\frac{dy}{dt} = f(t, y)$$

can be viewed as a formula for the slope of a function $y = y(t)$. Geometrically, the equation tells us that, at any point (t_0, y_0), the slope of a solution curve is given by $f(t_0, y_0)$. Suppose that our differential equation is defined on the rectangle $R = [a, b] \times [c, d]$, and let $y(t)$ be a solution curve for $y' = f(t, y)$ that passes through the point (t_0, y_0). Then the differential

equation tells us the slope of this solution curve at (t_0, y_0). We can indicate this on the (t, y)-plane by drawing a short line segment at the point (t, y) with slope $f(t_0, y_0)$. Thus, we can obtain a ***direction field*** or ***slope field*** for the differential equation. A solution curve must be tangent to its direction field at every point.

For example, consider the differential equation $y' = y^2/2 - t$. The direction field for this equation is given in Figure 1.18 along with several solution curves.

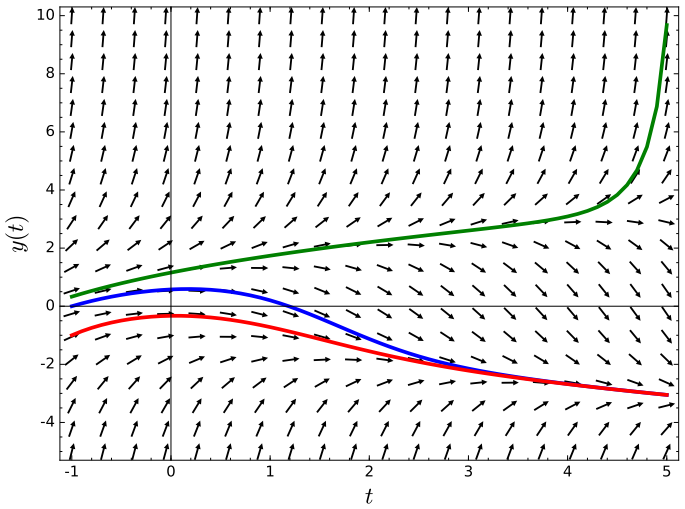

Figure 1.18: The direction field for $y' = y^2/2 - t$

Although direction fields can be tedious to compute using pencil and paper, we can easily generate direction fields for any differential equation with the use of computer software. Most computer algebra systems, including *Sage*, have facilities for generating and graphing direction fields. For example, the following *Sage* cell will generate the direction field for $y' = y^2/2 - t$ using the command `plot_slope_field`. We will learn how to add solution curves at the end of this section.

```
t,y = var('t,y')
f(t,y) = y^2/2 - t
plot_slope_field(f, (t,-1,5), (y,-5,10), headaxislength=3,
    headlength=3, axes_labels=['$t$','$y(t)$'])
```

1.3.3 RC Circuits Revisited

Now let us return to our RC circuit and consider different functions $E(t)$ for the differential equation

$$\frac{dE_C}{dt} = \frac{E(t) - E_C}{RC},$$

with $R = 1$ and $C = 1$. First suppose that there is no voltage source in the circuit. If we let $E(t) = 0$ for all $t \geq 0$, we will get the direction field of given in Figure 1.19. The direction field agrees with our analytic solution

$$E_C(t) = v_0 e^{-t},$$

where $v_0 = E_C(0)$.

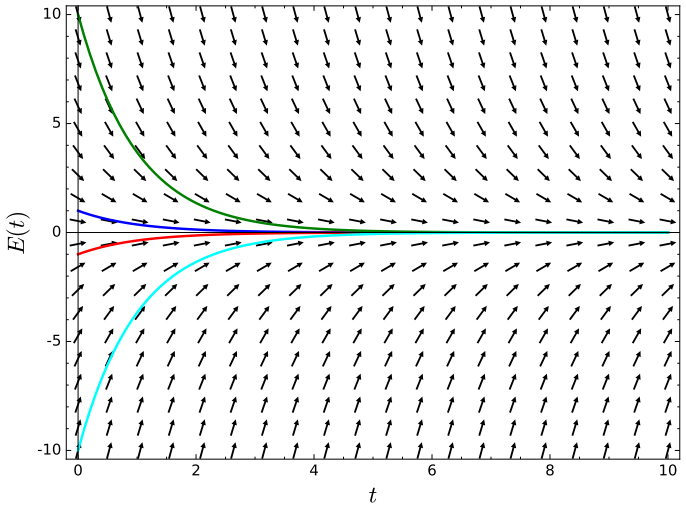

Figure 1.19: The direction field for no current

If we assume that we have a nonzero constant source of voltage, $E(t) = K$, in our circuit such as a battery, then we obtain the separable differential equation

$$\frac{dE_C}{dt} = K - E_C.$$

The direction field for this differential equation for $K = 10$ is given in Figure 1.20.

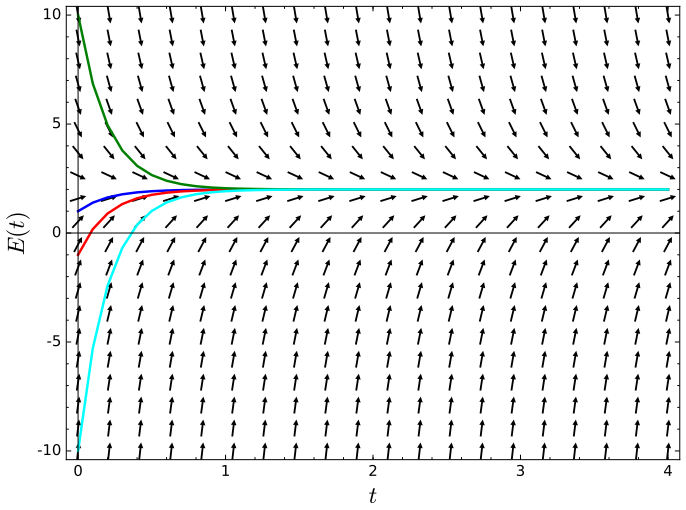

Figure 1.20: The direction field for a constant current

If we attach a battery to our circuit at time $t = 0$ and then disconnect the battery at $t = 4$, then we obtain a different solution. For example, if

$$E(t) = \begin{cases} 10 & 0 \le t \le 4, \\ 0 & t > 4, \end{cases}$$

we will obtain a different direction field (Figure 1.21).

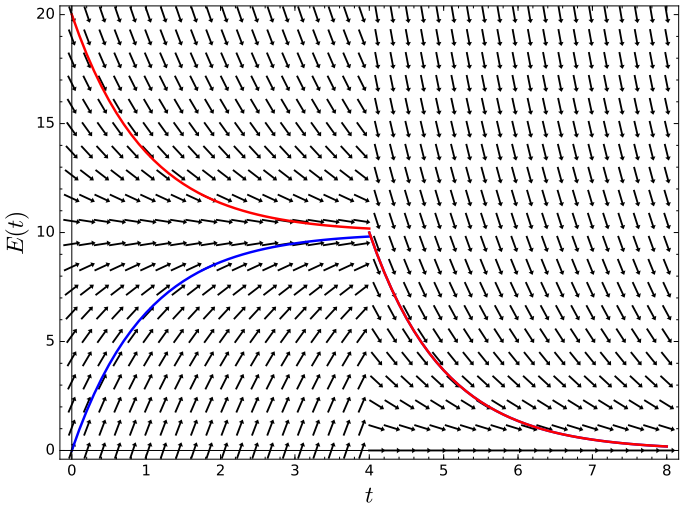

Figure 1.21: The direction field for a single pulse

If our voltage source emits a series of pulses, say

$$E(t) = \begin{cases} 10 & 0 \le t < 4, \\ 0 & 4 \le t < 8, \\ 10 & 8 \le t < 12, \\ & \vdots \end{cases}$$

then the direction field for our differential equation is given in Figure 1.22.

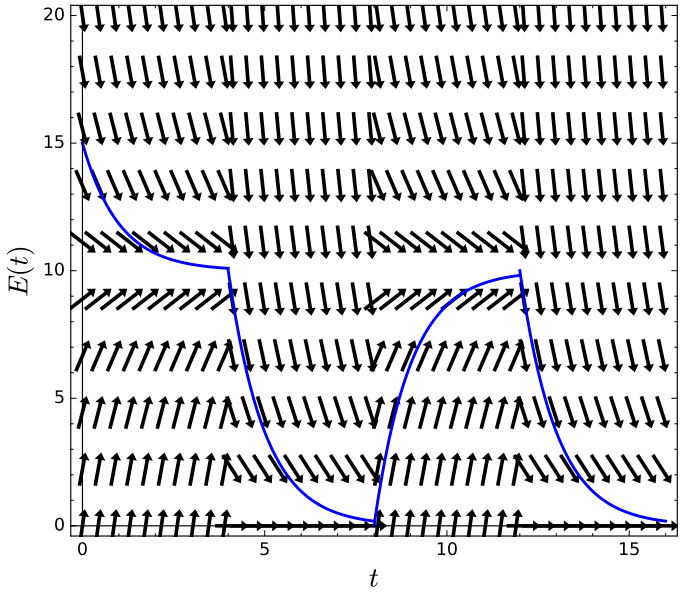

Figure 1.22: The direction field for a series of pulses

Finally, if we use a generator for a voltage source, the voltage source might be given by a function such as $E(t) = \sin(\pi t/2)$. The direction field for this circuit is given in Figure 1.23.

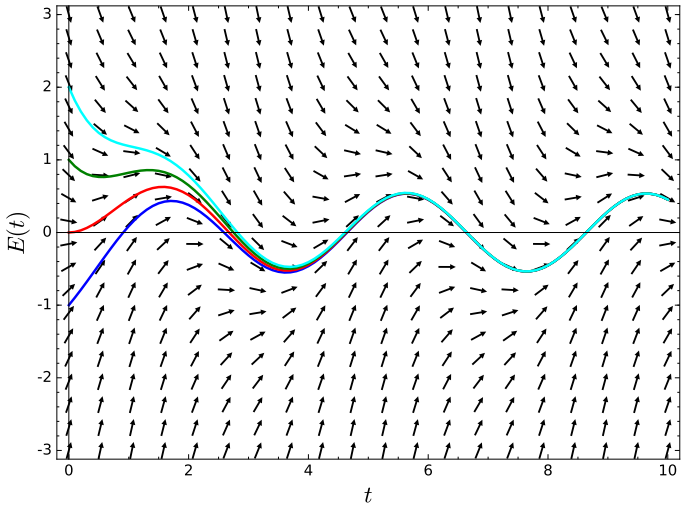

Figure 1.23: The direction field for an oscillating voltage

1.3.4 Plotting Direction Fields with Sage

It is very tedious to plot direction fields by hand. Fortunately, we have technology to help. The Sage interact below will help you plot direction fields for a given initial value problem. A Sage interact is a menu driven Sage applet. Simply evaluate the Sage cell. You can then change each field in the cell output to solve a particular initial value problem.

```
x, t = var('x,t')
@interact
def _(f = input_box(default = x^2/2 - t),
                    t_min = input_box(default = -1.0),
                    t_max = input_box(default = 6),
                    t_start = input_box(default = 1),
                    x_start = input_box(default = -0.1),
                    step_size = (0.1, (0.001, 1.0))):
    P = desolve_rk4(f,x,ics=[t_start,
        x_start],ivar=t,end_points=[t_min, t_max], step = step_size,
        thickness=3)
    p = line(P, thickness = 2, axes_labels = ['$t$','$x$'],)
    x_min = min([P[i][1]] for i in (0..len(P)-1))[0]
    x_max = max([P[i][1]] for i in (0..len(P)-1))[0]
    p += plot_slope_field(f, (t, t_min, t_max), (x, x_min, x_max),
        headaxislength=3, headlength=3)
    p += arrow(P[int(len(P)/2)], P[int(len(P)/2) + 1], width = 1,
        arrowsize = 8)
    p += point((t_start, x_start), size=60, color="red")
    p.show()
    pretty_print(html(r"$\displaystyle\frac{dx}{dt} = %s$"
        %latex(f)))
```

Of course, it is quite easy to give the interact a problem that yield unitelligible results.

1.3.5 Autonomous Differential Equations

An ***autonomous differential equation*** is one of the form

$$\frac{dx}{dt} = f(x).$$

In other words, a differential equation is autonomous if it does not depend on t. Since an autonomous differential equation is time independent, its direction field is particularly easy to draw. The slope only depends on x and is the same for all values of t.

Example 1.24 (A Logistic Model with Harvesting)**.** Let us consider a trout pond that has a carrying capacity of 200 fish. Suppose that the trout population can be modeled according to the logistic equation

$$\frac{dP}{dt} = P\left(1 - \frac{P}{200}\right),$$

where t is the time in years. If we allow the fish to be harvested at a constant rate of 32 per year, our equation becomes

$$\frac{dP}{dt} = P\left(1 - \frac{P}{200}\right) - 32.$$

The direction field for this equation is given in Figure 1.25.

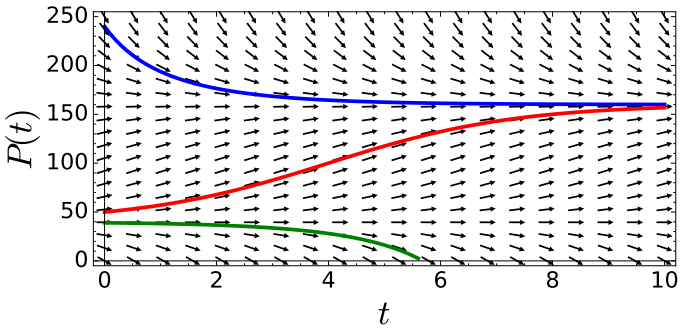

Figure 1.25: Logistic growth with harvesting

One of the basic questions that we can ask of our model is whether or not we have a sustainable population in our trout pond given this harvest rate. If so, under what conditions for sustainablility?

Since

$$\frac{dP}{dt} = P\left(1 - \frac{P}{200}\right) - 32 = -\frac{1}{200}(P - 40)(P - 160)$$

is an autonomous differential equation, the direction field does not depend on t. Consequently, we need only keep track of what happens on the vertical axis. We can do this with a ***phase line***. Instead of drawing the entire direction field, we can draw a single line containing the same information.

Notice that $dP/dt = 0$ when $P = 40$ or $P = 160$. Thus, the two constant solutions $P(t) = 40$ and $P(t) = 160$ are the same for all values of the independent variable t. We say that such a solution is an ***equilibrium solution***. Equilibrium solutions graph as horizontal lines. We can identify equilibrium solutions by setting the derivative of the function equal to zero.

On our phase line we will represent these solutions as equilibrium points (Figure 1.26). For values of P between 40 and 160, we know that $dP/dt > 0$. Thus, any solution curve must be increasing. We denote this property on the phase line by drawing an upward pointing arrow. On the other hand, we know that $dP/dt < 0$ when $P < 40$ or $P > 160$. In this case any solution curve will be decreasing, and we will indicate this by a downward pointing arrow.

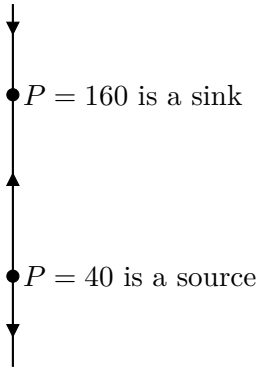

Figure 1.26: Phase line diagram

An equilibrium solution is ***stable*** if a small change in the initial conditions gives a solution which tends toward the equilibrium as the independent variable tends towards positive infinity. An equilibrium solution is ***unstable*** if a small change in the initial conditions gives a solution which veers away from the equilibrium as the independent variable tends towards positive infinity.

Let $y' = f(y)$ and suppose that $y = y_0$ is an equilibrium solution. We say this solution is a ***sink*** if for any solution $y(t)$ with initial condition sufficiently close to y_0, we have

$$\lim_{t\to\infty} y(t) = y_0.$$

We say that an equilibrium point is a ***source*** if all solutions that start sufficiently close to y_0 tend toward y_0 as $t \to -\infty$. An equilibrium solution that is neither a sink or a source is called a ***node*** (Figure 1.26).

Consider the differential equation

$$\frac{dy}{dt} = y^4 - 4y^2 = y^2(y+2)(y-2). \tag{1.8}$$

The graph of $f(y) = y^4 - 4y^2$ is given in Figure 1.27. If $y = -2$, we have a sink. If $y = 2$, we have a source. Finally, if $y = 0$, we have a node.

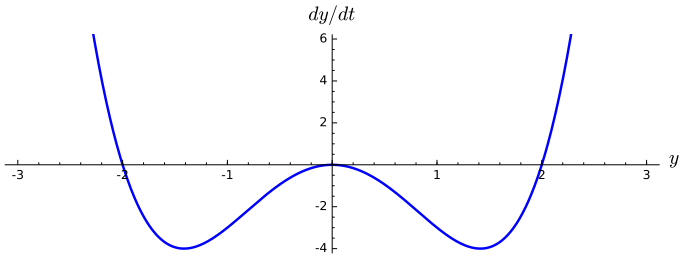

Figure 1.27: Sinks, sources, and nodes

It is easy to generate a phase line diagram for equation (1.8) from the graph of $f(y) = y^2(y+2)(y-2)$ (Figure 1.27). If the graph is above the y-axis, then y is increasing. If the graph is below the y-axis, then y is decreasing. Therefore, the phase line is easy to sketch (Figure 1.28).

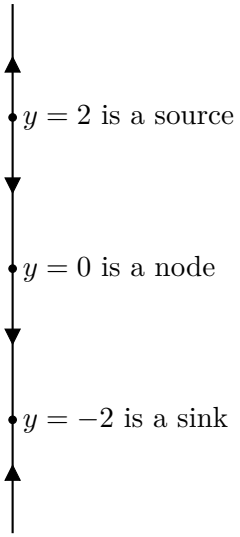

Figure 1.28: Phase line diagram for $y' = y^2(y+2)(y-2)$

One of the reasons why autonomous equations are so important is Taylor's theorem, which tells us that any function $f(x)$ can be approximated near a point x_0 by an nth degree polynomial,

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

near x_0. For example, if

$$\frac{dx}{dt} = f(x) = \cos(x^2 + \pi).$$

with $x(0) = x_0$ then we may approximate this initial value problem near x_0 with

$$\begin{aligned} \frac{dx}{dt} &= f(x_0) + f'(x_0)(x - x_0) = \cos(x_0^2 + \pi) + 2x_0 \sin(x_0^2 + \pi)(x - x_0) \\ x(0) &= x_0. \end{aligned}$$

Of course, this strategy might not work very well if $f(x_0) = \cos(x_0^2 + \pi) = 0$ or $f'(x_0) = 2x_0 \sin(x_0^2 + \pi) = 0$.

1.3.6 Important Lessons

- Direction fields and phase lines are a useful way of analyzing a differential equation from a geometric point of view, especially since not all differential equations can be solved analytically.

- An autonomous equation is a differential equation of the form $y' = f(y)$. We can use a phase line to analyze autonomous differential equations.

- Equilibrium solutions to a differential equation $y' = f(y)$ are those solutions given by $f(y) = 0$ for all y. In this case, any solution must be constant. We can classify equilibrium solutions according to whether they are stable or unstable. In particular, an equilibrium solution is either a sink, source, or node.

1.3.7 Exercises

1. Find all of the equilibrium solutions for each of the following differential equations. Use Sage to graph the direction field of each equation and then superimpose a plot of the equilibrium solution(s) on the direction field. Classify each equilibrium solution as a sink, a source, or a node.

(a) $y' = 2y(1-y)$

(b) $\dfrac{dP}{dt} = 0.1P(P-100)$

(c) $\dfrac{dx}{dt} = (2-x)\cos x$

(d) $\dfrac{dy}{dx} = (2-y)\cos x$

2. Consider the differential equation $y' = f(y)$, where the graph of $f(y)$ is given below. Identify any equilibrium solutions and classify each equilibrium solution as a sink, source, or node.

Hint.

3. What happens if we increase the harvest rate to 100 in Example 1.24.? What should be our strategy to maintain a viable population in the trout pond?

Hint.

1.3.8 Sage—Plotting Direction Fields and Solutions

Plotting direction fields

If we wish to plot a direction field for a differential equation, we can use the command `plot_slope_field`. Let us plot the direction field for the equation $y' = y^2/2 - t$.

```
t, y = var('t, y')
f(t, y) = y^2/2 - t
v = plot_slope_field(f, (t,-1,5), (y,-5,10), headaxislength=3,
    headlength=3, axes_labels=['$t$','$y(t)$'])
v
```

There are a few extra commands to specify the size of the arrows in the plot and to label the axes. Try changing or omitting these commands and see what happens.

```
t, y = var('t, y')
f(t, y) = y^2/2 - t
v = plot_slope_field(f, (t,-1,5), (y,-5,10), headaxislength=3,
    headlength=3, axes_labels=['$t$','$y(t)$'])
v
```

For more examples and options, see http://doc.sagemath.org/html/en/reference/plotting/sage/plot/plot_field.html

Plotting solutions

Now let us find a numerical solution to the equation using the command `desolve_rk4`. This is a fourth-order Runge-Kutta method, and returns a numerical solution (a table of values). Here, we must supply the dependent variable and initial conditions.

```
t, y = var('t, y')
f(t, y) = y^2/2 - t
p = desolve_rk4(f, y, ics=[-1,0], ivar=t, output='plot',
    end_points=[-1,5], thickness=2)
p
```

Of course, we can combine the two plots.

```
t, y = var('t, y')
f(t, y) = y^2/2 - t
p = plot_slope_field(f, (t,-1,5), (y,-5,10), headaxislength=3,
    headlength=3, axes_labels=['$t$','$y(t)$'], fontsize=12)
p += desolve_rk4(f, y, ics=[-1,0], ivar=t, output='plot',
    end_points=[-1,5], thickness=2)
p
```

There are many other commands and packages to solve ordinary differential equations using Sage. For more information, see http://www.sagemath.org/doc/reference/calculus/sage/calculus/desolvers.html. Below is an empty Sage cell, where you can practice.

```

```

Sage Exercises

1. Suppose that the population of a trout pond can be accurately modeled by the logistic equation

$$\frac{dp}{dt} = 0.4p\left(1 - \frac{p}{500}\right).$$

At time $t = 30$, a disease is introduced into the population that kills 10% of the population per year. To see how the disease affects the fish population, we will change our original model to the folllowing:

$$\frac{dp}{dt} = \begin{cases} 0.4p\left(1 - \dfrac{p}{500}\right) & \text{for } 0 \le t < 30; \\ 0.4p\left(1 - \dfrac{p}{500}\right) - 0.1p & \text{for } t > 30. \end{cases}$$

(a) Plot the direction field for this equation using Sage.

(b) Plot the graphs of two or three representative solutions to this equation on the direction field.

(c) Find formulas for the solutions of this equation for initial conditions $p(0) = 30$.

(d) Give a qualitative description of how the disease affects the population.

1.4 Analyzing Equations Numerically

Just as numerical algorithms are useful when finding the roots of polynomials, numerical methods will prove very useful in our study of ordinary differential equations. Consider the polynomial $f(x) = x^2 - 2$. We do not need a numerical algorithm to see that the roots roots of this polynomial are $x = \sqrt{2}$ and $x = -\sqrt{2}$. However, a numerical method such as the Newton-Raphson Algorithm[6] could be useful if we need a decimal approximation for $\sqrt{2}$. Similarly, it may be easier to generate a numerical solution for a differential equations if our goal is simply to plot a solution, and there will be differential equations for which we cannot find an analytic solution of course.

[6] See any calculus text for a description of the Newton-Raphson Algorithm.

1.4.1 Euler's Method

Suppose that we wish to solve the initial value problem

$$y' = f(t, y) = y + t, \tag{1.9}$$
$$y(0) = 1. \tag{1.10}$$

The equation $y' = y + t$ is not separable, which currently is the only analytic technique at our disposal; however, we can try to find an approximation for the solution. We will try to find an approximate solution on the interval $[0, 1]$.

Even with the use of a computer, we cannot approximate the solution at every single point on an interval. For the initial value problem

$$y' = f(t, y)$$
$$y(t_0) = y_0,$$

we might be able to find approximations at $a = t_0, t_1, t_2, \ldots, t_N = b$ in $[a, b]$ at best. If we choose $t_1, t_2, \ldots, t_N$ to be equally spaced on $[a, b]$, we can write

$$t_k = t_0 + kh,$$

where $h = 1/N$ and $k = 1, 2, \ldots, N$. We say that h is the ***step size*** for our approximation.

Given an approximation Y_k for the solution $y_k = y(t_k)$, the question is how to find an approximate solution Y_{k+1} at t_{k+1}. To generate the second approximation, we will construct a tangent line to the solution at $y(t_0) = y_0$. If we use slope of the solution curve at t_0, then

$$y'(t_0) = f(y_0, t_0).$$

Making use of the fact that

$$\frac{y(t_0 + h) - y(t_0)}{h} \approx y'(t_0, y(t_0)) = y'(t_0, y_0)$$

or equivalently

$$y(t_0 + h) = y(t_1) \approx y(t_0) + hy'(t_0, y_0),$$

the estimate for our solution at $t_1 = t_0 + h$ is

$$Y_1 = Y_0 + hf(t_0, Y_0).$$

Similarly, the approximation at $t_2 = t_0 + 2h$ will be

$$Y_2 = Y_1 + hf(t_1, Y_1).$$

Our general algorithm is

$$Y_{k+1} = Y_k + hf(t_k, Y_k).$$

The idea is to compute tangent lines at each step and use this information to get our next approximation.

The algorithm that we have described is known as ***Euler's method***. Let us estimate a solution to (1.9)–(1.10) on the interval $[0, 1]$ with step size $h = 0.1$. Since $y(0) = 1$, We can make our first approximation exact,

$$Y_0 = y(0) = 1.$$

To generate the second approximation, we will construct a tangent line to the solution at $y(0) = 1$. If we use slope of the solution curve at $t_0 = 0$,

$$y'(0) = f(y(0), 0) = y(0) + 0 = 1 + 0 = 1,$$

and make use of the fact that

$$\frac{y(h) - y(0)}{h} \approx y'(0, y(0)) \qquad \text{or} \qquad y(h) \approx y(0) + y'(0, y(0)),$$

the estimate for our solution at $t = 0.1$ is

$$\begin{aligned} Y_1 &= Y_0 + hf(t_0, Y_0) \\ &= Y_0 + h[Y_0 + t_0] \\ &= 1 + (0.1)[1 + 0] \\ &= 1.1000. \end{aligned}$$

Similarly, the approximation at $t = 0.2$ will be

$$\begin{aligned} Y_2 &= Y_1 + hf(t_1, Y_1) \\ &= Y_1 + h[Y_1 + t_1] \\ &= 1.1000 + (0.1)[1.1000 + 0.1] \\ &= 1.2200. \end{aligned}$$

Our general algorithm is

$$Y_{k+1} = Y_k + hf(t_k, Y_k) = Y_k + h[Y_k + t_k] = (1.1)Y_k + (0.01)k.$$

This equation is, in fact, solvable analytically with solution $y(t) = 2e^t - t - 1$. We can compare our approximation to the exact solution in Table 1.29. We can also see graphs of the approximate and exact solutions in Figure 1.31. Notice that the error grows as we get further away from our initial value. In fact, the graph of the approximation for $h = 0.001$ is obscured by the graph of the exact solution. In addition, a smaller step size gives us a more accurate approximation (Table 1.30).

k	t_k	Y_k	y_k	$\lvert y_k - Y_k \rvert$	Percent Error
0	0.0	1.0000	1.0000	0.0000	0.00%
1	0.1	1.1000	1.1103	0.0103	0.93%
2	0.2	1.2200	1.2428	0.0228	1.84%
3	0.3	1.3620	1.3997	0.0377	2.69%
4	0.4	1.5282	1.5836	0.0554	3.50%
5	0.5	1.7210	1.7974	0.0764	4.25%
6	0.6	1.9431	2.0442	0.1011	4.95%
10	1.0	3.1875	3.4366	0.2491	7.25%

Table 1.29: Euler's approximation for $y' = y + t$

t_k	$h = 0.1$	$h = 0.02$	$h = 0.001$	Exact Solution
0.1	1.1000	1.1082	1.1102	1.1103
0.2	1.2200	1.2380	1.2426	1.2428
0.3	1.3620	1.3917	1.3993	1.3997
0.4	1.5282	1.5719	1.5831	1.5836
0.5	1.7210	1.7812	1.7966	1.7974
0.6	1.9431	2.0227	2.0431	2.0442
0.7	2.1974	2.2998	2.3261	2.3275
0.8	2.4872	2.6161	2.6493	2.6511
0.9	2.8159	2.9757	3.0170	3.1092
1.0	3.1875	3.3832	3.4238	3.4366

Table 1.30: Step sizes for Euler's approximation

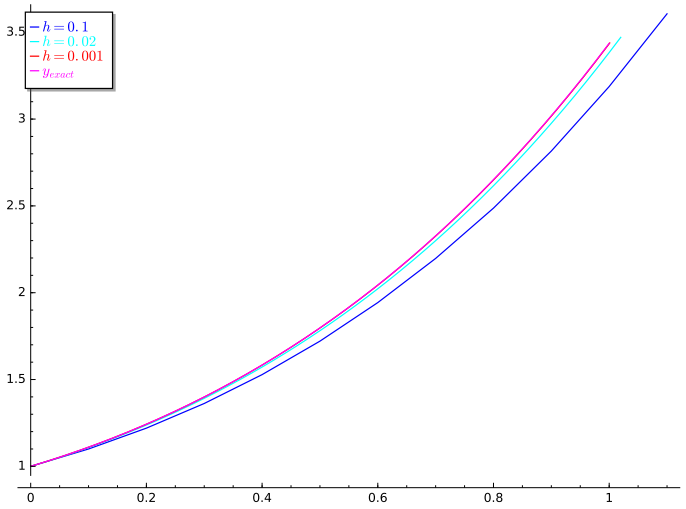

Figure 1.31: Euler's approximation for $y' = y + t$

1.4.2 Finding an Error Bound

To fully understand Euler's method, we will need to recall Taylor's theorem from calculus.

Theorem 1.32. *If $x > x_0$, then*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(\xi)}{n!}(x - x_0)^n,$$

where $\xi \in (x_0, x)$.

Given the initial value problem

$$y' = f(t, y),$$
$$y_0 = y(t_0),$$

choose $t_1, t_2, \ldots, t_N$ to be equally spaced on $[t_0, a]$, we can write

$$t_k = t_0 + kh,$$

where $h = (a - t_0)/N$ and $k = 1, 2, \ldots, N$. Taylor's Theorem tells us that

$$y(t_{k+1}) = y(t_k + h) = y(t_k) + y'(t_k)h + \frac{y''(t_k)}{2!}h^2 + \cdots .$$

If we know the values of y and its derivatives at t_k, then we can determine the value of y at t_{k+1}.

The simplest approximation can be obtained by taking the first two terms of the Taylor series. That is, we will use a linear approximation,

$$y_{k+1} = y(t_{k+1}) \approx y(t_k) + y'(t_k)h = y(t_k) + f(t_k, y_k)h.$$

This gives us Euler's method,

$$\begin{aligned} Y_0 &= y(t_0) \\ Y_1 &= Y_0 + hf(t_0, Y_0) \\ Y_2 &= Y_1 + hf(t_1, Y_1) \\ &\vdots \\ Y_{k+1} &= Y_k + hf(t_k, Y_k). \end{aligned}$$

The terms that we are omitting, all contain powers of h of at least degree two. If h is small, then h^n for $n = 2, 3, \ldots$ will be very small and these terms will not matter much.

We can actually estimate the error incurred by Euler's method if we make use of Taylor's Theorem.

Theorem 1.33. *Let y be the unique solution to the initial value problem*

$$\begin{aligned} y' &= f(t, y), \\ y(a) &= \alpha, \end{aligned}$$

where $t \in [a, b]$. Suppose that f is continuous and there exists a constant $L > 0$ such that

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|,$$

whenever (t, y_1) and (t, y_2) are in $D = [a, b] \times \mathbb{R}$. Also assume that there exists an M such that

$$|y''(t)| \leq M$$

for all $t \in [a, b]$. If $Y_0, \ldots, Y_N$ are the approximations generated by Euler's method for some positive integer N, then

$$|y(t_i) - Y_i| \leq \frac{hM}{2L}[e^{L(t_i - a)} - 1].$$

The condition that there exists a constant $L > 0$ such that

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|,$$

whenever (t, y_1) and (t, y_2) are in $D = [a, b] \times \mathbb{R}$ is called a ***Lipschitz condition***. Many of the functions that we will consider satisfy such a condition. If the condition is satisfied, we can usually say a great deal about the function.

k	t_k	Y_k	$y_k = y(t_k)$	$\lvert y_k - Y_k \rvert$	Estimated Error
0	0.0	1.0000	1.0000	0.0000	0.0000
1	0.1	1.1000	1.1103	0.0103	0.0286
2	0.2	1.2200	1.2428	0.0228	0.0602
3	0.3	1.3620	1.3997	0.0377	0.0951
4	0.4	1.5282	1.5836	0.0554	0.1337
5	0.5	1.7210	1.7974	0.0764	0.1763
6	0.6	1.9431	2.0442	0.1011	0.2235
7	0.7	2.1974	2.3275	0.1301	0.2756
8	0.8	2.4872	2.6511	0.1639	0.3331
9	0.9	2.8159	3.1092	0.2033	0.3968
10	1.0	3.1875	3.4366	0.2491	0.4671

Table 1.34: Error bound and actual error

We can now compare the estimated error from our theorem to the actual error of our example. We first need to determine M and L. Since

$$|f(t, y_1) - f(t, y_2)| = |(y_1 + t) - (y_2 + t)| = |y_1 - y_2|,$$

we can take L to be one. Since $y'' = 2e^t$, we can bound y'' on the interval $[0, 1]$ by $M = 2e$. Thus, we can bound the error by

$$|y(t_i) - Y_i| \leq \frac{hM}{2L}[e^{L(t_i - a)} - 1] = 0.1e(e^{t_i} - 1)$$

for $h = 0.1$. Our results are in Table 1.34.

1.4.3 Improving Euler's Method

If we wish to improve upon Euler's method, we could add more terms of Taylor series. For example, we can obtain a more accurate approximation by using a quadratic Taylor polynomial,

$$y(t_1) \approx y_0 + f(t_0, y_0)h + \frac{y''(t_0)}{2}h^2.$$

However, we need to know $y''(t_0)$ in order to use this approximation. Using the chain rule, we can differentiate both sides of $y' = f(t, y)$ to obtain

$$y'' = \frac{\partial f}{\partial t}\frac{dt}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_t + f f_y.$$

Thus, our approximation becomes

$$y(t_1) \approx y_0 + f(t_0, y_0)h + \frac{1}{2}\left(f_t(t_0, y_0) + f(t_0, y_0)f_y(t_0, y_0)\right)h^2.$$

The problem is that some preliminary analytic work must be done. That is, before we can write a program to compute our solution, we must find $\partial f/\partial t$ and $\partial f/\partial y$, although this is less of a problem with the availability of computer algebra systems such as Sage.

Around 1900, two German mathematicians, Carle Runge and Martin Kutta, independently invented several numerical algorithms to solve differential equations. These methods, known as ***Runge-Kutta methods***, estimate the higher-order terms of the Taylor series to find an approximation that does not depend on computing derivatives of $f(t, y)$.

If we consider the initial value problem

$$\begin{aligned} y' &= f(t,y), \\ y(t_0) &= y_0, \end{aligned}$$

then

$$y(t_1) = y(t_0) + \int_{t_0}^{t_1} f(s, y(s))\, ds$$

or

$$y_1 - y_0 = y(t_1) - y(t_0) = \int_{t_0}^{t_1} f(s, y(s))\, ds \tag{1.11}$$

by the Fundamental Theorem of Calculus. In Euler's method, we approximate the right-hand side of (1.11) by

$$y_1 - y_0 = f(t_0, y_0)h.$$

In terms of the definite integral, this is simply a left-hand sum. In the ***improved Euler's method*** or the ***second-order Runge-Kutta method*** we will estimate the right-hand side of (1.11) using the trapezoid rule from calculus,

$$\begin{aligned} y(t_1) - y(t_0) &= \int_{t_0}^{t_1} f(s, y(s))\, ds \\ &\approx \left(f(t_0, y_0) + f(t_1, y_1)\right) \frac{h}{2}. \end{aligned}$$

Thus, our algorithm becomes

$$y_1 = y_0 + \left(f(t_0, y_0) + f(t_1, y_1)\right) \frac{h}{2}. \tag{1.12}$$

However, we have a problem since y_1 appears in the right-hand side of our approximation. To get around this difficulty, we will replace y_1 in the right-hand side of (1.12) with the Euler approximation for y_1. Thus,

$$y_1 = y_0 + \left(f(t_0, y_0) + f(t_1, y_0 + f(t_0, y_0)h)\right) \frac{h}{2}.$$

To understand that the second-order Runge-Kutta method is actually an improvement over the traditional Euler's method, we will need to use the Taylor approximation for a function of two variables. Let us assume that $f(x, y)$ is defined on some rectangle and that all of the derivatives of f are continuously differentiable. Then

$$\begin{aligned} f(x+h, y+k) = f(x,y) &+ h\frac{\partial}{\partial x} f(x,y) + k\frac{\partial}{\partial y} f(x,y) \\ &+ \frac{1}{2!}\left(h^2 \frac{\partial^2}{\partial^2 x} f(x,y) + hk\frac{\partial^2}{\partial x \partial y} f(x,y) + k^2 \frac{\partial^2}{\partial^2 y} f(x,y)\right) \\ &+ \frac{1}{3!}\left(h^3 \frac{\partial^3}{\partial^3 x} f(x,y) + 3h^2 k \frac{\partial^3}{\partial^2 x \partial y} f(x,y)\right. \\ &\left. + hk^2 \frac{\partial^3}{\partial x \partial^2 y} f(x,y) + k^3 \frac{\partial^3}{\partial^3 y} f(x,y)\right) + \cdots . \end{aligned}$$

As in the case of the single variable Taylor series, we can write a Taylor polynomial if the Taylor series is truncated,

$$f(x+h, y+k) = \sum_{n=0}^{N} \frac{1}{n!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x,y)$$

$$+\frac{1}{(N+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{N+1} f(\overline{x},\overline{y}),$$

where the second term is the remainder term and $(\overline{x},\overline{y})$ lies on the line segment joining (x,y) and $(x+h,y+k)$.

In the Improved Euler's Method, we adopt a formula

$$y(t+h) = y(t) + w_1F_1 + w_2F_2,$$

where

$$\begin{aligned} F_1 &= hf(t,y)\\ F_2 &= hf(t+\alpha h, y+\beta F_1). \end{aligned}$$

That is,

$$y(t+h) = y(t) + w_1hf(t,y) + w_2hf(t+\alpha h, y+\beta hf(t,y)). \tag{1.13}$$

The idea is to choose the constants w_1, w_2, α, and β as accurately as possible in order to duplicate as many terms as possible in the Taylor series

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2!}y''(t) + \frac{h^3}{3!}y''(t) + \cdots. \tag{1.14}$$

We can make equations ((1.13)) and (1.14) agree if we choose $w_1 = 1$ and $w_2 = 0$. Since $y' = f$, we obtain Euler's method.

If we are more careful about choosing our parameters, we can obtain agreement up through the h^2 term. If we use the two variable Taylor series to expand $f(t+\alpha h, y+\beta hf)$, we have

$$f(t+\alpha h, y+\beta hf) = f + \alpha hf_t + \beta hff_y + \mathcal{O}(h^2),$$

where $\mathcal{O}(h^2)$ means that of the subsequent terms have a factor of h^n with $n \geq 2$. Using this expression, we obtain a new form for (1.13),

$$y(t+h) = y(t) + (w_1+w_2)hf + \alpha w_2h^2f_t + \beta w_2h^2ff_y + \mathcal{O}(h^3). \tag{1.15}$$

Since $y'' = f_t + f_yf$ by the chain rule, we can rewrite (1.14) as

$$y(t+h) = y(t) + hf + \frac{h^2}{2!}(f_t + ff_y) + \mathcal{O}(h^3). \tag{1.16}$$

We can make equations (1.15) and ((1.16)) agree up through the quadratic terms if we require that

$$\begin{aligned} w_1 + w_2 &= 1,\\ \alpha w_2 &= \frac{1}{2},\\ \beta w_2 &= \frac{1}{2}. \end{aligned}$$

If we choose $\alpha = \beta = 1$ and $w_1 = w_2 = 1/2$, these equations are satisfied and we obtain the improved Euler's method

$$y(t+h) = y(t) + \frac{h}{2}f(t,h) + \frac{h}{2}f(t+h, y+hf(t,y)).$$

The improved Euler's method or the second-order Runge-Kutta method is a more sophisticated algorithm that is less prone to error due to the step size h. Euler's method is based on truncating the Taylor series after the linear term. Since

$$y(t+h) = y(t) + hy'(t) + \mathcal{O}(h^2),$$

we know that the error depends on h. On the other hand, the error for the improved Euler's method depends on h^2, since

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2!}y''(t) + \mathcal{O}(h^3).$$

If we use Simpson's rule to estimate the integral in

$$y(t_1) - y(t_0) = \int_{t_0}^{t_1} f(s, y(s))\, ds$$

we can improve our accuracy up to h^4. The idea is exactly the same, but the algebra become much more tedious. This method is known as the Runge-Kutta method of order 4 and is given by

$$y(t+h) = y(t) + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4),$$

where

$$\begin{aligned} F_1 &= hf(t, y) \\ F_2 &= hf\left(t + \frac{1}{2}h, y + \frac{1}{2}F_1\right) \\ F_3 &= hf\left(t + \frac{1}{2}h, y + \frac{1}{2}F_2\right) \\ F_4 &= hf(t+h, y+F_3). \end{aligned}$$

1.4.4 Important Lessons

- We can use Euler's method to find an approximate solution to the initial value problem

$$\begin{aligned} y' &= f(t, y), \\ y(a) &= \alpha \end{aligned}$$

on an interval $[a, b]$. If we wish to find approximations at N equally spaced points $t_1, \ldots, t_N$, where $h = (b-a)/N$ and $t_i = a + ih$, our approximations should be

$$\begin{aligned} Y_0 &= \alpha, \\ Y_1 &= Y_0 + hf(\alpha, Y_0), \\ Y_2 &= Y_1 + hf(t_1, Y_1,) \\ &\vdots \\ Y_{k+1} &= Y_k + hf(t_k, Y_k), \\ Y_N &= Y_{N-1} + hf(t_{N-1}, Y_{N-1}). \end{aligned}$$

- Taylor's Theorem theorem is a very useful tool for studying differential equations. If $x > x_0$, then

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(\xi)}{n!}(x - x_0)^n,$$

where $\xi \in (x_0, x)$.

- Error analysis rate of convergence is very important for any numerical algorithm. Our approximation is more accurate for smaller values of h. Under reasonable conditions we can also bound the error by

$$|y(t_i) - Y_i| \leq \frac{hM}{2L}[e^{L(t_i - a)} - 1],$$

where y is the unique solution to the initial value problem

$$\begin{aligned} y' &= f(t, y), \\ y(a) &= \alpha. \end{aligned}$$

- The condition that there exists a constant $L > 0$ such that

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|,$$

whenever (t, y_1) and (t, y_2) are in $D = [a, b] \times \mathbb{R}$ is called a ***Lipschitz condition***.

- Using Taylor series, we can develop better numerical algorithms to compute solutions of differential equations. The Runge-Kutta methods are an important class of these algorithms.

- The improved Euler's method is given by

$$y(t + h) = y(t) + \frac{h}{2}f(t, h) + \frac{h}{2}f(t + h, y + hf(t, y))$$

with the error bound depending on h^2.

- The Runge-Kutta method of order 4 is given by

$$y(t + h) = y(t) + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4),$$

where

$$\begin{aligned} F_1 &= hf(t, y) \\ F_2 &= hf\left(t + \frac{1}{2}h, y + \frac{1}{2}F_1\right) \\ F_3 &= hf\left(t + \frac{1}{2}h, y + \frac{1}{2}F_2\right) \\ F_4 &= hf(t + h, y + F_3) \end{aligned}$$

with the error bound depending on h^4.

1.4.5 Exercises

1. Consider the differential equation

$$\frac{dy}{dt} = 3y - 1$$

with initial value $y(0) = 2$.

(a) Find the exact solution of the initial value problem.

(b) Use Euler's method with step size $h = 0.5$ to approximate the solution to the initial value problem on the interval $[0, 2]$ Your solution should include a table of approximate values of the dependent variable as well as the exact values of the dependent variable. Make sure that your approximations are accurate to *four* decimal places.

(c) Sketch the graph of the approximate and exact solutions.

(d) Use the error bound theorem (Theorem 1.33) to estimate the error at each approximation. Your solution should include a table of approximate values of the dependent variable the exact values of the dependent variable, the error estimates, and the actual error. Make sure that your approximations are accurate to *four* decimal places.

Hint.

2. In this series of exercises, we will prove the error bound theorem for Euler's method (Theorem 1.33).

(a) Use Taylor's Theorem to show that for all $x \geq -1$ and any positive m,

$$0 \leq (1 + x)^m \leq e^{mx}.$$

(b) Use part (1) and geometric series to prove the following statement: If s and t are positive real numbers, and $\{a_i\}_{i=0}^k$ is a sequence satisfying

$$a_0 \geq -\frac{t}{s}$$
$$a_{i+1} \leq (1 + s)a_i + t, \text{ for } i = 0, 1, 2, \ldots, k,$$

then

$$a_{i+1} \leq e^{(i+1)s}\left(\frac{t}{s} + a_0\right) - \frac{t}{s}.$$

(c) When $i = 0$, $y(t_0) = Y_0 = \alpha$ and the theorem is true. Use Euler's method and Taylor's Theorem to show that

$$|y_{i+1} - Y_{i+1}| \leq |y_i - Y_i| + h|f(t_i, y_i) - f(t_i, Y_i)| + \frac{h^2}{2}|y''(\xi_i)|$$

where $\xi_i \in (t_i, t_{i+1})$ and $y_k = y(t_k)$.

(d) Show that

$$|y_{i+1} - Y_{i+1}| \leq |y_i - Y_i|(1 + hL) + \frac{Mh^2}{2}.$$

(e) Let $a_j = |y_j - Y_j|$ for $j = 0, 1, \ldots, N$, $s = hL$, and $t = Mh^2/2$ and apply part (2) to show

$$|y_{i+1} - Y_{i+1}| \leq e^{(1+i)hL}\left(|y_0 - Y_0| + \frac{Mh^2}{2hL}\right) - \frac{Mh^2}{2hL}.$$

and derive that

$$|y_{i+1} - Y_{i+1}| \leq \frac{Mh}{2L}\left(e^{(t_{i+1}-a)L} - 1\right)$$

for each $i = 0, 1, \ldots, N - 1$.

Hint. Hints for part (2):

- For fixed i show that

$$\begin{aligned} a_{i+1} &\leq (1+s)a_i + t \\ &\leq (1+s)[(1+s)a_{i-1} + t] + t \\ &\leq (1+s)\{(1+s)[(1+s)a_{i-2} + t] + t\} + t \\ &\vdots \\ &\leq (1+s)^{i+1}a_0 + [1 + (1+s) + (1+s)^2 + \cdots + (1+s)^i]t. \end{aligned}$$

- Now use a geometric sum to show that

$$a_{i+1} \leq (1+s)^{i+1}a_0 + \frac{t}{s}[(1+s)^{i+1} - 1] = (1+s)^{i+1}\left(\frac{t}{s} + a_0\right) - \frac{t}{s}.$$

- Apply part (1) to derive

$$a_{i+1} \leq e^{(i+1)s}\left(\frac{t}{s} + a_0\right) - \frac{t}{s}.$$

1.4.6 Project—Numerical Routines for solving ODEs

Not all differential equations can be solved using algebra and calculus even if we are very clever. If we encounter an equation that we cannot solve or use *Sage* to solve, we must resort to numerical algorithms like Euler's method or one of the Runge-Kutta methods, which are best implemented using a computer. Fortunately, *Sage* has some very good numerical solvers. *Sage* will need to know the following to solve a differential equation:

- An abstract function
- A differential equation, *including an initial condition*
- A *Sage* command to solve the equation.

Suppose we wish to solve the initial value problem $dy/dx = x + y$, $y(0) = 1$. We can use *Sage* to find an algebraic solution.

```
y = function('y')(x)
de = diff(y,x) == x + y
h = desolve(de, y, ics = [0,1])
show(h)
plot(h, x, 0, 5)
```

We can also use Euler's method to find a solution for our initial value problem.

```
x,y = PolynomialRing(RR, 2, "xy").gens() #declare x, y as generators
    of a polynomial ring
eulers_method(x + y, 0,1, 0.5, 5)
```

The syntax of `eulers_method` for the inital value problem

$$\begin{aligned} y &= f(x,y) \\ f(x_0) &= y_0 \end{aligned}$$

with step size h on the interval $[x_0, x_1]$ is `eulers_method(f, x0, y0, h, x1)` Notice that we obtained a table of values. However, we can use the `line` command from *Sage* to plot the values (x, y).

```
x,y = PolynomialRing(RR, 2, "xy").gens()
pts = eulers_method(x + y, 0, 1, 1/2, 4.5, algorithm="none")
p = list_plot(pts, color="red")
p += line(pts)
p
```

As we pointed out, `eulers_method` is not very sophisticated. We have to use a very small step size to get good accuracy, and the method can generate errors if we are not careful. Fortunately, *Sage* has much better algorithms for solving initial value problems. One such algorithm is `desolve_rk4`, which implements the fourth order Runge-Kutta method.

```
x,y = var('x␣y')
desolve_rk4(x + y, y, ics=[0, 1], end_points = 5, step = 0.5)
```

Again, we just get a list of points. However, `desolve_rk4` has some nice built-in graphing utilities.

```
x,y = var('x␣y')
p = desolve_rk4(x + y, y, ics = [0,1], ivar = x, output =
    'slope_field', end_points = [0, 5], thickness = 2, color = 'red')
p
```

Write the *Sage* commands to compare the graphs obtained using `eulers_method` and `desolve` with the exact solution.

```
```

Not only is `desolve_rk4` more accurate, it is much easier to use. For more information, see http://www.sagemath.org/doc/reference/calculus/sage/calculus/desolvers.html.

1.5 First-Order Linear Equations

First-order differential equations occur naturally in many modeling problems. These are equations of the form

$$\frac{dy}{dt} + p(t)y = g(t). \tag{1.17}$$

This equation will not be separable if $p(t)$ in not a constant. We shall have to find a new approach to solving such an equation. We could, of course, use a numerical algorithm to solve (1.17); however, we shall find that we can always find an algebraic solution to a first-order linear differential equation. Moreover, the fact that can obtain such a solution analytically will prove very useful when we investigate more complicated equations and systems of equations.

1.5.1 Mine Tailings

In any mining operation, tailings are what is left after everything of value has been extracted. For example, in a hard rock mining operation, ore might be pulverized and then processed using chemicals to remove a certain mineral. Soft rock mining operations such as coal mining or extracting oil from tar sands might use solvents or water to extract any commodity of value. The material that is left over after the minerals, coal, or oil is extracted can often present huge environmental challenges. There are different ways of processing mine tailings, but one way is to store them in a pond, especially if water is used in the mining operation.

This method allows any particles that are suspended in the water to settle to the bottom of the pond. The water can then be treated and recycled.

Suppose that we have a gold mining operation and we are storing our tailings in a pond that has an initial volume of 20,000 cubic meters. When we begin our operation, the tailings pond is filled with clean water. The pond has a stream flowing into it, and water is also pumped out of the pond. Chemicals are used in processing gold ore. These chemicals such as sodium cyanide are often highly poisonous and dangerous to the environment, the water must be treated before it is released into the watershed. Suppose that 1500 cubic meters per day flow into the pond from stream and 1500 cubic meters are pumped from the pond each day to be processed and recycled. Thus, the water level of the pond remains constant.

At time $t = 0$, the water from stream becomes contaminated with chemicals from the mining operation, say at a rate of 5 kilograms of chemicals per 1000 cubic meters. We will assume that water in our tailings pond is well mixed so that the concentration of chemicals through out the pond is fairly uniform. In addition, any particulate matter pumped into the pond from the stream settles to the bottom of the pond at a rate of 50 cubic meters per day. Thus, the volume of our tailings pond is reduced by 50 cubic meters each day, and our tailings pond will become full after 400 days of operation. We shall assume that the particulate matter and the chemicals are included in the 1000 cubic meters that flow into the pond from the stream each day.

We wish to find a differential equation that will model the amount of chemicals in the tailings pond at any particular time. Let $x(t)$ be the amount of chemicals in the pond at time t. Then dx/dt is the difference between the rate at which the chemicals are entering the pond and the rate at which the chemicals leave the pond.

$$\frac{dx}{dt} = \text{rate in} - \text{rate out}.$$

Since water flows into the pond from the stream at a rate of 1000 cubic meters per day, the rate as which the chemicals enter the pond is 5 kilograms per day. On the other hand, the rate at which the chemicals leave the pond will depend on the amount of chemicals in the pond at time t. The volume of the pond is decreasing due to sediment, and at time t it is $V(t) = 20000 - 50t$. Thus, the concentration of chemicals in the pond at time t is $x/(20000 - 50t)$, and the rate at which the chemicals are flowing out of the pond to be recycled is

$$1000\left(\frac{x}{20000 - 50t}\right) = \frac{20x}{400 - t}.$$

Hence, the differential equation that models the amount of chemical in the tailings pond at time t is

$$\frac{dx}{dt} = 5 - \frac{20x}{400 - t}. \tag{1.18}$$

Of course, we will have to cease mining operations once the pond is full, since there will only be water in the pond if $V(t) = 20000 - 50t \geq 0$; that is, when $0 \leq t < 400$.

Notice that equation (1.18) is not autonomous. In fact, it is not even separable. We will have to use a different approach to find a solution. First, we will rewrite the equation in the form

$$\frac{dx}{dt} + \frac{20}{400 - t}x = 5.$$

If we multiply both sides of this equation by $(400 - t)^{-20}$, we obtain

$$(400 - t)^{-20}\frac{dx}{dt} + 20(400 - t)^{-21}x = 5(400 - t)^{-20}. \tag{1.19}$$

We now make the crucial observation that the product rule applies to the lefthand side of our equation,

$$(400-t)^{-20}\frac{dx}{dt} + 20(400-t)^{-21}x = \frac{d}{dt}\left((400-t)^{-20}x\right).$$

Thus, equation (1.19) becomes

$$\frac{d}{dt}\left((400-t)^{-20}x\right) = 5(400-t)^{-20}.$$

Integrating both sides of this equation, we have

$$(400-t)^{-20}x = 5\int (400-t)^{-20}\,dt = \frac{5(400-t)^{-19}}{19} + C,$$

where C is an arbitrary constant. Solving for x, we obtain

$$x(t) = \frac{5}{19}(400-t) + C(400-t)^{20}.$$

Since $x(0) = 0$, we can quickly determine that $C = -(5/19)400^{-19}$ and that the solution to our initial value problem is

$$x(t) = \frac{5}{19}(400-t)\left[1 - \left(\frac{400-t}{400}\right)^{19}\right].$$

The graph of the solution to our differential equation (Figure 1.35) fits the situation. Initially, there are no chemicals in the pond, but $x(t)$ quickly increases. However, the amount of chemicals decreases as the pond begins to fill with sediment. Eventually, there are no chemicals at $t = 400$.

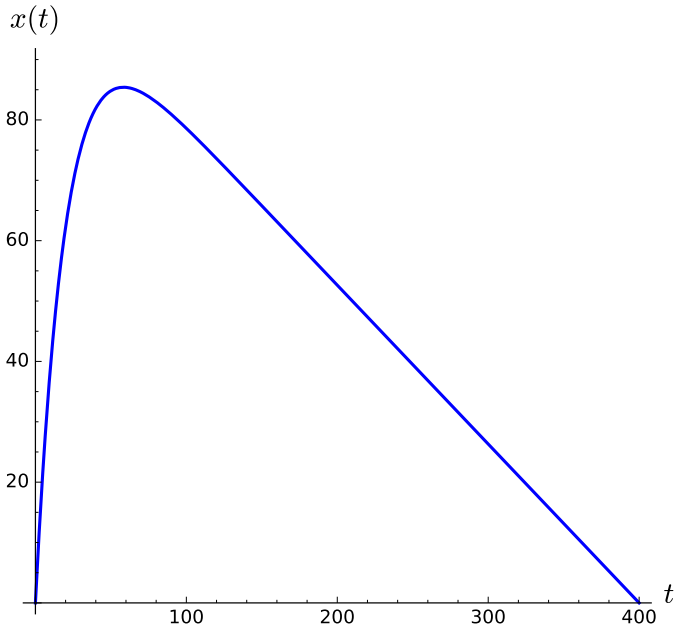

Figure 1.35: Chemical waste in a tailings pond

1.5.2 First-Order Linear Equations

The differential equation

$$\frac{dx}{dt} + \frac{20}{400 - t}x = 5$$

is an example of a first-order linear differential equation. More specifically, a ***first-order linear differential equation*** is an equation that can be written in the form

$$\frac{dx}{dt} + p(t)x = g(t).$$

Let us first show how to solve first-order linear equations when the coefficient functions are constant. If

$$\begin{aligned} \frac{dx}{dt} + px &= q, \\ x(t_0) &= x_0, \end{aligned}$$

then we first multiply both sides of the equation by e^{pt} to obtain

$$e^{pt}\frac{dx}{dt} + e^{pt}px = qe^{pt}.$$

The left-hand side of the equation is ***exact***. That is,

$$e^{pt}\frac{dx}{dt} + e^{pt}px = \frac{d}{dt}\left(x(t)e^{pt}\right).$$

If we integrate both sides of

$$\frac{d}{dt}\left(x(t)e^{pt}\right) = qe^{pt},$$

then

$$x(t)e^{pt} = \frac{q}{p}e^{pt} + C.$$

If we apply the initial condition, we can determine C,

$$C = \left(x_0 - \frac{q}{p}\right)e^{pt_0}.$$

Thus, the solution that we seek is

$$x(t) = \frac{q}{p} + \left(x_0 - \frac{q}{p}\right)e^{p(t_0 - t)}.$$

Example 1.36. Suppose we wish to solve the initial value problem

$$\frac{dx}{dt} - 2x = 3, \tag{1.20}$$

$$x(0) = 1. \tag{1.21}$$

Multiplying both sides of equation (1.20) by e^{-2t}, we obtain

$$\frac{d}{dt}\left(e^{-2t}x\right) = e^{-2t}\frac{dx}{dt} - 2e^{-2t}x = 3e^{-2t}.$$

Integrating both sides of this last equation, gives us the following

$$e^{-2t}x = 3\int e^{-2t}\,dt + C = -\frac{3}{2}e^{-2t} + C.$$

Applying the initial condition $x(0) = 1$, we can conclude that $C = 5/2$, and

$$x(t) = \frac{5}{2}e^{2t} - \frac{3}{2}.$$

Example 1.37. Now let us now solve a first-order linear differential equation where the coefficients are not constant. Suppose that

$$y' + 2xy = e^{-x^2},$$

where $y(0) = 1$. We will multiply both sides of the equation by e^{x^2} and use the product rule to obtain

$$\frac{d}{dx}\left(e^{x^2}y\right) = e^{x^2}y' + 2xe^{x^2}y = e^{x^2}e^{-x^2} = 1.$$

Integrating both sides, we get

$$e^{x^2}y = \int dx + C.$$

Thus, the general solution is

$$y = (x + C)e^{-x^2}.$$

Using the initial condition to solve for C, we find that $C = 1$ and

$$y = (x + 1)e^{-x^2}.$$

We can use Sage to check our solution.

```
x = var('x')
y = function('y')(x)
de = diff(y, x) + 2*x*y == exp(-x^2)
solution = desolve(de, y, ics = [0, 1])
solution
```

```
(x + 1)*e^(-x^2)
```

Surprisingly, the strategy in Example 1.37 will always work. Suppose that

$$y' + p(x)y = g(x). \tag{1.22}$$

If we choose $P(x)$ such that $P'(x) = p(x)$ and multiply both sides of the equation by $e^{P(x)}$, then

$$\left(e^{P(x)}y\right)' = e^{P(x)}(y'(x) + p(x)y(x) = e^{P(x)}g(x).$$

Integrating both sides,

$$e^{P(x)}y = \int e^{P(x)}g(x)\,dx + C$$

or

$$y = \frac{1}{e^{P(x)}}\left(\int e^{P(x)}g(x)\,dx + C\right).$$

The Fundamental Theorem of Calculus tells us that $P(x) = \int p(x)\,dx$. We say that

$$\mu(x) = e^{\int p(x)\,dx} = \exp\left(\int p(x)\,dx\right)$$

is an ***integrating factor*** for the differential equation (1.22).

Example 1.38. Consider the initial value problem

$$\frac{dy}{dx} - \frac{2}{x}y = 2x^2$$
$$y(-2) = 4.$$

Our integrating factor is

$$\mu(x) = \exp\left(-\int \frac{2}{x}\,dx\right) = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}.$$

Multiplying both of our differential equation by $\mu(x) = x^{-2}$, we obtain

$$x^{-2}\left(\frac{dy}{dx} - \frac{2}{x}y\right) = 2$$

or

$$\frac{d}{dx}\left(x^{-2}y\right) = x^{-2}\frac{dy}{dx} - 2x^{-3}y = 2.$$

We can now integrate this equation to get

$$x^{-2}y = 2x + C.$$

The initial condition $y(-2) = 4$ allows us to find $C = 5$. Therefore, the solution to our initial value problem is

$$y = 2x^3 + 5x^2.$$

1.5.3 Mixing Models

Many applications involve the mixing of two or more substances together. We can model how petroleum products are mixed together in a refinery, how various ingredients are mixed together in a brewery, or how greenhouse gases mix move across various layers of the earth's atmosphere.

Suppose that a 100-gallon tank initially contains 50 gallons of salt water containing five pounds of salt. A brine mixture containing one pound of salt per gallon flows into the top of the tank at a rate of 5 gallons per minute. A well mixed solution leaves the tank at rate of 4 gallons per minute. We wish to know how much salt is in the tank, when the tank is full.

To construct our model, we will let t be the time (measured in minutes) and set up a differential equation that will measure how fast the amount of salt at time t, $x(t)$, is changing. We have the initial condition $x(0) = 5$, and

$$\begin{aligned}\frac{dx}{dt} &= \text{amount of salt flowing in} - \text{amount of salt flowing out}\\ &= \underbrace{5}_{\text{in flow}} - \underbrace{4\frac{x}{V(t)}}_{\text{out flow}},\end{aligned}$$

where $V(t)$ is the volume at time t. The expression $x/V(t)$ is the amount of salt in one gallon at time t. We have $V(t) = 50 + t$, since the tank starts with 20 gallons and four gallons are pumped into the tank per minute while three gallons leaves the tank during the same time interval. Thus, our differential equation becomes

$$\frac{dx}{dt} = 5 - \frac{4}{50+t}x.$$

Our equation is linear since we can rewrite it as

$$\frac{dx}{dt} + \frac{4}{50+t}x = 5. \tag{1.23}$$

An integrating factor for this differential equation is

$$\mu(t) = \exp\left(\int \frac{4}{50+t}\,dt\right) = e^{4\ln(50+t)} = (50+t)^4.$$

Therefore, if we multiply both sides of equation (1.23) by $\mu(t)$, we get

$$(50+t)^4\frac{dx}{dt} + 4(t+50)^3 x = 5(50+t)^4.$$

We can now apply the product rule to obtain

$$\frac{d}{dt}[(50+t)^4 x] = 5(50+t)^4.$$

Integrating both sides and simplifying gives us

$$x = t + 50 + \frac{C}{(t+50)^4}.$$

Our initial condition, $x(0) = 5$ tells us that $C = 31,250,000$ and

$$x(t) = t + 50 - \frac{31250000}{(t+50)^4}.$$

Thus, when the tank is full, $t = 50$ and the amount of salt in the tank is $x(50) = 97.188$ pounds. We can use *Sage* to easily check the solution of our initial value problem.

```
t = var('t')
x = function('x')(t)
de = diff(x, t) + 4*x/(50 + t) == 5
solution(t) = desolve(de, x, ics=[0, 5])
solution(50).n(digits = 5)
```

```
97.188
```

1.5.4 Finance Models

There are a number of problems in finance that can be modeled using differential equations. Let $P(t)$ be the balance of an account at time t and suppose that the account pays interest at a rate of r percent per year compounded continuously. Suppose that we also allow withdrawals of W dollars per year. The net increase in the balance between times t and $t + \Delta t$ can now be described as

$$P(t+\Delta t) - P(t) \approx rP\Delta t - W\Delta t$$

Thus,

$$P'(t) = \lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t} = rP - W.$$

We can solve the equation

$$P' = rP - W$$

by multiplying both sides of the equation by the integrating factor

$$\mu(t) = \exp\left(-\int r\,dt\right) = e^{-rt}.$$

to obtain

$$\frac{d}{dt}[e^{-rt}P] = -We^{-rt}.$$

Integrating both sides of this equation, we have

$$e^{-rt}P = \frac{W}{r}e^{-rt} + C$$

or

$$P = \frac{W}{r} + Ce^{rt}.$$

If we know the initial balance in the account, say $P(0) = P_0$, we can determine C. That is,

$$P_0 = \frac{W}{r} + C$$

or

$$C = P_0 - \frac{W}{r}.$$

Thus, the solution to our initial value problem is

$$P = \frac{W}{r} + \left(P_0 - \frac{W}{r}\right)e^{rt}.$$

1.5.5 Existence and Uniqueness of Solutions

Several questions about the existence and uniqueness of solutions to first order linear differential equations now arise.

- Does an initial-value problem always have a solution?
- Is the solution unique?
- Is the solution globally defined or does it only hold for a small interval?

We can use the following theorem to answer these questions.

Theorem 1.39. *If*

$$y' + p(t)y = g(t)$$

is a differential equation such that $y(t_0) = y_0$, and $p(t)$ and $g(t)$ are continuous on the open interval $I = (\alpha, \beta)$, then there exists a unique function $y = \phi(t)$ satisfying the differential equation and the initial condition on I.

Proof. If

$$\mu(t) = \exp\left(\int p(t)\,dt\right),$$

then

$$\frac{d}{dt}(\mu(t)y) = \mu(t)\left(\frac{dy}{dt} + p(t)y\right) = \mu(t)g(t).$$

Integrating both sides of this equation and solving for y, we have

$$y = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)\,dt + C\right).$$

Since $y(t_0) = y_0$, the constant C is uniquely determined. Notice that we have used continuity to guarantee that the integrals exist. □

1.5.6 Important Lessons

- Any first order linear differential equation

$$y' + p(x)y = g(x)$$

can be solved by multiplying each side of the equation by an integrating factor

$$\mu(s) = e^{\int p(x)\,dx}.$$

In this case, we get

$$\left(e^{P(x)}y\right)' = e^{P(x)}(y'(x) + p(x)y(x) = e^{P(x)}g(x).$$

Integrating both sides,

$$e^{P(x)}y = \int e^{P(x)}g(x)\,dx$$

or

$$y = \frac{1}{e^{P(x)}}\int e^{P(x)}g(x)\,dx.$$

- If

$$y' + p(t)y = g(t)$$

is a differential equation such that $y(t_0) = y_0$, and $p(t)$ and $g(t)$ are continuous on the open interval $I = (\alpha, \beta)$, then there exists a unique function $y = \phi(t)$ satisfying the differential equation and the initial condition on I.

1.5.7 Exercises

1. Solve the differential equation

$$\frac{dy}{dx} + \left(\frac{2}{x}\right)y = \frac{\sin x}{x^2}.$$

2. Solve the initial value problem

$$\begin{aligned} y' &= -\frac{2y}{x+1} + 2x \\ y(0) &= 2. \end{aligned}$$

Hint. $y = \dfrac{3x^4 + 8x^3 + 6x^2 + 12}{6(x+1)^2}$

3. Solve the initial value problem

$$\begin{aligned} y' &= -\frac{2y}{1+x} + e^x \\ y(0) &= 1. \end{aligned}$$

Hint. $y = (e^x + 1)/(x+1)^2$

4. Solve the initial value problem

$$y' = -\frac{y}{x+2} + \frac{\cos x}{x+2}$$
$$y(0) = 1.$$

Hint. $y = \sin x/(x+2)$

5. Solve the initial value problem

$$y' = y\tan x + \frac{e^x}{\cos x}$$
$$y(0) = 1.$$

Hint. $y = e^x \sec x$

6. A 600-gallon tank initially contains 200 liters of water containing 10 kilograms of salt. Supposed that water containing 0.1 kilograms of salt flows into the top of the tank at a rate of 10 liters per minute. The water in the tank is kept well mixed, and 5 liters per minute are removed from the bottom of the tank. How much salt is in the tank when the tank is full?

Hint. If $x(t)$ is the amount of salt in the tank at time t, we know that $x(0) = 10$. The volume of the tank is $V = 600 + 5t$. We can model the amount of salt in the tank at time t with a differential equation,

$$\begin{aligned}\frac{dx}{dt} &= \text{rate in} - \text{rate out}\\ &= 10(0.1) - 5\frac{x}{V}\\ &= 1 - 5\frac{x}{600+5t}\\ &= 1 - \frac{x}{120+t}.\end{aligned}$$

The resulting equation

$$\frac{dx}{dt} + \frac{1}{120+t}x = 1$$

is a first order linear differential equation. An integrating factor for this equation is given by

$$\mu(t) = \exp\left(\int \frac{1}{120+t}\,dt\right) = 120 + t.$$

Multiplying both sides of the differential equation by $\mu(t)$, we have

$$\frac{d}{dt}[(120+t)x] = (120+t)\frac{dx}{dt} + x = (120+t)\left(\frac{dx}{dt} + \frac{1}{120+t}x\right) = 120 + t.$$

Integrating both sides of this equation, we obtain

$$(120+t)x = 120t + \frac{t^2}{2} + C.$$

Using the intial condition $x(0) = 10$, we can determine that $C = 1200$ or

$$x(t) = \frac{t^2 + 240t + 2400}{2t + 240}.$$

The tank is full at time $t = 400/5 = 80$, and the tank contains $x(80) = 70$ kilograms of salt when the tank is full.

7. A manager in a communications company contributes \$2400 per year into her retirement fund by making many small deposits throughout the year. The fund grows at a rate of 3.5% per year compounded continuously. After 35 years, she retires and begins and begins withdrawing from the retirement fund at a rate of \$3500 per month. If she does not make any deposits after she retires, how long will her retirement fund last? [*Hint*: Divide the problem into two smaller problems—one that deals with the situation before retirement and one that deals with the problem after retirement.]

Hint.

8. Lake Baikal, located in southern Siberia, is only the seventh largest lake in the world in terms of surface area; however, it is the world's deepest lake. The lake has a depth of 1,642 meters, and the bottom lies 1,186.5 meters below sea level. Lake Baikal has a volume of 23,600 cubic kilometers and contains 20% of the world's unfrozen fresh water. In contrast, Lake Superior, the largest of the Great Lakes, has a volume of only 12,100 cubic kilometers. Although 544 rivers flow into Lake Baikal, there is only a single outlet, the Angara River. The outflow of the lake is pretty constant at 60.4 cubic kilometers per year.
Pollution is an increasing concern in Lake Baikal. One of the major polluters has been Baykalsk Pulp and Paper Mill. The mill was constructed in 1966 on the shoreline of Lake Baikal and regularly discharged waste into the lake. The plant was closed in November 2008 due to unprofitability, but production resumed in January 2010. The mill underwent a final bankruptcy in September 2013, but the longterm fate of the mill is still undecided.[7] Suppose that we wish to understand how the pollution level changes in Lake Baikal over a period of years. Hypothetically, let us assume that the Baykalsk Pulp and Paper Mill has been responsible for most of the pollution in Lake Baikal for the last several decades. Suppose that at $t = 0$ years the mill ceases operation and there are no longer any pollutants discharged into the lake from the mill although there are still other sources of pollution. Let us assume that the lake is currently 6 times more polluted than these other sources of contaminants. We wish to know how long it will take for the pollution level to reduce to half of its current level of lake. Lake Baikal's waters are well-mixed and well-oxygenated in spite of its great depth, so we can model this situation as a simple mixing problem.

(a) The volume of water in Lake Baikal is $V = 23,600$ km^3, and

$$r_{\text{in}} = r_{\text{out}} = r = 60 \text{ km}^3/\text{year}$$

be the rates of inflow from the numerous rivers that feed the lake and outflow to the Angara River. Assume that $C = C_{\text{in}}$ is the pollutant concentration flowing into Lake Baikal and C_{out} is the concentration of the outflow into the Angara River (measured in metric tons per cubic km). If $x(t)$ is the amount of solute at time t in Lake Baikal, then

$$x_0 = x(0) = 6CV$$

is the initial amount of solute in the lake. Estimate Δx during the time interval $[t, t + \Delta t]$, where $\Delta t > 0$ is small.

(b) From your estimate of Δx in part (1), write an initial value problem that describes the amount of pollutants in the lake at time t.

(c) The equation that you found in part (2) is a first-order linear equation. Solve this equation

[7] Levy, Clifford J. (November 8, 2010). "Last Gasp for Factory Bequeathed by Soviets." The New York Times. Retrieved March 14, 2014 from http://www.nytimes.com/2010/11/09/world/europe/09baikal.html.

(d) Using part (3), predict how many years it will take to reduce the pollution in Lake Baikal to half of its current level.

Hint.

9. *Variation of Parameters.* Consider the following method of solving the general linear equation of the first order,

$$y' + p(t)y = g(t). \tag{1.24}$$

(a) If $g(t)$ is identically zero, show that the solution is

$$y = A \exp\left[-\int p(t)\,dt\right],$$

where A is a constant.

(b) If $g(t)$ is not identically zero, assume that the solution is of the form

$$y = A(t) \exp\left[-\int p(t)\,dt\right], \tag{1.25}$$

where A is now a function of t. By substituting for y in the given differential equation (A.1), show that $A(t)$ must satisfy the condition

$$A'(t) = g(t) \exp\left[\int p(t)\,dt\right]. \tag{1.26}$$

(c) Find $A(t)$ from Equation (A.3). Then substitute for $A(t)$ in equation (A.2) and determine y. Verify that the solution obtain in this manner agrees with the solution given in the proof of Theorem 1.39. That is, show that this solution is equivalent to the solution

$$y = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)\,dt + C\right),$$

where

$$\mu(t) = \exp\left(\int p(t)\,dt\right).$$

This technique is know as *variation of parameters*, which we will revisit when we study second order linear differential equations.

Hint.

10. *Bernoulli's equation* is an equation of the form

$$y' = a(t)y + f(t)y^n,$$

where $n \neq 0$ or 1. Bernoulli's equation is nonlinear and cannot be solved by the techniques that we have used to solve first order linear equations.[8]

(a) Using the substitution $z = y^{1-n}$, show that we can transform Bernoulli's equation into the linear equation

$$z' = (1-n)a(t)z + (1-n)f(t).$$

(b) Solve the equation $xy' + y = x^4y^3$.

Hint.

[8] Polking p. 63

11. The first-order nonlinear differential equation

$$y' = p(t) + q(t)y + r(t)y^2 \tag{1.27}$$

is known as the *Ricatti equation* and has some useful applications in control theory. If one solution $y_1(t)$ of the Ricatti equation is known, then a more general solution containing an arbitrary constant can be found by substituting $y = y_1(t) + 1/v(t)$ into equation (A.4) to find a first-order linear equation in v and t, which we can solve to find a general solution to the Ricatti equation.

(a) Show that this first-order linear equation is $v' + [q(t) + 2r(t)y_1(t)]v = -r(t)$.

(b) Find the solution to the Ricatti equation

$$y' = -\frac{1}{t^2} - \frac{1}{t}y + y^2$$

given the particular solution $y_1 = 1/t$.

(c) Find the solution to the Ricatti equation

$$y' = \cos t - y\tan t + y^2 \sec t$$

given the particular solution $y_1 = \sin t$.

(d) Find the solution to the Ricatti equation

$$y' = 2 - 3y + y^2$$

given the particular solution $y_1 \equiv 2$.

Hint.

(a) If $y = y_1 + 1/v$, then $y' = y_1' - v'/v^2$. Substituting into our original equation, we obtain

$$y' = y_1' - \frac{v'}{v^2} = p + qy_1 + ry_1^2 - \frac{v'}{v^2}.$$

On the other hand,

$$\begin{aligned} y' &= p + q\left(y_1 + \frac{1}{v}\right) + r\left(y_1 + \frac{1}{v}\right)^2 \\ &= p + qy_1 + \frac{q}{v} + ry_1^2 + \frac{2ry_1}{v} + \frac{r}{v^2} \\ &= y_1' + \frac{q}{v} + \frac{2ry_1}{v} + \frac{r}{v^2}. \end{aligned}$$

Therefore,

$$-\frac{v'}{v^2} = \frac{q}{v} + \frac{2ry_1}{v} + \frac{r}{v^2},$$

which is just the first-order linear equation

$$v' + [q(t) + 2r(t)y_1(t)]v = -r(t).$$

(b)

$$y = t + \frac{1}{C - t}$$

(c)

$$y(t) = \frac{1}{C\cos t - \sin t} + \sin t$$

(d)

$$y(t) = 2 + \frac{1}{Ce^t - 1}$$

12. Suppose that we have a population that not only grows logistically but also requires a minimum threshold population to survive. For example, the case of the North Pacific right whale, a species now very much on the endangered list. If the population drops too low, whales might not be able to find suitable mates and the species will eventually go extinct. In other words, the population will die out if it drops below a certain threshold. We can model this with the following equation,

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)(P - aN), \tag{1.28}$$

where P is the population of the whales at time t and N is the carrying capacity. The constants k and a are positive with $a < 1$.

(a) Find the equilibrium solutions of the this equation.

(b) Since equation (A.5) is autonomous, we can find a solution using separation of variables. Find this solution.

(c) Equation (A.5) is also a Ricatti equation (A.4). Since we know an equilibrium solution from part (1), we can use the method of the previous problem to find a general solution to (A.5). Find the general solution using the fact that we have a Ricatti equation and show that your solution agrees with the solution that you found in part (2)

Hint.

1.5.8 Project—Modeling

1.6 Existence and Uniqueness of Solutions

If $x' = f(t, x)$ and $x(t_0) = x_0$ is a linear differential equation, we have already shown that a solution exists and is unique. We will now take up the question of existence and uniqueness of solutions for all first-order differential equations. The existence and uniqueness of solutions will prove to be very important—even when we consider applications of differential equations.

1.6.1 The Existence and Uniqueness Theorem

The following theorem tells us that solutions to first-order differential equations exist and are unique under certain reasonable conditions.

Theorem 1.40 (Existence and Uniqueness Theorem)**.** *Let $x' = f(t, x)$ have the initial condition $x(t_0) = x_0$. If f and $\partial f/\partial x$ are continuous functions on the rectangle*

$$R = \{(t, x) : 0 \leq |t - t_0| \leq a, 0 \leq |x - x_0| \leq b\},$$

there exists a unique solution $u = u(t)$ for $x' = f(t, x)$ and $x(t_0) = x_0$ on some interval $|t - t_0| < h$ contained in the interval $|t - t_0| < a$.

Let us examine some consequences of the existence and uniqueness of solutions.

Example 1.41. Consider the initial value problem $y' = y^{1/3}$ with $y(0) = 0$ and $t \geq 0$. Separating the variables,

$$y^{-1/3}y' = dt.$$

Thus,

$$\frac{3}{2}y^{2/3} = t + C$$

or

$$y = \left(\frac{2}{3}(t + C)\right)^{3/2}.$$

If $C = 0$, the initial condition is satisfied and

$$y = \left(\frac{2}{3}t\right)^{3/2}$$

is a solution for $t \geq 0$. However, we can find two additional solutions for $t \geq 0$:

$$y = -\left(\frac{2}{3}t\right)^{3/2},$$
$$y \equiv 0.$$

This is especially troubling if we are looking for equilibrium solutions. Although $y' = y^{1/3}$ is an autonomous differential equation, there is no equilibrium solution at $y = 0$. The problem is that

$$\frac{\partial}{\partial y}y^{1/3} = \frac{1}{3}y^{-2/3}$$

is not defined at $y = 0$.

Example 1.42. Suppose that $y' = y^2$ with $y(0) = 1$. Since $f(t, y) = y^2$ and $\partial f/\partial y = 2y$ are continuous everywhere, a unique solution exists near $t = 0$. Separating the variables,

$$\frac{1}{y^2}\, dy = dt,$$

we see that

$$y = -\frac{1}{t + C}$$

or

$$y = \frac{1}{1 - t}.$$

Therefore, a solution also exists on $(-\infty, 1)$ if $y(0) = -1$. In the case that $y(0) = -1$, the solution is

$$y = -\frac{1}{t + 1},$$

and a solution exists on $(-1, \infty)$. Solutions are only guaranteed to exist on an open interval containing the initial value and are very dependent on the initial condition.

Remark 1.43 (Solutions Curves Cannot Cross)**.** The Existence and Uniqueness Theorem tells us that the integral curves of any differential equation satisfying the appropriate hypothesis, cannot cross. If the curves did cross, we could take the point of intersection as the initial value for the differential equation. In this case, we would no longer guaranteed unique solutions to a differential equation.

1.6.2 Picard Iteration

It was Emile Picard (1856–1941) who developed the method of successive approximations to show the existence of solutions of ordinary differential equations. He proved that it is possible to construct a sequence of functions that converges to a solution of the differential equation. One of the first steps towards understanding ***Picard iteration*** is to realize that an initial value problem can be recast in terms of an integral equation.

Theorem 1.44. *The function $u = u(t)$ is a solution to the initial value problem*

$$\begin{aligned} x' &= f(t,x) \\ x(t_0) &= x_0, \end{aligned}$$

if and only if u is a solution to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

PROOF. Suppose that $u = u(t)$ is a solution to

$$\begin{aligned} x' &= f(t,x) \\ x(t_0) &= x_0, \end{aligned}$$

on some interval I containing t_0. Since u is continuous on I and f is continuous on R, the function $F(t) = f(t, u(t))$ is also continuous on I. Integrating both sides of $u'(t) = f(t, u(t))$ and applying the Fundamental Theorem of Calculus, we obtain

$$u(t) - u(t_0) = \int_{t_0}^{t} u'(s)\, ds = \int_{t_0}^{t} f(s, u(s))\, ds$$

Since $u(t_0) = x_0$, the function u is a solution of the integral equation.

Conversely, assume that

$$u(t) = x_0 + \int_{t_0}^{t} f(s, u(s))\, ds.$$

If we differentiate both sides of this equation, we obtain $u'(t) = f(t, u(t))$. Since

$$u(t_0) = x_0 + \int_{t_0}^{t_0} f(s, u(s))\, ds = x_0,$$

the initial condition is fulfilled. □

To show the existence of a solution to the initial value problem

$$\begin{aligned} x' &= f(t,x) \\ x(t_0) &= x_0, \end{aligned}$$

we will construct a sequence of functions, $\{u_n(t)\}$, that will converge to a function $u(t)$ that is a solution to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

We define the first function of the sequence using the initial condition,

$$u_0(t) = x_0.$$

We derive the next function in our sequence using the right-hand side of the integral equation,

$$u_1(t) = x_0 + \int_{t_0}^{t} f(s, u_0(s))\, ds.$$

Subsequent terms in the sequence can be defined recursively,

$$u_{n+1} = x_0 + \int_{t_0}^{t} f(s, u_n(s))\, ds.$$

Our goal is to show that $u_n(t) \to u(t)$ as $n \to \infty$. Furthermore, we need to show that u is the continuous, unique solution to our initial value problem. We will leave the proof of Picard's Theorem to a series of exercises, but let us see how this works by developing an example.

Example 1.45. Consider the exponential growth equation,

$$\begin{aligned} \frac{dx}{dt} &= kx \\ x(0) &= 1. \end{aligned}$$

We already know that the solution is $x(t) = e^{kt}$. We define the first few terms of our sequence $\{u_n(t)\}$ as follows:

$$\begin{aligned} u_0(t) &= 1, \\ u_1(t) &= 1 + \int_0^t k u_0(s)\, ds \\ &= 1 + \int_0^t k\, ds \\ &= 1 + kt, \\ u_2(t) &= 1 + \int_0^t k u_1(s)\, ds \\ &= 1 + \int_0^t k(1 + ks)\, ds \\ &= 1 + kt + \frac{(kt)^2}{2}. \end{aligned}$$

The next term in the sequence is

$$u_3(t) = 1 + kt + \frac{(kt)^2}{2} + \frac{(kt)^3}{2 \cdot 3},$$

and the nth term is

$$\begin{aligned} u_n(t) &= 1 + 1 + \int_0^t k u_{n-1}(s)\, ds \\ &= 1 + \int_0^t k \left(1 + ks\frac{(kt)^2}{2!} + \frac{(kt)^3}{3!} + \cdots + \frac{(kt)^{n-1}}{(n-1)!} \right) ds \\ &= 1 + kt + \frac{(kt)^2}{2!} + \frac{(kt)^3}{3!} + \cdots + \frac{(kt)^n}{n!}. \end{aligned}$$

However, this is just the nth partial sum for the power series for $u(t) = e^{kt}$, which is what we expected.

1.6.3 Important Lessons

- Existence and uniqueness of solutions of differential has important implications. Let $x' = f(t,x)$ have the initial condition $x(t_0) = x_0$. If f and $\partial f/\partial x$ are continuous functions on the rectangle

$$R = \{(t,x) : 0 \leq |t - t_0| < a, 0 \leq |x - x_0| < b\},$$

there exists a unique solution $u = u(t)$ for $x' = f(t,x)$ and $x(t_0) = x_0$ on some interval $|t - t_0| < h$ contained in the interval $|t - t_0| < a$. In particular,

  - Solutions are only guaranteed to exist locally.
  - Uniqueness is especially important when it comes to finding equilibrium solutions.
  - Uniqueness of solutions tells us that the integral curves for a differential equation cannot cross.

- The function $u = u(t)$ is a solution to the initial value problem

$$\begin{aligned} x' &= f(t,x) \\ x(t_0) &= x_0, \end{aligned}$$

if and only if u is a solution to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

- Existence and uniqueness of solutions is proved by Picard iteration. This is of particular interest since the proof actually tells us how to construct as sequence of functions that converge to our solution.

1.6.4 Exercises

1. Which of the following initial value problems are guaranteed to have a unique solution by the Existence and Uniqueness Theorem (Theorem 1.40)? In each case, justify your conclusion.

(a) $y' = 4 + y^3$, $y(0) = 1$

(b) $y' = \sqrt{y}$, $y(1) = 0$

(c) $y' = \sqrt{y}$, $y(1) = 1$

(d) $x' = \dfrac{t}{x-2}$, $x(0) = 2$

(e) $x' = \dfrac{t}{x-2}$, $x(2) = 0$

(f) $y' = x \tan y$, $y(0) = 0$

(g) $y' = \dfrac{1}{t}y + 2t$, $y(0) = 1$

Hint.

(a) There exists a unique solution to $y' = 4 + y^3$, $y(0) = 1$, since $f(t,y) = 4 + y^3$ and $\partial f(t,y)/\partial y = 3y^2$ are continuous at the point $(0,1)$.

(b) The Existence and Uniqueness Theorem does not apply to $y' = \sqrt{y}$, $y(1) = 0$, since $f(t, y) = \sqrt{y}$ is not continuous at $(1, 0)$.

(c) There exists a unique solution to $y' = \sqrt{y}$, $y(1) = 1$, since $f(t, y) = \sqrt{y}$ and $\partial f(t, y)/\partial y = 1/(2\sqrt{y})$ are both continuous at the point $(1, 1)$.

(d) The Existence and Uniqueness Theorem does not apply to $x' = t/(x - 2)$, $x(0) = 2$, since $f(t, x) = t/(x - 2)$ is not continuous at $(0, 2)$.

(e) There exists a unique solution to $x' = t/(x - 2)$, $x(2) = 0$, since $f(t, x) = t/(x - 2)$ and $\partial f(t, x)/\partial x = -t/(x - 2)^2$ are both continuous at the point $(2, 0)$.

(f) There exists a unique solution to $y' = x \tan y$, $y(0) = 0$, since $f(x, y) = x \tan y$ and $\partial f(x, y)/\partial y = x \sec^2 y$ are both continuous at the point $(0, 0)$.

(g) The Existence and Uniqueness Theorem does not apply to $y' = y/t + 2t$, $y(0) = 1$, since $f(t, y) = y/t + 2t$ is not continuous at $(0, 1)$.

2. Find an explicit solution to the initial value problem

$$\begin{aligned} y' &= \frac{1}{(t-1)(y+1)} \\ y(0) &= 1. \end{aligned}$$

Use your solution to determine the interval of existence.

Hint.

3. Consider the initial value problem

$$\begin{aligned} y' &= 3y^{2/3} \\ y(0) &= 0. \end{aligned}$$

(a) Show that the constant function, $y(t) \equiv 0$, is a solution to the initial value problem.

(b) Show that

$$y(t) = \begin{cases} 0, & t \le t_0 \\ (t - t_0)^3, & t > t_0 \end{cases}$$

is a solution for the initial value problem, where t_0 is any real number. Hence, there exists an infinite number of solutions to the initial value problem. [*Hint*: Make sure that the derivative of $y(t)$ exists at $t = t_0$.]

(c) Explain why this example does not contradict the Existence and Uniqueness Theorem.

Hint.

4. Let $\phi_n(x) = x^n$ for $0 \le x \le 1$ and show that

$$\lim_{n\to\infty} \phi_n(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1. \end{cases}$$

This is an example of a sequence of continuous functions that does not converge to a continuous function, which helps explain the need for *uniform continuity* in the proof of the Existence and Uniqueness Theorem.

Hint.

5. Consider the initial value problem

$$\begin{aligned} y' &= 2ty + t \\ y(0) &= 1. \end{aligned}$$

(a) Use the fact that $y' = 2ty + t$ is a first-order linear differential equation to find a solution to the initial value problem.

(b) Let $\phi_0(t) = 1$ and use Picard iteration to find $\phi_n(t)$.

(c) Show that the sequence $\{\phi_n(t)\}$ converges to the exact solution that you found in part (a) as $n \to \infty$.

Hint.

6. *The following series of exercises, prove the Existence and Uniqueness Theorem for first-order differential equations.*
Use the Fundamental Theorem of Calculus to show that the function $u = u(t)$ is a solution to the initial value problem

$$\begin{aligned} x' &= f(t, x) \\ x(t_0) &= x_0, \end{aligned}$$

if and only if u is a solution to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

Hint.

7. If $\partial f/\partial x$ is continuous on the rectangle

$$R = \{(t, x) : 0 \le |t - t_0| < a, 0 \le |x - x_0| < b\},$$

prove that there exists a $K > 0$ such that

$$|f(t, x_1) - f(t, x_2)| \le K|x_1 - x_2|$$

for all (t, x_1) and (t, x_2) in R.

Hint.

8. Define the sequence $\{u_n\}$ by

$$\begin{aligned} u_0(t) &= x_0, \\ u_{n+1} &= x_0 + \int_{t_0}^{t} f(s, u_n(s))\, ds, \qquad n = 1, 2, \ldots. \end{aligned}$$

Use the result of the previous exercise to show that

$$|f(t, u_n(t)) - f(t, u_{n-1}(t))| \le K|u_n(t) - u_{n-1}(t)|.$$

Hint.

9. Show that there exists an $M > 0$ such that

$$|u_1(t) - x_0| \le M|t - t_0|.$$

Hint.

10. Show that

$$|u_2(t) - u_1(t)| \le \frac{KM|t - t_0|^2}{2}.$$

Hint.

11. Use mathematical induction to show that

$$|u_n(t) - u_{n-1}(t)| \leq \frac{K^{n-1}M|t-t_0|^n}{n!}.$$

Hint.

12. Since

$$u_n(t) = u_1(t) + [u_2(t) - u_1(t)] + \cdots + [u_n(t) - u_{n-1}(t)],$$

we can view $u_n(t)$ as a partial sum for the series

$$u_0(t) + \sum_{n=1}^{\infty}[u_n(t) - u_{n-1}(t)].$$

If we can show that this series converges absolutely, then our sequence will converge to a function $u(t)$. Show that

$$\sum_{n=1}^{\infty}|u_n(t) - u_{n-1}(t)| \leq \frac{M}{K}\sum_{n=1}^{\infty}\frac{(K|t-t_0|)^n}{n!} \leq \frac{M}{K}\left(e^{K|h|} - 1\right),$$

where h is the maximum distance between (t_0, x_0) and the boundary of the rectangle R. Since $|u_n(t) - u_{n-1}(t)| \to 0$, we know that $u_n(t)$ converges to a continuous function $u(t)$ that solves our equation.[9]

Hint.

13. To show uniqueness, assume that $u(t)$ and $v(t)$ are both solutions to

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

Show that

$$|u(t) - v(t)| \leq K\int_{t_0}^{t}|u(s) - v(s)|\, ds.$$

Hint.

14.

(a) Define[10]

$$\phi(t) = \int_{t_0}^{t}|u(s) - v(s)|\, ds,$$

then $\phi(t_0) = 0$ and $\phi(t) \geq 0$ for $t \geq t_0$. Show that

$$\phi'(t) = |u(t) - v(t)|.$$

(b) Since

$$|u(t) - v(t)| - K\int_{t_0}^{t}|u(s) - v(s)|\, ds \leq 0,$$

we know that

$$\phi'(t) - K\phi(t) \leq 0.$$

[9] We must a theorem from advanced calculus here to ensure uniform continuity. Any sequence of functions that converges uniformly, must converge to a continuous function.

[10] A similar argument will work for $t \leq t_0$.

Use this fact to show that

$$\frac{d}{dt}\left[e^{-Kt}\phi(t)\right] \leq 0.$$

Conclude that

$$\phi(t) = \int_{t_0}^{t} |u(s) - v(s)|\, ds = 0$$

for $t \geq t_0$ or for all $t \geq t_0$ and $u(t) = v(t)$.

Hint.

1.6.5 Project—Picard Iteration

1.7 Bifurcations

Many of the equations that we have examined have a ***parameter***, which means that we actually have a family of differential equations. For example,

$$\frac{dx}{dt} = kx$$

has the growth rate parameter k. The logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)$$

has two parameters, k and N. In this section we will investigate how the solutions of a differential equation vary as we change the value of a parameter.

1.7.1 The Logistic Model with Harvesting Revisited

Recall how we modeled logistic growth in a trout pond in Example 1.24 with the equation

$$\frac{dP}{dt} = P\left(1 - \frac{P}{200}\right).$$

If we allowed fishing in our pond at a rate of 32 fish per year, then the equation became

$$\frac{dP}{dt} = P\left(1 - \frac{P}{200}\right) - 32.$$

There are two equilibrium solutions for this equation, $P_1 = 160$ (a sink) and $P_2 = 40$ (a source). If the population of the pond falls below 40, then the fish will die out unless the pond is restocked or fishing is banned (Figure 1.46).

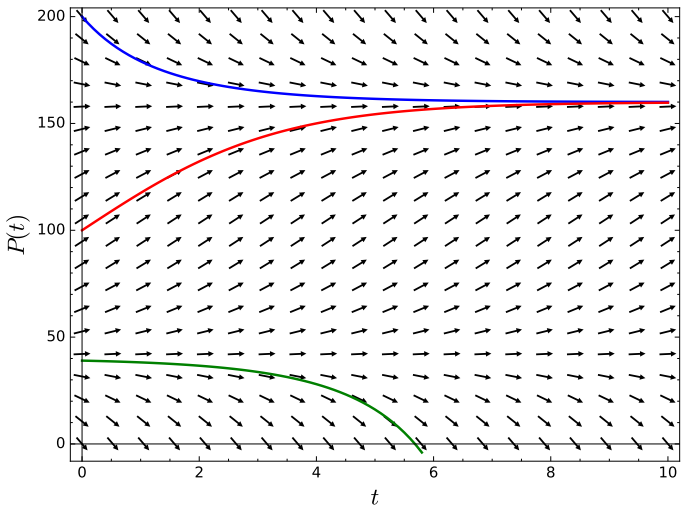

Figure 1.46: Harvesting with $H = 32$

Now let us see what happens when we allow more fishing in our pond, say $H = 100$. Our differential equation now becomes

$$\frac{dP}{dt} = P\left(1 - \frac{P}{200}\right) - 100.$$

To determine the equilibrium solutions, we must solve

$$\frac{dP}{dt} = P\left(1 - \frac{P}{200}\right) - 100 = 0 \tag{1.29}$$

for P. This last equation can be rewritten as $P^2 - 200P - 20000 = 0$. Thus,

$$P = \frac{200 \pm \sqrt{200^2 - 4 \cdot 20000}}{2} = 100 \pm \sqrt{-10000},$$

which means that equation (1.29) has no real solutions and that we have no equilbrium solutions. Furthermore, $dP/dt < 0$ for all values of P. This means that no matter how many fish are in the pond initially, the trout population will eventually die out due to over fishing (Figure 1.47).

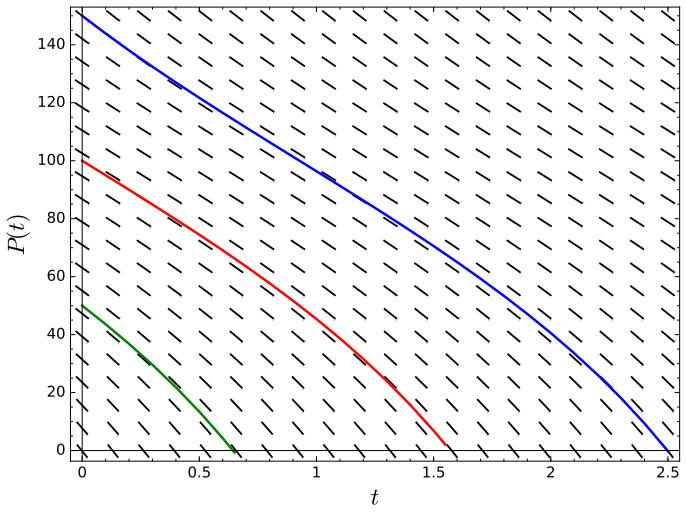

Figure 1.47: Harvesting with $H = 100$

Finally, we will let $H = 50$. In this case, we must solve

$$\frac{dP}{dt} = P\left(1 - \frac{P}{200}\right) - 50 = 0$$

in order to determine any equilibrium solutions. We now obtain a single equilibrium solution at $P = 50$. In fact, $P = 100$ will be a node. For values of $P > 100$ as well as values of $P < 100$, we have $dP/dt < 0$, and the number of fish in the pond will decrease (Figure 1.48).

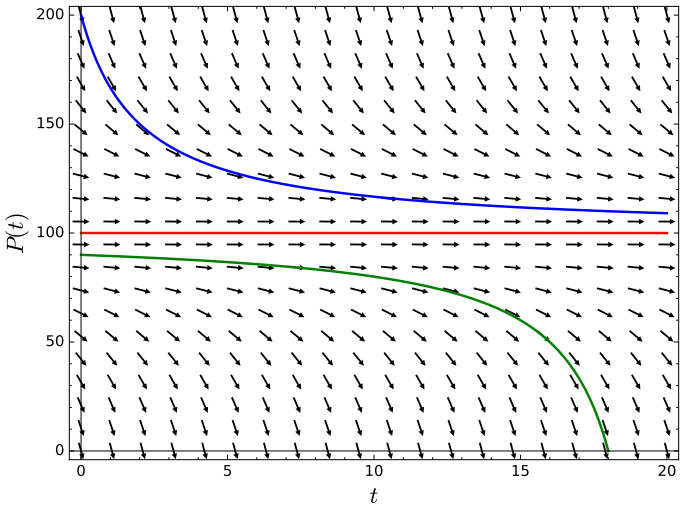

Figure 1.48: Harvesting with $H = 50$

To better understand what is happening, we will generalize our model. Suppose that a population with a limited carrying capacity N is modeled with the logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right).$$

If we allow harvesting at a constant rate H, our model now becomes

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - H.$$

To analyze our model, we will first find the equilibrium solutions. If we will let

$$f_H(P) = kP\left(1 - \frac{P}{N}\right) - H,$$

each equilibrium solution must satisfy $f_H(P) = 0$ or

$$-kP^2 + kNP - HN = 0.$$

Therefore, our equilibrium solutions are given by

$$P = \frac{-kN \pm \sqrt{k^2N^2 - 4kHN}}{-2k} = \frac{N}{2} \pm \sqrt{\frac{N^2}{4} - \frac{HN}{k}}.$$

The explanation of how our model behaves lies in the discriminant,

$$\frac{N^2}{4} - \frac{HN}{k}.$$

If

$$\frac{N^2}{4} - \frac{HN}{k} < 0$$

or, equivalently if $H > kN/4$, there are no equilibrium solutions and

$$\frac{dP}{dt} = f_H(P) < 0$$

for all values of P. In particular, all solutions of $dP/dt = f_H(P)$ tend towards negative infinity as $t \to \infty$. In this case, the population is doomed to extinction no matter how large the initial population is. Since negative populations do not make sense, we say that the population is extinct when $P = 0$.

On the other hand, if $H < kN/4$, we have equilibrium solutions at

$$P_1 = \frac{N}{2} + \sqrt{\frac{N^2}{4} - \frac{HN}{k}}$$

and

$$P_2 = \frac{N}{2} - \sqrt{\frac{N^2}{4} - \frac{HN}{k}}.$$

The first equilibrium solution, P_1 is a sink, while the second, P_2 is a source.

Finally, if $H = kN/4$, then we will have exactly one equilibrium solution at $P = N/2$. Although $dP/dt < 0$ for all $P \neq N/2$, we see that $P \to N/2$ as $t \to \infty$ for all initial values of P greater than $N/2$. For initial values of P less than $N/2$, solutions tend towards $-\infty$ as $t \to \infty$. Thus, the initial population of fish must be at least $kN/4$; otherwise, the fish will go extinct.

In our example, we have a family of differential equations—one for each value of H,

$$\frac{dP}{dt} = P\left(1 - \frac{P}{200}\right) - H. \tag{1.30}$$

A small change in H can have a dramatic effect on how the solutions of the differential equation behave. Changing the value of H from 50 to 50.1 will doom the population of fish to extinction no matter what the initial population is. As we increase the value of H, the number of equilibrium solutions changes from two to one and then to none. This change occurs exactly at $H = 50$. We say that a ***bifurcation*** occurs at $H = 50$ for equation (1.30).

1.7.2 One-Parameter Families

Let us consider the equation

$$\frac{dx}{dt} = x^2 - 4x + \lambda \tag{1.31}$$

as a family of differential equations indexed by the parameter λ. If we let $f_\lambda(x) = x^2 - 4x + \lambda$, then

$$\frac{dx}{dt} = f_\lambda(x)$$

is a called ***one-parameter family*** of differential equations. For each value of λ, we obtain an autonomous differential equation, and for each value of λ, we have a different phase line to examine.

For $\lambda = 0$, the differential equation

$$\frac{dx}{dt} = f_0(x) = x^2 - 4x = x(x-4),$$

there is a sink at $x = 0$ and a source at $x = 4$ (Figure 1.49).

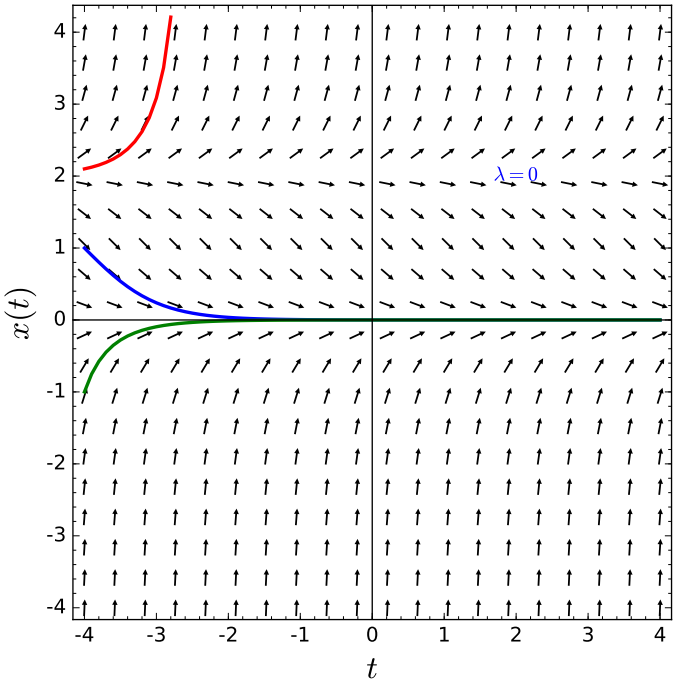

Figure 1.49: $x' = x^2 - 4x + \lambda$ for $\lambda = 0$

For $\lambda = 4$, the differential equation

$$\frac{dx}{dt} = f_4(x) = x^2 - 2x + 4 = (x-2)^2,$$

we have exactly one equilibrium solution, a node at $x = 2$ (Figure 1.50).

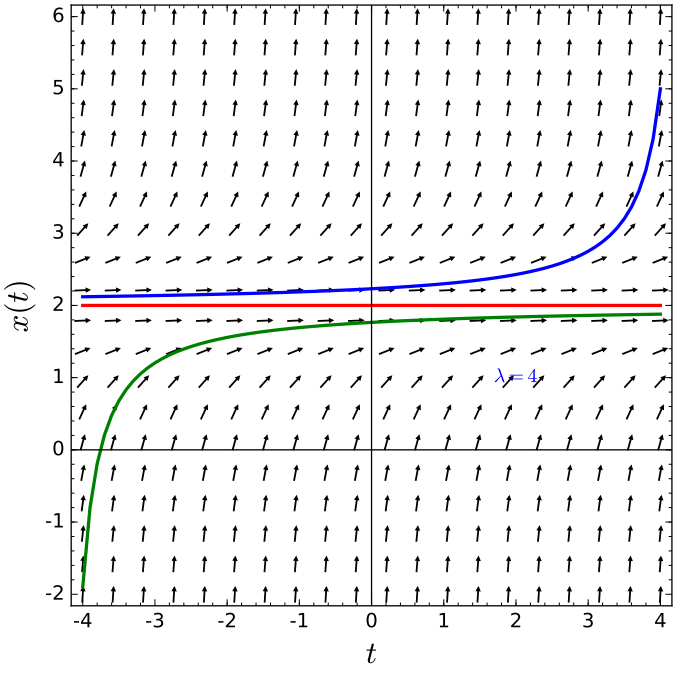

Figure 1.50: $x' = x^2 - 4x + \lambda$ for $\lambda = 4$

If $\lambda = 8$, then the differential equation

$$\frac{dx}{dt} = f_8(x) = x^2 - 2x + 8$$

has no equilibrium solutions (Figure 1.51).

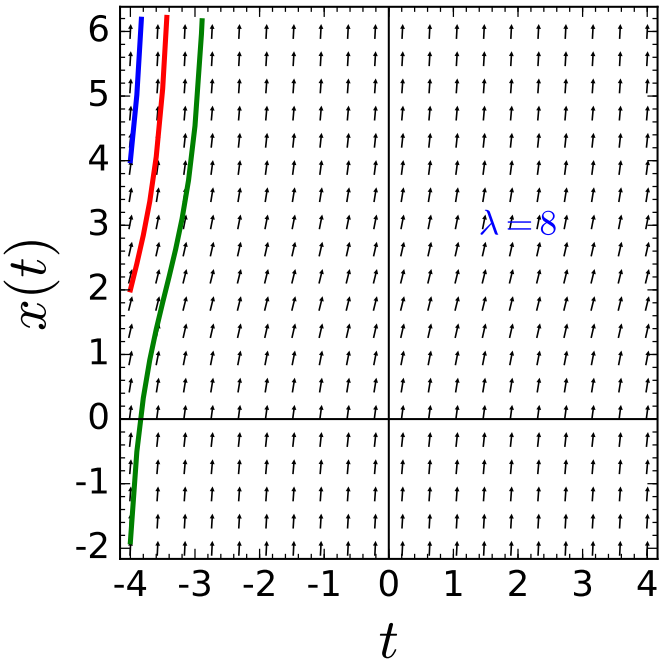

Figure 1.51: $x' = x^2 - 2x + \lambda$ for $\lambda = 8$

In fact, the number of equilibrium solutions for (1.31) changes at $\lambda = 4$. We say that $\lambda = 4$ is a ***bifurcation value*** for the differential equation

$$\frac{dx}{dt} = f_\lambda(x) = x^2 - 4x + \lambda. \tag{1.32}$$

For $\lambda < 4$, we have two equilibrium solutions.

$$x = 2 \pm \sqrt{4 - \lambda}.$$

For values of $\lambda > 1$, there are no equilibrium solutions. We can record all of the information for the various values in a graph called the ***bifurcation diagram***. The horizontal axis is λ and the vertical axis is x. Over each value of λ, we will plot the corresponding phase line. The curve in the graph represents the various equilibrium solutions for the different values of λ. The bifurcation diagram for equation (1.32) is a parabola (Figure 1.52). We have a phase line for each value of λ.

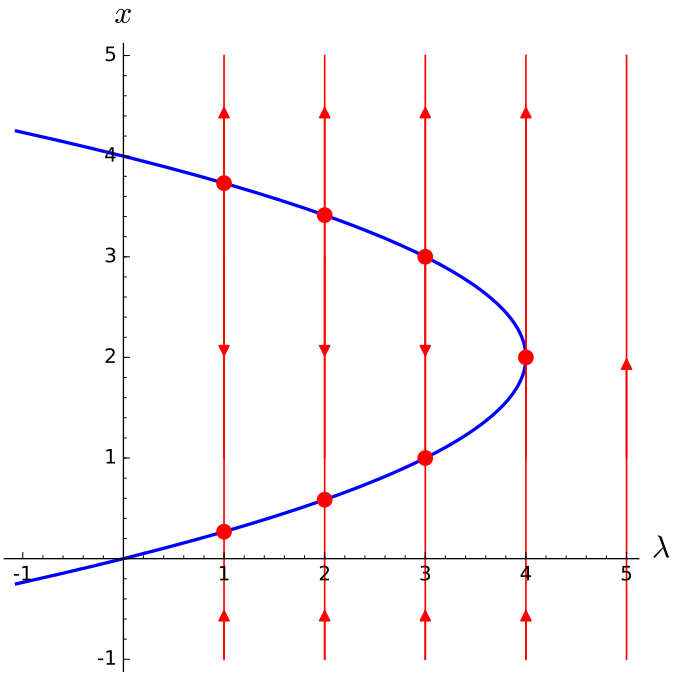

Figure 1.52: Bifurcation diagram for $x' = x^2 - 4x + \lambda$

Bifurcations for a one-parameter family of differential equations $dx/dt = f_\lambda(x)$ are, in fact, rare. Let us consider a bifurcation where a sink changes to a source as we vary the parameter λ. Suppose that for $\lambda = \lambda_0$, we have a sink at x_0. Then

$$\frac{dx}{dt} = f_{\lambda_0}(x_0) = 0.$$

Furthermore, the graph of $f_{\lambda_0}(x)$ must be decreasing for x near x_0, since $f_{\lambda_0}(x)$ must be postive for values of $x < x_0$ and negative for values of $x > x_0$. In other words, $f'_{\lambda_0}(x) < 0$ for x near x_0 with $f'_{\lambda_0}(x_0) < 0$, then for all λ_1 sufficiently close to λ_0, the differential equation

$$\frac{dx}{dt} = f_{\lambda_1}(x)$$

must have sink at a point $x = x_1$ very close to x_0. A similar situation hold if x_0 is a source and $f'_{\lambda_0}(x_0) > 0$. Thus, bifurcations can only occur when $f_{\lambda_0}(x_0) = 0$ and $f'_{\lambda_0}(x_0) = 0$.

Example 1.53. Now consider the one-parameter family

$$\frac{dy}{dt} = y^3 - \alpha y = y(y^2 - \alpha).$$

We will have an equilibrium solution at zero for all values of α and two additional equilibrium solutions at $\pm\sqrt{\alpha}$ for $\alpha > 0$. This type of bifurcation is a ***pitch fork bifurcation*** (Figure 1.54).

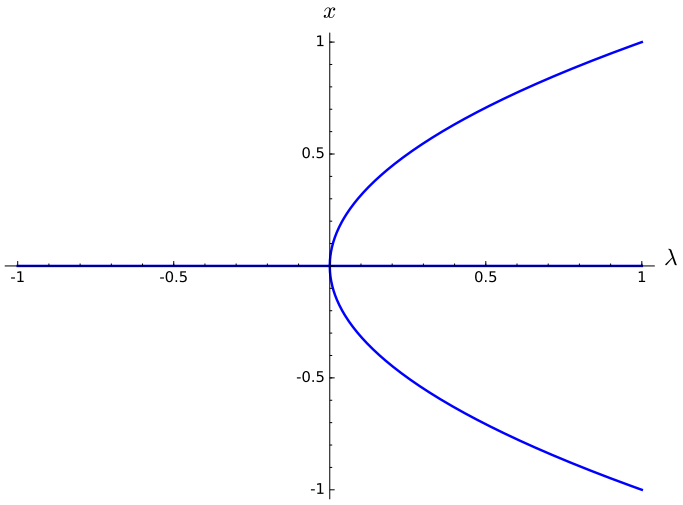

Figure 1.54: The bifurcation diagram for $y' = y^3 - \alpha y$

Example 1.55. Let us find the bifurcation values of the one-parameter family

$$\frac{dy}{dt} = y(y-2)^2 + \lambda. \tag{1.33}$$

If $g_\lambda(y) = y(y-2)^2 + \lambda$, then $g'_\lambda(y) = 3y^2 - 8y + 4$. The roots of $g'_\lambda(y) = 0$ are $y = 2$ and $y = 2/3$. In order for λ to be a bifurcation value, we must have $g_\lambda(2) = \lambda = 0$ or

$$g_\lambda(2/3) = \frac{32}{27} + \lambda = 0$$

Thus, equation (1.33) has two bifurcation values, $\lambda = -32/27$ and $\lambda = 0$. The bifurcation diagram for this one-parameter family is given in Figure 1.56.

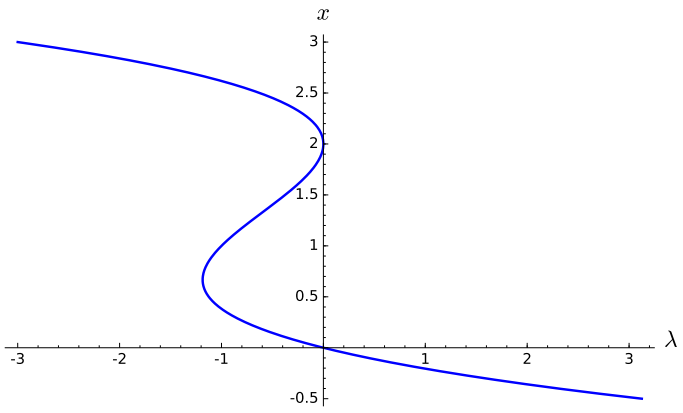

Figure 1.56: The bifurcation diagram for $y' = y(1-y)^2 + \lambda$

1.7.3 Important Lessons

- A one-parameter family of differential equations

$$\frac{dx}{dt} = f_\lambda(x)$$

has a bifurcation at $\lambda = \lambda_0$ if a change in the number of equilibrium solutions occurs.

- Bifurcation diagrams are an effective way of representing the nature of the solutions of a one-parameter family of differential equations.
- Bifurcations for a one-parameter family of differential equations $dx/dt = f_\lambda(x)$ are rare. Bifurcations occur when $f_{\lambda_0}(x_0) = 0$ and $f'_{\lambda_0}(x_0) = 0$.

1.7.4 Exercises

1. Describe the phase line portraits for each of the following equations and how they depend on the parameter λ. Draw the bifurcation diagram for each equation.

(a) $y' = \lambda y - \sin y$ for $\lambda > 2/\pi$.

(b) $y' = \lambda y^2 - 1$ for $\lambda \in \mathbb{R}$.

Hint.

2. Outbreaks of the spruce budworm have been responsible for some major deforestations in Canada and the United States. The equation

$$x' = r\left(1 - \frac{x}{K}\right)x - c\frac{x^2}{a + x^2}$$

has been used to describe the dynamics of spruce budworm populations, where the variable x denotes the population or density of the insect [16]. One explanation that has been given for the occurrence of outbreaks is based on the multiple bifurcations that occur with this differential equation.

(a) If $a = 0.01$, $c = 1$, and $K = 1$, we have a family of differential equations parameterized by r,

$$x' = rx(1 - x) - \frac{x^2}{0.01 + x^2}.$$

Solve the equation equation

$$rx(1 - x) - \frac{x^2}{0.01 + x^2} = 0$$

and plot the result in the xr-plane for $0 \leq r < 1$.

(b) To find the bifurcation diagram for the spruce budworm equation, reflect the graph obtained in part (1) about the line $r = x$ line.

(c) Estimate the two bifurcation values from your graph. Explain what happens to the population as r increases. That is, when does an outbreak occur? What happens after an outbreak?

Hint.

3. The differential equation

$$\frac{dy}{dt} = y - 4t + y^2 - 8yt + 16t^2 + 4.$$

is not autonomous, separable, or linear; however, we can solve this equation with a change of variable.

(a) Transform this equation into a new differential equation of the form

$$\frac{du}{dt} = f(u)$$

by letting $u = y - 4t$.

(b) Sketch the phase line for this new equation, $u' = f(u)$, and sketch several solutions.

(c) Find the solutions of the original differential equation that correspond to the equilibrium solutions of $u' = f(u)$. Graph these solutions in ty-plane. Also, sketch the graphs of the solutions that you plotted in part (b).

(d) Solve the differential new equation and use this information to solve the original differential equation.

Hint.

1.7.5 Project—Bifurcations

2

Systems of Differential Equations

2.1 Modeling with Systems

Many situations are best modeled with a system of differential equations rather than a single equation. We have already derived a model that describes how a population of snowshoe hares and and how they interact with one of their primary predators, the lynx (Section 1.1). We denoted the population of hares by $H(t)$ and the population of lynx by $L(t)$, where t is the time measured in years and derived the system of differential equations

$$\begin{aligned} \frac{dH}{dt} &= aH - bHL, \\ \frac{dL}{dt} &= -cL + dHL. \end{aligned}$$

Just as in first-order differential equations, we can examine the equilibrium solutions of a system. More specifically, we define an ***equilibrium solution*** for a system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned}$$

to be those values of x and y such that $f(x, y) = 0$ and $g(x, y) = 0$. That is, an equilibrium solution is a solution where neither $x(t)$ or $y(t)$ is changing.

2.1.1 Predator-Prey Systems

Suppose that we have a predator-prey system consisting of a population of foxes (F) and of rabbits (R).[1]

$$\begin{aligned} \frac{dR}{dt} &= 2R - RF, \\ \frac{dF}{dt} &= -5F + RF, \end{aligned}$$

we have an equilibrium solution at $R = 5$ and $F = 2$. That is, the system is in balance and there is just enough prey to support a constant population of predators at the point $(R, F) = (5, 2)$.

[1] Foxes are omnivores. Their diet consists of small mammals, including rabbits, as well as fruits, berries, and vegetables. They will even eat fish and crabs.

If the number of rabbits or foxes changes, then the system is no longer in balance. For example, if $R = 1$ and $F = 1$, then

$$\frac{dR}{dt} = 1,$$
$$\frac{dF}{dt} = -4,$$

and the rabbit population will be increasing while the fox population will decrease. If we assume that we have initial conditions

$$R_0 = R(0) = 1,$$
$$F_0 = F(0) = 1,$$

we can apply a numerical algorithm to generate a solution for our system.[2] The graphs of the solutions for $R(t)$ and $F(t)$ are given in Figure 2.1. Notice that the solutions are periodic with the same period. Observe that a peak in the rabbit population is followed by a peak in the fox population.

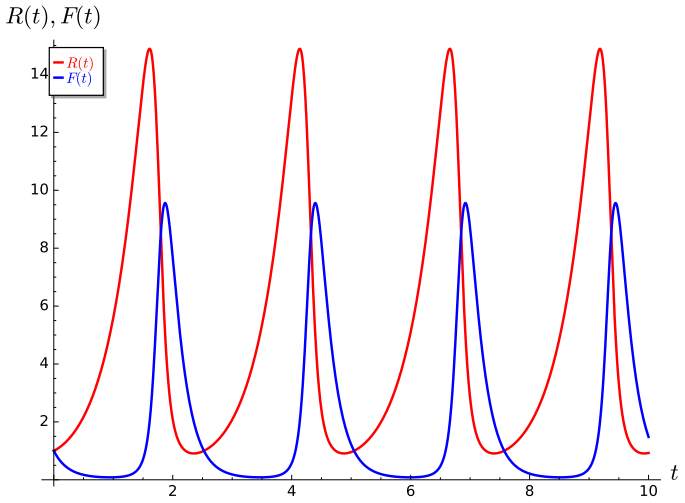

Figure 2.1: A simple predator-prey system

We can graph the solution to our system in a different manner—we can construct a parametric plot of our solution in the RF-plane. Thus, a point on the graph is given by $(R(t), F(t))$ at time t. We can view the solution curve of our system in the RF-plane in Figure 2.2. The RF-plane is called the ***phase plane*** for our system of differential equations and is analogous to the phase line that we used during our investigation of slope fields for autonomous differential equations. We can plot many solutions to our predator prey-system and even plot direction fields in the phase plane (Figure 2.3).

[2] You will find technology extremely useful when analyzing systems. We will introduce *Sage* commands for analyzing systems of equations at the end of this section.

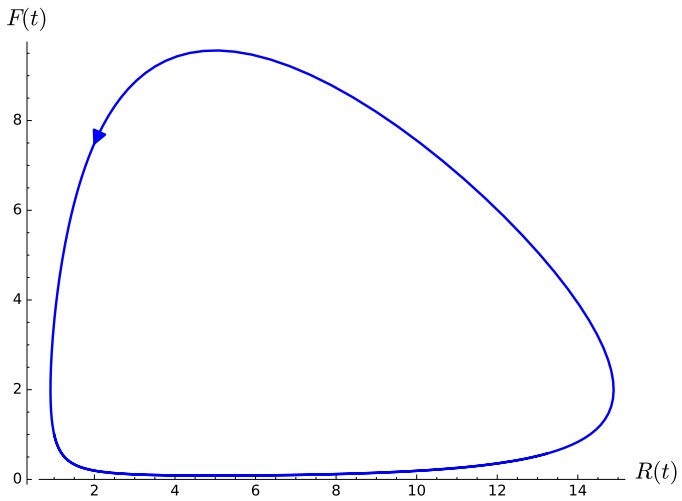

Figure 2.2: A solution curve in the RF-plane

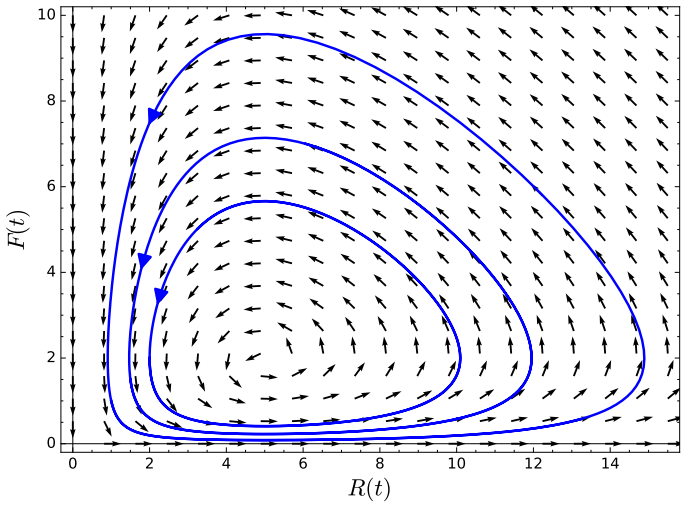

Figure 2.3: The phase plane for a predator-prey system

We will now modify our system by assuming that the rabbit population will grow logistically if there are no predators present. The system can now be written as

$$\frac{dR}{dt} = aR(1 - R/N) - bRF,$$
$$\frac{dF}{dt} = -cF + dRF,$$

where N is the carrying capacity. As a specific example, consider system

$$\frac{dR}{dt} = 2R\left(1 - \frac{R}{10}\right) - RF,$$
$$\frac{dF}{dt} = -5F + RF.$$

It is easy to see that we have two equilibrium solutions—one at $(R, F) = (0, 0)$ and one at $(R, F) = (5, 1)$. Our solutions now behave very differently from the assumption that the population of the prey grows exponentially. If we have initial values $R_0 = 1$ and $F_0 = 1$,

then our solution is no longer periodic (Figure 2.4). In fact, the solutions tend towards the equilibrium solution. The phase plane for our modified predator-prey system is given in Figure 2.5. The equilibrium solution $(5, 1)$ is an example of a ***stable equilibrium solution***.

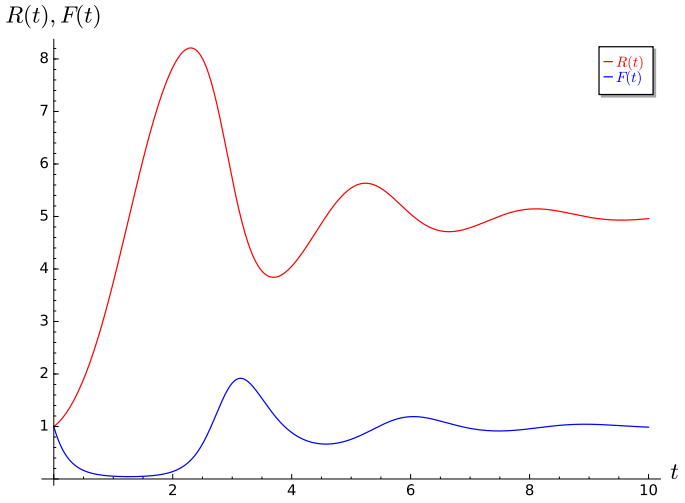

Figure 2.4: Solutions for a modified predator-prey system

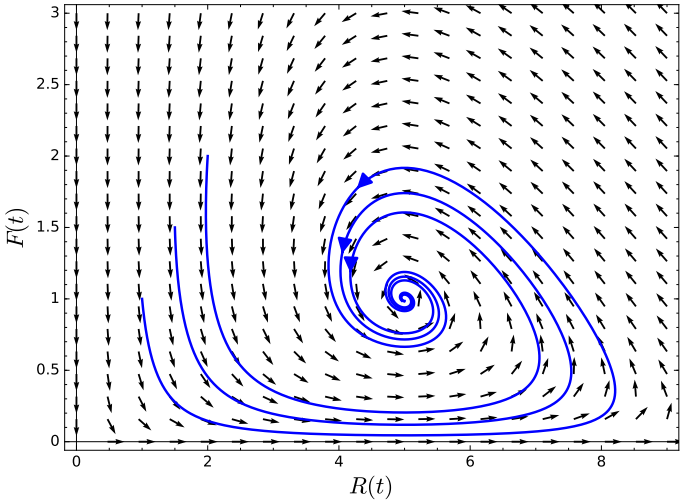

Figure 2.5: Phase plane for a modified predator-prey system

2.1.2 The Spring-Mass Model Revisited

Recall the spring-mass model from Section 1.1. We have a mass lying on a flat surface that is attached to one end of a spring with the other end of the spring attached to a wall. The spring displacement is denoted by x. If $x > 0$, then the spring is stretched. If $x < 0$, the spring is compressed. If $x = 0$, then the spring is in a state of equilibrium (Figure 1.4). If the surface is frictionless and we pull on the mass, then the mass will oscillate.

If we use a dashpot to add damping, the second-order differential equation for the spring-mass system is

$$mx'' = -bx' - kx.$$

or

$$mx'' + bx' + kx = 0,$$

where m, b, and k are all positive constants.

A second-order differential equation can be restated in terms of a system of first order differential equations. Given the differential equation

$$mx'' + bx' + kx = F(t),$$

with initial position $x(0) = x_0$ and initial velocity $x'(0) = x'_0$, we can rewrite this equation as a system of first-order differential equations by letting $v(t) = x'(t)$. In this case, the equation becomes

$$mv' + bv + kx = F(t).$$

We now have a system of first-order differential equations,

$$\begin{aligned} x' &= v, \\ v' &= \frac{1}{m}F(t) - \frac{b}{m}v - \frac{k}{m}x, \end{aligned}$$

with initial conditions

$$\begin{aligned} x(0) &= x_0 \\ v(0) &= v'_0. \end{aligned}$$

2.1.3 Modeling Epidemics

Systems of differential equations are very useful in epidemiology. Differential equations can be used to model various epidemics, including the bubonic plague, influenza, AIDS, and the 2015 ebola outbreak in west Africa. To understand how we might model an epidemic, we will consider a very simple situation. We will assume that we have a closed population of size N, where immigration, emigration, and birth do not play an important role. We will also ignore any deaths that are not related to our disease.

We will assume that each individual in the population falls into one of the following categories:

$$\begin{aligned} S(t) &= \text{Susceptible individuals} \\ I(t) &= \text{Infected individuals} \\ R(t) &= \text{Removed individuals} \end{aligned}$$

Susceptible individuals who do not yet have the disease and can catch the disease from infected individuals. Individuals enter the removed population by either recovering from the disease or dying. If an infected individual recovers, then the individual is immune to the disease. Schematically, we can represent the effect of the disease by the diagram

$$S \longrightarrow I \longrightarrow R.$$

Since the population is closed, we know that

$$S(t) + I(t) + R(t) = N.$$

This model is called an ***SIR model***.

We can model how the disease acts with the following system of equations,

$$\frac{dS}{dt} = -\alpha SI$$

$$\frac{dI}{dt} = \alpha SI - \beta I$$
$$\frac{dR}{dt} = \beta I.$$

We say that α is the rate of infection and β is the rate at which the infected are removed. That is, an infected individual either dies or recovers after $1/\beta$ days. Since

$$\frac{d}{dt}\left[S(t) + I(t) + R(t)\right] = \frac{d}{dt}N = 0,$$

we need only solve the system

$$\frac{dS}{dt} = -\alpha SI$$
$$\frac{dI}{dt} = \alpha SI - \beta I.$$

Letting $\alpha = 0.005$ and $\beta = 0.08$, we can see how the susceptible and removed populations interact in an SIR epidemic in Figure 2.6.

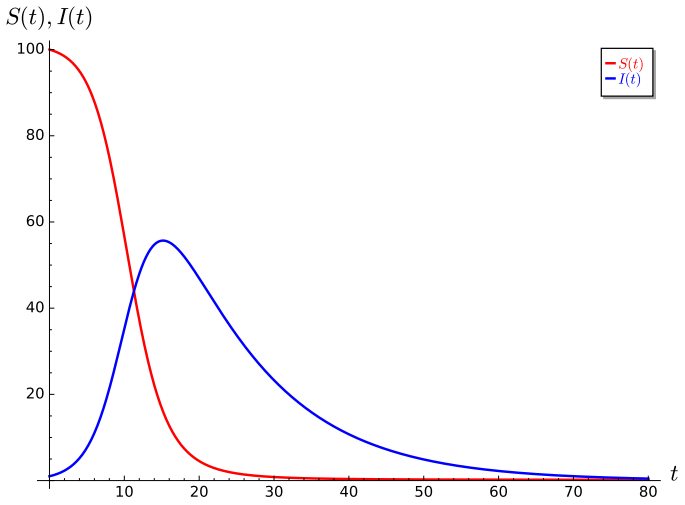

Figure 2.6: The susceptible and removed populations for an SIR epidemic

There are many questions associate with epidemic models.

- Will there be an epidemic?
- If there is an epidemic, how many individuals will be infected?
- Is there a period of time for which individuals are exposed to the disease but exhibit no symptoms and cannot infect others?
- If the disease is endemic, what is the prevalence of the infection?
- Can the disease be eradicated or controlled?
- What is the effect of the population age structure?

We will later explore how we can adjust the SIR model to address some of these issues.

2.1.4 Important Lessons

- A ***second-order linear equation***

$$a(t)x'' + b(t)x' + c(t)x = g(t)$$

can be written as a system of first-order equations by letting $v(t) = x'(t)$.

- An ***equilibrium solution*** for a system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned}$$

to be those values of x and y such that $f(x, y) = 0$ and $g(x, y) = 0$.

- We can graph solutions to the system

$$\begin{aligned} \frac{dx}{dt} &= f(x, y), \\ \frac{dy}{dt} &= g(x, y), \end{aligned}$$

as parametric plots in the xy-plane. A point on the graph is given by $(x(t), y(t))$ at time t. The xy-plane is called the ***phase plane*** for our system of differential equations.

- Solution curves in the phase plane of

$$\begin{aligned} \frac{dx}{dt} &= f(x, y), \\ \frac{dy}{dt} &= g(x, y), \end{aligned}$$

can act very differently with respect to equilibrium solutions. Consider the two predator-prey systems

$$\begin{aligned} \frac{dH}{dt} &= aH - bHL, \\ \frac{dL}{dt} &= -cL + dHL, \end{aligned}$$

and

$$\begin{aligned} \frac{dH}{dt} &= aH(1 - H/\alpha) - bHL, \\ \frac{dL}{dt} &= -cL + dHL. \end{aligned}$$

Solutions graph as closed orbits in our example of the first system. In the example of the second system, solutions approach the equilibrium solution as $t \to \infty$.

2.1.5 Exercises

1. Consider the following predator-prey system,

$$x' = 2x(1 - x/2) - 1.2xy$$
$$y' = -y + 0.9xy,$$

where $x(t)$ is the population of the prey and $y(t)$ is the population of the predator at time t.

(a) How would you modify this system to include the effect of hunting of the prey at a rate of α units of prey per unit of time?

(b) How would you modify this system to include the effect of hunting of the predators at a rate proportional to the number of predators?

(c) Suppose the predators discover a second, unlimited source of food, but they still prefer to eat the primary prey when they can catch them. How would you modify the system to include this assumption?

(d) Suppose the predators discover a second source of food that is limited in supply. How would you modify the system to include this assumption?

(e) Suppose that the predators migrate to an area if there is more than five times as many prey in that area ($x > 5y$), and that they move away if there are fewer than five times as many prey as predators. How would you modify the system to take this into account?

(f) Suppose that prey move out of an area at a rate proportional to the number of predators in the area. How would you modify the system to take this into account?

Hint.

2. Consider the equation

$$\frac{d^2}{dt^2} + \frac{k}{m}y = 0$$

for the motion of a simple harmonic oscillator.

(a) Consider the function $y(t) = \cos \beta t$. Under what conditions on β is $y(t)$ a solution to the differential equation?

(b) What initial condition ($t = 0$) in the yv-plane corresponds to this solution?

(c) In terms of k and m, what is the period of this solution?

(d) Sketch the solution curve in the yv-plane that corresponds to this solution.

Hint.

3. A mass weighing 4 pounds stretches a spring 4 inches.

(a) Consider the function $y(t) = \cos \beta t$. Under what conditions on β is $y(t)$ a solution to the differential equation?

(b) Formulate an initial-value problem that corresponds to the motion of this undamped mass-spring system if the mass is extended 1 foot from its rest position and released with no initial velocity.

(c) Using the result of the previous exercise, find the solution of this initial-value problem.

Hint.

4. SEIR model

Hint.

2.1.6 Project—Using *Sage* to Solve Systems

We can use *Sage* to plot the solution of the system

$$x' = -x - y \tag{2.1}$$
$$y' = x + y/5 \tag{2.2}$$
$$x(0) = 1 \tag{2.3}$$
$$y(0) = -1 \tag{2.4}$$

First, let us plot a direction field for our system.

```
x, y, t = var('x y t')
F = [-x - y, x + y/5]
p = plot_vector_field(F, (x,-4,4), (y,-4,4))
p
```

Notice the the vectors are different lengths depending on their magnitudes. If we wish all of the vectors to be the same length, we can divide each component by the length of the vector.

```
x, y, t = var('x y t')
F = [-x - y, x + y/5]
n = sqrt(F[0]^2 + F[1]^2)
F_unit = [F[0]/n, F[1]/n]
p = plot_vector_field(F_unit, (x,-4,4), (y,-4,4))
p
```

We can also add axes labels to our plot.

```
x, y, t = var('x y t')
F = [-x - y, x + y/5]
n = sqrt(F[0]^2 + F[1]^2)
F_unit = [F[0]/n, F[1]/n]
p = plot_vector_field(F_unit, (x,-4,4), (y,-4,4),
    axes_labels=['$x(t)$','$y(t)$'])
p
```

Now suppose that we wish to plot solutions $x(t)$ and $y(t)$ to our system. We can use the *Sage* command `desolve_system_rk4`.

```
x, y, t = var('x y t')
F = [-x - y, x + y/5]
P = desolve_system_rk4(F, [x, y], ics = [0,1,-1], ivar = t,
    end_points = 10, step = 0.01)
P
```

We new have a numerical approximation of the solution to the system (2.1)–(2.4). However, our approximation is just a very long list of points. In fact, we get a list of triples, (t, x, y). It would be much more useful if we could display a graph of the solution. In the code below, we grab pairs (t, x) and (t, y) and then plot the points using the `line` command.

```
x, y, t = var('x y t')
F = [-x - y, x + y/5]
P = desolve_system_rk4(F, [x, y], ics = [0,1,-1], ivar = t,
    end_points = 10, step = 0.01)
Q = [ [i, j] for i,j,k in P]
```

```
R = [ [i, k] for i,j,k in P]
p = line(Q)
p += line(R)
p
```

We can also add colors, axes labels, and legend colors so that the plot makes more sense.

```
x, y, t = var('x␣y␣t')
F = [-x - y, x + y/5]
P = desolve_system_rk4(F, [x, y], ics = [0,1,-1], ivar = t,
    end_points = 10, step = 0.01)
Q = [ [i, j] for i,j,k in P]
R = [ [i, k] for i,j,k in P]
p = line(Q, color='red', axes_labels=['$t$','$x(t),␣y(t)$'],
    legend_label='$x(t)$', legend_color='red', fontsize=12)
p += line(R, color='blue', legend_label='$y(t)$', legend_color='blue')
p
```

To plot the solution in the xy–plane, we will need to select the second two entries in each triple.

```
x, y, t = var('x␣y␣t')
F = [-x - y, x + y/5]
P = desolve_system_rk4(F, [x, y], ics = [0,1,-1], ivar = t,
    end_points = 10, step = 0.01)
Q = [ [j,k] for i,j,k in P]
p = line(Q, axes_labels=['$x(t)$','$y(t)$'], fontsize=12)
p
```

We can adjust the thickness of the plot add an arrow to indicate the direction of the solution.

```
x, y, t = var('x␣y␣t')
F = [-x - y, x + y/5]
P = desolve_system_rk4(F, [x, y], ics = [0,1,-1], ivar = t,
    end_points = 10, step = 0.01)
Q = [ [j,k] for i,j,k in P]
p = line(Q, axes_labels=['$x(t)$','$y(t)$'], fontsize=12, thickness=3)
p += arrow(Q[int(len(Q)/5)], Q[int(len(Q)/5) + 1])
p
```

Now let us add a vector field to our plot.

```
x, y, t = var('x␣y␣t')
F = [-x - y, x + y/5]
P = desolve_system_rk4(F, [x, y], ics = [0,1,-1], ivar = t,
    end_points = 10, step = 0.01)
Q = [ [j,k] for i,j,k in P]
p = line(Q, axes_labels=['$x(t)$','$y(t)$'], fontsize=12, thickness=3)
p += arrow(Q[int(len(Q)/5)], Q[int(len(Q)/5) + 1])
n = sqrt(F[0]^2 + F[1]^2)
F_unit = [F[0]/n, F[1]/n]
p += plot_vector_field(F_unit, (x,-1.5,1.5), (y,-1.5,1.5),
    axes_labels=['$x(t)$','$y(t)$'])
p
```

To learn more about how to use Sage to solve systems of differential equations, see http://www.sagemath.org/doc/reference/calculus/sage/calculus/desolvers.html. The Sage cell below can be used to make your own computations.

2.2 The Geometry of Systems

We can use ***direction fields*** in the phase plane to represent ***autonomous systems***

$$\frac{dx}{dt} = f(x, y), \tag{2.5}$$
$$\frac{dy}{dt} = g(x, y). \tag{2.6}$$

Equation (2.5) tells us how a solution curve changes in the x direction, while equation (2.6) tells us how a solution curve changes in the y direction.

2.2.1 Direction Fields

Example 2.7. Consider the differential equation for a simple harmonic oscillator that we developed in Section 1.1,

$$mx'' + kx = 0.$$

If we assume that k and m are both equal to one and let $x' = v$, we can rewrite this equation as the first order system,

$$x' = v,$$
$$v' = -x.$$

The direction field is relatively easy to understand. After plotting only few vectors, we can very quickly see that the vectors are tangent to circles at the origin (Figure 2.8). Since the solutions to the undamped harmonic oscillator $x'' + x = 0$ are of the form

$$x(t) = A\cos t + B\sin t$$

for arbitrary constants A and B, this should not be too surprising.

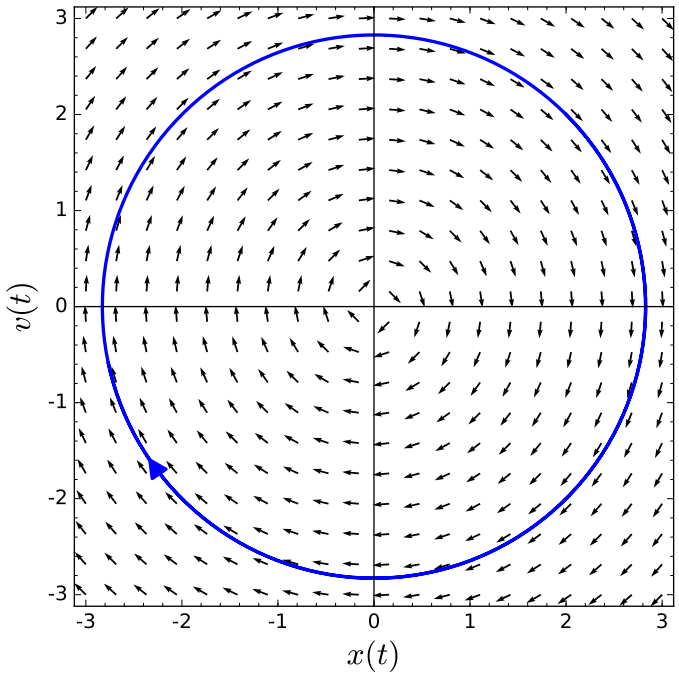

Figure 2.8: The direction field for a harmonic oscillator

Let us examine some systems of equations with direction fields that are easily plotted.

Example 2.9. The system

$$x' = x$$
$$y' = y$$

gives us a direction field where the vectors point away from the origin (Figure 2.10).

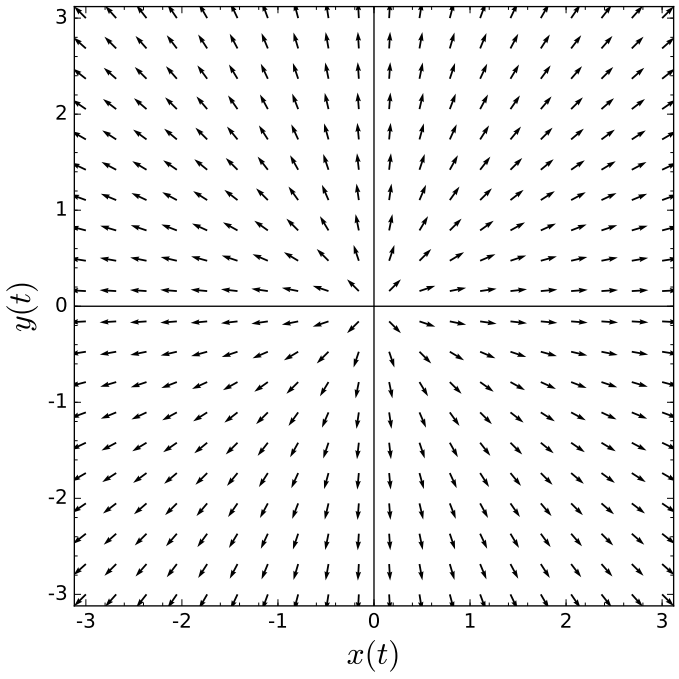

Figure 2.10: The direction field for $x' = x$ and $y' = y$

The system

$$x' = -x,$$
$$y' = -y.$$

gives us a direction field where the vectors point towards the origin (Figure 2.11).

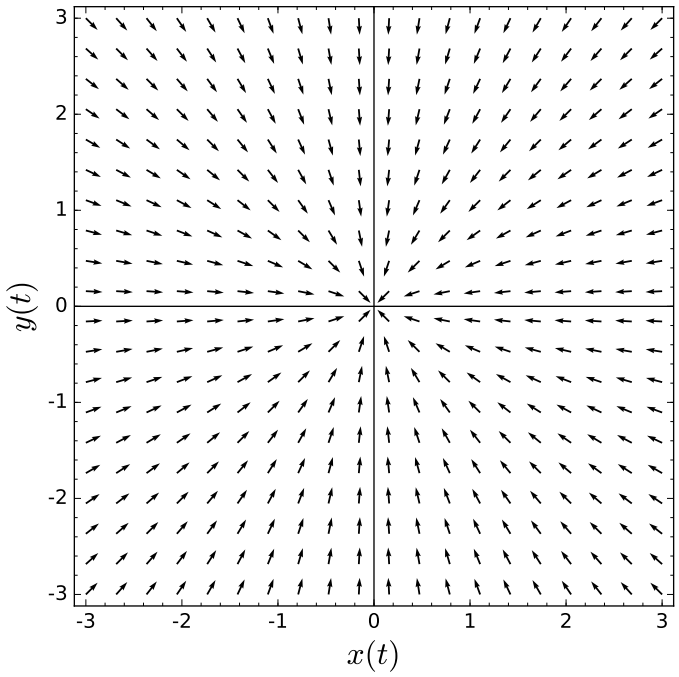

Figure 2.11: The direction field for $x' = x$ and $y' = y$

The system

$$x' = -x,$$
$$y' = -5y.$$

also gives us a direction field where the vectors point towards the origin; however, we shall soon see that there are important differences between this direction field and the direction field of the previous system (Figure 2.11).

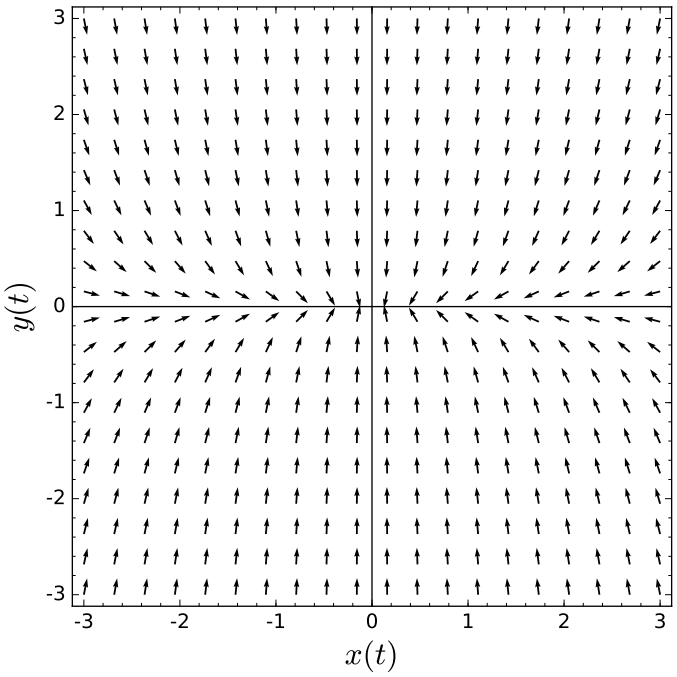

Figure 2.12: The direction field for $x' = -x$ and $y' = -5y$

2.2.2 Modified Predator-Prey System

Let us recall the modified predator-prey system that we developed in the last section. That is, we will assume that the prey in our model has logistic growth,

$$\begin{aligned} \frac{dR}{dt} &= aR(1 - R/\alpha) - bRF, \\ \frac{dF}{dt} &= -cF + dRF, \end{aligned}$$

where N is the carrying capacity. In order to investigate the geometric properties of our system, we will rewrite our system in vector form. For each value of t, we will use $\mathbf{x}(t)$ to denote the vector-valued function

$$\mathbf{x}(t) = \begin{pmatrix} R(t) \\ F(t) \end{pmatrix}.$$

This vector-valued function, $\mathbf{x}(t)$ corresponds to our solution curve $(R(t), F(t))$ in the RF-plane. Now we can write our predator-prey model as a single vector equation,

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} R'(t) \\ F'(t) \end{pmatrix} = \begin{pmatrix} aR(1 - R/N) - bRF, \\ -cF + dRF \end{pmatrix}.$$

We can view the right side of the equation as a ***vector field***. The specific example that we examined was

$$\begin{aligned} \frac{dH}{dt} &= 2R\left(1 - \frac{R}{10}\right) - RF, \\ \frac{dL}{dt} &= -5F + RF. \end{aligned}$$

The vector field form of this system is

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2R(1 - R/10) - RF \\ -5F + RF \end{pmatrix}.$$

We can associate a vector in the RF-plane for each value of R and F. For example, if we let $(R, F) = (10, 10)$, we have $(R', F') = (-100, 50)$. At this particular point, the population of rabbits is falling rapidly while the number of foxes is climbing very quickly, We can represent this vector in the phase plane by drawing an arrow in the proper direction. Thus, we obtain an direction field for our system of differential equations (Figure 2.13).

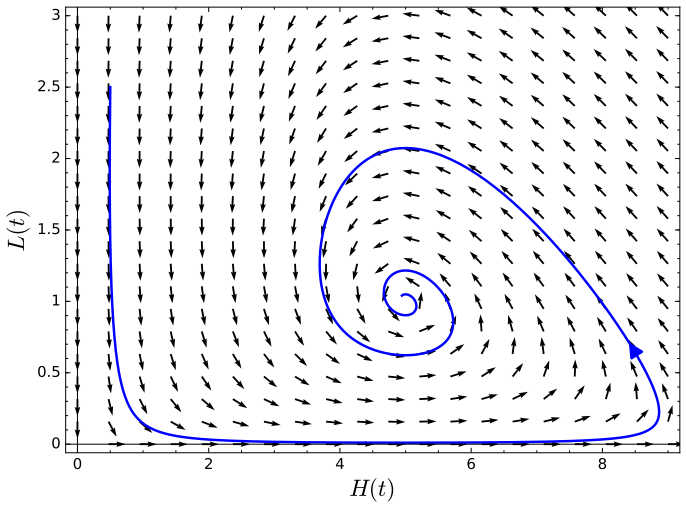

Figure 2.13: A vector field for the predator-prey system

2.2.3 A Competing Species Model

Suppose that x and y are the population of two distinct species that compete for the same resources. For example, two species of fish may compete for the same food in a lake or cattle and sheep may compete for the same grazing land. We can model two competing species using the following system of first-order differential equations,

$$\frac{dx}{dt} = \alpha x \left(1 - \frac{x}{M}\right) - \beta xy$$
$$\frac{dy}{dt} = \gamma y \left(1 - \frac{y}{N}\right) - \delta xy.$$

The first term in each equation is the logistic growth of each species. The second term is how the species is affected by interacting with the competing species.

Example 2.14. Suppose that $x = x(t)$ and $y = y(t)$ are two populations who compete for the same resources are governed by the following system of differential equations,

$$\frac{dx}{dt} = x(1 - x) - \alpha xy$$
$$\frac{dy}{dt} = y(1 - y) - \alpha xy.$$

If we study how the two populations interact, we will discover two very different cases depending on the value of the parameter α.

If we let $\alpha = 2$, then there is a strong interaction between the two species and our system becomes

$$\begin{aligned} \frac{dx}{dt} &= x(1-x) - 2xy \\ \frac{dy}{dt} &= y(1-y) - 2xy. \end{aligned}$$

We are interested in what happens as $t \to \infty$. First we will examine what happens when $dx/dt = 0$ and $dy/dt = 0$:

$$\begin{aligned} \frac{dx}{dt} &= x(1-x-2y) \\ \frac{dy}{dt} &= y(1-y-2x). \end{aligned}$$

If $dx/dt = 0$, then

$$x = 0 \tag{2.7}$$
$$y = (1-x)/2. \tag{2.8}$$

Similarly, if $dy/dt = 0$, we have

$$y = 0 \tag{2.9}$$
$$y = 1 - 2x. \tag{2.10}$$

The lines in equations (2.7)–(2.10) are called ***nullclines***. In general, if we are given the system

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y), \end{aligned}$$

then the values of x and y satisfying $dx/dt = f(x,y) = 0$ are called the x***-nullclines*** of the system and the values of x and y satisfying the equation $dy/dt = g(x,y) = 0$ are called the y***-nullclines*** of the system.

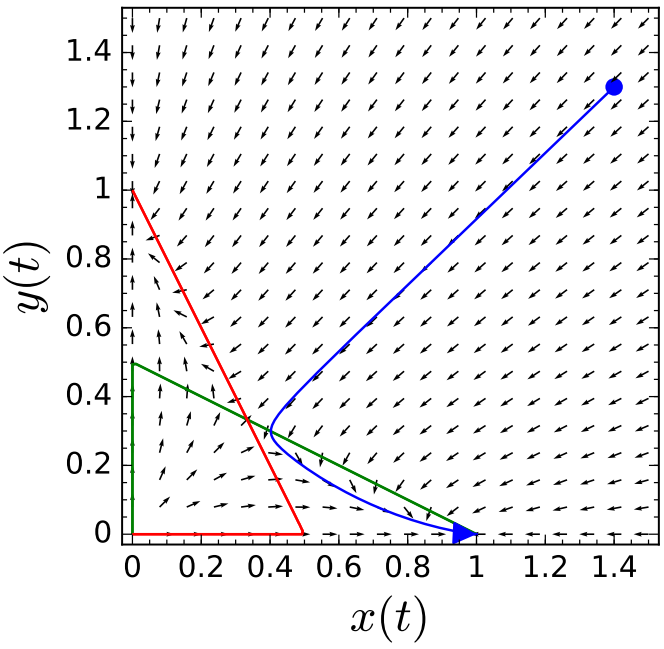

Figure 2.15: Nullclines for the case $\beta = 2$

In our example, we can plot the x and y-nullclines to help us understand our system (Figure 2.15). Of course, we have an equilibrium solution whenever an x-nullcline intersects with a y-nullcline. Thus, the equilibrium points for our particular system are $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1/3, 1/3)$. Furthermore, we can choose a representative point in each region to find how the direction field is oriented. If we are given the initial condition $(1.4, 1.3)$ for example, we can see that

$$\frac{dx}{dt} = -4.2$$
$$\frac{dy}{dt} = -4.03.$$

Since both of these numbers are negative, we can see that our initial trajectory is headed down and to the left—slightly more to the left than down. However, we have no guarantee that the trajectory will continue in this direction.

Before we proceed further with our analysis, let us determine what happens on the nullclines themselves. That is, we will examine the case when $dx/dt = 0$ or $dy/dt = 0$. If $dx/dt = 0$, then any solution crossing this x-nullcline must be moving vertically. There can be no right or left movement at this point in the phase portrait. We can indicate this fact along with the direction of vertical movement by drawing a vertical slash with an arrow on the x-nullcline (Figure 2.15). For example, the point $(0.5, 0.25)$ lies on the x-nullcline and

$$\frac{dy}{dt}(0.5, 0.25) < 0.$$

Therefore, the trajectory that crosses the R-nullcline at $(0.5, 0.25)$ is moving down. As we move along the x-nullcline, the direction of this arrow can only change when we cross an y-nullcline.

Similarly, there can be no vertical motion on an y-nullcline. We will indicate the direction of horizontal motion by drawing a horizontal line with an arrow pointing to the right if $dx/dt > 0$ on the y-nullcline and an arrow pointing to the left if $dx/dt < 0$ on the y-nullcline.

We can also determine the basic direction of the solution curve by checking what happens at a point in each of the regions bounded by the nullclines. For example, at the point $(0.1, 0.1)$ we find that

$$\frac{d}{dt}x(0.1, 0.1) = 0.7$$
$$\frac{d}{dt}y(0.1, 0.1) = 0.7.$$

Thus, the general direction of any solution curve in this region is up and right (Figure 2.15).

What happens to the initial condition $(1.4, 1.3)$? We see three possible scenarios if we follow the nullclines for large values of t.

- Only species x survives and species y becomes extinct.
- Only species y survives and species x becomes extinct.
- There are essentially equal numbers of species x and y.

Example 2.16. There is no reason why our nullclines should be limited to straight lines. The system

$$x' = y - x^2$$
$$y' = x - 2$$

has an x-nullcline $y = x^2$ and a y-nullcline $x = 2$. These two nullclines intersect at $(2, 4)$, and the system has equilibrium solution. The nullclines divide the plane into four basic regions (Figure 2.17). By choosing a point in each of these regions and determining the direction of the slope field at that point, we can decide the direction of the slope field in the entire region.

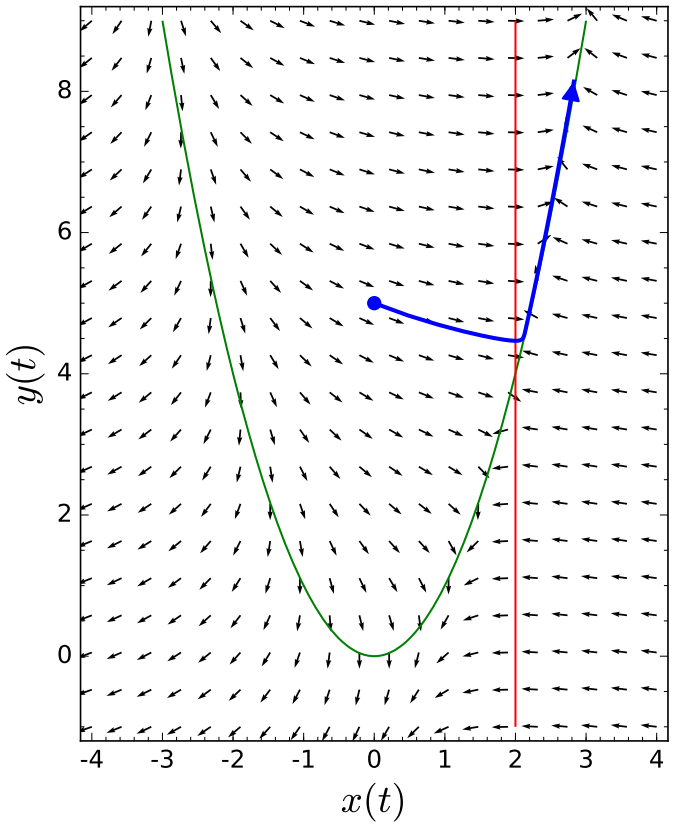

Figure 2.17: The system $x' = y - x^2$ and $y' = x - 2$

The only equilibrium solution of our system occurs at $(2, 4)$.

2.2.4 Plotting Direction Fields with *Sage*

It is easy to plot direction fields using *Sage*. We can plot the direction field for the system

$$x' = y - 2x,$$
$$y' = x - xy - x^3$$

using the following *Sage* interact. You can evaluate the cell below to get a menu driven applet for drawing direction fields for systems of two differential equations.

```
x,y,t = var('x,y,t')
@interact
def _(f = input_box(default = x*(1 - x) - 2*x*y),
      g = input_box(default = y*(1-y) - 2*x*y),
      x_min = input_box(default = 0), x_max = input_box(default =
          1.5),
      y_min = input_box(default = 0), y_max = input_box(default =
          1.5),
      t_start = input_box(default = 0.0), x_start = input_box(default
          = 1.4), y_start = input_box(default = 1.3),
      step_size = (0.01, (0.001, 1.0)), t_end = (10, (5, 50)),
      x_null = checkbox(False, "Show␣\(x\)-nullcline␣(green)"),
      y_null = checkbox(False, "Show␣\(y\)-nullcline␣(red)")):
    n = sqrt(f^2 + g^2)
```

```
    p = plot_vector_field( (f/n, g/n), (x, x_min, x_max), (y, y_min,
        y_max))
    p += point((x_start, y_start), pointsize=50)
    P = desolve_system_rk4([f, g], [x, y],ics = [t_start, x_start,
        y_start], ivar = t, end_points = t_end, step = step_size)
    Q = [ [j,k] for i,j,k in P]
    p += line(Q, axes_labels = ['$x(t)$','$y(t)$'])
    p += arrow(Q[int(len(Q)) - 2], Q[int(len(Q)) - 1], width = 1,
        arrowsize = 4)
    x_nullcline = implicit_plot(f, (x, x_min, x_max), (y, y_min,
        y_max), color = 'green', legend_label = '\(x\)-nullcline')
    y_nullcline = implicit_plot(g, (x, x_min, x_max), (y, y_min,
        y_max), color = 'red', legend_label = '\(y\)-nullcline')
    if x_null:
        p += x_nullcline
    if y_null:
        p += y_nullcline
    p.show(xmin = x_min, xmax = x_max, ymin = y_min, ymax = y_max)
    pretty_print(html("<br␣/>"))
    pretty_print(html(r"$\displaystyle\frac{dx}{dt}␣=␣%s$"
        %latex(f)))
    pretty_print(html("<br␣/>"))
    pretty_print(html(r"$\displaystyle\frac{dy}{dt}␣=␣%s$"
        %latex(g)))
```

2.2.5 A Summary of Phase Plane Analysis

We can use the following series of steps to summarize phase plane analysis for the nonlinear system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y).$$

- *Step 1.* Draw the curves where $f(x, y) = 0$. These curves are called the ***x-nullclines***. When a solution curve $\mathbf{x}(t)$ lies on one of these curves, $dx/dt = 0$, draw vertical slash marks on the x-nullclines to remind yourself that a trajectory crossing the nullcline can only do so if it is moving in a vertical direction at the instant of crossing.

- *Step 2.* Draw the curves where $g(x, y) = 0$. These curves are called the ***y-nullclines***. When $\mathbf{x}(t)$ lies on one of these curves, $dg/dt = 0$. Draw horizontal slash marks on the y-nullclines to remind yourself that a trajectory crossing the nullcline can only do so if it is moving in a horizontal direction at the instant of crossing.

- *Step 3.* Label the points where the x and y-nullclines intersect. These intersections are the ***equilibrium points***. If $\mathbf{x}(t)$ is ever at one of these points, then both dx/dt and dy/dt vanish. This means that the trajectory stays at the equilibrium point for all time. If our system is going to tend towards as steady state, then $\mathbf{x}(t)$ will approach on of the equilibrium points as $t \to \infty$.

- *Step 4.* Label the regions of the xy-plane where $dx/dt < 0$ and where $dx/dy > 0$. These regions are always separated by x-nullclines. Likewise, label the regions where dy/dt is positive and negative.

- *Step 5.*Go back and put arrows on the vertical hash marks of the x null clines. These arrows indicate whether the motion across the null cline is up or down. The arrows are up on the parts of the x-nullclines that are in the $dy/dt > 0$ region, and down on those parts of the x-nullclines in the $dy/dt < 0$ regions. Likewise, draw arrows on the horizontal slash marks of the y-nullclines. These arrows are pointing right on the parts of the y-nullclines in the $dx/dt > 0$ regions and left point on the parts in the $dx/dt < 0$ regions.

- *Step 6.*

  - If $dx/dt > 0$ and $dy/dt > 0$, then both $x(t)$ and $y(t)$ are increasing and the trajectory moves up and right.
  - If $dx/dt > 0$ and $dy/dt < 0$, the trajectory moves down and right.
  - If $dx/dt < 0$ and $dy/dt > 0$, the trajectory moves up and left.
  - If $dx/dt < 0$ and $dy/dt < 0$, the trajectory moves down and left.

2.2.6 Exercises

1. Consider the competing species model

$$\frac{dx}{dt} = x(1-x) - \beta xy$$
$$\frac{dy}{dt} = y(1-y) - \beta xy,$$

where the species interact weakly, say $\beta = 1/2$.

(a) Find the x and y-nullclines for this system.

(b) Find all equilibrium points for this system.

(c) Sketch the phase plane for this system.

(d) If the inital populations are given by $x(0) = 2.4$ and $y(0) = 0.3$, what happens to the two populations as $t \to \infty$?

Hint.

2. For the following two systems of equations

- Find the equilibrium points of the system.
- Sketch the phase plane and direction field for each system using technology.
- Briefly describe the behavior of typical solutions.

(a)

$$x' = \cos y$$
$$y' = -x + y$$

(b)

$$x' = y$$
$$y' = x - x^3 - y$$

Hint.

3. Consider an epidemic that moves through an isolated population. We will make the following assumptions about the epidemic.

- Individuals are infected at a rate proportional to the product of the number of infected and susceptible individuals. We assume that the constant of proportionality is α.
- The length of the incubation period is negligible, and infectious individuals are immediately infectious.
- On the average, an infected individual dies after 10 days.
- Only a single individual is initially ill.
- Infected individuals do not give birth, but susceptible individuals have a birth rate of 0.0003 per individual per year. Newborns are susceptible.

If $S(t)$ is the number of susceptible and $I(t)$ is the number of infected people, then

$$\frac{dS}{dt} = -\alpha SI + 0.0003S \tag{2.11}$$

$$\frac{dI}{dt} = \alpha SI - 0.1I. \tag{2.12}$$

The constant α is a measure of the relative infectivity of the disease. Some diseases such as Ebola, a viral hemorrhagic fever, are extremely infectious with a mortaility rate of up to 90%. On the other hand, AIDS which is caused by the HIV virus, has a much lower transmission rate. The goal of this exercise is to examine the differences between the two.

(a) If $\alpha = 0.05$, draw the phase portrait. Be sure to label all nullclines and equilibrium solutions. Suppose that $S(0) = 1000$ and $I(0) = 1$. What happens to the solution curve as $t \to \infty$?

(b) If $\alpha = 0.000001$, draw the phase portrait. Be sure to label all nullclines and equilibrium solutions. Suppose that $S(0) = 30000$ and $I(0) = 1$. What happens to the solution curve as $t \to \infty$?

(c) What conclusions can you draw about the behavior of the two different epidemics?

Hint.

2.2.7 Project—Plotting Nullclines with *Sage*

Let us use *Sage* to analyze the system

$$x' = x + y,$$
$$y' = -2x + y.$$

First let us plot the phase plane of the system without nullclines.

```
x, y, t = var('x y t')
F = [x + y, -2*x + y]
P = desolve_system_rk4(F,[x,
    y],ics=[0,0.55,0],ivar=t,end_points=2,step=0.01)
Q = [ [j,k] for i,j,k in P]
p = line(Q, axes_labels=['$x(t)$','$y(t)$'], fontsize=18, thickness=2)
n = sqrt(F[0]^2 + F[1]^2)
F_unit = [F[0]/n, F[1]/n]
p += plot_vector_field(F_unit, (x,-4,4), (y,-4,4),
    axes_labels=['$x(t)$','$y(t)$'])
p
```

If these *Sage* commands seem unfamiliar to you, you may want to refer to Subsection 2.1.6

We can add nullclines to our plot using the `implicit_plot` function from *Sage*.

```
x, y, t = var('x y t')
F = [x + y, -2*x + y]
P = desolve_system_rk4(F,[x,
    y],ics=[0,0.55,0],ivar=t,end_points=2,step=0.01)
Q = [ [j,k] for i,j,k in P]
p = line(Q, axes_labels=['$x(t)$','$y(t)$'], fontsize=18, thickness=2)
n = sqrt(F[0]^2 + F[1]^2)
F_unit = [F[0]/n, F[1]/n]
p += plot_vector_field(F_unit, (x,-4,4), (y,-4,4),
    axes_labels=['$x(t)$','$y(t)$'])
p += implicit_plot(F[0], (x,-4,4), (y,-4,4), color='green')
p += implicit_plot(F[1], (x,-4,4), (y,-4,4), color='red')
p
```

The x-nullclines are plotted in green and the y-nullclines are in red.

2.3 Numerical Techniques for Systems

If we are unable to find an analytic solution to a first-order differential equation $y' = f(t, y)$, there is no reason to expect that it will be any easier to solve a system of equations. However, many of the numerical techniques that are used to solve a first-order equation can be extended to solve a system.

2.3.1 Duffing's Equation

Large mobile cranes can reach up to 700 feet (Figure 2.18). This is about the same height as a 50 story building. At such heights, cranes are susceptible to winds and the end of the boom can move back and forth several feet even in moderate winds. We can model the motion as a harmonic oscillator if the side-to-side motion is not too great. However, if the wind becomes too strong and the crane moves too much, gravity will become a factor and the crane may topple due to its weight.

$$\frac{dv}{dt} = -qx - pv.$$

In the case of our swaying crane, we will let $x = x(t)$ be the displacement of the end of the boom from the perfect vertical position. When $x(t) \neq 0$, the boom is bent, and the structure of the boom supplies a strong restorative force to bring everything back to a true vertical position. Thus, the swaying of our crane's boom might be described by an equation such as

$$\frac{d^2x}{dt^2} + 0.1\frac{dx}{dt} + 0.3x = 0$$

or the equivalent system,

$$\begin{aligned}\frac{dx}{dt} &= v,\\ \frac{dv}{dt} &= -0.3x - 0.1v.\end{aligned}$$

We will learn how to solve such systems later, but it is easy to check that

$$x(t) = c_1 e^{-t/20}\cos\left(\frac{\sqrt{119}}{20}t\right) + c_2 e^{-t/20}\sin\left(\frac{\sqrt{119}}{20}t\right)$$

is a solution to our equation. The *Sage* code for solving this system is given below.

```
t = var('t')
x = function('x')(t)
b = 0.1
k = 0.3
position = desolve(diff(x,t,2)+b*diff(x,t)+ k*x == 0,x)
velocity = diff(position, t)
position.show()
velocity.show()
```

The constants c_1 and c_2 can be determined if we know the initial position and initial velocity of the end of the crane's boom. We show several solutions to our equation in Figure 2.19.

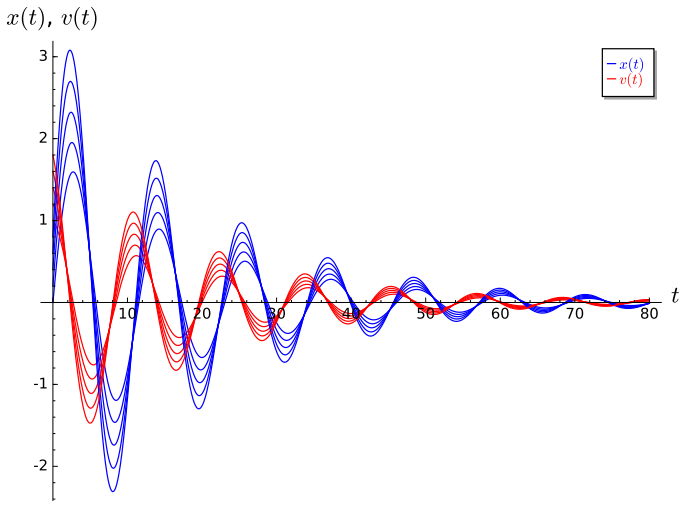

Figure 2.19: Solutions to $x'' + 0.1x' + 0.3x = 0$

Modeling the swaying motion of a crane with a harmonic oscillator might work only if the side-to-side motion is small. If the motion is larger, we must account for the effect of

gravity in our model. When $x(t)$ is large, part of the crane will not be above any other part of the crane. Thus, gravity will pull downward on that part of the crane will cause the crane to bend even further. We can model this effect by setting the equation for our harmonic oscillator equal to a factor of x^3,

$$\frac{d^2x}{dt^2} + 0.1\frac{dx}{dt} + 0.3x = 0.02x^3. \tag{2.13}$$

When x is small, this forcing term will not contribute much to the motion of the building. However, when x is large, the term will contribute a great deal. The equation $x'' + 0.1x' + 0.3x = 0.02x^3$ is an example of ***Duffing's equation***.

We can rewrite equation (2.13) as a system of first-order differential equations by letting $dx/dt = v$,

$$\frac{dx}{dt} = v \tag{2.14}$$

$$\frac{dv}{dt} = -0.1v - 0.3x + 0.02x^3. \tag{2.15}$$

Thus, one of our main objectives should be finding and analyzing solutions to such a system.

2.3.2 Euler's Method for Systems

The system of equations (2.14)–(2.14) is nonlinear due the x^3 term. There is little hope to finding an analytic solution to such as system. We can use software to plot the phase plane in order to learn something about the solutions (Figure 2.20). However, we still need to numerically generate solutions in order to plot the phase plane.

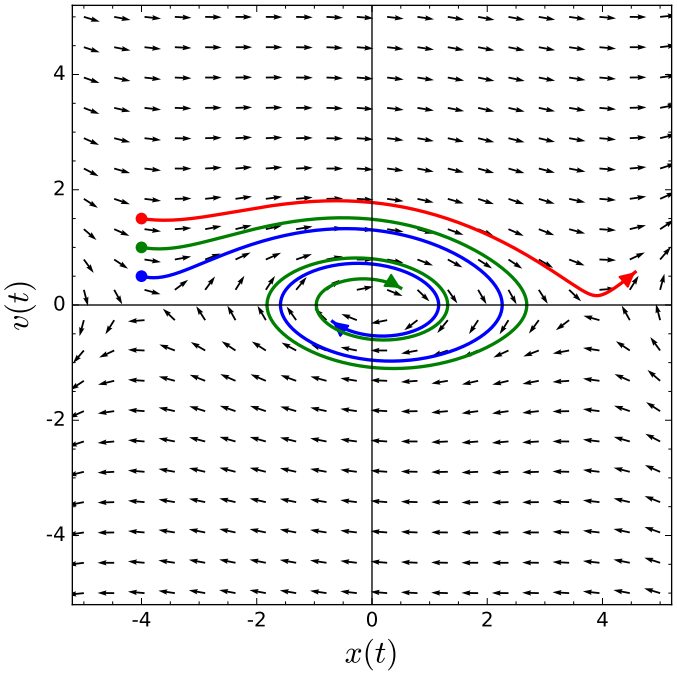

Figure 2.20: The phase plane for Duffing's equation

Let us see how we might find a numerical solution for a system. Consider the system

$$\frac{dx}{dt} = f(t, x, y)$$

$$\frac{dy}{dt} = g(t, x, y),$$

with initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$. We can rewrite our system in vector form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x})$$
$$\mathbf{x}(t_0) = \mathbf{x}_0$$

where $\mathbf{x} = (x, y)$, $d\mathbf{x}/dt = (dx/dt, dy/dt)$, $\mathbf{f} = (f, g)$, and $\mathbf{x}_0 = (x_0, y_0)$. We wish to find approximate values $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ for the solution $x(t)$ and $y(t)$ at the points

$$t_k = t_0 + kh, \qquad k = 1, 2, \ldots, n,$$

where h is the step size. If we let $\mathbf{x}_k = (x_k, y_k)$ and $\mathbf{f}_k = (f(x_k, y_k), g(x_k, y_k))$, then Euler's method now becomes

$$\mathbf{x}_{k+1} = \mathbf{x}_k + h\mathbf{f}_k$$

or

$$x_{k+1} = x_k + hf(t_k, x_k, y_k)$$
$$y_{k+1} = y_k + hg(t_k, x_k, y_k).$$

The initial conditions are used to determine $\mathbf{f}_0$, which is the tangent vector to the graph of the solution $\mathbf{x}(t)$ in the xy-plane (Figure 2.20). We can move in the direction of this tangent vector for time h in order to find the next point $\mathbf{x}_1$. We then calculate a new tangent vector $\mathbf{f}_1$ and then move along this new vector for a time step h to find $\mathbf{x}_2$. We can repeat this technique to generate an approximate solution curve in the phase plane.

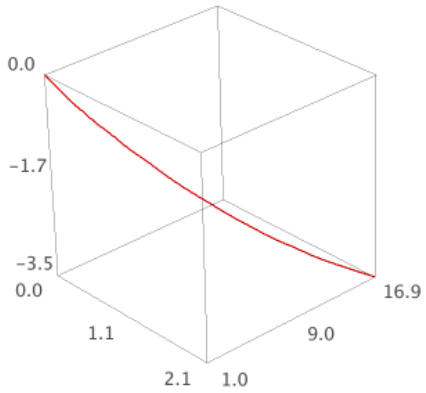

Figure 2.21: A solution curve for Duffing's equation

Example 2.22 (Numerical Solutions for Duffing's Equation)**.** We are now in a position to calculate some solutions to Duffing's equation. Suppose that

$$\frac{dx}{dt} = v,$$

$$\frac{dv}{dt} = -0.3x + 0.02x^3 - 0.1v,$$

with initial conditions $(t_0, x_0, v_0) = (0, 0, 1.7652)$. Using a numerical algorithm, we can generate enough points to generate a graph of the solution (Figure 2.23).

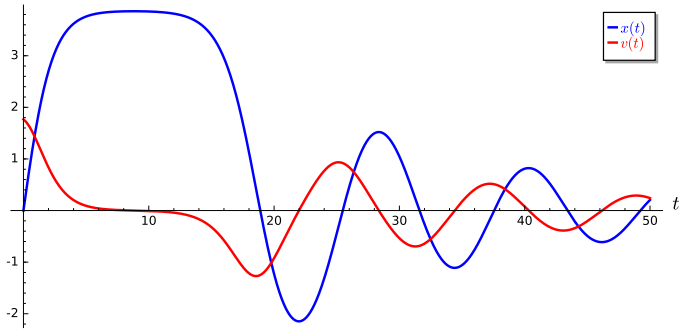

Figure 2.23: Solutions for Duffing's equation with initial conditions $x(0) = 0$ and $v(0) = 1.7652$

The surprising thing about Duffing's equation is that it is so sensitive to initial conditions. A slight change in the initial conditions can yield dramatically different solutions If we change the initial velokcity to $v(0) = 1.7653$, we obtain a very different graph of the solution (Figure 2.23). For small initial velocities, solutions spiral towards the origin. However, a larger initial velocity will send the solution in the phase plane away from the origin. If the crane sways too violently, we will have a disaster.

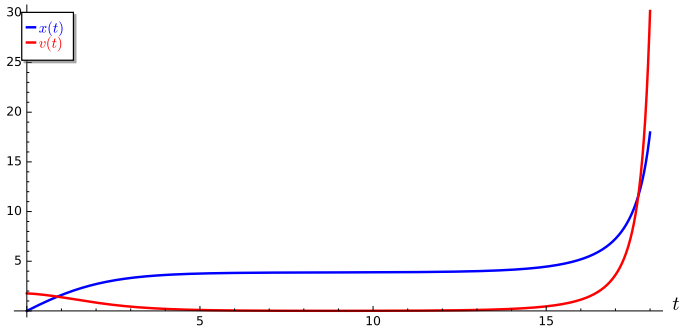

Figure 2.24: Solutions for Duffing's equation with initial conditions $x(0) = 0$ and $v(0) = 1.7653$

Example 2.25. Let us consider the system

$$\begin{aligned} x' &= x - y + t \\ y' &= -x + y^2 - t^2 \\ x(0) &= 1 \\ y(0) &= 0. \end{aligned}$$

Then $(x_0, y_0) = (1, 0)$. Letting the step size be $h = 0.1$, we obtain and

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + h \begin{pmatrix} f(t_0, x_0, y_0) \\ g(t_0, x_0, y_0) \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (0.1) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$= \begin{pmatrix} 1.1 \\ -0.1 \end{pmatrix}.$$

Similarly,

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + h \begin{pmatrix} f(t_1, x_1, y_1) \\ g(t_1, x_1, y_1) \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (0.1) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$= \begin{pmatrix} 1.1 \\ -0.1 \end{pmatrix}.$$

Using this procedure, we can generate a list of values that will approximate the solution to our system (Table 2.26).

k	t_k	x_k	$hf(t_k, x_k, y_k)$	y_k	$hg(t_k, x_k, y_k)$
0	0.0	1.0000	0.1000	0.0000	−0.1000
1	0.1	1.1000	0.1300	−0.1000	−0.0990
2	0.2	1.2300	0.1629	−0.1990	−0.0990
3	0.3	1.3930	0.1991	−0.2980	−0.1004
4	0.4	1.5920	0.2390	−0.3984	−0.1033
5	0.5	1.8310	0.2833	−0.5018	−0.1079
6	0.6	2.1143	0.3324	−0.6097	−0.1143
7	0.7	2.4467	0.3871	−0.7240	−0.1223
8	0.8	2.8338	0.4480	−0.8462	−0.1318
9	0.9	3.2818	0.5160	−0.9780	−0.1425
10	1.0	3.7978	0.5918	−1.1205	−0.1542

Table 2.26: Euler's approximation for a system of equations

Notice that our system is not autonomous and depends on time. Therefore, we cannot graph the phase plane of this system; however, we can graph a solution curve (Figure 2.21).

2.3.3 Taylor Series Methods

Just as in the case of a single first-order differential equation, we can think of Euler's approximation as the first two terms of the Taylor series expansion,

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} + h \begin{pmatrix} x_k' \\ y_k' \end{pmatrix} + \frac{h^2}{2} \begin{pmatrix} x_k'' \\ y_k'' \end{pmatrix} + \cdots$$
$$= \begin{pmatrix} x_k \\ y_k \end{pmatrix} + h \begin{pmatrix} f(t_k, x_k, y_k) \\ g(t_k, x_k, y_k) \end{pmatrix} + \frac{h^2}{2} \begin{pmatrix} x_k'' \\ y_k'' \end{pmatrix} + \cdots.$$

To get a more accurate approximation, we can take the first two terms of the Taylor series. In this case, we must compute $\mathbf{x}''$. If we return our example,

$$x' = x - y + t$$
$$y' = -x + y^2 - t^2$$

$$x(0) = 1$$
$$y(0) = 0.$$

we can see how this is done. First, note that

$$\begin{aligned}
\mathbf{x}'' &= \begin{pmatrix} x'' \\ y'' \end{pmatrix} \\
&= \begin{pmatrix} x' - y' + 1 \\ -x' + 2yy' - 2t \end{pmatrix} \\
&= \begin{pmatrix} (x - y + t) - (-x^2 + y^2 - t^2) \\ -(x - y + t) + 2y(-x + y^2 - t^2) - 2t \end{pmatrix} \\
&= \begin{pmatrix} 2x - y - y^2 + t + t^2 \\ -x + y - 2xy + 2y^3 - 2t^2 y - 3t \end{pmatrix}.
\end{aligned}$$

Our algorithm now becomes clear,

$$\begin{aligned}
\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} &= \begin{pmatrix} x_k \\ y_k \end{pmatrix} + h \begin{pmatrix} f(t_k, x_k, y_k) \\ g(t_k, x_k, y_k) \end{pmatrix} + \frac{h^2}{2} \begin{pmatrix} x_k'' \\ y_k'' \end{pmatrix} \\
&= \begin{pmatrix} x_k + h(x_k - y_k + t_k) + h^2(2x_k - y_k - y_k^2 + t_k + t_k^2)/2 \\ y_k + h(-x_k + y_k^2 - t_k^2) + h^2(-x_k + y_k - 2x_k y_k + 2y_k^3 - 2t_k^2 y_k - 3t_k)/2 \end{pmatrix}.
\end{aligned}$$

Of course, this algorithm requires us to compute some derivatives.

2.3.4 A Word about Existence and Uniqueness

Theorem 2.27. *Let*

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x})$$

be a system of differential equations such that $\mathbf{f}$ *is continuously differentiable. If* $\mathbf{x}_0$ *is the initial value at time* t_0*, there exists a unique solution for the initial value problem on some open interval* $t_0 - \epsilon < t < t_0 + \epsilon$ *for some* $\epsilon > 0$*.*

The consequences of the Existence and Uniqueness Theorem are much the same as they were for first-order differential equations.

- If we are interested in a certain system of differential equations, it is very nice to know that a unique solution exists.
- Uniqueness tells us that two solutions cannot start at the same place. Geometrically, this implies that solution curves cannot cross.
- If we have an autonomous system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, our solution does not depend on time. Thus, we obtain the same solution curve if we start at the same point even though we might start at different times.

For a proof of existence and uniqueness for systems of differential equations, see [12].

2.3.5 Important Lessons

- A damped harmonic oscillator can be described by the second-order equation

$$\frac{d^2x}{dt^2} + p\frac{dx}{dt} + qx = 0.$$

We can rewrite this equation as a first-order system,

$$\begin{aligned}\frac{dx}{dt} &= v,\\ \frac{dv}{dt} &= -qx - pv.\end{aligned}$$

- *Duffing's equation*,

$$\frac{d^2x}{dt^2} + p\frac{dx}{dt} + qx = x^3,$$

is an example of a differential equation that is very sensitive to initial conditions.

- Many systems of equations cannot be solved analytically. However, we can use numerical techniques such as *Euler's Method* to find approximate solutions for the system. Given the system

$$\begin{aligned}\frac{dx}{dt} &= f(x,y)\\ \frac{dy}{dt} &= g(x,y),\end{aligned}$$

with initial condition (x_0, y_0) and step size h, we can approximate a solution to the system by

$$\begin{aligned}x_{k+1} &= x_k + f(x_k, y_k)h\\ y_{k+1} &= y_k + g(x_k, y_k)h.\end{aligned}$$

We can extend this method to the improved Euler's method, Taylor series methods, or Runge-Kutta methods.

- Provided certain conditions are met, we are guaranteed unique local solutions to any system of first-order differential equations. Some of the following are consequences of existence and uniqueness.

- Uniqueness tells us that two distinct solutions cannot start at the same place. Geometrically, this implies that solution curves cannot cross.

- If we have an autonomous system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, our solution does not depend on time. Thus, we obtain the same solution curve if we start at the same point even though we might start at different times.

2.3.6 Exercises

1. Consider the system

$$\begin{aligned}x' &= 2x\\ y' &= y\end{aligned}$$

with initial conditions $x(0) = 1$ and $y(0) = 3$.[3]

[3] You will find Sage very useful for this exercise

(a) Show that $\mathbf{x}(t) = (e^{2t}, 3e^t)$ satisfies the initial value problem.

(b) Use Euler's method with step size $\Delta t = 0.5$ to approximate this solution, and check how close the approximation is to the real solution when $t = 2$, $t = 4$, and $t = 6$.

(c) Use Euler's method with step size $\Delta t = 0.1$ to approximate this solution, and check how close the approximation is to the real solution when $t = 2$, $t = 4$, and $t = 6$.

(d) Discuss how and why the Euler approximations differ from the real solution.

Hint.

2.3.7 Project—Comparing Numerical Methods

2.4 Solving Systems Analytically

Mixing problems model how substances flow back and forth between two or more compartments. These problems often arise in applications—for example, we might want to model how greenhouse gases flow back and forth between different layers of the earth's atmosphere[17], how chemicals move between tanks in a refinery or a brewery, or how pollutants move between a series of lakes or pounds. Systems of differential equations can prove very useful when it comes to modeling such situations.

2.4.1 Mixing Problems

We will use linear systems of differential equations such as

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned}$$

to illustrate how we can use systems of differential equations to model how substances flow back and forth between two or more compartments. Suppose that we have two tanks (A and B) between which a mixture of brine flows (Figure 2.28). Tank A contains 300 liters of water in which 100 kilograms of salt has been dissolved and Tank B contains 300 liters of pure water. Fresh water is pumped into Tank A at the rate of 500 liters per hour, and brine is pumped into Tank B from Tank A at the rate of 500 liters per hour. Brine is also drained at a rate 500 liters of brine per hour from Tank B. All brine mixtures are well-stirred. If we let $x = x(t)$ be the amount of salt in Tank A at time t and $y = y(t)$ be the amount of salt in Tank B at time t, then we know that

$$\begin{aligned} x(0) &= 100 \\ y(0) &= 0 \end{aligned}$$

We know that the salt concentrations in the two tanks are $x/300$ kilograms per liter and $y/300$ kilograms per liter. Thus, we can describe the rate of change in each tank with a differential equation,

$$\begin{aligned} \frac{dx}{dt} &= -500 \cdot \frac{x}{300} = -\frac{5}{3}x, \\ \frac{dy}{dt} &= 500 \cdot \frac{x}{300} - 500 \cdot \frac{y}{300} - 500 \cdot \frac{y}{300} = \frac{5}{3}x - \frac{5}{3}y. \end{aligned}$$

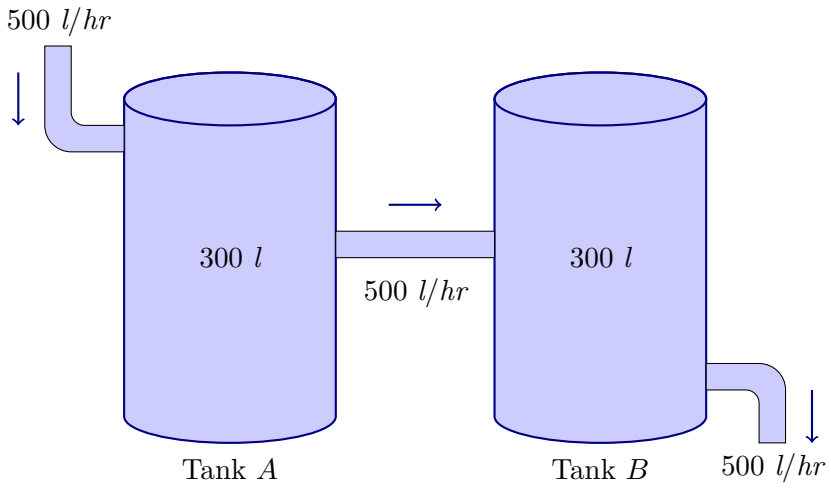

Figure 2.28: Mixing example with two tanks

We can now ask how we might solve the system of equations

$$\begin{aligned}\frac{dx}{dt} &= -\frac{5}{3}x,\\ \frac{dy}{dt} &= \frac{5}{3}x - \frac{5}{3}y.\end{aligned}$$

The task of solving the system

$$\begin{aligned}\frac{dx}{dt} &= f(x,y),\\ \frac{dy}{dt} &= g(x,y),\end{aligned}$$

may be quite difficult or even impossible. However, we can find solutions in certain cases. For example, if we have a system of the form

$$\begin{aligned}\frac{dx}{dt} &= f(x),\\ \frac{dy}{dt} &= g(y),\end{aligned}$$

then each equation is an autonomous first-order equation. To solve our system, we only need to solve two first-order equations. Such a system is said to be ***decoupled***. Generalizing slightly, we say the a ***partially coupled system*** is a system of the form

$$\begin{aligned}\frac{dx}{dt} &= f(x),\\ \frac{dy}{dt} &= g(x,y).\end{aligned}$$

Since the first equation is an autonomous first-order equation in x, we can solve this equation separately, and substitute our solution into the second equation.

Consider the system

$$\begin{aligned}\frac{dx}{dt} &= x\\ \frac{dy}{dt} &= x + y.\end{aligned}$$

We can easily solve the first equation, $dx/dt = x$, to obtain $x = ae^t$. Using this information in the second equation, we have

$$\frac{dy}{dt} - y = ae^t$$

which is a first-order linear equation. This equation has an integrating factor $\mu(t) = e^{-t}$, and

$$\frac{d}{dt}(e^{-t}y) = \mu(t)\left(\frac{dy}{dt} - y\right) = ae^t\mu(t) = a.$$

Therefore, the solution to our second equation is

$$y(t) = ate^t + be^t.$$

Revisiting the mixing problem that we posed at the beginning of this section, we have the following initial value problem,

$$\begin{aligned} \frac{dx}{dt} &= -\frac{5}{3}x, \\ \frac{dy}{dt} &= \frac{5}{3}x - \frac{5}{3}y. \\ x(0) &= 100, \\ y(0) &= 0. \end{aligned}$$

Solving $dx/dt = -(5/3)x$ is easy. We can quickly determine that $x(t) = c_1e^{-5t/3}$. Applying the initial condition $x(0) = 100$, we can determine that $c_1 = 100$ and $x(t) = 100e^{-5t/3}$. Our second equation now becomes

$$\frac{dy}{dt} = \frac{5}{3}x - \frac{5}{3}y = \frac{500}{3}e^{-5t/3} - \frac{5}{3}y.$$

This last equation is a first-order linear equation

$$\frac{dy}{dt} + \frac{5}{3}y = \frac{500}{3}e^{-5t/3}.$$

Multiplying both sides of this last equation by the integrating factor $\mu(t) = e^{5t/3}$ yields

$$\frac{d}{dt}(e^{5t/3}y) = e^{5t/3}\frac{dy}{dt} + e^{5t/3}\frac{5}{3}y = \frac{500}{3}.$$

Integrating both sides of this last equation gives us

$$e^{5t/3}y = \frac{500}{3}t + c_2.$$

Using our initial condition $y(0) = 0$, we can determine that $c_2 = 0$. Thus,

$$y = \frac{500}{3}te^{-5t/3}$$

2.4.2 Using *Sage* to Solve Linear Systems

Solving linear systems such as

$$x' = ax + by$$
$$y' = cx + dy$$

is much more difficult, since we cannot use the same strategies that we used to solve partially-coupled systems. We will devote all of Chapter 3 to find an answer. However, we can use *Sage* to solve linear systems for the moment. The following is a *Sage* interact that will solve the initial-value problem

$$x' = 3x - 4y$$
$$y' = 2x + y$$
$$x(0) = 1$$
$$y() = 1;$$

however, we can change the coefficients and initial value to be anything that we like.

```
t = var('t')
x = function('x')(t)
y = function('y')(t)
@interact
def _(f = 3*x - 4*y, g = 2*x + y,
      t_start = input_box(default = 0), x_start = input_box(default =
          1), y_start = input_box(default = 1)):
    de1 = diff(x,t) == f
    de2 = diff(y,t) == g
    sol = desolve_system([de1, de2], [x,y], ics=[t_start, x_start,
        y_start])
    sol[0].show()
    sol[1].show()
    pretty_print(html(r"$\displaystyle\frac{dx}{dt} = %s$"
        %latex(f)))
    pretty_print(html(r"$\displaystyle\frac{dy}{dt} = %s$"
        %latex(g)))
```

2.4.3 Harmonic Oscillators

The equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$$

is a specific case of a damped harmonic oscillator, where $m = 1$, the spring constant is 2, and the damping constant is 3. We can rewrite this equation as a first-order linear system,

$$\frac{dx}{dt} = v,$$
$$\frac{dv}{dt} = -2x - 3v.$$

Suppose that $x(0) = 0$ is the initial position of the mass and $v(0) = 1$ is the initial velocity. We can use Sage to verify that the solution to our system is

$$x(t) = e^{-t} - e^{-2t},$$

$$v(t) = -e^{-t} + 2e^{-2t}.$$

This is an example of an over-damped harmonic oscillator (Figure 2.29). In other words, a spring-mass system that is modeled by this system of equations has so much damping that the mass will not oscillate.

```
t = var('t')
x = function('x')(t)
v = function('v')(t)
t_start = 0
x_start = 0
v_start =  1
de1 = diff(x,t) == v
de2 = diff(v,t) == -2*x - 3*v
sol = desolve_system([de1, de2], [x,v], ics=[t_start, x_start,
    v_start])
sol[0].show()
sol[1].show()
```

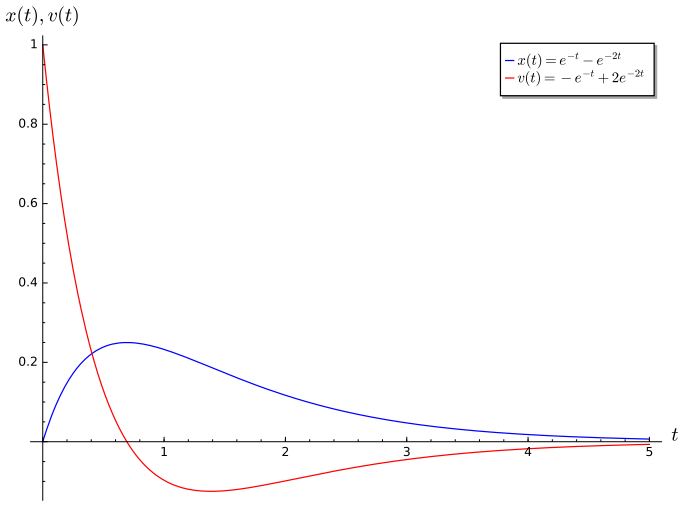

Figure 2.29: An over-damped harmonic oscillator

Now let us relax the damping and increase the spring constant on our harmonic oscillator,

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 0.$$

The corresponding linear system is

$$\begin{aligned} \frac{dx}{dt} &= v, \\ \frac{dv}{dt} &= -10x - 2v \\ x(0) &= 0 \\ v(0) &= 1. \end{aligned}$$

Notice that our initial conditions have not changed. We again use *Sage* to solve our system.

```
t = var('t')
```

```
x = function('x')(t)
v = function('v')(t)
t_start = 0
x_start = 0
v_start =  1
de1 = diff(x,t) == v
de2 = diff(v,t) == -10*x - 2*v
sol = desolve_system([de1, de2], [x,v], ics=[t_start, x_start,
    v_start])
sol[0].show()
sol[1].show()
```

The solution to our system is

$$x(t) = \frac{1}{3}e^{-t}\sin t,$$
$$v(t) = e^{-t}\cos 3t - \frac{1}{3}e^{-t}\sin 3t.$$

Notice that the system oscillates due to the sine and cosine terms in the solution. This system is underdamped (Figure 2.30).

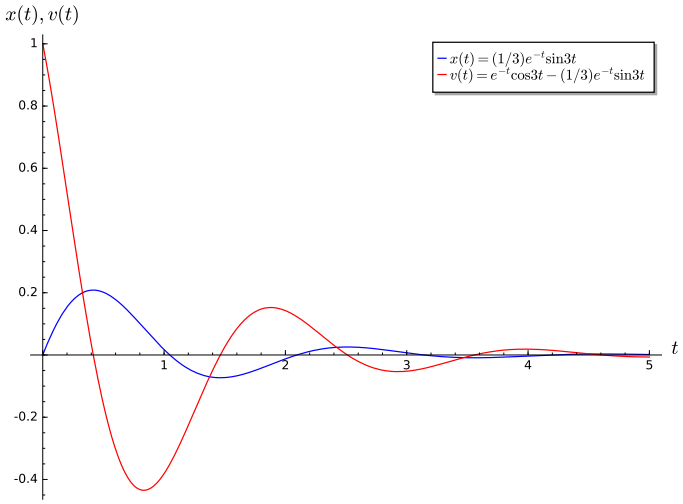

Figure 2.30: An under-damped harmonic oscillator

2.4.4 Important Lessons

- A system of the form

$$\frac{dx}{dt} = f(x),$$
$$\frac{dy}{dt} = g(y)$$

  is said to be *decoupled*. Such a system can be solved by solving each equation independently.

- A system of the form

$$\frac{dx}{dt} = f(x),$$

$$\frac{dy}{dt} = g(x, y)$$

a *partially coupled system*. Since the first equation is an autonomous first-order equation in x, we can solve this equation separately, and substitute our solution into the second equation.

- We can use *Sage* to solve systems of the form

$$x' = ax + by$$
$$y' = cx + dy.$$

- *Sage* is useful for investigating the behavior of harmonic oscillators.

2.4.5 Exercises

1. This is an exercise.

Hint.

3

Linear Systems

3.1 Linear Algebra in a Nutshell

Linear algebra and matrices provide a convenient notation for representing the 2×2 system

$$\frac{dx}{dt} = ax + by,$$
$$\frac{dy}{dt} = cx + dy..$$

If we let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

then we can rewrite our system as

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} ax(t) + by(t) \\ cx(t) + dy(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

In other words, we can write our system as

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

where

$$\mathbf{x}' = \frac{d\mathbf{x}}{dt} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}.$$

3.1.1 Matrices and Systems of Linear Equations

A short review of linear algebra and 2×2 matrices is useful at this point. Recall that any system of two equations in two variables,

$$ax + by = \alpha,$$
$$cx + dy = \beta,$$

can be written as a matrix equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{3.1}$$

We will denote the 2×2 ***coefficient matrix*** by A. That is,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If a solution for the system (3.1) exists, it is easy to find. A unique solution will occur exactly when the matrix A is ***invertible*** (or ***nonsingular***). The unique solution is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

where

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The matrix A is invertible if and only if its ***determinant*** is nonzero,

$$\det(A) = ad - bc \neq 0.$$

If $\det(A) = 0$, then we either have no solution or infinitely many solutions.

Let us consider the special case

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If $\det(A) \neq 0$, we have exactly one solution, $x = 0$ and $y = 0$. On the other hand, if $\det(A) = 0$, we have infinitely many solutions. Suppose that $a \neq 0$. Then $x = -(b/a)y$, and

$$-c \left(\frac{b}{a} \right) y + dy = 0.$$

Therefore, $(ad - bc)y = 0$. Since $\det(A) = ad - bc = 0$, the variable y can assume any value and $x = -(b/a)y$. Thus, the solutions to our system lie along a line through the origin. In fact, we will always get a line of solutions through the origin as long as at least one entry in our matrix is nonzero.[1]

3.1.2 Linear Independence

We say that two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$ are ***linearly independent*** if they do not lie on the same line through the origin. If, on the other hand, they do lie on the same line, then the vectors are ***linearly dependent***. Equivalently, two vectors are linearly dependent if one vector is a multiple of the other. We leave the proof of the following theorem as an exercise.

Theorem 3.1. *Let* $\mathbf{x} = (x_1, x_2)$ *and* $\mathbf{y} = (y_1, y_2)$. *Then* $\mathbf{x}$ *and* $\mathbf{y}$ *are linearly independent if and only if*

$$\det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \neq 0.$$

If we have a pair of linearly independent vectors in $\mathbb{R}^2$, then we can write any vector in $\mathbb{R}^2$ as a unique ***linear combination*** of the two vectors. That is, given two linearly independent vectors $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, we can write $\mathbf{z} = (z_1, z_2)$ as

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \beta \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where α and β are unique. To see why this is true, we must solve the equations

$$z_1 = \alpha x_1 + \beta y_1$$

[1] We will not worry about the 2×2 zero matrix, since it will not play a role in our study of linear equations.

$$z_2 = \alpha x_2 + \beta y_2$$

for α and β. However, this system has a unique solution since

$$\det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \neq 0.$$

Two vectors are said to be a ***basis*** for $\mathbb{R}^2$ if we can write any vector in $\mathbb{R}^2$ as a linear combination of these two vectors. By our arguments above, any two linearly independent vectors will form a basis for $\mathbb{R}^2$.

Example 3.2. The vectors $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ form a basis for $\mathbb{R}^2$. Indeed, if $\mathbf{z} = (z_1, z_2)$, then we can write

$$\mathbf{z} = z_1\mathbf{e}_1 + z_2\mathbf{e}_2.$$

The vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are called the ***standard basis*** for $\mathbb{R}^2$.

Example 3.3. Let $\mathbf{v}_1 = (2, 1)$ and $\mathbf{v}_2 = (3, 2)$. Since

$$\det \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \neq 0,$$

these vectors form a basis for $\mathbb{R}^2$. If $\mathbf{z} = (-5, -4)$, then we can write

$$\mathbf{z} = 2\mathbf{v}_1 - 3\mathbf{v}_2.$$

We say that the ***coordinates*** of $\mathbf{z}$ are $(2, -3)$ with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.

Example 3.4. The vectors $(1, 1)$ and $(-1, -1)$ do not form a basis for $\mathbb{R}^2$ since these two vectors lie along the same line.

If 2×2 matrices and the rest of what we have described above make you nervous, you should work through the exercises at the end of this section.

3.1.3 Finding Eigenvalues and Eigenvectors

A nonzero vector $\mathbf{v}$ is an ***eigenvector*** of A if $A\mathbf{v} = \lambda\mathbf{v}$ for some $\lambda \in \mathbf{R}$. The constant λ is called an ***eigenvalue*** of A. Letting

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} \neq \mathbf{0},$$

we have $A\mathbf{x} = \lambda\mathbf{x}$ or $A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$. In matrix form this is

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \lambda \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

This matrix equation is certainly true if $(x, y) = (0, 0)$. However, we seek nonzero solutions to this system. This will occur exactly when the determinant of

$$A - \lambda I = \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}$$

is zero. In this case

$$\det(A-\lambda I) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = \lambda^2 - (a+d)\lambda + (ad-bc).$$

We say that

$$\det(A-\lambda I) = \lambda^2 - (a+d)\lambda + (ad-bc)$$

is the ***characteristic polynomial*** of A. We summarize the results of this discussion in the following theorem.

Theorem 3.5. *The roots of the characteristic polynomial of A are the eigenvalues of A.*

Example 3.6. Suppose that we wish to find the eigenvalues and associated eigenvectors of

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}.$$

To find the eigenvalues and eigenvectors for A, we must solve the equation

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}.$$

If we let I denote the 2×2 identity matrix,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we can rewrite this equation in the form

$$(A-\lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.2}$$

We know that $A-\lambda I$ is a 2×2 matrix and that this system will only have nonzero solutions if $\det(A-\lambda I) = 0$. In our example,

$$\begin{aligned} \det(A-\lambda I) &= \det\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix} \\ &= (1-\lambda)(3-\lambda) - 8 \\ &= \lambda^2 - 4\lambda - 5 \\ &= (\lambda - 5)(\lambda + 1). \end{aligned}$$

Thus, $\lambda = 5$ or -1.

To see this from a different perspective, we will rewrite equation (3.2) as

$$\begin{aligned} x + 2y &= \lambda x \\ 4x + 3y &= \lambda y. \end{aligned}$$

This system is equivalent to

$$\begin{aligned} (1-\lambda)x + 2y &= 0 \\ 4x + (3-\lambda)y &= 0 \end{aligned}$$

which can be reduced to

$$\begin{aligned} (1-\lambda)x + 2y &= 0 \\ (\lambda^2 - 4\lambda - 5)y &= 0. \end{aligned}$$

Therefore, either $\lambda = 5$ or $\lambda = -1$ to obtain a nonzero solution.

- If $\lambda = 5$, the first equation in the system becomes $-2x + y = 0$, and the eigenvectors corresponding to this eigenvalue are the nonzero solutions of this equation. That is, a vector must be a nonzero multiple of $(1, 2)$ to be an eigenvector of A corresponding to $\lambda = 5$.
- If $\lambda = -1$, then the corresponding eigenvectors are the nonzero multiples of $(1, -1)$.

3.1.4 Important Lessons

- The roots of the characteristic polynomial, $\det(A - \lambda I)$, of a matrix A are the eigenvalues of A. Given a specific eigenvalue, λ, for a matrix A, the eigenvectors associated with A are the nonzero solutions of the system of equations

$$(A - \lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- If $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of two distinct real eigenvalues of a matrix A, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

3.1.5 Exercises

1. If

$$A = \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4 & 1 \\ -1 & -2 \end{pmatrix},$$

find each of the following.

(a) $A + B$

(b) $2A - 3B$

(c) AB

(d) BA

(e) A^{-1}

(f) B^{-1}

Hint.

2. If

$$A = \begin{pmatrix} 2 & 1-i \\ 2-i & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4i & 1-i \\ 1+3i & -2-i \end{pmatrix},$$

find each of the following.

(a) $A + B$

(b) $3A - 2B$

(c) AB

(d) BA

Hint.

3. If

$$A = \begin{pmatrix} 3 & -2 \\ 0 & -1 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} -2 \\ 3 \end{pmatrix},$$

find each of the following.

(a) $A\mathbf{x}$

(b) $A\mathbf{y}$

(c) $\mathbf{x}^T\mathbf{y}$

(d) $\mathbf{y}^T\mathbf{x}$

Hint.

4. Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. Prove that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent if and only if

$$\det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \neq 0.$$

Hint.

3.1.6 Project—Finding Eigenvalues and Eigenvectors

Sage can be used to find eigenvalues and eigenvectors for a matrix A for now. Consider the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}.$$

Using *Sage*, we would enter the matrix A as follows.

```
A = matrix([[1, 3], [1, -1]])
A
```

```
[ 1  3] [ 1 -1]
```

We can use the following command to find the eigenvalues of A.

```
A = matrix([[1, 3], [1, -1]])
A.eigenvalues()
```

```
[2, -2]
```

Sage will also allow us find eigenvectors for each of the eigenvalues of A.

```
A = matrix([[1, 3], [1, -1]])
A.eigenvectors_right()
```

```
[(2, [(1, 1/3)], 1), (-2, [(1, -1)], 1)]
```

Thus, the matrix A has two eigenvalues: $\lambda_1 = 2$ with eigenvector $\mathbf{v}_1 = (1, 1/3)$ and $\lambda_2 = -2$ with eigenvector $\mathbf{v}_2 = (1, -1)$.

There is a third entry in the *Sage* output which refers to the multiplicity of the eigenvalue. In the previous example, the multiplicity is 1. In the matrix

$$B = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}.$$

in the *Sage* cell below, we obtain a single repeated eigenvalue $\lambda = 2$ and only one eigenvector $\mathbf{v} = (1, 1)$. The multiplicity of this eigenvalue is 2. In our previous examples, we obtained two linearly independent eigenvalues allowing us to solve initial value problems given a general solution.

```
B = matrix([[1, 1],[-1, 3]])
B.eigenvectors_right()
```

```
[(2, [(1, 1)], 2)]
```

We may also matrices such as

$$C = \begin{pmatrix} 4 & 1 \\ -1 & 4 \end{pmatrix}$$

has complex eigenvalues, $\lambda = 4 - i$ and $\mu = 4 + i$. Eigenvectors for λ and μ are $\mathbf{u} = (1, -i)$ and $\mathbf{v} = (1, i)$, respectively.

```
C = matrix([[4, 1],[-1, 4]])
C.eigenvectors_right()
```

```
[(4 - 1*I, [(1, -1*I)], 1), (4 + 1*I, [(1, 1*I)], 1)]
```

3.2 Planar Systems

A ***first-order linear system of*** n ***equations and*** n ***variables*** is any system that can be written in the form

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}(t)x_1(t) + \cdots + a_{1n}(t)x_n(t) + f_1(t), \\
\frac{dx_2}{dt} &= a_{21}(t)x_1(t) + \cdots + a_{2n}(t)x_n(t) + f_2(t), \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}(t)x_1(t) + \cdots + a_{nn}(t)x_n(t) + f_n(t).
\end{aligned}$$

If each of the coefficients is constant and the functions f_i vanish, then we have a ***homogeneous first-order linear system with constant coefficients***,

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + \cdots + a_{2n}x_n, \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + \cdots + a_{nn}x_n.
\end{aligned}$$

We will concentrate on 2×2 homogeneous first-order linear systems or ***planar systems*** for the time being,

$$\frac{dx}{dt} = ax + by, \tag{3.3}$$

$$\frac{dy}{dt} = cx + dy. \tag{3.4}$$

3.2.1 Planar Systems and 2×2 Matrices

We will use linear systems of differential equations to illustrate how we can use systems of differential equations to model how subtances flow back and forth between two or more compartments. Suppose that we have two tanks (A and B) between which a mixture of brine flows (Figure 3.7). Tank A contains 300 liters of water in which 100 kilograms of salt

has been dissolved and Tank B contains 300 liters of pure water. Fresh water is pumped into Tank A at the rate of 500 liters per hour, and brine is pumped into Tank B from Tank A at the rate of 900 liters per hour. Brine is also pumped back into Tank A from Tank B at the rate of 400 liters per hour, and an additional 500 liters of brine per hour is drained from Tank B. All brine mixtures are well-stirred. If we let $x = x(t)$ be the amount of salt in Tank A at time t and $y = y(t)$ be the amount of salt in Tank B at time t, then we know that

$$x(0) = 100$$
$$y(0) = 0$$

We know that the salt concentrations in the two tanks are $x/300$ kilograms per liter and $y/300$ kilograms per liter. Thus, we can describe the rate of change in each tank with a differential equation,

$$\frac{dx}{dt} = -900 \cdot \frac{x}{300} + 400 \cdot \frac{y}{300} = -3x + \frac{4}{3}y,$$
$$\frac{dy}{dt} = 900 \cdot \frac{x}{300} - 400 \cdot \frac{y}{300} - 500 \cdot \frac{y}{300} = 3x - 3y.$$

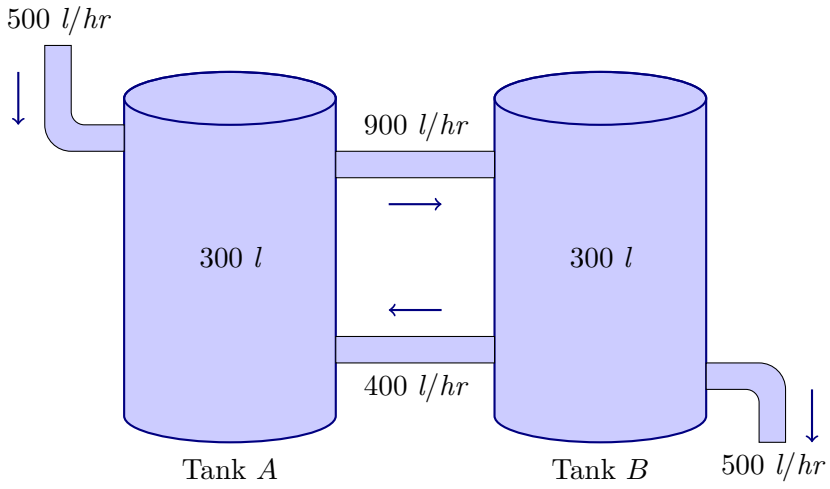

Figure 3.7: Mixing example with two tanks

Matrix notation gives us a convenient way of representing the 2×2 system (3.3)–(3.3). If we let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

then we can rewrite our system as

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} ax(t) + by(t) \\ cx(t) + dy(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

In other words, we can write our system as

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

where

$$\mathbf{x}' = \frac{d\mathbf{x}}{dt} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}.$$

3.2.2 Systems of Differential Equations

A linear planar system

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$

has an ***equilibrium solution*** at (x_0, y_0) if

$$\begin{aligned} ax_0 + by_0 &= 0, \\ cx_0 + dy_0 &= 0. \end{aligned}$$

The following proposition tells us exactly where to find the equilibrium solutions of a linear system with constant coefficients.

Theorem 3.8. *Let*

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

be a 2×2 linear system, where A is not the zero matrix.

1. *If $det(A) \neq 0$, then $(x, y) = (0, 0)$ is the unique equilibrium solution for the system.*
2. *If $det(A) = 0$, then the equilibrium solutions for the system form a straight line in $\mathbb{R}^2$.*

Now let us attack the problem of finding all of the solutions of the system $\mathbf{x}' = A\mathbf{x}$. Suppose that we can find a nonzero vector $\mathbf{v}_0$ such that $A\mathbf{v}_0 = \lambda\mathbf{v}_0$ for some real number λ. In this case, the matrix A just sends the vector $\mathbf{v}_0$ to a vector on the same line through the origin, $\lambda\mathbf{v}_0$. This is a very special case of course; however, we claim that

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}_0$$

is a solution for our linear system if we can find such a vector. To see that this is indeed the case, we compute

$$\begin{aligned} \mathbf{x}'(t) &= \lambda e^{\lambda t}\mathbf{v}_0 \\ &= e^{\lambda t}(\lambda\mathbf{v}_0) \\ &= e^{\lambda t}(A\mathbf{v}_0) \\ &= A(e^{\lambda t}\mathbf{v}_0) \\ &= A\mathbf{x}(t). \end{aligned}$$

In other words, the key to solving a linear system $\mathbf{x}' = A\mathbf{x}$ is to be able to find eigenvalues and eigenvectors for the matrix A. We are now ready to state the results of our discussion in a theorem.

Theorem 3.9. *Let $\mathbf{v}_0$ be an eigenvector for the matrix A with associated eigenvalue λ. Then the function $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}_0$ is a solution of the system $\mathbf{x}' = A\mathbf{x}$.*

We say that the solution $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}_0$ is a ***straight-line solution***. The vector $e^{\lambda t}\mathbf{v}_0$ lies on the same line for each value of t. Note that if $\mathbf{v}_0$ is an eigenvector for A, then any nonzero multiple of $\mathbf{v}_0$ is also an eigenvector for A,

$$A(\alpha\mathbf{v}_0) = \alpha A\mathbf{v}_0 = \alpha(\lambda\mathbf{v}_0) = \lambda(\alpha\mathbf{v}_0).$$

Example 3.10. Consider the system

$$x' = x + 3y$$
$$y' = x - y.$$

We can rewrite this system in matrix form as $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}.$$

The matrix A has an eigenvector $\mathbf{u} = (3, 1)$ with associated eigenvalue $\lambda = 2$, since

$$A\mathbf{u} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \lambda\mathbf{u}.$$

Similarly, $\mathbf{v} = (1, -1)$ is an eigenvalue for A with associated eigenvector $\mu = -2$. Thus, we have two solutions for our system: the equilibrium solution at the origin, the solution

$$\mathbf{x}_1(t) = e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

and the solution

$$\mathbf{x}_2(t) = e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Since

$$\begin{aligned} \frac{d}{dt}(c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)) &= c_1\frac{d}{dt}\mathbf{x}_1(t) + c_2\frac{d}{dt}\mathbf{x}_2(t) \\ &= c_1A\mathbf{x}_1(t) + c_2A\mathbf{x}_2(t) \\ &= A(c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)), \end{aligned}$$

any linear combination of solutions to a linear system is also a solution. Thus, a general solution to our system is

$$\mathbf{x}(t) = c_1e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

or

$$x(t) = 3c_1e^{2t} + c_2e^{-2t}$$
$$y(t) = c_1e^{2t} - c_2e^{-2t}.$$

If we are given initial conditions, say $x(0) = 0$ and $y(0) = 1$, then we can determine c_1 and c_2 by solving the linear system of equations

$$3c_1 + c_2 = 0$$
$$c_1 - c_2 = 1$$

to get $c_1 = 1/4$ and $c_2 = -3/4$. Thus, the solution to our initial value problem is

$$x(t) = \frac{3}{4}e^{2t} - \frac{3}{4}e^{-2t}$$
$$y(t) = \frac{1}{4}e^{2t} + \frac{3}{4}e^{-2t}.$$

If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions to the linear system $\mathbf{x}' = A\mathbf{x}$, then

$$\begin{aligned}\mathbf{x}_1' &= A\mathbf{x}_1\\ \mathbf{x}_2' &= A\mathbf{x}_2.\end{aligned}$$

Thus, for any two real numbers c_1 and c_2,

$$\begin{aligned}\frac{d}{dt}(c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)) &= \alpha\frac{d}{dt}\mathbf{x}_1(t) + c_2\frac{d}{dt}\mathbf{x}_2(t)\\ &= c_1A\mathbf{x}_1(t) + c_2A\mathbf{x}_2(t)\\ &= A(c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)).\end{aligned}$$

We state this result in the following theorem.

Theorem 3.11 (Principle of Superposition)**.** *If A is a 2×2 matrix, then any linear combination of solutions to the linear system $\mathbf{x}' = A\mathbf{x}$ is also a solution.*

Revisiting the mixing problem that we posed at the beginning of this section, we have the following initial value problem,

$$\begin{aligned}\frac{dx}{dt} &= -3x + \frac{4}{3}y,\\ \frac{dy}{dt} &= 3x - 3y,\\ x(0) &= 100,\\ y(0) &= 0.\end{aligned}$$

If we write our system in matrix form, $\mathbf{x}' = A\mathbf{x}$, then

$$A = \begin{pmatrix}-3 & 4/3\\ 3 & -3\end{pmatrix}.$$

It is easy to check that we have eigenvectors $\mathbf{u} = (2, 3)$ and $\mathbf{v} = (-2, 3)$ with eigenvectors $\lambda = -1$ and $\mu = -5$, respectively. Thus, we have two solutions to our system,

$$\begin{aligned}\mathbf{x}_1(t) &= e^{-t}\mathbf{u},\\ \mathbf{x}_2(t) &= e^{-5t}\mathbf{v}.\end{aligned}$$

Since any linear combination of solutions is also a solution,

$$\mathbf{x}(t) = c_1\begin{pmatrix}2e^{-t}\\ 3e^{-t}\end{pmatrix} + c_2\begin{pmatrix}-2e^{-5t}\\ 3e^{-5t}\end{pmatrix}$$

is a solution to our system. Using the initial values $x(0) = 100$ and $y(0) = 0$, we can determine that $c_1 = 25$ and $c_2 = -25$. We now have the solution that we seek,

$$\begin{aligned}x(t) &= 50e^{-t} + 50e^{-5t}\\ y(t) &= 75e^{-t} - 75e^{-5t}.\end{aligned}$$

3.2.3 Solving Linear Systems

Our goal is to prove the following theorem.

Theorem 3.12. *Suppose that A has a pair of distinct real eigenvalues, λ_1 and λ_2, with associated eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Then the general solution of the linear system $\mathbf{x}' = A\mathbf{x}$ is given by*

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2.$$

Lemma 3.13. *Let A be an 2×2 matrix with a pair of distinct real eigenvalues, λ_1 and λ_2 and eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.*

Proof. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent, then there exists $\alpha \neq 0$ such that

$$\mathbf{v}_1 = \alpha\mathbf{v}_2. \tag{3.5}$$

Multiplying both sides of this equation by A, we have

$$\lambda_1\mathbf{v}_1 = A\mathbf{v}_1 = \alpha A\mathbf{v}_2 = \alpha\lambda_2\mathbf{v}_2. \tag{3.6}$$

One the other hand, we obtain

$$\lambda_2\mathbf{v}_1 = \alpha\lambda_2\mathbf{v}_2 \tag{3.7}$$

if we multiply both sides of (3.5) by λ_2. Using (3.6) and (3.7), we can conclude that

$$(\lambda_1 - \lambda_2)\mathbf{v}_1 = \alpha(\lambda_2 - \lambda_2)\mathbf{v}_2 = 0\mathbf{v}_2 = \mathbf{0}.$$

However, this contradicts the assumption that λ_1 and λ_2 are distinct. □

We can now proceed to the proof of the theorem. Suppose that we have a linear system $\mathbf{x}' = A\mathbf{x}$ such that A has a pair of distinct real eigenvalues, λ_1 and λ_2, with associated eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. By the Principle of Superposition, we know that

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2.$$

is a solution to the linear system $\mathbf{x}' = A\mathbf{x}$. To show that this is the general solution, we must show that we can choose c_1 and c_2 to satisfy a given initial condition $\mathbf{x}_0 = \mathbf{x}(0) = (x_0, y_0)$. By Lemma 3.13, we know that $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\mathbb{R}^2$. That is, we can write $\mathbf{x}_0$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. In other words, we can find c_1 and c_2 such that

$$\mathbf{x}_0 = \mathbf{x}(0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2.$$

It remains to show that $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$ is the unique solution to the system

$$\begin{aligned}\mathbf{x}'(t) &= A\mathbf{x}(t),\\ \mathbf{x}(0) &= \mathbf{x}_0.\end{aligned}$$

Suppose that there is another solution $\mathbf{y}(t)$ such that $\mathbf{y}(0) = \mathbf{x}_0$. Then we can write

$$\mathbf{y}(t) = f(t)\mathbf{v}_1 + g(t)\mathbf{v}_2,$$

where

$$\begin{aligned}f(0) &= c_1,\\ g(0) &= c_2.\end{aligned}$$

Since $\mathbf{y}(t)$ is a solution to our system of equations, we know that

$$A\mathbf{y}(t) = \mathbf{y}'(t) = f'(t)\mathbf{v}_1 + g'(t)\mathbf{v}_2.$$

On the other hand,

$$A\mathbf{y}(t) = f(t)A\mathbf{v}_1 + g(t)A\mathbf{v}_2 = \lambda_1 f(t)\mathbf{v}_1 + \lambda_2 g(t)\mathbf{v}_2.$$

Consequently, we have two first-order initial value problems,

$$\begin{aligned} f'(t) &= \lambda_1 f(t), \\ f(0) &= c_1, \end{aligned}$$

and

$$\begin{aligned} g'(t) &= \lambda_2 g(t), \\ g(0) &= c_2. \end{aligned}$$

The solutions of these initial value problems are

$$\begin{aligned} f(t) &= c_1 e^{\lambda_1 t}, \\ g(t) &= c_2 e^{\lambda_2 t}, \end{aligned}$$

respectively. Thus, $\mathbf{y}(t) = \mathbf{x}(t)$, and proof our theorem is complete.

3.2.4 Important Lessons

- If $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of two distinct real eigenvalues of a matrix A, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.
- The Principle of Superposition tells us that any linear combination of solutions to the linear system $\mathbf{x}' = A\mathbf{x}$ is also a solution.
- Let A be a 2×2 matrix. If A has a pair of distinct real eigenvalues, λ_1 and λ_2, with associated eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then the general solution of the linear system $\mathbf{x}' = A\mathbf{x}$ is given by
$$\mathbf{x}(t) = \alpha e^{\lambda_1 t}\mathbf{v}_1 + \beta e^{\lambda_2 t}\mathbf{v}_2.$$

3.2.5 Exercises

1. Solve the following initial-value problem,

$$\begin{aligned} x' &= x + 2y \\ y' &= -x + 4y \\ x(0) &= 3 \\ y(0) &= 2. \end{aligned}$$

2. Solve the following initial-value problem,

$$\begin{aligned} x' &= -x + 3y \\ y' &= -3x - y \\ x(0) &= 3 \\ y(0) &= 2. \end{aligned}$$

3. Solve the following initial-value problem,

$$\begin{aligned} x' &= -2x + y \\ y' &= -9x + 4y \\ x(0) &= 5 \\ y(0) &= 3. \end{aligned}$$

4. Consider the system of differential equations,

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{f},$$

where $\mathbf{x}(t) = (x(t), y(t))$, $\mathbf{f}(t) = (f(t), g(t))$, and

$$A = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}.$$

Even though this system is not linear, we are often able to find solutions to this system. Let us assume that the solution of $d\mathbf{x}/dt = A\mathbf{x}$ is given by

$$\mathbf{x}_h = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} + c_2 \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}.$$

If A has constant entries, we can certainly determine this solution if A has distinct real eigenvalues. Now, suppose that $\mathbf{x}_p = (\phi_1(t), \phi_2(t))$ is a solution to $d\mathbf{x}/dt = A\mathbf{x} + \mathbf{f}$. Show that

$$\mathbf{x}_h + \mathbf{x}_p = \begin{pmatrix} x(t) + \phi_1(t) \\ y(t) + \phi_2(t) \end{pmatrix}$$

is the general solution to the system. Consequently, to solve a nonhomogeneous system of linear differential equations, we need to find the solution of the corresponding homogeneous system and one particular solution of the nonhomogeneous system.

5. Consider the linear system

$$\begin{aligned} x' &= x + 3y + (t - 3t^2) \\ y' &= x - y + (2 - t + t^2) \\ x(0) &= 1 \\ y(0) &= -1. \end{aligned}$$

(a) Find the general solution of the homogeneous system

$$\begin{aligned} x' &= x + 3y \\ y' &= x - y \end{aligned}$$

(b) Find a particular solution for

$$\begin{aligned} x' &= x + 3y + (t - 3t^2) \\ y' &= x - y + (1 + t - t^2) \end{aligned}$$

(c) Find the solution of

$$\begin{aligned} x' &= x + 3y + (t - 3t^2) \\ y' &= x - y + (1 + t - t^2) \\ x(0) &= 1 \\ y(0) &= -1. \end{aligned}$$

Hint. Assume that your solution must be of the form

$$\mathbf{x}_p = \begin{pmatrix} a_2t^2 + a_1t + a_0 \\ b_2t^2 + b_1t + b_0. \end{pmatrix}$$

This is called the ***method of undetermined coefficients***.

6. Consider the system

$$\begin{aligned} x' &= ax + y \\ y' &= 2ax + 2y, \end{aligned}$$

where $a \in \mathbb{R}$. For what values of a do you find a bifurcation (a change in the type of phase portrait)? Sketch typical phase portraits for a values of a above and below the bifurcation point.

7. Prove that

$$\alpha e^{\lambda t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda t}\begin{pmatrix} t \\ 1 \end{pmatrix}$$

is the general solution of

$$\mathbf{x}' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\mathbf{x}.$$

3.2.6 Project

3.3 Phase Plane Analysis of Linear Systems

In Section 3.2, we learned how to solve the system

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$$

provided the system has distinct real eigenvalues. If A has distinct real eigenvalues λ and μ with eigenvectors $\mathbf{u}$ and $\mathbf{v}$, respectively, then the general solution of the system is

$$\mathbf{x}(t) = c_1e^{\lambda t}\mathbf{u} + c_2e^{\mu t}\mathbf{v}.$$

Furthermore, we can use the general solution of such a system to find the straight-line solutions to the system. If $c_2 = 0$, then all solutions will lie along the line in the xy-plane that contains the vector u. Similarly, if $c_1 = 0$, then all solutions will lie long the line in the xy-plane that contains the vector u.

3.3.1 The Case $\lambda_1 < 0 < \lambda_2$

Example 3.14. The system

$$\begin{aligned} x' &= x + 3y \\ y' &= x - y \end{aligned}$$

can be written in matrix form $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}.$$

The eigenvalues of A are $\lambda = -2$ or $\lambda = 2$ with eigenvectors $\mathbf{u} = (1, -1)$ and $\mathbf{v} = (3, 1)$, respectively. Therefore, the straightline solutions must lines containing $\mathbf{u}$ and $\mathbf{v}$ (Figure 3.15).

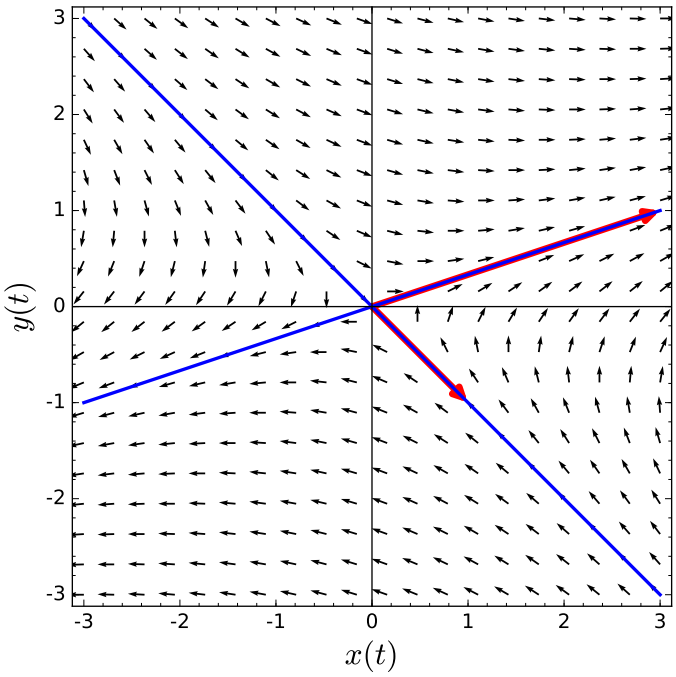

Figure 3.15: Straightline solutions

Let us consider the special case of the system $\mathbf{x}' = A\mathbf{x}$, where $\lambda_1 < 0 < \lambda_2$ and

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Since this is a decoupled system,

$$\begin{aligned} \frac{dx}{dt} &= \lambda_1 x \\ \frac{dy}{dt} &= \lambda_2 y, \end{aligned}$$

we already know how to find the solutions. However, in keeping with the spirit of our investigation, we will find the eigenvalues of A. The characteristic equation of A is

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0,$$

and our eigenvalues are λ_1 and λ_2. It is easy to see that we can associate eigenvectors $(1, 0)$ and $(0, 1)$ to λ_1 and λ_2, respectively. Thus, our general solution is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

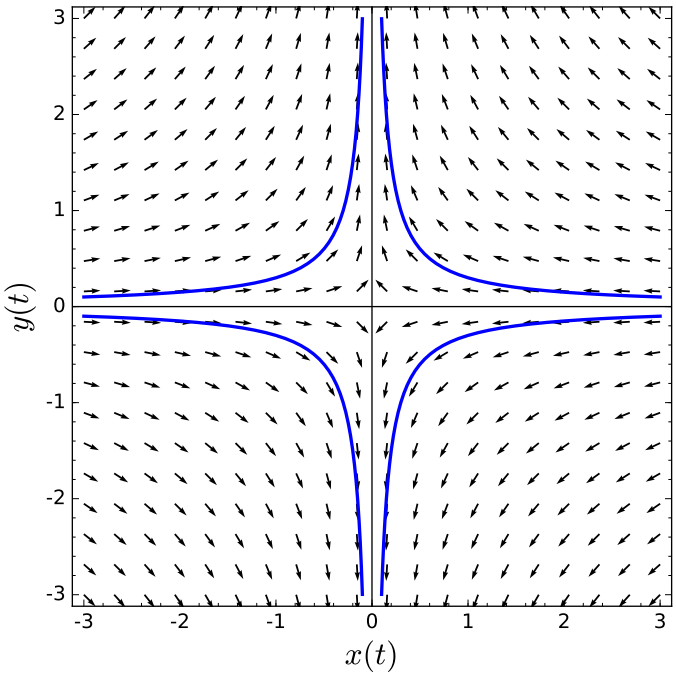

Figure 3.16: Saddle phase portrait

Since $\lambda_1 < 0$, the straight-line solutions of the form $c_1 e^{\lambda_1 t}(1,0)$ lie on the x-axis. These solutions approach zero as $t \to \infty$. On the other hand, the solutions $c_2 e^{\lambda_2 t}(0,1)$ line on the y-axis and approach infinity as $t \to \infty$. The x-axis is called the ***stable line***, and the y-axis is called the ***unstable line***. All other solutions

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(with $c_1, c_2 \neq 0$) tend to infinity in the direction of the unstable line, since $\mathbf{x}(t)$ approaches $(0, c_2 e^{\lambda_2 t})$ as $t \to \infty$. The phase portrait for the system

$$\begin{aligned} x' &= -x \\ y' &= y \end{aligned}$$

is given in Figure 3.16. The equilibrium point of such systems is called a ***saddle***.

Example 3.17. For the system in Example 3.14, the unstable line of solutions is

$$\mathbf{x}_1(t) = c_1 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Each solution tends away from the origin as $t \to \infty$. The stable line of solutions is given by

$$\mathbf{x}_2(t) = c_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

and each solution on this line approaches the origin as $t \to \infty$. By the Principle of Superposition, the general solution to the system is

$$\mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

If $c_1 \neq 0$, we have $\mathbf{x}(t) \to \mathbf{x}_1(t)$ as $t \to \infty$. If $c_2 \neq 0$, we have $\mathbf{x}(t) \to \mathbf{x}_2(t)$ as $t \to -\infty$. Thus, we have the phase portrait in Figure 3.18.

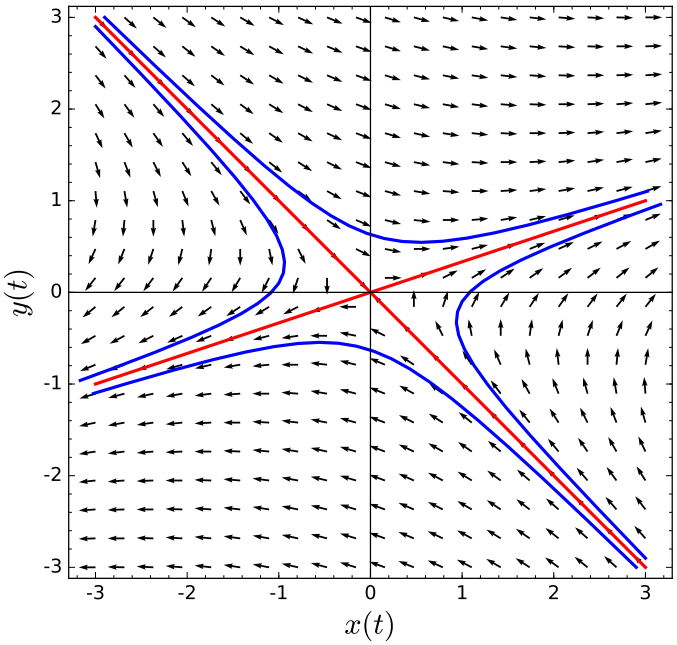

Figure 3.18: Saddle phase portrait

For the general case, where A has eigenvalues $\lambda_1 < 0 < \lambda_2$, we always have a stable line of solutions and an unstable line of solutions. All other solutions approach the unstable line as $t \to \infty$ and the stable line as $t \to -\infty$.

3.3.2 The Case $\lambda_1 < \lambda_2 < 0$

Suppose $\lambda_1 < \lambda_2 < 0$ and consider the diagonal system

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

The general solution of this system is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

but unlike the case of the saddle, all solutions tend towards the origin as $t \to \infty$. To see how they solutions approach the origin, we will compute dy/dx for $c_2 \neq 0$. If

$$\begin{aligned} x(t) &= c_1 e^{\lambda_1 t} \\ y(t) &= c_2 e^{\lambda_2 t}, \end{aligned}$$

then

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\lambda_2 c_2 e^{\lambda_2 t}}{\lambda_1 c_1 e^{\lambda_1 t}} = \frac{\lambda_2 c_2}{\lambda_1 c_1} e^{(\lambda_2 - \lambda_1)t}.$$

Since $\lambda_2 - \lambda_1 > 0$, the derivative, dy/dx, must approach $\pm\infty$, provided $c_2 \neq 0$. Therefore, the solutions tend towards the origin tangentially to the y-axis (Figure 3.19). We say that the equilibrium point for this system is a ***sink***.

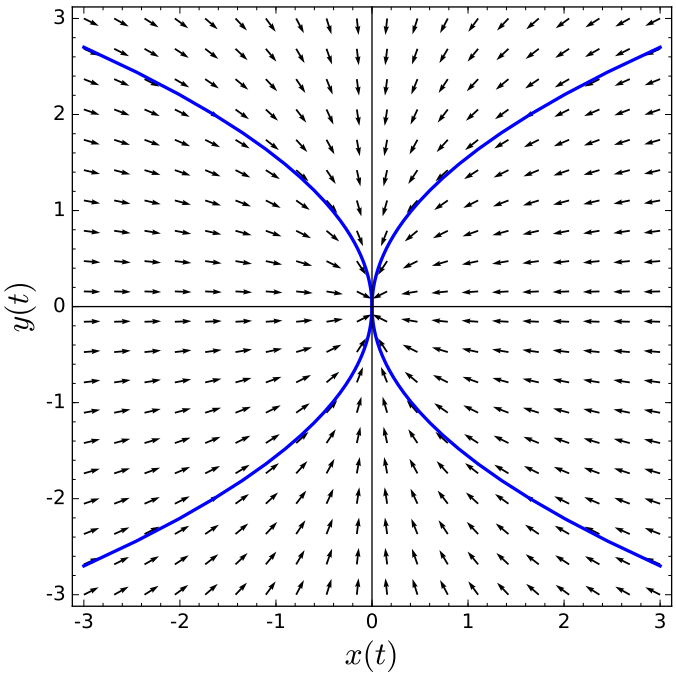

Figure 3.19: Sink phase portrait

Since $\lambda_1 < \lambda_2 < 0$, we say that λ_1 is the ***dominant eigenvalue***. The x-coordinates of the solutions approach the origin much faster than the y-coordinates.

To see what happens in the general case, suppose that $\lambda_1 < \lambda_2 < 0$, the eigenvectors associated with λ_1 and λ_2 are (u_1, u_2) and (v_1, v_2), respectively. The general solution of our system is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

The slope of a solution curve at (x, y) is given by

$$\begin{aligned} \frac{dy}{dx} &= \frac{\lambda_1 c_1 e^{\lambda_1 t} u_2 + \lambda_2 c_2 e^{\lambda_2 t} v_2}{\lambda_1 c_1 e^{\lambda_1 t} u_1 + \lambda_2 c_2 e^{\lambda_2 t} v_1} \\ &= \left(\frac{\lambda_1 c_1 e^{\lambda_1 t} u_2 + \lambda_2 c_2 e^{\lambda_2 t} v_2}{\lambda_1 c_1 e^{\lambda_1 t} u_1 + \lambda_2 c_2 e^{\lambda_2 t} v_1} \right) \frac{e^{-\lambda_2 t}}{e^{-\lambda_2 t}} \\ &= \frac{\lambda_1 c_1 e^{(\lambda_1 - \lambda_2) t} u_2 + \lambda_2 c_2 v_2}{\lambda_1 c_1 e^{(\lambda_1 - \lambda_2) t} u_1 + \lambda_2 c_2 v_1}. \end{aligned}$$

This last expression tends toward the slope v_2/v_1 of the eigenvector of λ_2 (unless $c_2 = 0$). If $c_2 = 0$, then we have the straight-line solution corresponding to the eigenvalue λ_1. Hence, all the solutions for this case (except those on the straight-line belonging to the dominant eigenvalue) tend toward the origin tangentially to the straight-line solution corresponding to the weaker eigenvalue, λ_2.

Example 3.20. Consider the system

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -5 & -2 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

The eigenvalues of this system are $\lambda_1 = -6$ and $\lambda_2 = -3$ with eigenvectors $\mathbf{v}_1 = (2, 1)$ and $\mathbf{v}_2 = (1, -1)$, respectively. Since the dominant eigenvalue is $\lambda_2 = -6$, solutions tend towards the straightline solution containing the vector $\mathbf{v}_2 = (1, -1)$ more quickly (Figure 3.21).

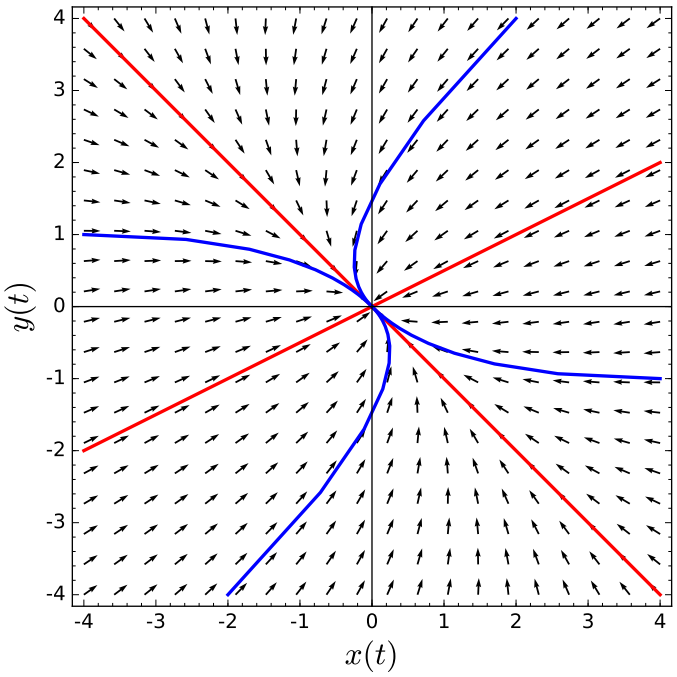

Figure 3.21: Source phase portrait

3.3.3 The Case $\lambda_1 > \lambda_2 > 0$

If $\lambda_1 > \lambda_2 > 0$, we can regard our direction field as the negative of the direction field of the previous case. The general solution and the direction field are the same, but the arrows are reversed (Figure 3.22). In this case, we say that the equilibrium point is a ***source***.

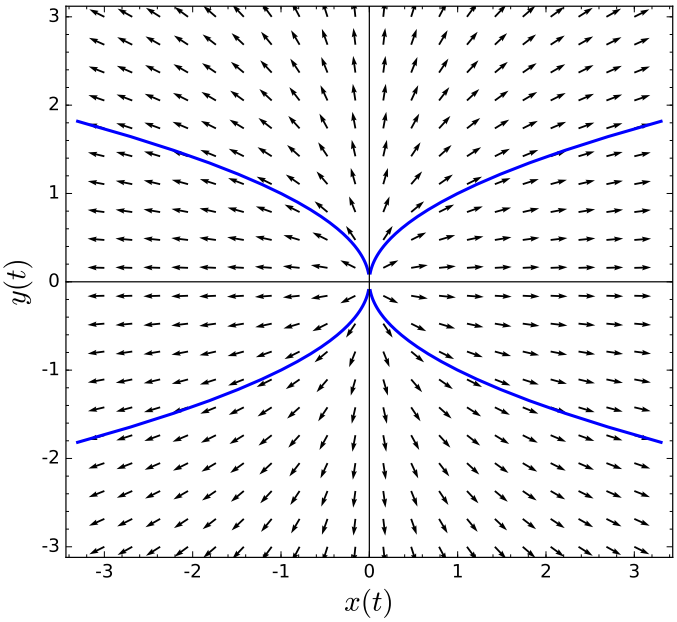

Figure 3.22: Source phase portrait

Example 3.23. Consider the system

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -5 & -2 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

The eigenvalues of this system are $\lambda_1 = 5$ and $\lambda_2 = 1$ with eigenvectors $\mathbf{v}_1 = (3, -4)$ and $\mathbf{v}_2 = (1, 1)$, respectively. Since the dominant eigenvalue is $\lambda_1 = 5$, solutions are closer to the straightline solution containing the vector $\mathbf{v}_2 = (3, -4)$ more as $t \to \infty$ (Figure 3.24).

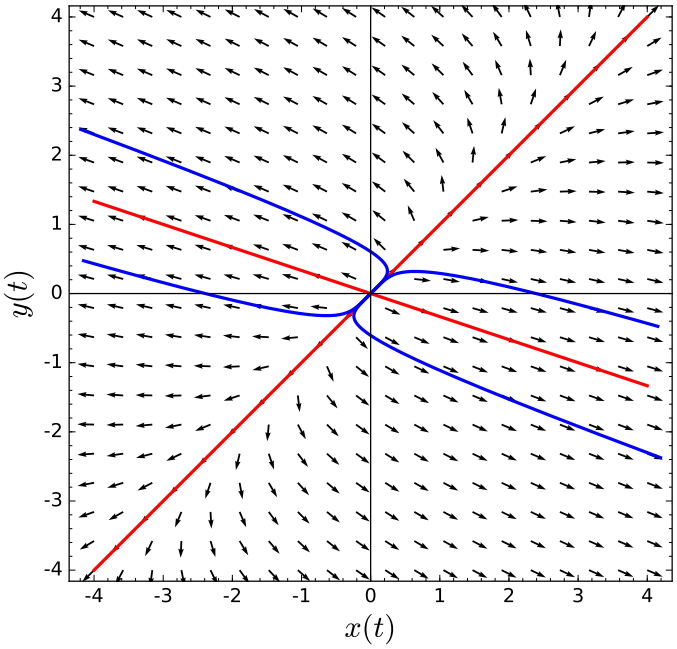

Figure 3.24: Source phase portrait

3.3.4 Important Lessons

- Given a system of linear differential equations

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix},$$

  we can use the eigenvalues of A to find and classify the solutions of the system.

- If

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

  then A has two distinct real eigenvalues. The general solution to the system $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = \alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

  - For the case $\lambda_1 < 0 < \lambda_2$, the equilibrium point of the system $\mathbf{x}' = A\mathbf{x}$ is a saddle.
  - For the case $\lambda_1 < \lambda_2 < 0$, the equilibrium point of the system $\mathbf{x}' = A\mathbf{x}$ is a sink.
  - For the case $0 < \lambda_1 < \lambda_2$, the equilibrium point of the system $\mathbf{x}' = A\mathbf{x}$ is a source.

3.3.5 Exercises

1. This is an exercise.

Hint.

3.3.6 Project

3.4 Complex Eigenvalues

Consider the following system,

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.8}$$

The characteristic polynomial of the system (3.8) is $\lambda^2+4\lambda+5$. The roots of this polynomial are $\lambda_1 = -2+i$ and $\lambda_2 = -2-i$ with eigenvectors $\mathbf{v}_1 = (1, 1+i)$ and $\mathbf{v}_2 = (1, 1-i)$, respectively. It is clear from the phase portrait of the system that there are no straightline solutions (Figure 3.25). However, we would like to real solutions to a linear system with real coefficients.

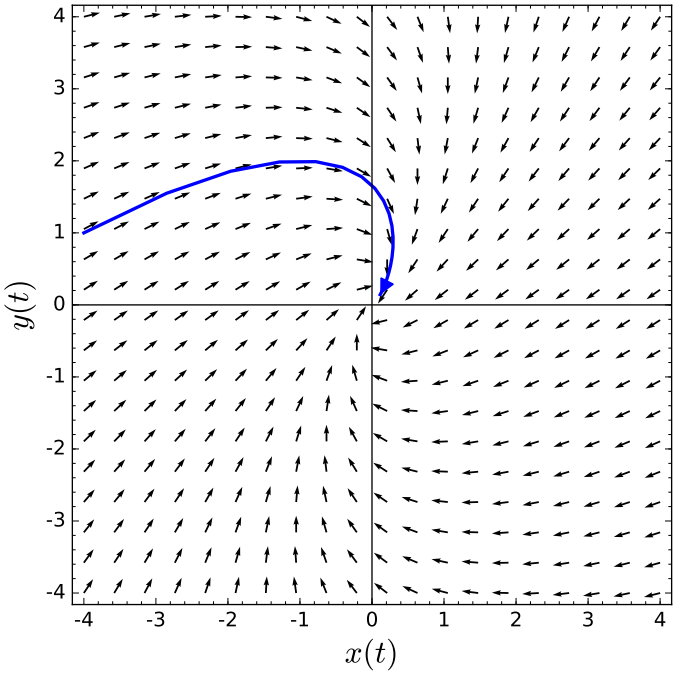

Figure 3.25: A system with no straightline solutions

3.4.1 Complex Eigenvalues

Suppose that we have the system

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix},$$

where $\beta \neq 0$. The characteristic polynomial of this system is $\det(A-\lambda I) = \lambda^2+\beta^2$, and so we have imaginary eigenvalues $\pm i\beta$. To find the eigenvector corresponding to $\lambda = i\beta$, we

must solve the system

$$\begin{pmatrix} -i\beta & \beta \\ -\beta & -i\beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

however, this reduces to solving the equation $i\beta x = \beta y$. Thus, we can find a complex eigenvector $(1, i)$. Consequently,

$$\mathbf{x}(t) = e^{i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

is a complex solution to the system $\mathbf{x}' = A\mathbf{x}$. The problem is that we have a real system of differential equations and would like real solutions. We can remedy the situation if we use ***Euler's formula***, [2]

$$e^{i\beta t} = \cos \beta t + i \sin \beta t.$$

Let us rewrite our solution as

$$\begin{aligned} \mathbf{x}(t) &= e^{i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix} \\ &= \begin{pmatrix} \cos \beta t + i \sin \beta t \\ i(\cos \beta t + i \sin \beta t) \end{pmatrix} \\ &= \begin{pmatrix} \cos \beta t + i \sin \beta t \\ -\sin \beta t + i \cos \beta t \end{pmatrix} \\ &= \begin{pmatrix} \cos \beta t \\ -\sin \beta t \end{pmatrix} + i \begin{pmatrix} \sin \beta t \\ \cos \beta t \end{pmatrix} \end{aligned}$$

and consider the real and imaginary parts of the solution:

$$\mathbf{x}_{\text{Re}} = \begin{pmatrix} \cos \beta t \\ -\sin \beta t \end{pmatrix} \qquad \text{and} \qquad \mathbf{x}_{\text{Im}} = \begin{pmatrix} \sin \beta t \\ \cos \beta t \end{pmatrix}.$$

Since

$$\begin{aligned} \mathbf{x}'_{\text{Re}}(t) + i\mathbf{x}'_{\text{Im}}(t) &= \mathbf{x}'(t) \\ &= A\mathbf{x}(t) \\ &= A(\mathbf{x}_{\text{Re}}(t) + i\mathbf{x}_{\text{Im}}(t) \\ &= A\mathbf{x}_{\text{Re}}(t) + iA\mathbf{x}_{\text{Im}}(t). \end{aligned}$$

we know that $\mathbf{x}'_{\text{Re}}(t) = A\mathbf{x}_{\text{Re}}(t)$ and $\mathbf{x}'_{\text{Im}}(t) = A\mathbf{x}_{\text{Im}}(t)$ by setting the real and imaginary parts equal. Thus, both $\mathbf{x}_{\text{Re}}(t)$ and $\mathbf{x}_{\text{Im}}(t)$ are solutions to our system. Moreover, since

$$\mathbf{x}_{\text{Re}}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{x}_{\text{Im}}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we know that the appropriate linear combination of $\mathbf{x}_{\text{Re}}(t)$ and $\mathbf{x}_{\text{Im}}(t)$ will provide a solution to any initial value problem.

We claim that

$$\mathbf{x}(t) = c_1\mathbf{x}_{\text{Re}}(t) + c_2\mathbf{x}_{\text{Im}}(t) \tag{3.9}$$

is a general solution to our system. That is, we must be able to write every solution of our system as a linear combination of $\mathbf{x}_{\text{Re}}(t)$ and $\mathbf{x}_{\text{Im}}(t)$. If

$$\mathbf{y}(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

[2] If you are unfamiliar with Euler's formula, try expanding both sides as a power series to check that we do indeed have a correct identity.

is another solution to the system $\mathbf{x}' = A\mathbf{x}$, then

$$\mathbf{y}'(t) = \begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \beta v(t) \\ -\beta u(t) \end{pmatrix}.$$

In other words, $u'(t) = \beta v(t)$ and $v'(t) = -\beta u(t)$. Now, define f by

$$f(t) = (u(t) + iv(t))e^{i\beta t}.$$

The derivative of f is

$$\begin{aligned} f'(t) &= (u'(t) + iv'(t))e^{i\beta t} + i\beta(u(t) + iv(t))e^{i\beta t} \\ &= (\beta v(t) - i\beta u(t))e^{i\beta t} + (i\beta u(t) + i^2\beta v(t))e^{i\beta t} \\ &= 0. \end{aligned}$$

Therefore, $f(t)$ is a complex constant and $f(t) = (u(t) + iv(t))e^{i\beta t} = a + bi$. We can now write $u(t) + iv(t) = (a + ib)e^{-i\beta t}$. Thus,

$$\begin{aligned} u(t) + iv(t) &= (a + ib)e^{-i\beta t} \\ &= (a + ib)(\cos\beta t - i\sin\beta t) \\ &= (a\cos\beta t + b\sin\beta t) + i(b\cos\beta t - a\sin\beta t). \end{aligned}$$

Therefore,

$$\begin{aligned} u(t) &= a\cos\beta t + b\sin\beta t \\ v(t) &= b\cos\beta t - a\sin\beta t. \end{aligned}$$

Consequently,

$$\begin{aligned} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} &= \begin{pmatrix} a\cos\beta t + b\sin\beta t \\ b\cos\beta t - a\sin\beta t \end{pmatrix} \\ &= a\begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix} + b\begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix} \\ &= a\mathbf{x}_{\mathrm{Re}}(t) + b\mathbf{x}_{\mathrm{Im}}(t). \end{aligned}$$

Notice that the solutions

$$\mathbf{x}(t) = c_1\begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix} + c_2\begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix}$$

are periodic with period $2\pi/\beta$. This type of system is called a ***center***.

Example 3.26. Consider the initial value problem

$$\begin{aligned} \frac{dx}{dt} &= 2y \\ \frac{dy}{dt} &= -2x \\ x(0) &= 1 \\ y(0) &= 2. \end{aligned}$$

The eigenvalues of this system are $\lambda = \pm 2i$. Therefore, the general solution to the system is

$$x(t) = c_1\cos 2t + c_2\sin 2t$$

$$y(t) = -c_1 \sin 2t + c_2 \cos 2t.$$

Using the initial conditions to solve for c_1 and c_2, the solution to our initial value problem is

$$\begin{aligned} x(t) &= \cos 2t + 2 \sin 2t \\ y(t) &= -\sin 2t + 2 \cos 2t. \end{aligned}$$

The phase portrait is a circle of radius 2 about the origin (Figure 3.27).

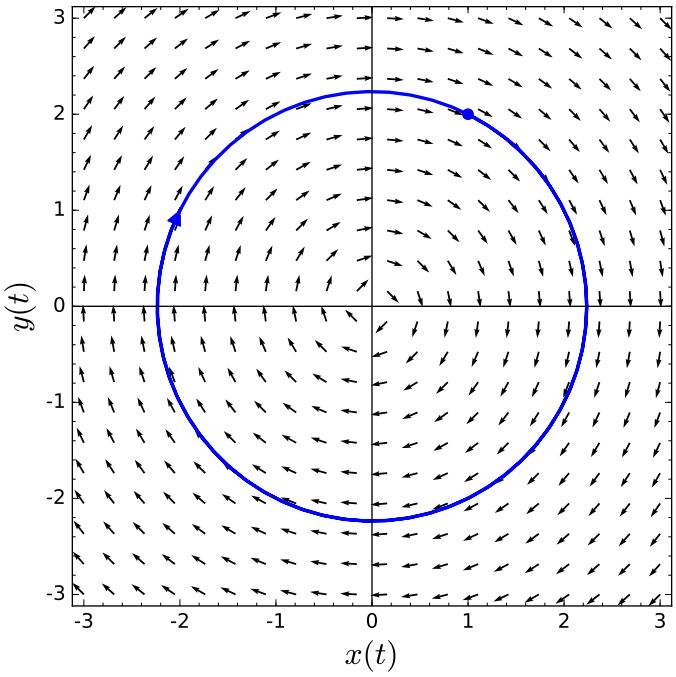

Figure 3.27: Phase portrait for a center

3.4.2 Spiral Sinks and Sources

Now let us consider the system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

and α and β are nonzero real numbers. The characteristic equation of A is

$$\lambda^2 - 2\alpha\lambda + (\alpha^2 + \beta^2) = 0,$$

so the eigenvalues are $\lambda = \alpha \pm i\beta$. We can use the equation

$$(\alpha - (\alpha + i\beta))x + \beta y = 0$$

to show that $(1, i)$ is an eigenvector for $\alpha + i\beta$. Therefore, we have a complex solution of the form

$$\mathbf{x}(t) = e^{(\alpha+i\beta)t} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\begin{aligned} &= e^{\alpha t}\begin{pmatrix}\cos\beta t\\ -\sin\beta t\end{pmatrix} + ie^{\alpha t}\begin{pmatrix}\sin\beta t\\ \cos\beta t\end{pmatrix} \\ &= \mathbf{x}_{\mathrm{Re}}(t) + i\mathbf{x}_{\mathrm{Im}}(t). \end{aligned}$$

As before, both

$$\mathbf{x}_{\mathrm{Re}}(t) = e^{\alpha t}\begin{pmatrix}\cos\beta t\\ -\sin\beta t\end{pmatrix} \quad \text{and} \quad \mathbf{x}_{\mathrm{Im}}(t) = e^{\alpha t}\begin{pmatrix}\sin\beta t\\ \cos\beta t\end{pmatrix}$$

are real solutions to $\mathbf{x}' = A\mathbf{x}$. Furthermore, these solutions are linearly independent. Indeed, $\mathbf{x}_{\mathrm{Re}}$ cannot be a multiple of $\mathbf{x}_{\mathrm{Im}}$ for all values of t. Thus, we have the general solution

$$\mathbf{x}(t) = c_1 e^{\alpha t}\begin{pmatrix}\cos\beta t\\ -\sin\beta t\end{pmatrix} + c_2 e^{\alpha t}\begin{pmatrix}\sin\beta t\\ \cos\beta t\end{pmatrix}.$$

The $e^{\alpha t}$ factor tells us that the solutions either spiral into the origin if $\alpha < 0$ or spiral out to infinity if $\alpha > 0$. In this case we say that the equilibrium points are ***spiral sinks*** and ***spiral sources***, respectively.

Example 3.28. Consider the initial value problem

$$\begin{aligned} \frac{dx}{dt} &= -x/10 + y \\ \frac{dy}{dt} &= -x - y/10 \\ x(0) &= 2 \\ y(0) &= 2. \end{aligned}$$

The matrix

$$\begin{pmatrix}-1/10 & 1\\ -1 & -1/10\end{pmatrix}$$

has eigenvalues $\lambda = -1/10 \pm i$. The eigenvalue $\lambda = -1/10 + i$ has an eigenvector $\mathbf{v} = (1, i)$. The complex solution of our system is

$$\begin{aligned} \mathbf{x}(t) &= e^{(-1/10+i)t}\begin{pmatrix}1\\ i\end{pmatrix} \\ &= e^{-t/10}\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} + ie^{-t/10}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix} \\ &= e^{-t/10}(\cos t + i\sin t)\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} + ie^{-t/10}(\cos t + i\sin t)\begin{pmatrix}\sin t\\ \cos t\end{pmatrix} \\ &= e^{-t/10}\begin{pmatrix}\cos^2 t + i\cos t\sin t\\ -\cos t\sin t - i\sin^2 t\end{pmatrix} + e^{-t/10}\begin{pmatrix}-\sin^2 t + i\cos t\sin t\\ -\cos t\sin t + i\cos^2 t\end{pmatrix} \\ &= e^{-t/10}\begin{pmatrix}(\cos^2 t - \sin^2 t) + 2i\ \sin t\cos t\\ -2\sin t\cos t + i(\cos^2 t - \sin^2 t)\end{pmatrix} \\ &= e^{-t/10}\begin{pmatrix}\cos 2t + i\ \sin 2t\\ -\sin 2t + i\cos 2t\end{pmatrix} \\ &= e^{-t/10}\begin{pmatrix}\cos 2t\\ -\sin 2t\end{pmatrix} + ie^{-t/10}\begin{pmatrix}\sin 2t\\ \cos 2t\end{pmatrix} \end{aligned}$$

The real and imaginary parts of this solution are

$$\mathbf{x}_{\text{Re}}(t) = e^{-t/10}\begin{pmatrix}\cos 2t\\ -\sin 2t\end{pmatrix} \quad \text{and} \quad \mathbf{x}_{\text{Im}}(t) = e^{-t/10}\begin{pmatrix}\sin 2t\\ \cos 2t\end{pmatrix},$$

respectively. Thus, we have the general solution

$$\mathbf{x}(t) = c_1 e^{-t/10}\begin{pmatrix}\cos 2t\\ -\sin 2t\end{pmatrix} + c_2 e^{-t/10}\begin{pmatrix}\sin 2t\\ \cos 2t\end{pmatrix}.$$

Applying our initial conditions, we can determine that $c_1 = 2$ and $c_2 = 0$; hence, the solution to our initial value problem is

$$\mathbf{x}(t) = 2e^{-t/10}\begin{pmatrix}\cos 2t\\ -\sin 2t\end{pmatrix}.$$

The phase portrait of this solution indicates that we do indeed have a spiral sink (Figure 3.29).

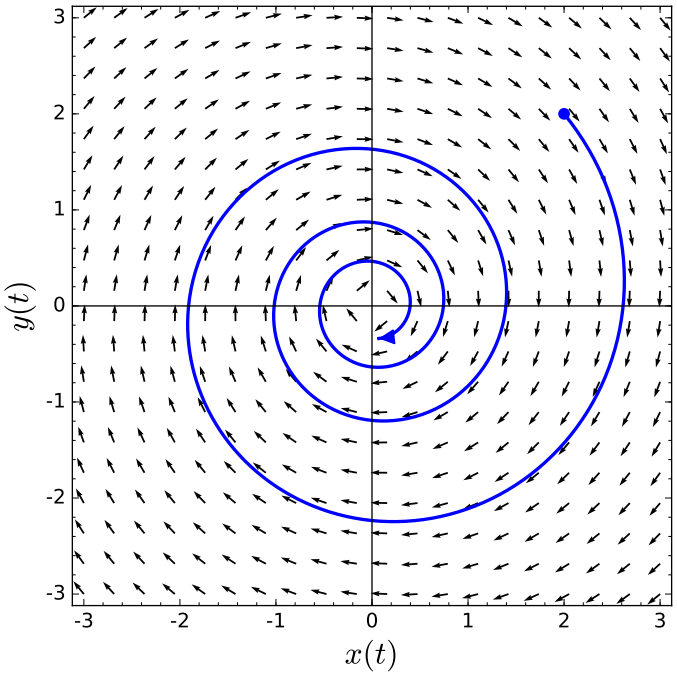

Figure 3.29: Phase portrait for a spiral sink

Example 3.30. The initial value problem

$$\begin{aligned}\frac{dx}{dt} &= x/10 + y\\ \frac{dy}{dt} &= -x + y/10\\ x(0) &= 0\\ y(0) &= 1/2.\end{aligned}$$

The matrix

$$\begin{pmatrix}1/10 & 1\\ -1 & 1/10\end{pmatrix}$$

has an eigenvector $(1, -i)$ with eigenvalue $\lambda = 1/10 - i$. Thus, the complex solution is

$$\mathbf{x}(t) = e^{(1/10-i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Following the procedure that we used in the previous example, the solution to our initial value problem is

$$\mathbf{x}(t) = \frac{1}{2} e^{t/10} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix},$$

and he phase portrait is a spiral source (Figure 3.31).

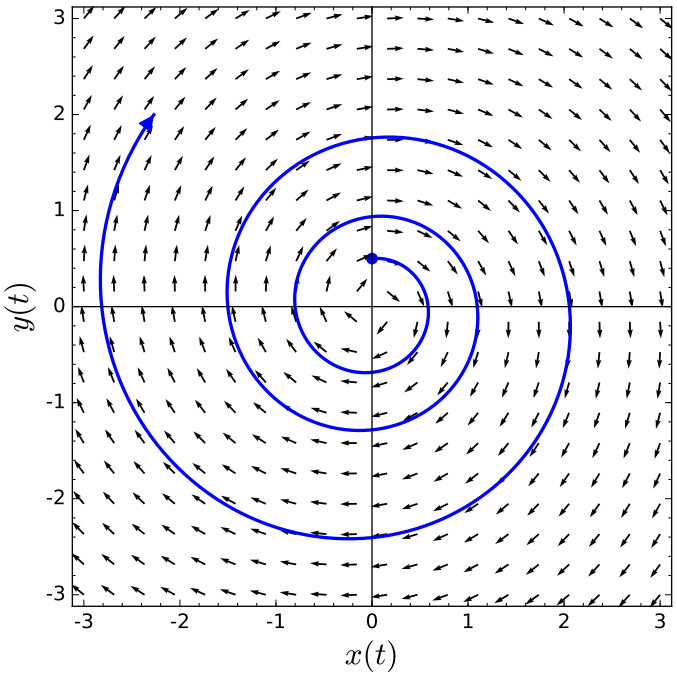

Figure 3.31: Phase portrait for a spiral source

3.4.3 Solving Systems with Complex Eigenvalues

Suppose that we have the linear system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The charactistic polynomial of A is

$$p(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc).$$

If $(a+d)^2 - 4(ad-bc) < 0$, then the eigenvalues of A are complex, and we cannot apply the strategy that we used to determine the general solution in the case of distinct real roots.

Example 3.32. The system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} -1 & -2 \\ 4 & 3 \end{pmatrix}.$$

The characteristic polynomial of A is $\lambda^2 - 2\lambda + 5$ and so the eigenvalues are complex conjugates, $\lambda = 1 + 2i$ and $\overline{\lambda} = 1 - 2i$. It is easy to show that an eigenvector for $\lambda = 1 + 2i$ is $\mathbf{v} = (1, -1 - i)$. Recalling that $e^{i\theta} = \cos\theta + i\sin\theta$,

$$\begin{aligned}
\mathbf{x}(t) &= e^{(1+2i)t}\mathbf{v} \\
&= e^{(1+2i)t}\begin{pmatrix} 1 \\ -1-i \end{pmatrix} \\
&= e^t e^{2it}\begin{pmatrix} 1 \\ -1-i \end{pmatrix} \\
&= e^t(\cos 2t + i\sin 2t)\begin{pmatrix} 1 \\ -1-i \end{pmatrix} \\
&= e^t\begin{pmatrix} 2\cos 2t + 2i\sin 2t \\ (-1-i)(\cos 2t + i\sin 2t) \end{pmatrix} \\
&= e^t\begin{pmatrix} 2\cos 2t + 2i\sin 2t \\ (-\cos 2t + \sin 2t) + i(-\cos 2t - \sin 2t) \end{pmatrix} \\
&= e^t\begin{pmatrix} 2\cos 2t \\ -\cos 2t + \sin 2t \end{pmatrix} + ie^t\begin{pmatrix} 2\sin 2t \\ -\cos 2t - \sin 2t \end{pmatrix}
\end{aligned}$$

is a complex solution to our system. Taking the real and imaginary parts of this solution, we obtain the general solution to our system

$$\mathbf{x}(t) = c_1 e^t\begin{pmatrix} 2\cos 2t \\ -\cos 2t + \sin 2t \end{pmatrix} + c_2 e^t\begin{pmatrix} 2\sin 2t \\ -\cos 2t - \sin 2t \end{pmatrix}.$$

The nature of the equilibrium solution is determined by the factor $e^{\alpha t}$ in the solution. If $\alpha < 0$, the equilibrium point is a ***spiral sink***. If $\alpha > 0$, the equilibrium point is a ***spiral source***. If $\alpha = 0$, the equilibrium point is a ***center***.

Although we have outlined a procedure to find the general solution of $\mathbf{x}' = A\mathbf{x}$ if A has complex eigenvalues, we have not shown that this method will work in all cases. We will do so in Section 3.6.

3.4.4 Important Lessons

- If
$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$
then A has two complex eigenvalues, $\lambda = \alpha \pm i\beta$. The general solution to the system $\mathbf{x}' = A\mathbf{x}$ is
$$\mathbf{x}(t) = c_1 e^{\alpha t}\begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix} + c_2 e^{\alpha t}\begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix}.$$
If $\alpha < 0$, the equilibrium point is a spiral sink. If $\alpha > 0$, the equilibrium point is a spiral source.

3.4.5 Exercises

1. Solve each of the following linear systems for the given initial values. Sketch the phase plane of each system and determine the nature of the equilibrium solution(s). Do this first without the use of technology, then check your answer with *Sage*.

(a)

$$\begin{aligned} x' &= 2x + 2y \\ y' &= -4x + 6y \\ x(0) &= 2 \\ y(0) &= -3 \end{aligned}$$

(b)

$$\begin{aligned} x' &= 2x - 6y \\ y' &= 2x + y \\ x(0) &= 2 \\ y(0) &= 1 \end{aligned}$$

(c)

$$\begin{aligned} x' &= 2x + y \\ y' &= -x + 4y \\ x(0) &= 1 \\ y(0) &= -1 \end{aligned}$$

(d)

$$\begin{aligned} x' &= 4x + 2y \\ y' &= -2x - y \\ x(0) &= 1 \\ y(0) &= -1 \end{aligned}$$

Hint.

2. Determine a computable condition that guarantees that, if a 2×2 matrix A has complex eigenvalues, then the solutions of $\mathbf{x}' = A\mathbf{x}$ travel around the origin in a counterclockwise direction.

Hint.

3.4.6 Project

3.5 Repeated Eigenvalues

Consider the following system

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.10}$$

The characteristic polynomial of the system (3.10) is $\lambda^2 - 6\lambda + 9$ and $\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$. This polynomial has a single root $\lambda = 3$ with eigenvector $\mathbf{v} = (1, -1)$. This there is a single straightline solution for this system (Figure 3.33). The strategy that we used in find the general solution to a system with distinct real eigenvalues will clearly have to be modified if we are to find a general solution to a system with a single eigenvalue.

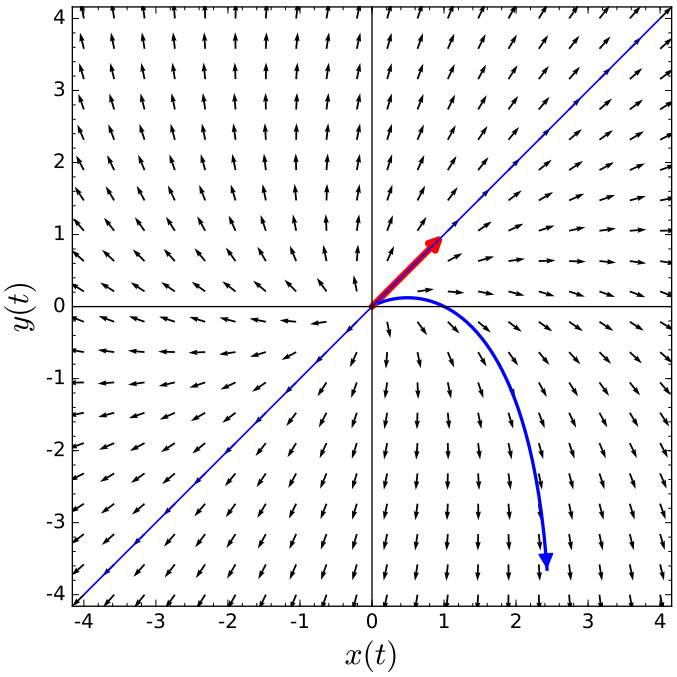

Figure 3.33: A system with one straightline solution

3.5.1 Repeated Eigenvalues

The remaining case the we must consider is when the characteristic equation of a matrix A has repeated roots. The simplest such case is

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}.$$

The eigenvalues of A are both λ. Since $A\mathbf{v} = \lambda\mathbf{v}$, any nonzero vector in $\mathbb{R}^2$ is an eigenvector for λ. Thus, solutions to this system are of the form

$$\mathbf{x}(t) = \alpha e^{\lambda t}\mathbf{v}.$$

Each solution to our system lies on a straight line through the origin and either tends to the origin if $\lambda < 0$ or away from zero if $\lambda > 0$.

A more interesting case occurs if

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Again, both eigenvalues are λ; however, there is only one linearly independent eigenvector, which we can take to be $(1, 0)$. Therefore, we have a single straight-line solution

$$\mathbf{x}_1(t) = \alpha e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

To find other solutions, we will rewrite the system as

$$\begin{aligned} x' &= \lambda x + y \\ y' &= \lambda y. \end{aligned}$$

This is a partially coupled system. If $y \neq 0$, the solution of the second equation is

$$y(t) = \beta e^{\lambda t}.$$

Therefore, the first equation becomes

$$x' = \lambda x + \beta e^{\lambda t},$$

which is a first-order linear differential equation with solution

$$x(t) = \alpha e^{\lambda t} + \beta t e^{\lambda t}.$$

Consequently, a solution to our system is

$$\alpha e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda t} \begin{pmatrix} t \\ 1 \end{pmatrix}.$$

Example 3.34. Consider the linear system

$$\begin{aligned} x' &= -x + y \\ y' &= -y \\ x(0) &= 1 \\ y(0) &= 3. \end{aligned}$$

The matrix that corresponds to this system is

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

has a single eigenvalue, $\lambda = -1$. An eigenvector for λ is $\mathbf{v} = (1, 0)$. Thus, the general solution to our system is

$$\begin{aligned} x(t) &= c_1 e^{-t} + c_2 t e^{-t} \\ y(t) &= c_2 e^{-t}. \end{aligned}$$

Applying the initial conditions $x(0) = 1$ and $y(0) = 3$, the solution to our initial value problem is

$$\begin{aligned} x(t) &= e^{-t} + 3te^{-t} \\ y(t) &= 3e^{-t}. \end{aligned}$$

Notice that we have only one straightline solution (Figure 3.35).

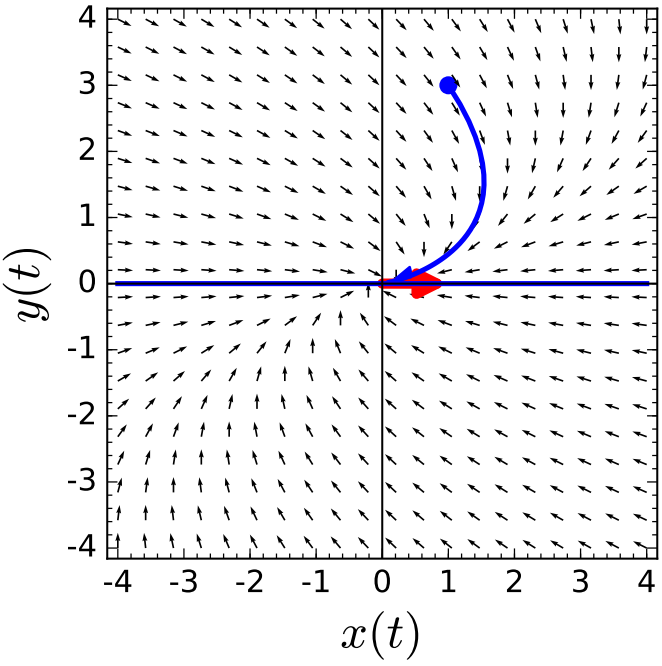

Figure 3.35: Phase portrait for repeated eigenvalues

3.5.2 Solving Systems with Repeated Eigenvalues

If the characteristic equation has only a single repeated root, there is a single eigenvalue. If this is the situation, then we actually have two separate cases to examine, depending one whether or not we can find two linearly independent eigenvectors.

Example 3.36. Suppose we have the system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

The single eigenvalue is $\lambda = 2$, but there are two linearly independent eigenvectors, $\mathbf{v}_1 = (1, 0)$ and $\mathbf{v}_2 = (0, 1)$. In this case our solution is

$$\mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This is not two surprising since the system

$$\begin{aligned} x' &= 2x \\ y' &= 2y \end{aligned}$$

is uncoupled and each equation can be solved separately.

Example 3.37. Now let us consider the example $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 5 & 1 \\ -4 & 1 \end{pmatrix}.$$

Since the characteristic polynomial of A is $\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$, we have only a single eigenvalue $\lambda = 3$ with eigenvector $\mathbf{v}_1 = (1, -2)$. This gives us one solution to our system, $\mathbf{x}_1(t) = e^{3t}\mathbf{v}_1$; however, we still need a second solution.

Since all other eigenvectors of A are a multiple of $\mathbf{v}$, we cannot find a second linearly independent eigenvector and we need to obtain the second solution in a different manner. We must find a vector $\mathbf{v}_2$ such that $(A-\lambda I)\mathbf{v}_2 = \mathbf{v}_1$. To do this we can start with any nonzero vector $\mathbf{w}$ that is not a multiple of $\mathbf{v}_1$, say $\mathbf{w} = (1,0)$. We then compute

$$(A-\lambda I)\mathbf{w} = (A-3I)\mathbf{w} = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} = 2\mathbf{v}_1.$$

Thus, we can take $\mathbf{v}_2 = (1/2)\mathbf{w} = (1/2, 0)$, and our second solution is

$$\mathbf{x}_2 = e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1) = e^{3t}\begin{pmatrix} 1/2+t \\ -2t \end{pmatrix}$$

Thus, our general solution is

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1e^{3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2e^{3t}\begin{pmatrix} 1/2+t \\ -2t \end{pmatrix}.$$

If the eigenvalue is positive, we will have a nodal source. If it is negative, we will have a nodal sink. Notice that we have only given a recipe for finding a solution to $\mathbf{x}' = A\mathbf{x}$, where A has a repeated eigenvalue and any tow eigen vectors are linearly dependent. We will justify our procedure in the next section (Section 3.6).

3.5.3 Important Lessons

- If
$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$
then A has one repeated real eigenvalue. The general solution to the system $\mathbf{x}' = A\mathbf{x}$ is
$$\mathbf{x}(t) = \alpha e^{\lambda t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda t}\begin{pmatrix} t \\ 1 \end{pmatrix}.$$
If $\lambda < 0$, then the solutions tend towards the origin as $t \to \infty$. For $\lambda > 0$, the solutions tend away from the origin.

3.5.4 Exercises

1. Hint.

3.5.5 Project

3.6 Changing Coordinates

In the beginning sections of this chapter, we outlined procedures for solving systems of linear differential equations of the form

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$$

by determining the eigenvalues of A, but we have only justified the case where A has distinct real eigenvalues. However, we have considered the following special cases for A

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Although it may seem that we have limited ourselves by attacking only a very small part of the problem of finding solutions for $\mathbf{x}' = A\mathbf{x}$, we are actually very close to providing a complete classification of all solutions. We will now show that we can transform any 2×2 system of first-order linear differential equations with constant coefficients into one of these special systems by using a ***change of coordinates.***

3.6.1 Linear Maps

A ***linear map*** or ***linear transformation*** on $\mathbb{R}^2$ is a function $T : \mathbb{R}^2 \to \mathbb{R}^2$ that is defined by a matrix. That is,

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

When there is no confusion, we will think of the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ and the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

as interchangeable.

We will say that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an ***invertible linear map*** if we can find a second linear map S such that $T \circ S = S \circ T = I$, where I is the identity transformation. In terms of matrices, this means that we can find a matrix S such that

$$TS = ST = I,$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

is the 2×2 identity matrix. We write T^{-1} for the inverse matrix of T. It is easy to check that the inverse of

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is

$$T^{-1} = \frac{1}{\det T}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Theorem 3.38. *A linear map T is invertible if and only if* $\det T \neq 0$.

Proof. If $\det T = 0$, then there are infinitely many nonzero vectors $\mathbf{x}$ such that $T\mathbf{x} = \mathbf{0}$. Suppose that T^{-1} exists and $\mathbf{x} \neq \mathbf{0}$ such that $T\mathbf{x} = \mathbf{0}$. Then

$$\mathbf{x} = T^{-1}T\mathbf{x} = T^{-1}\mathbf{0} = \mathbf{0},$$

which is a contradiction. On the other hand, we can certainly compute T^{-1}, at least in the 2×2 case, if the determinant is nonzero. □

3.6.2 Changing Coordinates

Suppose that we consider a linear system

$$\mathbf{y}' = (T^{-1}AT)\mathbf{y} \tag{3.11}$$

where T is an invertible matrix. If $\mathbf{y}(t)$ is a solution of (3.11), we claim that $\mathbf{x}(t) = T\mathbf{y}(t)$ solves the equation $\mathbf{x}' = A\mathbf{x}$. Indeed,

$$\begin{aligned}\mathbf{x}'(t) &= (T\mathbf{y})'(t)\\ &= T\mathbf{y}'(t)\\ &= T((T^{-1}AT)\mathbf{y}(t))\\ &= A(T\mathbf{y}(t))\\ &= A\mathbf{x}(t).\end{aligned}$$

We can think of this in two ways.

1. A linear map T converts solutions of $\mathbf{y}' = (T^{-1}AT)\mathbf{y}$ to solutions of $\mathbf{x}' = A\mathbf{x}$.
2. The inverse of a linear map T takes solutions of $\mathbf{x}' = A\mathbf{x}$ to solutions of $\mathbf{y}' = (T^{-1}AT)\mathbf{y}$.

We are now in a position to solve our problem of finding solutions of an arbitrary linear system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

3.6.3 Distinct Real Eigenvalues

Consider the system $\mathbf{x}' = A\mathbf{x}$, where A has two real, distinct eigenvalues λ_1 and λ_2 with eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. Let T be the matrix with columns $\mathbf{v}_1$ and $\mathbf{v}_2$. If $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$, then $T\mathbf{e}_i = \mathbf{v}_i$ for $i = 1,2$. Consequently, $T^{-1}\mathbf{v}_i = \mathbf{e}_i$ for $i = 1,2$. Thus, we have

$$\begin{aligned}(T^{-1}AT)\mathbf{e}_i &= T^{-1}A\mathbf{v}_i\\ &= T^{-1}(\lambda_i\mathbf{v}_i)\\ &= \lambda_i T^{-1}\mathbf{v}_i\\ &= \lambda_i\mathbf{e}_i\end{aligned}$$

for $i = 1,2$. Therefore, the matrix $T^{-1}AT$ is in canonical form,

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

The eigenvalues of the matrix $T^{-1}AT$ are λ_1 and λ_2 with eigenvectors $(1,0)$ and $(0,1)$, respectively. Thus, the general solution of

$$\mathbf{y}' = (T^{-1}AT)\mathbf{y}$$

is

$$\mathbf{y}(t) = \alpha e^{\lambda_1 t}\begin{pmatrix}1\\0\end{pmatrix} + \beta e^{\lambda_2 t}\begin{pmatrix}0\\1\end{pmatrix}.$$

Hence, the general solution of

$$\mathbf{x}' = A\mathbf{x}$$

is

$$T\mathbf{y}(t) = T\left(\alpha e^{\lambda_1 t}\begin{pmatrix}1\\0\end{pmatrix} + \beta e^{\lambda_2 t}\begin{pmatrix}0\\1\end{pmatrix}\right)$$

$$= \alpha e^{\lambda_1 t} T \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t} T \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= \alpha e^{\lambda_2 t} \mathbf{v}_1 + \beta e^{\lambda_2 t} \mathbf{v}_2.$$

Example 3.39. Suppose $d\mathbf{x}/dt = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}.$$

The eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = -1$ and the associated eigenvectors are $(1, 2)$ and $(1, -1)$, respectively. In this case, our matrix T is

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}.$$

If $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$, then $T\mathbf{e}_i = \mathbf{v}_i$ for $i = 1, 2$. Consequently, $T^{-1}\mathbf{v}_i = \mathbf{e}_i$ for $i = 1, 2$, where

$$T^{-1} = \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{pmatrix}.$$

Thus,

$$T^{-1}AT = \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues of the matrix

$$\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$$

are $\lambda_1 = 5$ and $\lambda_2 = -1$ with eigenvectors $(1, 0)$ and $(0, 1)$, respectively. Thus, the general solution of

$$\mathbf{y}' = (T^{-1}AT)\mathbf{y}$$

is

$$\mathbf{y}(t) = \alpha e^{5t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence, the general solution of

$$\mathbf{x}' = A\mathbf{x}$$

is

$$T\mathbf{y}(t) = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \left(\alpha e^{5t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$
$$= \alpha e^{5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The linear map T converts the phase portrait of the system $\mathbf{y}' = (T^{-1}AT)\mathbf{y}$ (Figure 3.40) to the phase portrait of the system $\mathbf{x}' = A\mathbf{x}$ (Figure 3.41).

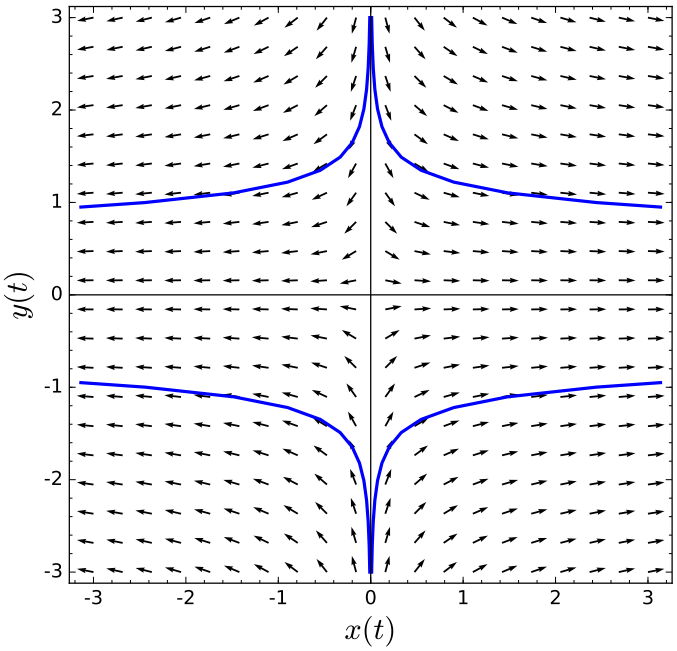

Figure 3.40: Phase portrait for $\mathbf{y}' = (T^{-1}AT)\mathbf{y}$

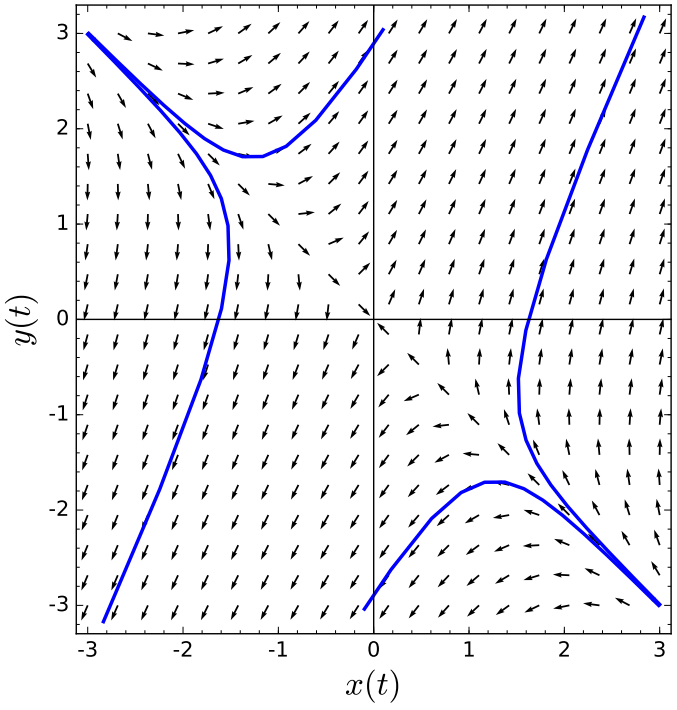

Figure 3.41: Phase portrait for $\mathbf{x}' = A\mathbf{x}$

3.6.4 Complex Eigenvalues

Suppose the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in system $\mathbf{x}' = A\mathbf{x}$ has complex eigenvalues. In this case, the characteristic polynomial $p(\lambda) = \lambda^2 - (a+d)\lambda + (ad-bc)$ will have roots $\lambda = \alpha + i\beta$ and $\overline{\lambda} = \alpha - i\beta$, where

$$\alpha = \frac{a+d}{2}$$
$$\beta = \frac{\sqrt{4bc-(a-d)^2}}{2}.$$

The eigenvalues λ and $\overline{\lambda}$ are complex conjugates. Now, suppose that the eigenvalue $\lambda = \alpha + i\beta$ has an eigenvector of the form

$$\mathbf{v} = \mathbf{v}_1 + i\mathbf{v}_2,$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are real vectors. Then $\overline{\mathbf{v}} = \mathbf{v}_1 - i\mathbf{v}_2$ is an eigenvector for $\overline{\lambda}$, since

$$A\overline{\mathbf{v}} = \overline{A\mathbf{v}} = \overline{\lambda\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}.$$

Consequently, if A is a real matrix with complex eigenvalues, one of the eigenvalues determines the other.

Proposition 3.42. *If $\lambda = \alpha + i\beta$ is an eigenvalue of a real matrix A with $\beta \neq 0$ and eigenvector the form*

$$\mathbf{v}_1 + i\mathbf{v}_2,$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are real vectors, then the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Proof. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are not linearly independent, then $\mathbf{v}_1 = c\mathbf{v}_2$ for some $c \in \mathbb{R}$. On one hand, we have

$$A(\mathbf{v}_1 + i\mathbf{v}_2) = A(c\mathbf{v}_2 + i\mathbf{v}_2) = (c+i)A\mathbf{v}_2.$$

However,

$$\begin{aligned} A(\mathbf{v}_1 + i\mathbf{v}_2) &= (\alpha + i\beta)(\mathbf{v}_1 + i\mathbf{v}_2) \\ &= (\alpha + i\beta)(c+i)\mathbf{v}_2 \\ &= (c+i)(\alpha + i\beta)\mathbf{v}_2 \end{aligned}$$

In other words, $A\mathbf{v}_2 = (\alpha + i\beta)\mathbf{v}_2$. However, this is a contradiction since the left-side of the equation says that we have real eigenvector while the right-side of the equation is complex. Thus, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. □

Proposition 3.43. *Let A be a real matrix with eigenvalue $\lambda = \alpha + i\beta$, where $\beta \neq 0$. If*

$$\mathbf{v}_1 + i\mathbf{v}_2,$$

is an eigenvector for λ, then there exists a matrix T such that

$$T^{-1}AT = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

Proof. Since $\mathbf{v}_1 + i\mathbf{v}_2$ is an eigenvector associated to the eigenvalue $\alpha + i\beta$, we have

$$A(\mathbf{v}_1 + i\mathbf{v}_2) = (\alpha + i\beta)(\mathbf{v}_1 + i\mathbf{v}_2).$$

Equating the real and imaginary parts, we find that

$$A\mathbf{v}_1 = \alpha\mathbf{v}_1 - \beta\mathbf{v}_2$$

$$A\mathbf{v}_2 = \beta\mathbf{v}_1 + \alpha\mathbf{v}_2.$$

If T is the matrix with columns $\mathbf{v}_1$ and $\mathbf{v}_2$, then

$$\begin{aligned} T\mathbf{e}_1 &= \mathbf{v}_1 \\ T\mathbf{e}_2 &= \mathbf{v}_2. \end{aligned}$$

Thus, we have

$$(T^{-1}AT)\mathbf{e}_1 = T^{-1}(\alpha\mathbf{v}_1 - \beta\mathbf{v}_2) = \alpha\mathbf{e}_1 - \beta\mathbf{e}_2.$$

Similarly,

$$(T^{-1}AT)\mathbf{e}_2 = \beta\mathbf{e}_1 + \alpha\mathbf{e}_2.$$

Therefore, we can write the matrix $T^{-1}AT$ as

$$T^{-1}AT = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \qquad \square$$

The system $\mathbf{y}' = (T^{-1}AT)\mathbf{y}$ is in one of the canonical forms and has a phase portrait that is a spiral sink ($\alpha < 0$), a center ($\alpha = 0$), or a spiral source ($\alpha > 0$). After a change of coordinates, the phase portrait of $\mathbf{x}' = A\mathbf{x}$ is equivalent to a sink, center, or source.

Example 3.44. Suppose that we wish to find the solutions of the second order equation

$$2x'' + 2x' + x = 0.$$

This particular equation might model a damped harmonic oscillator. If we rewrite this second-order equation as a first-order system, we have

$$\begin{aligned} x' &= y \\ y' &= -\frac{1}{2}x - y, \end{aligned}$$

or equivalently $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 0 & 1 \\ -1/2 & -1 \end{pmatrix}.$$

The eigenvalues of A are

$$-\frac{1}{2} \pm i\frac{1}{2}.$$

The eigenvalue $\lambda = (1+i)/2$ has an eigenvector

$$\mathbf{v} = \begin{pmatrix} 2 \\ -1+i \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

respectively. Therefore, we can take T to be the matrix

$$T = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}.$$

Consequently,

$$T^{-1}AT = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1/2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1/2 & 1/2 \\ -1/2 & -1/2 \end{pmatrix},$$

which is in the canonical form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

The general solution to $\mathbf{y}' = (T^{-1}AT)\mathbf{y}$ is

$$\mathbf{y}(t) = c_1 e^{-t/2} \begin{pmatrix} \cos(t/2) \\ -\sin(t/2) \end{pmatrix} + c_2 e^{-t/2} \begin{pmatrix} \sin(t/2) \\ \cos(t/2) \end{pmatrix}.$$

The phase portrait of $\mathbf{y}' = (T^{-1}AT)\mathbf{y}$ is given in Figure 3.45.

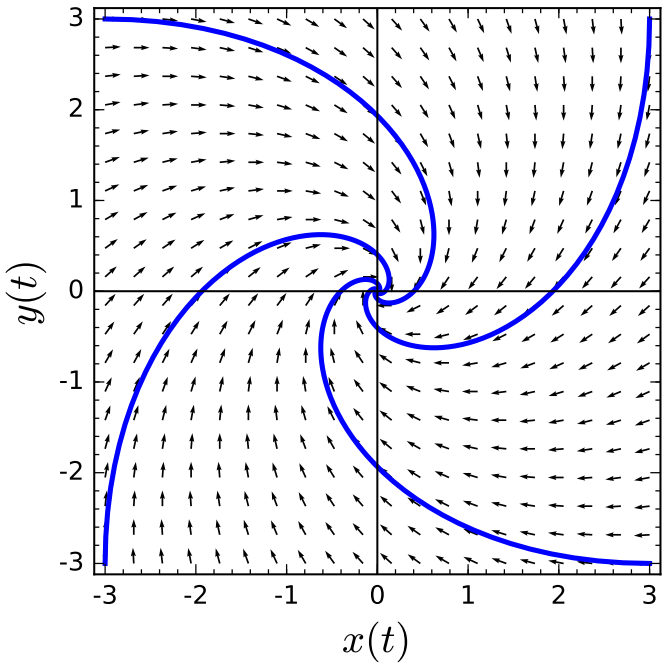

Figure 3.45: Phase portrait for $\mathbf{y}' = (T^{-1}AT)\mathbf{y}$

The general solution of $\mathbf{x}' = A\mathbf{x}$ is

$$\begin{aligned} T\mathbf{y}(t) &= \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \left[c_1 e^{-t/2} \begin{pmatrix} \cos(t/2) \\ -\sin(t/2) \end{pmatrix} + c_2 e^{-t/2} \begin{pmatrix} \sin(t/2) \\ \cos(t/2) \end{pmatrix} \right] \\ &= c_1 e^{-t/2} \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \cos(t/2) \\ -\sin(t/2) \end{pmatrix} + c_2 e^{-t/2} \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \sin(t/2) \\ \cos(t/2) \end{pmatrix} \\ &= c_1 e^{-t/2} \begin{pmatrix} 2\cos(t/2) \\ -\cos(t/2) - \sin(t/2) \end{pmatrix} + c_2 e^{-t/2} \begin{pmatrix} 2\sin(t/2) \\ -\sin(t/2) + \cos(t/2) \end{pmatrix}. \end{aligned}$$

The phase portrait for this system is given in Figure 3.46.

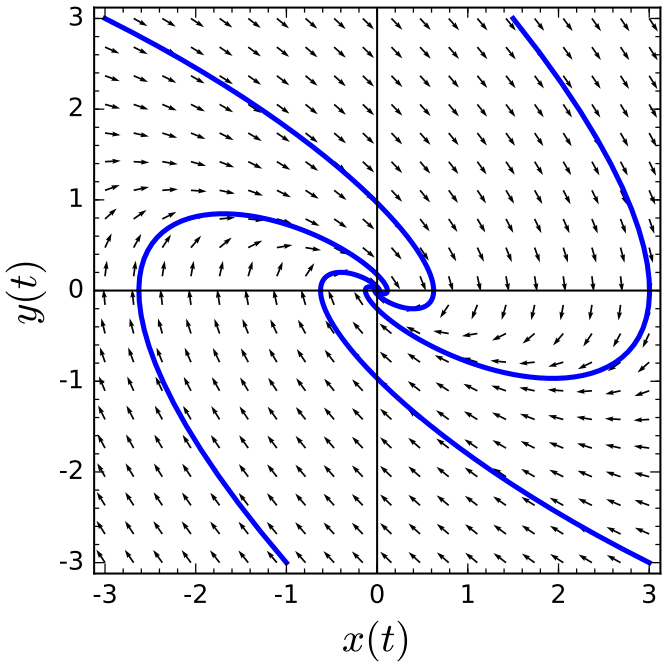

Figure 3.46: Phase portrait of $\mathbf{x}' = A\mathbf{x}$

Remark 3.47. Of course, we have a much more efficient way of solving the system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 0 & 1 \\ -1/2 & -1 \end{pmatrix}.$$

Since A has eigenvalue $\lambda = (-1+i)/2$ with an eigenvector $\mathbf{v} = (2, -1+i)$, we can apply Euler's formula and write the solution as

$$\begin{aligned}
\mathbf{x}(t) &= e^{(-1+i)t/2}\mathbf{v} \\
&= e^{-t/2}e^{it/2}\begin{pmatrix} 2 \\ -1+i \end{pmatrix} \\
&= e^{-t/2}(\cos(t/2) + i\sin(t/2))\begin{pmatrix} 2 \\ -1+i \end{pmatrix} \\
&= e^{-t/2}\begin{pmatrix} 2\cos(t/2) \\ -\cos(t/2) - \sin(t/2) \end{pmatrix} + ie^{-t/2}\begin{pmatrix} 2\sin(t/2) \\ -\sin(t/2) + \cos(t/2) \end{pmatrix}.
\end{aligned}$$

Taking the real and the imaginary parts of the last expression, the general solution of $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{-t/2}\begin{pmatrix} 2\cos(t/2) \\ -\cos(t/2) - \sin(t/2) \end{pmatrix} + c_2 e^{-t/2}\begin{pmatrix} 2\sin(t/2) \\ -\sin(t/2) + \cos(t/2) \end{pmatrix},$$

which agrees with the solution that we found by transforming coordinates.

3.6.5 Repeated eigenvalues

Now suppose that A has a single real eigenvalue λ. Then the characteristic polynomial of A is $p(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc)$, then A has an eigenvalue $\lambda = (a+d)/2$.

Proposition 3.48. *If A has a single eigenvalue and a pair of linearly independent eigenvectors, then A must be of the form*

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

PROOF. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are linearly indeendent eigenvectors for A, and let T be the matrix whose first column is $\mathbf{u}$ and second column is $\mathbf{v}$. That is, $T\mathbf{e}_1 = \mathbf{u}$ and $T\mathbf{e}_2 = \mathbf{v}$. Since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, $\det(T) \neq 0$ and T is invertible. So, it must be the case that

$$AT = (A\mathbf{u}, A\mathbf{v}) = (\lambda\mathbf{u}, \lambda\mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) = \lambda IT,$$

or

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}. \qquad \square$$

In this case, the system is uncoupled and is easily solved. That is, we can solve each equation in the system

$$\begin{aligned} x' &= \lambda x \\ y' &= \lambda y \end{aligned}$$

separately to obtain the general solution

$$\begin{aligned} x &= c_1 e^{\lambda t} \\ y' &= c_2 e^{\lambda t}. \end{aligned}$$

Proposition 3.49. *Suppose that A has a single eigenvalue λ. If $\mathbf{v}$ is an eigenvector for λ and any other eigenvector for λ is a multiple of $\mathbf{v}$, then there exists a matrix T such that*

$$T^{-1}AT = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

PROOF. If $\mathbf{w}$ is another vector in $\mathbb{R}^2$ such that $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, then $A\mathbf{w}$ can be written as a linear combination of $\mathbf{v}$ and $\mathbf{w}$,

$$A\mathbf{w} = \alpha\mathbf{v} + \beta\mathbf{w}.$$

We can assume that $\alpha \neq 0$; otherwise, we would have a second linearly independent eigenvector. We claim that $\beta = \lambda$. If this were not the case, then

$$\begin{aligned} A\left(\mathbf{w} + \left(\frac{\alpha}{\beta - \lambda}\right)\mathbf{v}\right) &= A\mathbf{w} + \left(\frac{\alpha}{\beta - \lambda}\right)A\mathbf{v} \\ &= \alpha\mathbf{v} + \beta\mathbf{w} + \lambda\left(\frac{\alpha}{\beta - \lambda}\right)\mathbf{v} \\ &= \beta\mathbf{w} + \alpha\left(1 + \frac{\lambda}{\beta - \lambda}\right)\mathbf{v} \\ &= \beta\mathbf{w} + \alpha\left(\frac{\beta - \lambda + \lambda}{\beta - \lambda}\right)\mathbf{v} \\ &= \beta\left(\mathbf{w} + \left(\frac{\alpha}{\beta - \lambda}\right)\mathbf{v}\right) \end{aligned}$$

and β would be an eigenvalue distinct from λ. Thus, $A\mathbf{w} = \alpha\mathbf{v} + \lambda\mathbf{w}$. If we will let $\mathbf{u} = (1/\alpha)\mathbf{w}$, then

$$A\mathbf{u} = \mathbf{v} + \frac{\lambda}{\alpha}\mathbf{w} = \mathbf{v} + \lambda\mathbf{u}.$$

We now define $T\mathbf{e}_1 = \mathbf{v}$ and $T\mathbf{e}_2 = \mathbf{u}$. Since

$$AT = A\mathbf{u} + A\mathbf{v} = \mathbf{v} + \lambda\mathbf{u} + \lambda\mathbf{v}$$

$$T\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix} = T(\lambda\mathbf{e}_1) + T\mathbf{e}_1 + T(\lambda\mathbf{e}_2) = \mathbf{v} + \lambda\mathbf{u} + \lambda\mathbf{v},$$

we have

$$T^{-1}AT = \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}.$$

Therefore, $\mathbf{x}' = A\mathbf{x}$ is in canonical form after a change of coordinates. □

Example 3.50. Consider the system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix}5 & 1\\ -4 & 1\end{pmatrix}.$$

The characteristic polynomial of A is $\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$, we have only a single eigenvalue $\lambda = 3$ with eigenvector $\mathbf{v} = (1, -2)$. Any other eigenvector for λ is a multiple of $\mathbf{v}$. If we choose $\mathbf{w} = (1, 0)$, then $\mathbf{v}$ and $\mathbf{w}$ are linearly independent. Furthermore,

$$A\mathbf{w} = \begin{pmatrix}5\\ -4\end{pmatrix} = 2\begin{pmatrix}1\\ -2\end{pmatrix} + \lambda\begin{pmatrix}1\\ 0\end{pmatrix} = 2\begin{pmatrix}1\\ -2\end{pmatrix} + 3\begin{pmatrix}1\\ 0\end{pmatrix}.$$

So we can let $\mathbf{u} = (1/2)\mathbf{w} = (1/2, 0)$. Therefore, the matrix that we seek is

$$T = \begin{pmatrix}1 & 1/2\\ -2 & 0\end{pmatrix},$$

and

$$T^{-1}AT = \begin{pmatrix}-1/2 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}5 & 1\\ -4 & 1\end{pmatrix}\begin{pmatrix}1 & 1/2\\ -2 & 0\end{pmatrix} = \begin{pmatrix}3 & 1\\ 0 & 3\end{pmatrix}.$$

From Section 3.3, we know that the general solution to the system

$$\begin{pmatrix}dx/dt\\ dy/dt\end{pmatrix} = \begin{pmatrix}3 & 1\\ 0 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

is

$$\mathbf{y}(t) = c_1e^{3t}\begin{pmatrix}1\\ 0\end{pmatrix} + c_2e^{3t}\begin{pmatrix}t\\ 1\end{pmatrix}.$$

Therefore, the general solution to

$$\begin{pmatrix}dx/dt\\ dy/dt\end{pmatrix} = \begin{pmatrix}5 & 1\\ -4 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

is

$$\begin{aligned}\mathbf{x}(t) &= T\mathbf{y}(t)\\ &= c_1e^{3t}T\begin{pmatrix}1\\ 0\end{pmatrix} + c_2e^{3t}T\begin{pmatrix}t\\ 1\end{pmatrix}\\ &= c_1e^{3t}\begin{pmatrix}1\\ -2\end{pmatrix} + c_2e^{3t}\begin{pmatrix}1/2 + t\\ -2t\end{pmatrix}.\end{aligned}$$

This solution agrees with the solution that we found in Example 3.37.

In practice, we find solutions to linear systems using the methods that we outlined in Sections 3.2–3.4. What we have demonstrated in this section is that those solutions are exactly the ones that we want.

3.6.6 Important Lessons

- A linear map T is invertible if and only if $\det T \neq 0$.
- A linear map T converts solutions of $\mathbf{y}' = (T^{-1}AT)\mathbf{y}$ to solutions of $\mathbf{x}' = A\mathbf{x}$.
- The inverse of a linear map T takes solutions of $\mathbf{x}' = A\mathbf{x}$ to solutions of $\mathbf{y}' = (T^{-1}AT)\mathbf{y}$.
- A change of coordinates converts the system $\mathbf{x}' = A\mathbf{x}$ to one of the following special cases,

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

3.6.7 Exercises

1. This is an exercise.

Hint.

3.6.8 Project

3.7 The Trace-Determinant Plane

Suppose that we have two tanks, Tank A and Tank B, that both have a volume of V liters and are both filled with a brine solution. Suppose that pure water enters Tank A at a rate of r_{in} liters per minute, and a salt mixture enters Tank A from Tank B at a rate of r_B liters per minute. Brine also enters Tank B from Tank A at a rate of r_A liters per minute. Finally, brine is drained from Tank B at a rate of r_{out} so that the volume in each tank is constant (Figure 3.51).

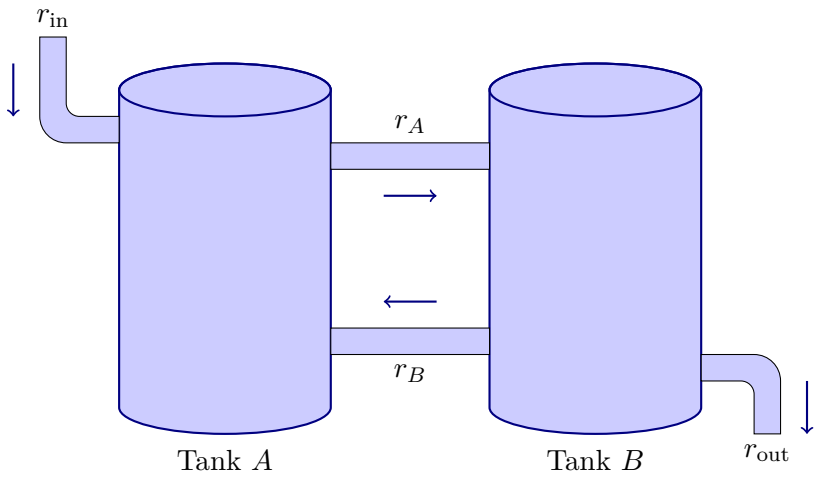

Figure 3.51: Mixing example with two tanks

If $x(t)$ and $y(t)$ are the amounts of salt in Tank A and Tank B, respectively, then our problem can be modeled with a linear system of two equations,

$$\frac{dx}{dt} = \text{rate in} - \text{rate out} = -r_A\frac{x}{V} + r_B\frac{y}{V}$$
$$\frac{dy}{dt} = \text{rate in} - \text{rate out} = r_A\frac{x}{V} - r_B\frac{y}{V} - r_{\text{out}}\frac{y}{V}.$$

Furthermore, $r_A = r_B + r_{\text{out}}$, since the volume in Tank B is constant. Consequently, our system now becomes

$$\frac{dx}{dt} = -r_A\frac{x}{V} + r_B\frac{y}{V}$$
$$\frac{dy}{dt} = r_A\frac{x}{V} - r_A\frac{y}{V}.$$

If we have initial conditions $x(0) = x_0$ and $y(0) = y_0$, it is not too difficult to deduce that the amount of salt in each tank will approach zero as $t \to \infty$, and we will have a stable equilibrium solution at $(0, 0)$. Determining the nature of the equilibrium solution is a more difficult question. For example, is it ever possible that the equilibrium solution is a spiral sink? One solution is provided by studying the ***trace-determinant plane***.

3.7.1 The Trace-Determinant Plane

The key to solving the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$$

is determining the eigenvalues of A. To find this eigenvalues, we need to derive the characteristic polynomial of A,

$$\det(A - \lambda I) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = \lambda^2 - (a+d)\lambda + (ad - bc).$$

Of course, $D = \det(A) = ad - bc$ is the determinant of A. The quantity $T = a + d$ is the sum of the diagonal elements of the matrix A. We call this quantity the ***trace*** of A and write $\text{tr}(A)$. Thus, we can rewrite the characteristic polynomial as

$$\det(A - \lambda I) = \lambda^2 - T\lambda + D.$$

We can use the trace and determinant to establish the nature of a solution to a linear system.

Theorem 3.52. *If a 2×2 matrix A has eigenvalues λ_1 and λ_2, then the trace of A is $\lambda_1 + \lambda_2$ and $\det(A) = \lambda_1\lambda_2$.*

Proof. The proof follows from a direct computation. Indeed, we can rewrite the characteristic polynomial as

$$\det(A - \lambda I) = \lambda^2 - T\lambda + D.$$

The eigenvalues of A are now given by

$$\lambda_1 = \frac{T + \sqrt{T^2 - 4D}}{2} \quad \text{and} \quad \lambda_2 = \frac{T - \sqrt{T^2 - 4D}}{2}.$$

Hence, $T = \lambda_1 + \lambda_2$ and $D = \lambda_1\lambda_2$. □

Theorem 3.52 tells us that we can determine the determinant and trace of a 2×2 matrix from its eigenvalues. Thus, we should be able to determine the phase portrait of a system $\mathbf{x}' = A\mathbf{x}$ by simply examining the trace and determinant of A. Since the eigenvalues of A are given by

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2},$$

we can immediately see that the expression $T^2 - 4D$ determines the nature of the eigenvalues of A.

- If $T^2 - 4D > 0$, we have two distinct real eigenvalues.
- If $T^2 - 4D < 0$, we have two complex eigenvalues, and these eigenvalues are complex conjugates.
- If $T^2 - 4D = 0$, we have repeated eigenvalues.

If $T^2 - 4D = 0$ or equivalently if $D = T^2/4$, we have repeated eigenvalues. In fact, we can represent those systems with repeated eigenvalues by graphing the parabola $D = T^2/4$ on the TD-plane or ***trace-determinant plane*** (Figure 3.53). Therefore, points on the parabola correspond to systems with repeated eigenvalues, points above the parabola ($D > T^2/4$ or equivalently $T^2 - 4D < 0$) correspond to systems with complex eigenvalues, and points below the parabola ($D < T^2/4$ or equivalently $T^2 - 4D > 0$) correspond to systems with real eigenvalues.

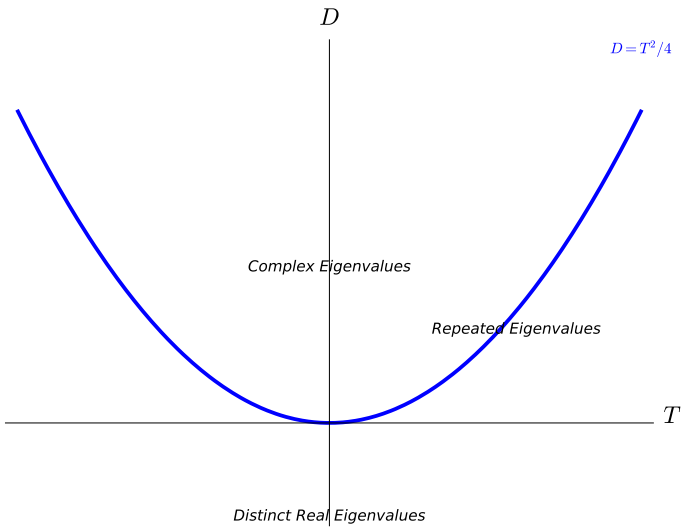

Figure 3.53: The trace-determinant plane

Theorem 3.54. *The trace and determinant of a 2×2 matrix are invariant under a change of coordinates. That is,* $\det(T^{-1}AT) = \det(A)$ *and* $\operatorname{tr}(T^{-1}AT) = \operatorname{tr}(A)$ *for any 2×2 matrix A and any invertible 2×2 matrix T.*

Proof. It is straightforward to verify that $\det(AB) = \det(A)\det(B)$ and $\det(T^{-1}) = 1/\det(T)$ for 2×2 matrices A and B. Therefore,

$$\det(T^{-1}AT) = \det(T^{-1})\det(A)\det(T) = \frac{1}{\det(T)}\det(A)\det(T) = \det(A).$$

A direct computation shows that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. Thus,

$$\operatorname{tr}(T^{-1}AT) = \operatorname{tr}(T^{-1}TA) = \operatorname{tr}(A).$$

□

Furthermore, each of the expression $T^2 - 4D$ is not affected by a change of coordinates by Theorem 3.54. That is, we only need to consider systems $\mathbf{x}' = A\mathbf{x}$, where A is one of the following matrices:

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

The system

$$\mathbf{x}' = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \mathbf{x}$$

has eigenvalues $\lambda = \alpha \pm i\beta$. The general solution to this system is

$$\mathbf{x}(t) = c_1 e^{\alpha t} \begin{pmatrix} \cos \beta t \\ -\sin \beta t \end{pmatrix} + c_2 e^{\alpha t} \begin{pmatrix} \sin \beta t \\ \cos \beta t \end{pmatrix}.$$

The $e^{\alpha t}$ factor tells us that the solutions either spiral into the origin if $\alpha < 0$, spiral out to infinity if $\alpha > 0$, or stay in a closed orbit if $\alpha = 0$. The equilibrium points are ***spiral sinks*** and ***spiral sources***, or ***centers***, respectively.

The eigenvalues of A are given by

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

If $T^2 - 4D < 0$, then we have a complex eigenvalues, and the type of equilibrium point depends on the real part of the eigenvalue. The sign of the real part is determined solely by T. If $T > 0$ we have a source. If $T < 0$, we have a sink. If $T = 0$, we have a center. See Figure 3.55.

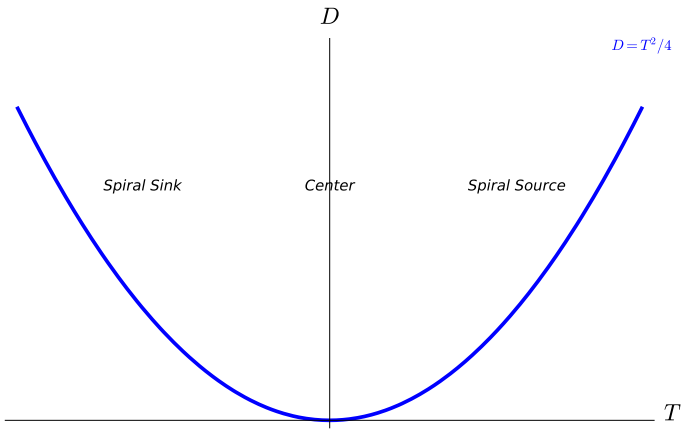

Figure 3.55: $D > T^2/4$

The situation for distinct real eigenvalues is a bit more complicated. Suppose that we have a system

$$\mathbf{x}' = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \mathbf{x}$$

with distinct eigenvalues λ and μ. We will have three cases to consider if none of our eigenvalues are zero:

- Both eigenvalues are positive (source).

- Both eigenvalues are negative (sink).

- One eigenvalue is negative and the other is positive (saddle).

Our two eigenvalues are given by

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

If $T > 0$, then the eigenvalue

$$\frac{T + \sqrt{T^2 - 4D}}{2}$$

is positive and we need only determine the sign of the second eigenvalue

$$\frac{T - \sqrt{T^2 - 4D}}{2}$$

If $D < 0$, we have one positive and one zero eigenvalue. That is, we have a ***saddle*** if $T > 0$ and $D < 0$.

If $D > 0$, then

$$T^2 - 4D < T^2.$$

Since we are considering the case $T > 0$, we have

$$\sqrt{T^2 - 4D} < T$$

and the value of the second eigenvalue $(T - \sqrt{T^2 - 4D}\,)/2$ is postive. Therefore, any point in the first quadrant below the parabola corresponds to a system with two positive eigenvalues and must correspond to a ***nodal source***.

One the other hand, suppose that $T < 0$. Then the eigenvalue $(T - \sqrt{T^2 - 4D}\,)/2$ is always negative, and we need to determine if other eigenvalue is positive or negative. If $D < 0$, then

$$T^2 - 4D > T^2$$

and

$$\sqrt{T^2 - 4D} > T$$

Therefore, the other eigenvalue $(T - \sqrt{T^2 - 4D}\,)/2$ is positive, telling us that any point in the fourth quadrant must correspond to a saddle. If $D > 0$, then $\sqrt{T^2 - 4D} < T$ and the second eigenvalue is negative. In this case, we will have a ***nodal sink***. We summarize our findings in Figure 3.56.

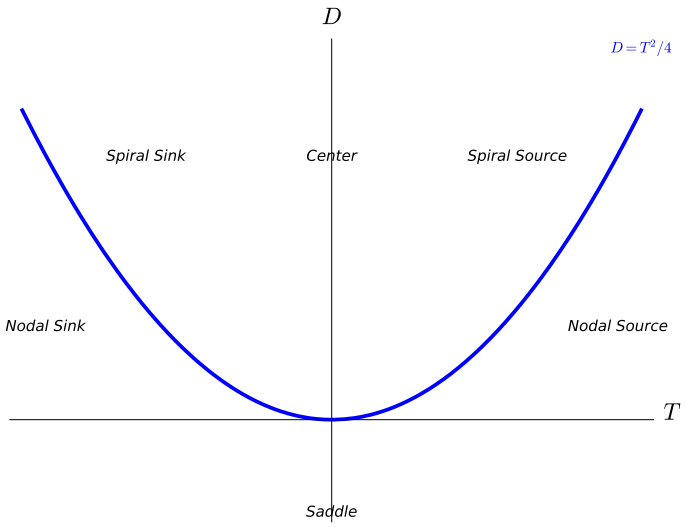

Figure 3.56: The trace-determinant plane for real and complex eigenvalues

For repeated eigenvalues, the analysis depends only on T. Since

$$T^2 - 4D = 0,$$

the only eigenvalue is $T/2$. Thus, we have sources if $T > 0$ and sinks if $T < 0$ (Figure 3.57).

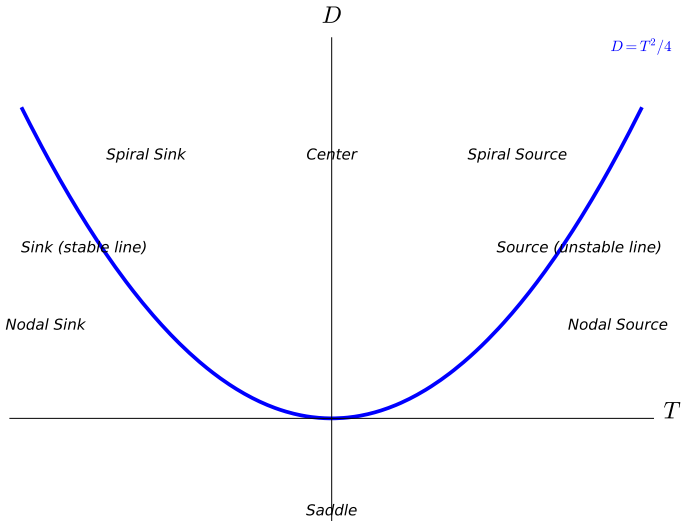

Figure 3.57: $D = T^2/4$

Example 3.58. Let us return to the mixing problem that we proposed at the beginning of this section. The problem could be modeled by the system of equations

$$\begin{aligned}
\frac{dx}{dt} &= -r_A\frac{x}{V} + r_B\frac{y}{V} \\
\frac{dy}{dt} &= r_A\frac{x}{V} - r_A\frac{y}{V} \\
x(0) &= x_0 \\
y(0) &= y_0.
\end{aligned}$$

The matrix corresponding to this system is

$$A = \begin{pmatrix} -r_A/V & +r_B/V \\ r_A/V & -r_A/V \end{pmatrix}.$$

Computing the trace and determinant of the matrix yields $T = -2r_A/V$ and $D = (r_A^2 - r_A r_B)/V^2$, where r_A and r_B are both positive. Certainly, $T < 0$ and

$$D = \frac{r_A^2 - r_A r_B}{V^2} = \frac{r_A(r_A - r_B)}{V^2} = \frac{r_A r_{\text{out}}}{V^2} > 0.$$

Therefore, any solution must be stable. Finally, since

$$4D - T^2 = 4\frac{r_A^2 - r_A r_B}{V^2} - \left(\frac{-2r_A}{V}\right)^2 = -\frac{4r_A r_B}{V^2} < 0,$$

we are below the parabola in the trace-determinant plane and know that our solution must be a nodal sink.

3.7.2 Parameterized Families of Linear Systems

The trace-determinant plane is an example of a ***parameter plane***. We can adjust the entries of a matrix A and, thus, change the value of the trace and the determinant.

Example 3.59. Consider the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A\mathbf{x} \begin{pmatrix} -2 & a \\ -2 & 0 \end{pmatrix} \mathbf{x}.$$

The trace of A is always $T = -2$, but $D = \det(A) = 2a$. We are on the parabola if

$$T^2 - 4D = 4 - 8a = 0 \qquad \text{or} \qquad a = \frac{1}{2}.$$

Thus, a bifurcation occurs at $a = 1/2$. If $a > 1/2$, we have a spiral sink. If $a < 1/2$, we have a sink with real eigenvalues. Further more, if $a < 0$, our sink becomes a saddle (Figure 3.60).

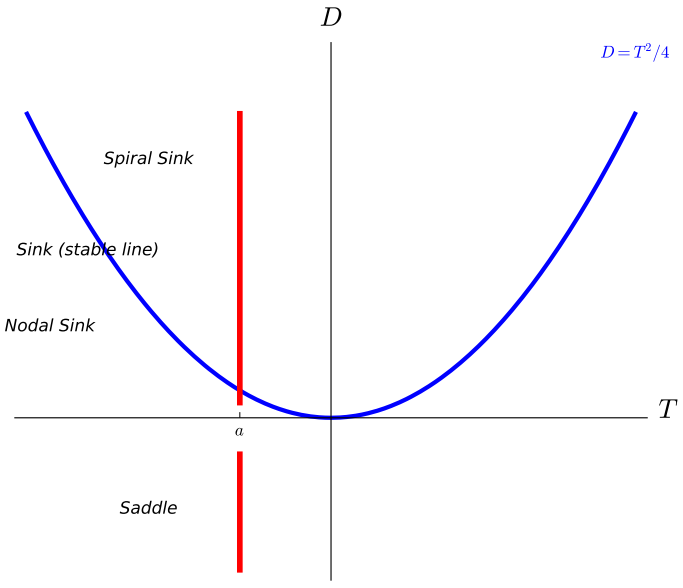

Figure 3.60: A one-parameter family of linear systems

Recall that a harmonic oscillator can be modeled by the second-order equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0,$$

where $m > 0$ is the mass, $b \geq 0$ is the damping coefficient, and $k > 0$ is the spring constant. If we rewrite this equation as a first-order system, we have

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -k/m & -b/m \end{pmatrix} \mathbf{x}.$$

Thus, for the harmonic oscillator $T = -b/m$ and $D = k/m$. If we use the trace-determinant plane to analyze the harmonic oscillator, we need only concern ourselves with the second quadrant (Figure 3.61).

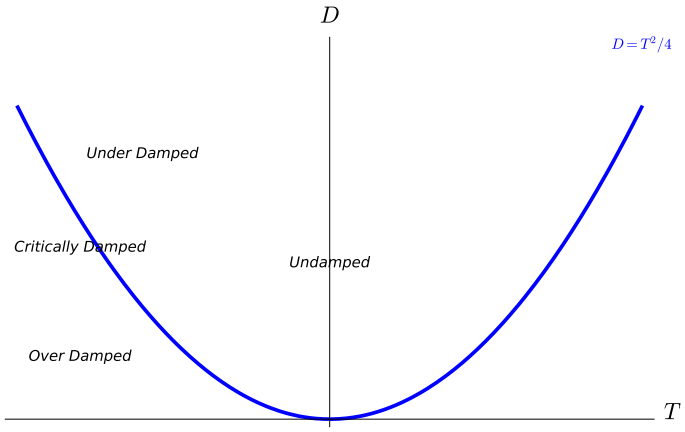

Figure 3.61: A one-parameter family for a harmonic oscillator

If $(T, D) = (-b/m, k/m)$ lies above the parabola, we have an underdamped oscillator. If $(T, D) = (-b/m, k/m)$ lies below the parabola, we have an overdamped oscillator. If $(T, D) = (-b/m, k/m)$ lies on the parabola, we have a critically damped oscillator. If $b = 0$, we have an undamped oscillator.

Example 3.62. Now let us see what happens to our harmonic oscillator when we fix $m = 1$ and $k = 3$ and let the damping b vary between zero and infinity. We can rewrite our system as

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -3x - by. \end{aligned}$$

Thus, $T = -b$ and $D = 3$. We can see how the phase portrait varies with the parameter b in Figure 3.63.

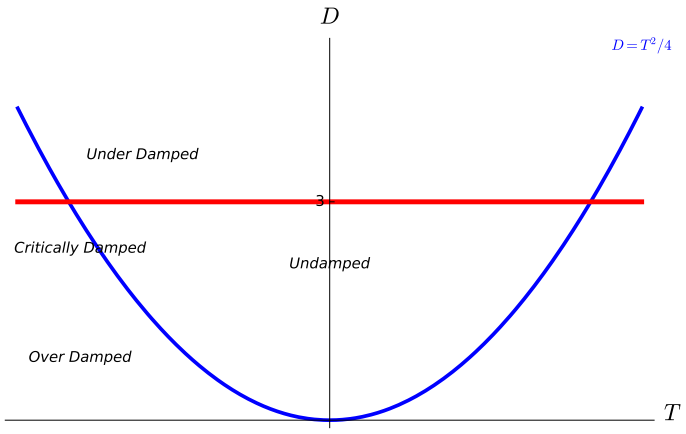

Figure 3.63: The trace-determinant plane for varying damping

The line $D = 3$ in the trace-determinant plane crosses the repeated eigenvalue parabola, $D = T^2/4$ if $b^2 = 12$ or when $b = 2\sqrt{3}$. If $b = 0$, we have purely imaginary eigenvalues. This is the undamped harmonic oscillator. If $0 < b < 2\sqrt{3}$, the eigenvalues are complex with a nonzero real part—the underdamped case. If $b = 2\sqrt{3}$, the eigenvalues are negative and repeated—the critically damped case. Finally, if $b > 2\sqrt{3}$, we have the overdamped case. In this case, the eigenvalues are real, distinct, and negative.

Example 3.64. Although the trace-determinant plane gives us a great deal of information about our system, we can not determine everything from this parameter plane. For example, the matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

both have the same trace and determinant, but the solutions to $\mathbf{x}' = A\mathbf{x}$ wind around the origin in a clockwise direction while those of $\mathbf{x}' = B\mathbf{x}$ wind around in a counterclockwise direction.

3.7.3 Important Lessons

- The characteristic polynomial of a 2×2 matrix can be written as

  $$\lambda^2 - T\lambda + D,$$

  where $T = \text{tr}(A)$ and $D = \det(A)$.

- If a 2×2 matrix A has eigenvalues λ_1 and λ_2, then $\text{tr}(A)$ is $\lambda_1 + \lambda_2$ and $\det(A) = \lambda_1\lambda_2$.

- The trace and determinant of a 2×2 matrix are invariant under a change of coordinates.

- The trace-determinant plane is determined by the graph of the parabola $D = T^2/4$ on the TD-plane. Points on the trace-determinant plane correspond to the trace and determinant of a linear system $\mathbf{x}' = A\mathbf{x}$. Since the trace and the determinant of a matrix determine the eigenvalues of A, we can use the trace-determinant plane to parameterize the phase portraits of linear systems.

- The trace-determinant plane is useful for studying bifurcations.

3.7.4 Exercises

1. Consider the one-parameter family of linear systems given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & \sqrt{2}+a/2 \\ \sqrt{2}-a/2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(a) Sketch the path traced out by this family of linear systems in the trace-determinant plane as a varies.

(b) Discuss any bifurcations that occur along this path and compute the corresponding values of a.

Hint.

2. Consider the two-parameter family of linear systems

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Identify all of the regions in the ab-plane where this system possesses a saddle, a sink, a spiral sink, and so on.

Hint.

3.7.5 Project—

3.8 Linear Systems in Higher Dimensions

Suppose that we have two masses on a table, m_1 and m_2, connected by three springs with the outside springs connected to two walls (Figure 3.65), and the masses are free to move horizontally. We will assume that the springs are uniform and all have the same spring constant k. The horizontal displacements of the springs are denoted by $x_1(t)$ and $x_2(t)$ for the masses m_1 and m_2, respectively. Assuming that there is no damping, the only forces acting on mass m_1 at time t are those of left and middle springs. The force from the left spring will be $-kx_1$ while the force from middle spring will be $k(x_2 - x_1)$. By Newton's Second Law of motion, we have

$$m_1 x_1'' = -kx_1 + k(x_2 - x_1).$$

Similarly, the only forces acting on the second mass, m_2, will come from middle and right springs. Again using Newton's Second Law,

$$m_2 x_2'' = -k(x_2 - x_1) - kx_2.$$

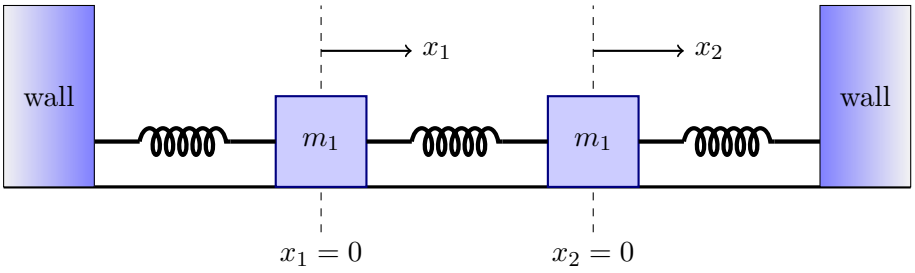

Figure 3.65: A double spring-mass system

We now have a system of two second-order linear equations

$$m_1 x_1'' = -2kx_1 + kx_2$$
$$m_2 x_2'' = kx_1 - 2kx_2.$$

If we define x_3 and x_4 by $x_3 = x_1'$ and $x_4 = x_2'$, we now have first-order linear system of four equations,

$$\begin{aligned} x_1' &= x_3 \\ x_2' &= x_4 \\ x_3' &= -2\frac{k}{m_1}x_1 + \frac{k}{m_1}x_2 \\ x_4' &= \frac{k}{m_2}x_1 - 2\frac{k}{m_2}x_2 \end{aligned}$$

We can represent this system in the matrix form $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2k/m_1 & k/m_1 & 0 & 0 \\ k/m_2 & -2k/m_2 & 0 & 0 \end{pmatrix}.$$

We will learn how to analyze and solve such systems in the next two sections.

3.8.1 Higher-Order Linear Systems

We can write the system

$$\begin{aligned} x_1' &= a_{11}x_1 + \cdots a_{1n}x_n \\ x_2' &= a_{21}x_1 + \cdots a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + \cdots a_{nn}x_n \end{aligned}$$

in matrix form $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad \text{and} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

The strategy for finding solutions to the system $\mathbf{x}' = A\mathbf{x}$ is the same as for systems of two equations. If λ is an eigenvalue of A with eigenvector $\mathbf{v}$,

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$$

is a solution for our system. Indeed,

$$\mathbf{x}'(t) = \lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}(\lambda \mathbf{v}) = e^{\lambda t}A\mathbf{v} = A(e^{\lambda t}\mathbf{v}) = A\mathbf{x}(t).$$

Example 3.66. The system

$$x' = -5x - 8y - 2z$$

$$y' = 5x + 12y + 4z$$
$$z' = -11x - 19y - 5z$$

can be rewritten as

$$\mathbf{x}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -5 & -8 & -2 \\ 5 & 12 & 4 \\ -11 & -19 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\mathbf{x}.$$

We can compute the eigenvalues of A by finding the roots of its characteristic polynomial

$$\det(A - \lambda I) = -\lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda - 1)(\lambda + 1)(\lambda - 2).$$

Thus, the eigenvalues of A are $\lambda = 2$, $\lambda = 1$, and $\lambda = -1$. To find an eigenvector for $\lambda = 2$, we must find a nontrivial solution for system of equations $(A - 2I)\mathbf{x} = \mathbf{0}$,

$$-7x - 8y - 2z = 0$$
$$-5x + 10y + 4z = 0$$
$$-11x - 19y - 7z = 0.$$

It is easy to check that $(2, -3, 5)$ is a solution. Similarly, we can determine that $(5, -7, 13)$ is an eigenvector for $\lambda = 1$ and $(1, -1, 2)$ is an eigenvector for $\lambda = -1$. Thus, we have found three solutions for the system $\mathbf{x}' = A\mathbf{x}$,

$$\mathbf{x}_1(t) = e^{2t} \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}, \mathbf{x}_2(t) = e^{t} \begin{pmatrix} 5 \\ -7 \\ 13 \end{pmatrix}, \quad \text{and} \quad \mathbf{x}_3(t) = e^{-t} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

The Principle of Superposition also holds for higher-order systems. If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions for $\mathbf{x}' = A\mathbf{x}$, then

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$$

is a solution for the system, since

$$\begin{aligned} \frac{d}{dt}[c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)] &= c_1\frac{d}{dt}\mathbf{x}_1(t) + c_2\frac{d}{dt}\mathbf{x}_2(t) \\ &= c_1A\mathbf{x}_1(t) + c_2A\mathbf{x}_2(t) \\ &= A[c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)]. \end{aligned}$$

Consequently,

$$\mathbf{x}(t) = c_1e^{2t} \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + e^{t} \begin{pmatrix} 5 \\ -7 \\ 13 \end{pmatrix} + c_3e^{-t} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

is a solution for our system. This is, in fact, the general solution for the system.

Although we shall not cover the notions of linear independence, canonical matrices, and change of coordinates for $\mathbb{R}^n$, the same ideas that we used for systems first-order linear differential equations in $\mathbb{R}^2$ carry over to $\mathbb{R}^n$.The necessary linear algebra is covered in any good linear algebra course. Also, see Chapters 5 and 6 in [12].

Example 3.67. Let us see how the linear algebra works in the previous example. If we form the matrix

$$T = \begin{pmatrix} 2 & 5 & 1 \\ -3 & -7 & -1 \\ 5 & 13 & 2 \end{pmatrix}$$

from the eigenvectors of A, we can convert the system $\mathbf{x}' = A\mathbf{x}$ to the system

$$\begin{aligned}\mathbf{y}' &= (T^{-1}AT)\mathbf{y}\\ &\begin{pmatrix}1 & -3 & -2\\ -1 & 1 & 1\\ 4 & 1 & -1\end{pmatrix}\begin{pmatrix}-5 & -8 & -2\\ 5 & 12 & 4\\ -11 & -19 & -5\end{pmatrix}\begin{pmatrix}2 & 5 & 1\\ -3 & -7 & -1\\ 5 & 13 & 2\end{pmatrix}\mathbf{y},\\ &= \begin{pmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\mathbf{y},\end{aligned}$$

which we can immediately solve:

$$\mathbf{y}(t) = c_1e^{2t}\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2e^{t}\begin{pmatrix}0\\1\\0\end{pmatrix} + c_3e^{-t}\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Multiplying our solution by T yields the general solution

$$\mathbf{x}(t) = T\mathbf{y} = c_1e^{2t}\begin{pmatrix}2\\-3\\5\end{pmatrix} + e^{t}\begin{pmatrix}5\\-7\\13\end{pmatrix} + c_3e^{-t}\begin{pmatrix}1\\-1\\2\end{pmatrix},$$

the solution to the system $\mathbf{x}'A\mathbf{x}$.

3.8.2 The Geometry of Solutions

In Section 3.6, we classified all of the geometry of the solutions for planar systems using the trace-determinant plane. The geometry for linear systems in three variables is a bit more complicated. For a system

$$\begin{aligned}x' &= a_1x + a_1y + a_3z\\ y' &= b_1x + b_2y + b_3z\\ z' &= c_1x + c_2y + c_3z,\end{aligned}$$

our solutions curves live in $\mathbb{R}^3$, and there is simply a lot more room to move around in three dimensions than in two dimensions. The origin is still an equilibrium solution for a system of linear differential equations in three variables. The origin is a ***stable*** equilibrium solution if any solution $\mathbf{x}(t)$ approaches $\mathbf{0} = (0, 0, 0)$ as $t \to \infty$; otherwise, $\mathbf{0}$ is an ***unstable*** equilibrium solution. In the case of planar systems, an unstable solution is a nodal saddle, a nodal source, a spiral source, or a source with a single unstable line. In the case of $\mathbb{R}^3$, we could have a ***stable line*** of solutions and an ***unstable plane*** of solutions. In this case, all solutions of the system with initial condition lying on the stable line would approach the origin as $t \to \infty$, but all solutions with initial conditions that are a nonzero point on the unstable plane would move away from the origin.

Example 3.68. In Example 3.66, we had solutions

$$\mathbf{x}_1(t) = e^{2t}\begin{pmatrix}2\\-3\\5\end{pmatrix}, \mathbf{x}_2(t) = e^{t}\begin{pmatrix}5\\-7\\13\end{pmatrix}, \quad \text{and} \quad \mathbf{x}_3(t) = e^{-t}\begin{pmatrix}1\\-1\\2\end{pmatrix}.$$

The straight line through the origin and the point $(1, -1, 2)$ is a stable line. That is, for any initial condition $\mathbf{x}(0) = (x_0, y_0, z_0)$ lying on this line, our solution will tend toward the

origin as $t \to \infty$. On the other hand, the plane spanned by $(2, -3, 5)$ and $(5, -7, 13)$ is unstable plane. Solutions on this plane move away from the origin as $t \to \infty$. Of course, $(0, 0, 0)$ is an equlibrium solution for our system. We say that the origin is a ***saddle*** in this example (Figure 3.69).

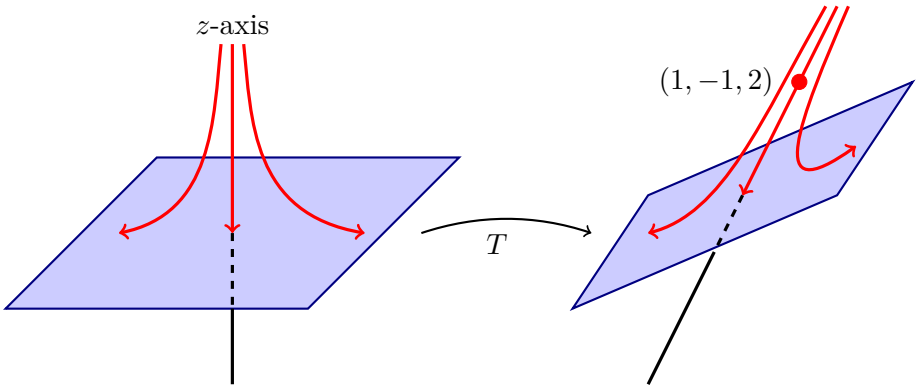

Figure 3.69: A saddle in $\mathbb{R}^3$

Example 3.70. For the system

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathbf{x}$$

we have a very different type of unstable equilibrium solution. The eigenvalues of this matrix are $\lambda = \pm i$ and $\lambda = -1$. Thus, a solution satisfying the initial condition $\mathbf{x}(0) = (x_0, y_0, z_0)$ is given by

$$\mathbf{x}(t) = x_0 \begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix} + y_0 \begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix} + z_0 e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

If $z_0 = 0$, then our initial condition is in the xy-plane and all of the solutions lie on circles centered at the origin. On the other hand, if $x_0 = 0$ and $y_0 = 0$, we have a stable line of solutions lying along the z-axis. In fact, each solution that does not lie on the stable line lies on a cylinder in $\mathbb{R}^3$ given by $x^2 + y^2 = r^2$ for some constant $r > 0$. These solutions spiral towards the circular solution of radius r in the xy-plane if $z_0 \neq 0$ (Figure 3.71).

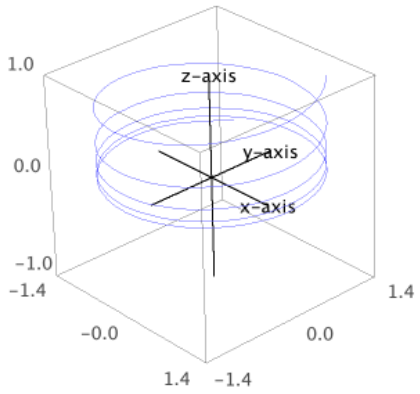

Figure 3.71: A spiral center in $\mathbb{R}^3$

Example 3.72. For an example of a stable plane and an unstable line, let us consider the system

$$\mathbf{x}' = \begin{pmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x} = A\mathbf{x}.$$

The characteristic equation of the matrix A is

$$\lambda^3 + \lambda^2 - 2 = (\lambda - 1)(\lambda^2 + 2\lambda + 2) = 0.$$

Thus, the eigenvalues of A are $\lambda = 1$ and $\lambda = -1 \pm i$. Solving

$$[A - (-1 + i)]\mathbf{x} = \mathbf{0}$$

gives us an eigenvector $(1, i, 0)$ for $\lambda = -1 + i$. Since

$$\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

we will let $\mathbf{v}_1 = (1, 0, 0)$ and $\mathbf{v}_2 = (0, 1, 0)$. Since $\mathbf{v}_3 = (0, 0, 1)$ is an eigenvector for $\lambda = 1$, our system has solution

$$\mathbf{x}(t) = x_0 e^{-t} \begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix} + y_0 e^{-t} \begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix} + z_0 e^{t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where $\mathbf{x}(0) = (x_0, y_0, z_0)$. If $z_0 = 0$, our initial condition lies in the xy-plane and solution curves sprial in towards the origin. Thus, we have a stable plane. On the other hand, if $x_0 = 0$ and $y_0 = 0$ but $z_0 \neq 0$, then our solution approaches $\pm\infty$ as $t \to \infty$. In this case, the z-axis is an unstable line (Figure 3.73).

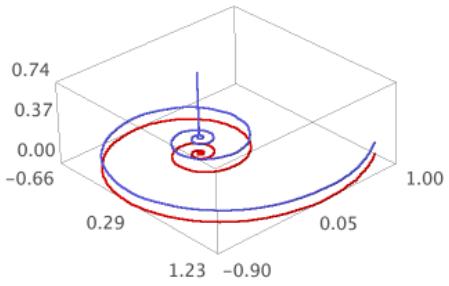

Figure 3.73: A spiral saddle in $\mathbb{R}^3$

For an example of a stable equilibrium solution at the origin, consider the system

$$\mathbf{x}' = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \mathbf{x},$$

where $\lambda_3 < \lambda_2 < \lambda_1 < 0$. For an initial condition (x_0, y_0, z_0) with at least one coordinate nonzero, the corresponding solution tends towards the origin tangentially to the x-axis as $t \to \infty$ (Figure 3.74).

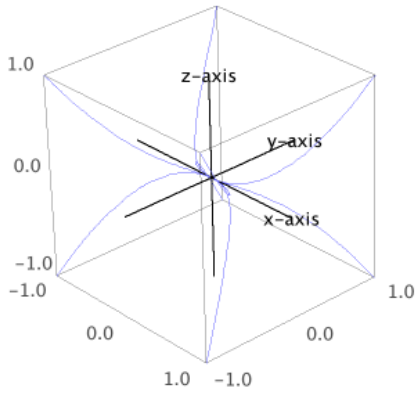

Figure 3.74: A sink in $\mathbb{R}^3$

Changing the system that in Example3.72 to be

$$\mathbf{x}' = \begin{pmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathbf{x},$$

we obtain the solution satisfying the initial condition $\mathbf{x}(0) = (x_0, y_0, z_0)$ to be

$$\mathbf{x}(t) = x_0 e^{-t} \begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix} + y_0 e^{-t} \begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix} + z_0 e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where $\mathbf{x}(0) = (x_0, y_0, z_0)$. In this case, all solutions will approach the origin as $t \to \infty$.

3.8.3 The Double Spring-Mass Systems Revisited

Let us return to our spring-mass system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2k/m_1 & k/m_1 & 0 & 0 \\ k/m_2 & -2k/m_2 & 0 & 0 \end{pmatrix}.$$

To keep matter simple, we will assume that $m_1 = m_2 = k = 1$. Thus, our matrix now becomes

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial of A is

$$\det(A - \lambda I) = \lambda^4 + 4\lambda^2 + 3 = (\lambda^2 + 1)(\lambda^2 + 3).$$

Thus the eigenvalues of A are $\lambda = \pm i$ and $\lambda = \pm i\sqrt{3}$. We can find eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} i/\sqrt{3} \\ -i/\sqrt{3} \\ -1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -i/\sqrt{3} \\ i/\sqrt{3} \\ -1 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -i \\ -i \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} i \\ i \\ 1 \\ 1 \end{pmatrix},$$

corresponding to the eigenvalues $\lambda_1 = i\sqrt{3}$, $\lambda_2 = -i\sqrt{3}$, $\lambda_3 = i$, $\lambda_4 = -i$, respectively. Consequently, the general solution to our system is

$$\mathbf{x}(t) = c_1 e^{i\sqrt{3}t}\mathbf{v}_1 + c_1 e^{-i\sqrt{3}t}\mathbf{v}_2 + c_1 e^{it}\mathbf{v}_3 + c_1 e^{-it}\mathbf{v}_4;$$

however, this form of the solution is not very useful. By examining real and imaginary parts of $e^{i\sqrt{3}t}\mathbf{v}_1$ and $c_1 e^{it}\mathbf{v}_3$, we can rewrite the solution as

$$\mathbf{x}(t) = c_1 \begin{pmatrix} \cos\sqrt{3}t \\ -\cos\sqrt{3}t \\ -\sqrt{3}\sin\sqrt{3}t \\ \sqrt{3}\sin\sqrt{3}t \end{pmatrix} + c_2 \begin{pmatrix} -\sin\sqrt{3}t \\ \sin\sqrt{3}t \\ -\sqrt{3}\cos\sqrt{3}t \\ \sqrt{3}\cos\sqrt{3}t \end{pmatrix} + c_3 \begin{pmatrix} \cos t \\ \cos t \\ -\sin t \\ -\sin t \end{pmatrix} + c_4 \begin{pmatrix} \sin t \\ \sin t \\ \cos t \\ \cos t \end{pmatrix}.$$

If we have the following initial conditions,

$$\begin{aligned} x_1(0) &= 0 \\ x_2(0) &= 0 \\ x_1'(0) = x_3(0) &= 2 \\ x_2'(0) = x_4(0) &= 2, \end{aligned}$$

we can determine $c_1 = c_2 = c_3 = 0$ and $c_4 = 2$. Thus, the solution to our initial value problem is

$$\mathbf{x}(t) = \begin{pmatrix} 2\sin t \\ 2\sin t \\ 2\cos t \\ 2\cos t \end{pmatrix},$$

and the two masses will oscillate with a frequency of one and an amplitude of two. We leave the details as an exercise.

3.8.4 Important Lessons

- $$\begin{aligned} x_1' &= a_{11}x_1 + \cdots a_{1n}x_n \\ x_2' &= a_{21}x_1 + \cdots a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + \cdots a_{nn}x_n \end{aligned}$$

can be written in matrix form $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad \text{and} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

- As in the case of $\mathbb{R}^2$, we can solve the system $\mathbf{x}' = A\mathbf{x}$ by finding eigenvalues and eigenvectors for A.
- The Principle of Superposition holds for higher-order systems. If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions for $\mathbf{x}' = A\mathbf{x}$, then
$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$$
is a solution for the system.
- The geometry for system in $\mathbb{R}^3$ is more complicated than the planar case. However, the solutions are usually characterized by stable lines or stable planes.

3.8.5 Exercises

1. Solve the system

$$\begin{aligned} x' &= -5x + 15y - 4z \\ y' &= -x + 5y - z \\ z' &= 4x - 6y + 3z \\ x(0) &= -1 \\ y(0) &= 2 \\ z(0) &= 0 \end{aligned}$$

Hint.

2. Find the general solution of the system

$$\begin{aligned} x_1' &= 6x_1 + 9x_2 + x_3 + 4x_4 \\ x_2' &= -x_1 + x_2 + x_3 \\ x_3' &= 3x_1 + 8x_2 + 4x_3 + 4x_4 \\ x_4' &= -3x_1 - 8x_2 - 7x_3 - 5x_4 \end{aligned}$$

Hint.

3. Let $ax'' + bx' + cx = 0$, where $a \neq 0$ and $b^2 - 4ac = 0$.

(a) Show that $x_1(t) = e^{-bt/2a}$ is a solution to $ax'' + bx' + cx = 0$.

(b) Assume that
$$y = v(t)x_1(t) = v(t)e^{-bt/2a}$$
is a solution to $ax'' + bx' + cx = 0$ and show that $v(t) = c_1 + c_2 t$. Thus,
$$x(t) = c_1 e^{-bt/2a} + c_2 te^{-bt/2a}$$
is a general solution for $ax'' + bx' + cx = 0$.

Hint.

4. Consider the equation

$$y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0.$$

Determine all values of α, if any, for which all solutions tend toward zero as $t \to \infty$. Also, determine the values of α, if any, for which all nonzero solutions become unbounded as $t \to \infty$.

Hint.

5. Solve each of the following initial value problems.

(a)

$$\begin{aligned} y'' - 2y' + 5y &= 0 \\ y(\pi/2) &= 0 \\ y'(\pi/2) &= 2 \end{aligned}$$

(b)

$$\begin{aligned} 9y'' - 12y' + 4y &= 0 \\ y(0) &= 2 \\ y'(0) &= -1 \end{aligned}$$

(c)

$$\begin{aligned} y'' + 8y' - 9y &= 0 \\ y(1) &= 1 \\ y'(1) &= 0 \end{aligned}$$

(d)

$$\begin{aligned} y'' + 2ay' + (a^2 + 1)y &= 0 \\ y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

Hint.

6. *Reduction of Order.* Suppose that $x_1(t)$ is a solution (not identically zero) to the equation

$$x'' + p(t)x' + q(t)x = 0.$$

(a) Assume that $x(t) = v(t)x_1(t)$ is a solution to $x'' + p(t)x' + q(t)x = 0$ and derive the equation

$$x_1 v'' + (2x_1' + px_1)v' = 0. \tag{3.12}$$

(b) Let $u = v'$ and show that ((A.10)) is a first-order linear differential equation in u.

(c) Show that $x_1(t) = 1/t$ is a solution to

$$2t^2 x'' + 3tx' - x = 0 \tag{3.13}$$

for $t > 0$ and find a second linearly independent solution to ((A.11)).

Hint.

7. *Euler Equations.* An important class of second-order linear differential equations are equations of the form

$$t^2y'' + \alpha ty' + \beta y = 0,$$

where $t > 0$ and α and β are real constants. An equation of this form is called an *Euler equation.*

(a) Show that the substitution $x = \ln t$ transforms an Euler equation into an equation of constant coefficients. *Hint*: Show that

$$\frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx}$$
$$\frac{d^2y}{dt^2} = \left(\frac{dx}{dt}\right)^2 \frac{d^2y}{dx^2} + \frac{d^2x}{dt^2}\frac{dy}{dx}.$$

(b) Solve the equation

$$t^2y'' + 4ty' + 2y = 0.$$

Hint.

3.8.6 Project

3.9 The Matrix Exponential

Consider the linear system

$$x' = -17x - 46y - 7z$$
$$y' = 8x + 21y + 3z$$
$$z' = -7x - 15y - z$$

The matrix associated with this system is

$$A = \begin{pmatrix} -17 & -46 & -7 \\ 8 & 21 & 3 \\ -7 & -15 & -1 \end{pmatrix}.$$

The characteristic polynomial of A is

$$p(\lambda) = \det(A - \lambda I) = \lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3$$

hence, there is only a single eigenvalue $\lambda = 1$. Moreover, we can only find a single linearly independent eigenvector $(1, -1, 4)$. Thus,

$$\mathbf{x}(t) = ce^{-t}\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$

is a solution to our system. However, this is not the general solution to the system. We can only solve initial value problems where the initial condition lies on the line through the origin containing the vector $(1, -1, 4)$. To construction a general solution to our system, we will need two other linearly independent solutions.

3.9.1 The Exponential of a Matrix

Our goal is to construct a solution to the initial value problem

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} \\ \mathbf{x}(0) &= \mathbf{x}_0, \end{aligned}$$

where A is an $n \times n$ matrix. Recalling that the solution to the initial value problem

$$\begin{aligned} x' &= kx \\ x(0) &= x_0 \end{aligned}$$

is $x(t) = x_0 e^{kt}$, we might guess that a solution to the initial value problem

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned}$$

has the form

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0$$

if we can make sense of the expression e^{tA}.

We will define the ***exponential*** of A using the power series for e^t. Thus,

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k,$$

where A is an $n \times n$ matrix, where $A^0 = I$ by convention. Each term makes sense in our definition since each is an expression of $n \times n$ matrices; however, there are some issues surrounding the convergence of the power series. A series, even a series whose individual terms are matrix expressions, converges if and only if its partial sums,

$$S_N = \sum_{k=0}^{N} \frac{1}{k!}A^k$$

converge. Although we shall not provide a proof, the matrix exponential e^A converges for all A.

Example 3.75. Let us compute the exponential of

$$A = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}.$$

Actually, this is quite easy,

$$\begin{aligned} e^A &= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} s^2 & 0 \\ 0 & t^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} s^3 & 0 \\ 0 & t^3 \end{pmatrix} + \cdots \\ &= \begin{pmatrix} 1 + s + s^2/2! + s^3/3! + \cdots & 0 \\ 0 & 1 + t + t^2/2! + t^3/3! + \cdots \end{pmatrix} \\ &= \begin{pmatrix} e^s & 0 \\ 0 & e^t \end{pmatrix}. \end{aligned}$$

Theorem 3.76. *If A is an $n \times n$ matrix, then*

$$\frac{d}{dt}e^{tA} = Ae^{tA}.$$

PROOF. We simply need to differentiate

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

term by term.[3] However,

$$\begin{aligned}\frac{d}{dt}e^{tA} &= \frac{d}{dt}\left(I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots\right)\\ &= A + tA^2 + \frac{t^2}{2!}A^3 + \cdots\\ &= A\left(I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!} + \cdots\right)\\ &= Ae^{tA}.\end{aligned}$$

□

Corollary 3.77. *Let A be an $n \times n$ matrix. Then $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$ is the solution to the initial value problem*

$$\begin{aligned}\mathbf{x}' &= A\mathbf{x}\\ \mathbf{x}(0) &= \mathbf{x}_0\end{aligned}$$

PROOF. The corollary follows immediately from Theorem 3.76. If $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$, then

$$\mathbf{x}'(t) = \frac{d}{dt}\left(e^{tA}\mathbf{x}_0\right) = \frac{d}{dt}\left(e^{tA}\right)\mathbf{x}_0 = Ae^{tA}\mathbf{x}_0 = A\mathbf{x}(t).$$

□

Thus, solving linear systems is simply a matter of computing matrix exponentials. The problem is that matrix exponentials may not be so easy to compute.

Example 3.78. The matrix

$$A = \begin{pmatrix} 4 & 4\\ -9 & -8\end{pmatrix}$$

has repeated eigenvalues $\lambda = -2$. If we try to compute e^{tA}, then

$$\begin{aligned}e^{tA} &= I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots\\ &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + t\begin{pmatrix}4 & 4\\ -9 & -8\end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix}4 & 4\\ -9 & -8\end{pmatrix}^2 + \frac{t^3}{3!}\begin{pmatrix}4 & 4\\ -9 & -8\end{pmatrix}^3 + \cdots\\ &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + t\begin{pmatrix}4 & 4\\ -9 & -8\end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix}-20 & -16\\ -36 & 28\end{pmatrix} + \frac{t^3}{3!}\begin{pmatrix}64 & 48\\ -108 & -80\end{pmatrix} + \cdots\\ &= \begin{pmatrix}1 + t - 20t^2/2! + 64t^3/3! + \cdots & 4t - 16t^2/2! + 48t^3/3! + \cdots\\ -9t - 36t^2/2! - 108t^3/3! + \cdots & 1 - 8t + 28t^2/2! - 80t^3/3! + \cdots\end{pmatrix}.\end{aligned}$$

It is not at all clear that this series converges to a matrix whose entries can be expressed in terms of elementary functions.

[3] Since we are differentiating an infinite series, we still need to show that differentiating term by term is something that can be done. We will, however, leave the details to a course in advanced calculus.

Now let us see how we can use the matrix exponential to solve a linear system as well as invent a more direct way to compute the matrix exponential.

Theorem 3.79. *Let λ be an eigenvalue of an $n \times n$ matrix A. If $\mathbf{v}$ is an eigenvector for λ, then $e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$.*

Proof. Since $\mathbf{v}$ is an eigenvalue for λ, we know that $A\mathbf{v} = \lambda\mathbf{v}$. Using mathematical induction, we can show that A^n has eigenvalue λ^n with associated eigenvector $\mathbf{v}$. Indeed,

$$A^n\mathbf{v} = A(A^{n-1}\mathbf{v}) = A(\lambda^{n-1}\mathbf{v}) = \lambda^{n-1}A\mathbf{v} = \lambda^n\mathbf{v}.$$

Hence,

$$\begin{aligned} e^{tA}\mathbf{v} &= \left(I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots\right)\mathbf{v} \\ &= \mathbf{v} + tA\mathbf{v} + \frac{t^2}{2!}A^2\mathbf{v} + \frac{t^3}{3!}A^3\mathbf{v} + \cdots \\ &= \mathbf{v} + t\lambda\mathbf{v} + \frac{t^2}{2!}\lambda^2\mathbf{v} + \frac{t^3}{3!}\lambda^3\mathbf{v} + \cdots \\ &= \left(I + t\lambda + \frac{t^2}{2!}\lambda^2 + \frac{t^3}{3!}\lambda^3 + \cdots\right)\mathbf{v} \\ &= e^{\lambda t}\mathbf{v}. \end{aligned}$$

□

The matrix exponential shares several properties with the function exponential function e^x that we studied in calculus.

Theorem 3.80. *Let A and B be $n \times n$ matrices. Then*

1. *$Ae^A = e^A A$;*
2. *if $AB = BA$, then $e^A e^B = e^{A+B}$;*
3. *e^A has inverse e^{-A}.*

Proof. To prove (1), we can simply expand both sides of the equality in a power series,

$$\begin{aligned} Ae^A &= A\left(I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots\right) \\ &= A + A^2 + \frac{1}{2!}A^3 + \frac{1}{3!}A^4 + \cdots \\ &= \left(I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots\right)A \\ &= e^A A. \end{aligned}$$

Proving (2) is a only bit more complicated if we notice that the binomial expansion holds for matrices,

$$(A + B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k},$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

providing $AB = BA$.

$$\begin{aligned} e^{A+B} &= \sum_{n=0}^{\infty} \frac{1}{n!}(A+B)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!(n-k)!} A^k B^{n-k} \right) \\ &= \left(\sum_{n=0}^{\infty} \frac{1}{n!} A^n \right) \left(\sum_{n=0}^{\infty} \frac{1}{n!} B^n \right) \\ &= e^A e^B. \end{aligned}$$

Simply expand each series out to see that this is true. Part (3) follows directly from Part (2), since A and $-A$ commute. □

Example 3.81. Now let us compute e^{tA} once again for

$$A = \begin{pmatrix} 4 & 4 \\ -9 & -8 \end{pmatrix}.$$

First notice that

$$A = \lambda I + (A - \lambda I).$$

Since the identity matrix I commutes with every matrix, we know that

$$e^{tA} = e^{t(\lambda I + (A - \lambda I))} = e^{t\lambda I} e^{t(A-\lambda I)}.$$

We also know that $e^{\lambda t I} = e^{\lambda t} I$ by Example 3.75. Thus,

$$e^{tA} = e^{t(\lambda I + (A - \lambda I))} = e^{t\lambda I} e^{t(A-\lambda I)} = e^{\lambda t} e^{t(A-\lambda I)}.$$

The matrix A has repeated eigenvalue $\lambda = -2$. Consequently,

$$A - \lambda I = A + 2I = \begin{pmatrix} 6 & 4 \\ -9 & -6 \end{pmatrix},$$

and $(A - \lambda I)^2$ is the zero matrix. Thus,

$$\begin{aligned} e^{tA} &= e^{\lambda t} e^{t(A-\lambda I)} \\ &= e^{\lambda t} \left(I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \frac{t^2}{3!}(A - \lambda I)^3 + \cdots \right) \\ &= e^{-2t} \left(I + t(A + 2I) + \frac{t^2}{2!}(A + 2I)^2 + \frac{t^2}{3!}(A + 2I)^3 + \cdots \right) \\ &= e^{-2t} \left(I + t(A + 2I) \right) \\ &= e^{-2t} \begin{pmatrix} 1 + 6t & 4t \\ -9t & -6t \end{pmatrix}. \end{aligned}$$

Our example suggests at the following proposition. We leave the proof of this proposition as an exercise.

Proposition 3.82. *If A is an $n \times n$ matrix with a single eigenvalue λ, then there exists nonnegative integer $k < n$ such that*

$$e^{tA} = e^{\lambda t}\left(I + t(A-\lambda I) + \frac{t^2}{2!}(A-\lambda I)^2 + \cdots + \frac{t^k}{k!}(A-\lambda I)^k\right).$$

Example 3.83. We are now ready return to our original system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} -17 & -46 & -7 \\ 8 & 21 & 3 \\ -7 & -15 & -1 \end{pmatrix}.$$

This matrix has a single eigenvalue $\lambda = 1$. It is easy to show that the only nonzero powers of $A - \lambda I = A - I$ are

$$(A+I) = \begin{pmatrix} -18 & -46 & -7 \\ 8 & 20 & 3 \\ -7 & -15 & -2 \end{pmatrix}$$
$$(A+I)^2 = \begin{pmatrix} 5 & 13 & 2 \\ -5 & -13 & -2 \\ 20 & 52 & 8 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} e^{tA} &= e^t\left(I + t(A+I) + \frac{t^2}{2!}(A+I)^2\right) \\ &= e^t \begin{pmatrix} 1-18t+5t^2/2 & -46t+13t^2/2 & -7t+t^2 \\ 8t-5t^2/2 & 1+20t-13t^2/2 & 3t-t^2 \\ -7t+10t^2 & -15t+26t^2 & 1-2t+4t^2 \end{pmatrix}. \end{aligned}$$

Now, to compute three linearly independent solutions for $\mathbf{x}' = A\mathbf{x}$, we simply compute $e^{tA}\mathbf{v}$ for three linearly independent vectors. We will use the standard basis vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, the general solution to our system is

$$\mathbf{x}(t) = c_1 e^{-t}\begin{pmatrix} 1-18t+5t^2/2 \\ 8t-5t^2/2 \\ -7t+10t^2 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -46t+13t^2/2 \\ 1+20t-13t^2/2 \\ -15t+26t^2 \end{pmatrix} + c_3 e^{-t}\begin{pmatrix} -7t+t^2 \\ 3t-t^2 \\ 1-2t+4t^2 \end{pmatrix}.$$

3.9.2 Generalized Eigenvalues

Example 3.84. Consider the system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 7 & -1 & 2 \\ 18 & -2 & 6 \\ -9 & 2 & -1 \end{pmatrix}.$$

The characteristic polynomial of A is

$$\det(A - \lambda I) = \lambda^3 - 4\lambda^2 + 5\lambda - 2 = (\lambda - 1)^2(\lambda - 2).$$

The eigenvalue $\lambda_1 = 1$ has eigenvector $\mathbf{v}_1 = (2, 6, -3)$ and the eigenvalue $\lambda_2 = 2$ has eigenvector $\mathbf{v}_2 = (1, 3, -1)$. Thus, we can find two linearly independent solutions in this case

$$\mathbf{x}_1(t) = e^t \begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix} \qquad \text{and} \qquad \mathbf{x}_2(t) = e^{2t} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}.$$

Since $\lambda_1 = 1$ has multiplicity two and we can find only one linearly independent eigenvector, it is not possible to apply Proposition 3.82 in this case.

If we consider the exponential

$$\begin{aligned} e^{tA}\mathbf{v} &= e^{\lambda_1 t} e^{t(A-\lambda_1 I)}\mathbf{v} \\ &= e^{\lambda_1 t}\left(\mathbf{v} + t(A-\lambda_1 I)\mathbf{v} + \frac{t^2}{2!}(A-\lambda_1 I)^2\mathbf{v} + \frac{t^2}{3!}(A-\lambda_1 I)^3\mathbf{v} + \cdots\right) \end{aligned}$$

where $\mathbf{v}, \mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, our goal is to choose $\mathbf{v}$ for which the series truncates. That is, we must look for vectors $\mathbf{v}$ such that $(A-\lambda_1 I)^k\mathbf{v} = (A-I)^k\mathbf{v} = \mathbf{0}$. If $k = 1$, then $(A-\lambda_1 I)\mathbf{v} = (A-I)\mathbf{v} = \mathbf{0}$, which means that $\mathbf{v}$ is an eigenvector. Thus, $\mathbf{v}$ must be a multiple of $\mathbf{v}_1 = (2, 6, -3)$ in this case. Since we already know that the eigenspace associated with this eigenvector has dimension one and is generated by $\mathbf{v}_1$, we must consider higher powers.

Since

$$(A-\lambda I) = (A-I) = \begin{pmatrix} 6 & -1 & 2 \\ 18 & -3 & 6 \\ -9 & 2 & -2 \end{pmatrix},$$

we have

$$(A-\lambda I)^2 = (A-I)^2 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 3 & 6 \\ 0 & -1 & -2 \end{pmatrix}.$$

The nullspace of this matrix has dimension two. Certainly, $\mathbf{v}_1 = (2, 6, -3)$ is in the nullspace of $(A-I)^2$, since it is the nullspace of $A - I$. We wish to find a vector in the nullspace that is not a multiple of the vector $\mathbf{v}_1$ that is also in the nullspace of $A - I$. The vector $\mathbf{v} = (0, 1, 0)$ will do. Now our series truncates,

$$\begin{aligned} \mathbf{x}_3(t) &= e^{tA}\mathbf{v} \\ &= e^{\lambda_1 t} e^{t(A-\lambda_1 I)}\mathbf{v} \\ &= e^{\lambda_1 t}\left(\mathbf{v} + t(A-\lambda_1 I)\mathbf{v}\right) \\ &= e^{\lambda_1 t}\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t(A-I)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right] \\ &= e^t \begin{pmatrix} t \\ 1+3t \\ -t \end{pmatrix} \end{aligned}$$

We now have a general solution for our system,

$$\mathbf{x}(t) = c_1 e^t \begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + c_3 e^t \begin{pmatrix} t \\ 1+3t \\ -t \end{pmatrix}.$$

If λ is an eigenvalue of A and $(A-\lambda I)^p\mathbf{v}=\mathbf{0}$ for some $p\geq 1$, then $\mathbf{v}$ is called a ***generalized eigenvector*** of A. When eigenvalues have algebraic multiplicity greater than one, we can compute extra solutions by looking for vectors in the nullspace of $(A-\lambda I)^p$ for $p>1$. The following theorem tells us that this is always possible. We leave the proof of the theorem as an exercise in linear algebra.

Theorem 3.85. *Suppose that λ is an eigenvalue of A with multiplicity q. Then there exists an integer $p\leq q$ such that the dimension of the nullspace of $(A-\lambda I)^p$ is q.*

We now have a procedure for finding q linearly independent solutions corresponding to an eigenvalue λ of multiplicity q.

1. Find the smallest integer p such that the nullspace of $(A-\lambda I)^p$ has dimension q.
2. Find a basis $\{\mathbf{v}_1,\ldots,\mathbf{v}_q\}$ for the nullspace of $(A-\lambda I)^p$.
3. For each $\mathbf{v}_j$ $(j=1,2,\ldots,q)$, we have the solution

$$\begin{aligned}\mathbf{x}_j(t) &= e^{tA}\mathbf{v}_j \\ &= e^{\lambda t}\left(\mathbf{v}_j + t(A-\lambda I)\mathbf{v}_j + \cdots + \frac{t^{p-1}}{(p-1)!}(A-\lambda I)^{p-1}\mathbf{v}_j.\right).\end{aligned}$$

This procedure works for complex eigenvalues as well as real. If $\lambda=\alpha+i\beta$ has eigenvector $\mathbf{z}=\mathbf{x}+i\mathbf{y}$, then set $\mathbf{x}=\operatorname{Re}\mathbf{z}$ and $\mathbf{y}=\operatorname{Im}\mathbf{z}$.

3.9.3 Important Lessons

- If A is an $n\times n$ matrix, we define the ***exponential*** of A to be

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{k=0}^{\infty}\frac{1}{k!}A^k.$$

- If A is an $n\times n$ matrix, then

$$\frac{d}{dt}e^{tA} = Ae^{tA}.$$

- Let A be an $n\times n$ matrix. Then $\mathbf{x}(t)=e^{tA}\mathbf{x}_0$ is the solution to the initial value problem

$$\begin{aligned}\mathbf{x}' &= A\mathbf{x} \\ \mathbf{x}(0) &= \mathbf{x}_0\end{aligned}$$

- Let λ be an eigenvalue of an $n\times n$ matrix A. If $\mathbf{v}$ is an eigenvector for λ, then $e^{tA}\mathbf{v}=e^{\lambda t}\mathbf{v}$.
- Let A and B be $n\times n$ matrices. Then
  - $Ae^A=e^AA$;
  - if $AB=BA$, then $e^Ae^B=e^{A+B}$;
  - e^A is nonsingular with inverse e^{-A}.

- If A is an $n \times n$ matrix with a single eigenvalue λ, then there exists nonnegative integer $k < n$ such that

$$e^{tA} = e^{\lambda t}\left(I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \cdots + \frac{t^k}{k!}(A - \lambda I)^k\right).$$

- Suppose that λ is an eigenvalue of A with multiplicity q. Then there exists an integer $p \leq q$ such that the dimension of the nullspace of $(A - \lambda I)^p$ is q.

- The procedure for finding q linearly independent solutions corresponding to an eigenvalue λ of multiplicity q is the following.

  - Find the smallest integer p such that the nullspace of $(A - \lambda I)^p$ has dimension q.
  - Find a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_q\}$ for the nullspace of $(A - \lambda I)^p$.
  - For each $\mathbf{v}_j$ $(j = 1, 2, \ldots, q)$, we have the solution

$$\begin{aligned}\mathbf{x}_j(t) &= e^{tA}\mathbf{v}_j \\ &= e^{\lambda t}\left(\mathbf{v}_j + t(A - \lambda I)\mathbf{v}_j + \cdots + \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}\mathbf{v}_j.\right).\end{aligned}$$

  The procedure works for complex eigenvalues as well as real. If $\lambda = \alpha + i\beta$ has eigenvector $\mathbf{z}$, then set $\mathbf{x} = \operatorname{Re}\mathbf{z}$ and $\mathbf{y} = \operatorname{Im}\mathbf{z}$.

3.9.4 Exercises

1. Solve the system

$$\begin{aligned} x' &= -5x + 15y - 4z \\ y' &= -x + 5y - z \\ z' &= 4x - 6y + 3z \\ x(0) &= -1 \\ y(0) &= 2 \\ z(0) &= 0 \end{aligned}$$

Hint.

2. Find the general solution of the system

$$\begin{aligned} x_1' &= 6x_1 + 9x_2 + x_3 + 4x_4 \\ x_2' &= -x_1 + x_2 + x_3 \\ x_3' &= 3x_1 + 8x_2 + 4x_3 + 4x_4 \\ x_4' &= -3x_1 - 8x_2 - 7x_3 - 5x_4 \end{aligned}$$

Hint.

3. Let $ax'' + bx' + cx = 0$, where $a \neq 0$ and $b^2 - 4ac = 0$.

(a) Show that $x_1(t) = e^{-bt/2a}$ is a solution to $ax'' + bx' + cx = 0$.

(b) Assume that

$$y = v(t)x_1(t) = v(t)e^{-bt/2a}$$

is a solution to $ax'' + bx' + cx = 0$ and show that $v(t) = c_1 + c_2 t$. Thus,

$$x(t) = c_1 e^{-bt/2a} + c_2 t e^{-bt/2a}$$

is a general solution for $ax'' + bx' + cx = 0$.

Hint.

4. Consider the equation

$$y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0.$$

Determine all values of α, if any, for which all solutions tend toward zero as $t \to \infty$. Also, determine the values of α, if any, for which all nonzero solutions become unbounded as $t \to \infty$.

Hint.

5. Solve each of the following initial value problems.

(a)

$$\begin{aligned} y'' - 2y' + 5y &= 0 \\ y(\pi/2) &= 0 \\ y'(\pi/2) &= 2 \end{aligned}$$

(b)

$$\begin{aligned} 9y'' - 12y' + 4y &= 0 \\ y(0) &= 2 \\ y'(0) &= -1 \end{aligned}$$

(c)

$$\begin{aligned} y'' + 8y' - 9y &= 0 \\ y(1) &= 1 \\ y'(1) &= 0 \end{aligned}$$

(d)

$$\begin{aligned} y'' + 2ay' + (a^2 + 1)y &= 0 \\ y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

Hint.

6. *Reduction of Order.* Suppose that $x_1(t)$ is a solution (not identically zero) to the equation

$$x'' + p(t)x' + q(t)x = 0.$$

(a) Assume that $x(t) = v(t)x_1(t)$ is a solution to $x'' + p(t)x' + q(t)x = 0$ and derive the equation

$$x_1 v'' + (2x_1' + px_1)v' = 0. \tag{3.14}$$

(b) Let $u = v'$ and show that ((A.10)) is a first-order linear differential equation in u.

(c) Show that $x_1(t) = 1/t$ is a solution to

$$2t^2 x'' + 3tx' - x = 0 \tag{3.15}$$

for $t > 0$ and find a second linearly independent solution to ((A.11)).

Hint.

7. *Euler Equations.* An important class of second-order linear differential equations are equations of the form

$$t^2y'' + \alpha ty' + \beta y = 0,$$

where $t > 0$ and α and β are real constants. An equation of this form is called an *Euler equation.*

(a) Show that the substitution $x = \ln t$ transforms an Euler equation into an equation of constant coefficients. *Hint*: Show that

$$\frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx}$$
$$\frac{d^2y}{dt^2} = \left(\frac{dx}{dt}\right)^2 \frac{d^2y}{dx^2} + \frac{d^2x}{dt^2}\frac{dy}{dx}.$$

(b) Solve the equation

$$t^2y'' + 4ty' + 2y = 0.$$

Hint.

3.9.5 Project

4

Second-Order Linear Equations

4.1 Homogeneous Linear Equations

A differential equation of the form

$$a(t)x'' + b(t)x' + c(t)x = g(t)$$

is called a ***second-order linear differential equation***. We will first consider the case

$$ax'' + bx' + cx = 0,$$

where a, b, and c are constants and $a \neq 0$. An equation of this form is said to be ***homogeneous*** with ***constant coefficients***. We already know how to solve such equations since we can rewrite them as a system of first-order linear equations. Thus, we can find the general solution of a homogeneous second-order linear differential equation with constant coefficients by computing the eigenvalues and eigenvectors of the matrix of the corresponding system.

4.1.1

Recall the RC circuits that we studied earlier (see Section 1.3). Such circuits contained a voltage source, a capacitor, and a resistor. A battery or generator is an example of a voltage source, and a toaster or an electric stove is an example of something that might provide a resistance in a circuit. Capacitors store an electrical charge and are used in electronic flashes for cameras. We will now add an inductor such as a solenoid, a coil that generates a magnetic field. Inductor applications include transformers, power supplies, televisions, radios. Our new circuit is called an RLC circuit (Figure 4.1).

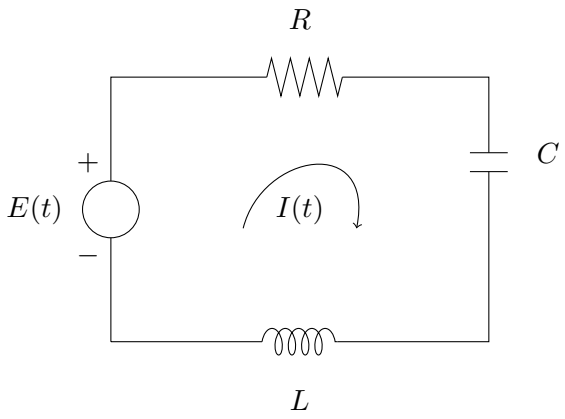

Figure 4.1: An RLC Circuit

Current, $I(t)$, is the rate at which a charge flows through this circuit and is measured in amperes or amps. We assign a direction to the current, and a current flowing in the opposite direction will be given negative values. The impressed voltage, $E(t)$, is measured in volts, the resistance R is measured in ohms, and the capacitance C is measured in farads. The charge on the capacitor $Q(t)$ at time t is measured in coulombs. Inductance on the coil, L, is measured in henrys.

The following laws from physics govern how our circuit behaves.

- $I = \dfrac{dQ}{dt}$.
- The voltage drop across a resistor is IR (Ohm's Law).
- The voltage drop across a capacitor is Q/C.
- The voltage drop across an inductor is $L(dI/dt)$.
- In a closed circuit the impressed voltage is equal to the sum of the voltage drops in the rest of the circuit (Kirchhoff's Second Law).

Applying Kirchhoff's Second Law to our circuit, we have the differential equation

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t) \tag{4.1}$$

or

$$LQ'' + RQ' + \frac{1}{C}Q = E(t).$$

Differentiating both sides of (4.1), we have

$$LI'' + RI' + \frac{1}{C}I = E'(t).$$

For example, we might consider an RLC circuit with $R = 1$, $L = 1$, and $C = 1$. At $t = 0$ when both $I(0) = 0$ and $I'(0) = Q(0) = 0$, the impressed voltage on the circuit is given by $E(t) = \sin(t)$. Our equation becomes

$$I'' + I' + I = E'(t) = \cos t.$$

This is an example of a ***second-order linear differential equation***.

4.1.2 Second-Order Linear Equations

Suppose that we have a homogeneous second-order linear differential equation with constant coefficients,

$$ax'' + bx' + cx = 0. \tag{4.2}$$

Since we can rewrite this equation as a system of first-order linear equations,

$$\begin{aligned} x' &= y \\ y' &= -\frac{c}{a}x - \frac{b}{a}y. \end{aligned}$$

we can find the general solution by computing the eigenvalues and eigenvectors of the matrix of the corresponding system. Solutions of a linear system $\mathbf{x}' = A\mathbf{x}$ often include terms of the form e^{rt}. It makes sense that solutions to equation (4.2) take the same form.

Example 4.2. Consider the equation

$$x'' + 7x' + 10x = 0. \tag{4.3}$$

If we assume that a solution is of the form e^{rt}, we can substitute this expression into the left-hand side of (4.3) to obtain

$$\begin{aligned}\frac{d^2}{dt^2}e^{rt} + \frac{d}{dt}7e^{rt} + 10e^{rt} &= r^2e^{rt} + 7re^{rt} + 10e^{rt}\\ &= (r^2 + 7r + 10)e^{rt}\\ &= (r+5)(r+2)e^{rt}.\end{aligned}$$

Since e^{rt} is never zero, we find that $(r+5)(r+2) = 0$ or $r = -5$ or -2. Thus, we have two solutions

$$x_1(t) = e^{-5t} \text{ and } x_2(t) = e^{-2t}.$$

By the Principle of Superposition,

$$x(t) = c_1x_1(t) + c_2x_2(t) = c_1e^{-5t} + c_2e^{-2t} \tag{4.4}$$

is a solution to $x'' + 7x' + 10x = 0$.

Indeed, this is the general solution of our second-order equation since we have a one-to-one correspondence between the solutions of

$$x'' + 7x' + 10x = 0$$

and the system

$$\begin{aligned}x' &= y\\ y' &= -10x - 7y.\end{aligned}$$

The matrix associated with this system

$$A = \begin{pmatrix} 0 & 1\\ -10 & -7\end{pmatrix}$$

has eigenvalues $\lambda_1 = -5$ and $\lambda_2 = -2$ with eigenvectors $\mathbf{v}_1 = (1, -5)$ and $\mathbf{v}_2 = (1, -2)$, respectively. Consequently, the solution to our system is

$$\begin{pmatrix} x(t)\\ y(t)\end{pmatrix} = c_1e^{-5t}\begin{pmatrix} 1\\ -5\end{pmatrix} + c_2e^{-2t}\begin{pmatrix} 1\\ -2\end{pmatrix},$$

which agrees with (4.4).

In general, suppose that

$$ax'' + bx' + cx = 0,$$

where $a \neq 0$. Applying the startegy in Example 4.2, we can find the general solution for this equation by finding the roots of the quadratic polynomial $a\lambda^2 + b\lambda + c$,

$$\lambda = -\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac > 0$, we have real roots

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

and the solution to our second-order differential equation is

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \tag{4.5}$$

where c_1 and c_2 are arbitrary constants.

To prove that equation (4.5) is indeed the general solution to the second-order equation $ax'' + bx' + cx = 0$, we can study the equivalent system of linear equations. If we let $y = x'$, the corresponding linear system is $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}.$$

The characteristic polynomial of A is

$$p(\lambda) = \det(A - \lambda I) = \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a}.$$

The roots of $p(\lambda)$ are the same as the roots of $a\lambda^2 + b\lambda + c$. If $b^2 - 4ac > 0$, we have real roots

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

We can find eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ (-b + \sqrt{b^2 - 4ac})/2a \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$$
$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ (-b - \sqrt{b^2 - 4ac})/2a \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$$

for λ_1 and λ_2, respectively. Thus, the general solution to the system of differential equations $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix},$$

which agrees with (4.5).

Example 4.3. Now let us solve the initial value problem

$$\begin{aligned} x'' + 4x' + 5x &= 0 \\ x(0) &= 1 \\ x'(0) &= 1. \end{aligned}$$

Again, we will assume that our solution has the form $x(t) = e^{rt}$. Substituting this function into our differential equation, we find that

$$0 = x'' + 4x' + 5x = r^2 e^{rt} + 4re^{rt} + 5e^{rt} = (r^2 + 4r + 5)e^{rt}.$$

As with the real case, $r^2 + 4r + 5 = 0$ or

$$r = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm i.$$

Thus, we can find a solution

$$x(t) = e^{(-2+i)t} = e^{-2t} e^{it} = e^{-2t}(\cos t + i \sin t).$$

The real and imaginary parts of our solution are

$$x_1(t) = e^{-2t}\cos t$$
$$x_2(t) = e^{-2t}\sin t,$$

respectively. We claim that both $x_1(t)$ and $x_2(t)$ are solutions to our differential equation. Indeed, since $x(t) = x_1(t) + ix_2(t)$ is a solution,

$$\begin{aligned} 0 &= ax'' + bx' + cx \\ &= a(x_1 + ix_2)'' + b(x_1 + ix_2)' + c(x_1 + ix_2) \\ &= (ax_1'' + bx_1' + cx_1) + i(ax_2'' + bx_2' + cx_2). \end{aligned}$$

Since the real part and the imaginary part of $x(t)$ must both be zero, we can conclude that $ax_1'' + bx_1' + cx_1 = 0$ and $ax_2'' + bx_2' + cx_2 = 0$. Therefore, the general solution to our equation is

$$x(t) = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t.$$

To apply our initial conditions $x(0) = 1$ and $x'(0) = 1$, we first calculate

$$x'(t) = -2e^{-2t}(c_1\cos t + \sin t) + e^{-2t}(-c_1\sin t + c_2\cos t).$$

Thus,

$$1 = x(0) = c_1$$
$$1 = x'(0) = -2c_1 + c_2,$$

and $c_1 = 1$ and $c_2 = 3$. Hence, the solution to our initial value problem is

$$x(t) = e^{-2t}\cos t + 3e^{-2t}\sin t.$$

As before, the corresponding linear sytem is $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}.$$

If $b^2 - 4ac < 0$, the eigenvalue of A are $\lambda = \alpha + i\beta$ and $\overline{\lambda} = \alpha - i\beta$, where

$$\alpha = -\frac{b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}.$$

The vector $\mathbf{v} = (1, \alpha + i\beta)$ is an eigenvector for λ. Thus, a solution to our system of differential equations is

$$\begin{aligned} \mathbf{x}(t) &= \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \\ &= \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} \\ &= e^{(\alpha+i\beta)t}\begin{pmatrix} 1 \\ \alpha + i\beta \end{pmatrix} \\ &= e^{\alpha t}(\cos\beta t + i\sin\beta t)\begin{pmatrix} 1 \\ \alpha + i\beta \end{pmatrix} \\ &= e^{\alpha t}\begin{pmatrix} \cos\beta t \\ \alpha\cos\beta t - \beta\sin\beta t \end{pmatrix} + ie^{\alpha t}\begin{pmatrix} \sin\beta t \\ \alpha\sin\beta t + \beta\cos\beta t \end{pmatrix}. \end{aligned}$$

Taking the real and imaginary parts of $x(t)$, we obtain two real solutions to the system, $x_1(t) = e^{\alpha t}\cos\beta t$ and $x_2(t) = e^{\alpha t}\sin\beta t$. Therefore, the general solution to $ax_1'' + bx_1' + cx_1 = 0$ is

$$x(t) = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t.$$

4.1.3 Reduction of Order

Given a second-order linear differential equation with constant coefficients, $ax'' + bx' + cx = 0$, our strategy has been to solve the ***characteristic equation*** $a\lambda^2 + b\lambda + c = 0$ to obtain two linearly independent solutions. We have covered the case where this equation has two distinct real solutions as well as when there are complex soluitons, but what if there is only a single real solution $\lambda = -b/2a$.

Example 4.4. Consider the equation

$$x'' + 2x' + x = 0.$$

If we choose $e^{\lambda t}$ as our guess, we find

$$\begin{aligned} x'' + 2x' + x &= \lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} + e^{\lambda t} \\ &= e^{\lambda t}(\lambda + 1)^2 \\ &= 0. \end{aligned}$$

Thus, $\lambda = -1$ and we have a solution $x_1(t) = e^{-t}$. In order to find a general solution to $x'' + 2x' + x = 0$, we must find a second solution that is not a multiple of $x_1(t) = e^{-t}$.

In order to find a general solution to $x'' + 2x' + x = 0$, we must find a second solution that is not a multiple of $x_1(t) = e^{-t}$. Since we already know that $cx_1(t)$ is a solution to our differential equation, we will try to generalize this observation by replacing c with a nonconstant function $v(t)$ and then try to determine $v(t)$ so that $v(t)x_1(t)$ is a solution to $x'' + 2x' + x = 0$. Indeed, if

$$x(t) = v(t)x_1(t) = v(t)e^{-t},$$

then

$$x'(t) = v(t)x_1'(t) + v'(t)x_1(t) = -v(t)e^{-t} + v'(t)e^{-t}$$

and

$$\begin{aligned} x''(t) &= v''(t)x_1(t) + 2v'(t)x_1'(t) + v(t)x_1''(t) \\ &= v''(t)e^{-t} - 2v'(t)e^{-t} + v(t)e^{-t}. \end{aligned}$$

Consequently,

$$\begin{aligned} x'' + 2x' + x &= [v''e^{-t} - 2v'e^{-t} + ve^{-t}] + 2[-ve^{-t} + v'e^{-t}] + [ve^{-t}] \\ &= e^{-t}v'' \\ &= 0, \end{aligned}$$

and $v'' = 0$. Therefore, $v = c_1 t + c_2$. Letting $c_1 = 1$ and $c_2 = 0$, we can assume that $v(t) = t$, and the second solution to our equation is $x = te^{-t}$. Hence, the general solution to $x'' + 2x' + x = 0$ is

$$x(t) = c_1 e^{-t} + c_2 te^{-t}.$$

We leave it as an exercise to show that our solution agrees with the solution that we would obtain from solving the equivalent first-order linear system.

The technique that we have used in Example 4.4 is called ***reduction of order***. We leave it as an exercise to show that this technique works in general. That is, given a second-order linear differential equation

$$ax'' + bx' + cx = 0$$

such that $b^2 - 4ac = 0$, then the general solution is given by

$$x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t},$$

where $\lambda = -b/2a$.

We did a great deal of work finding unique solutions to first-order linear systems of equations,

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy. \end{aligned}$$

Our efforts are now rewarded. Since each second-order homogeneous system with constant coefficients can be rewritten as a first-order linear system, we are guaranteed the existence and uniqueness of solutions.

4.1.4 Important Lessons

- A second-order linear differential equation with constant coefficients is an equation of the form
$$ax'' + bx' + cx = 0.$$
We can guess the solution to this equation. Since we can rewrite this equation as a system of first-order linear differential equations, we can determine the general solution to $ax'' + bx' + cx = 0$.

- Suppose that
$$ax'' + bx' + cx = 0,$$
where $a \neq 0$ and $b^2 - 4ac > 0$. If the roots of $ar^2 + br + c$ are r_1 and r_2, the general solution to this differential equation is
$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

- If $b^2 - 4ac < 0$, the differential equation
$$ax'' + bx' + cx = 0$$
has a general solution
$$x(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t,$$
where $\alpha \pm i\beta$ are the roots of $ar^2 + br + c = 0$.

- If $b^2 - 4ac = 0$, the differential equation
$$ax'' + bx' + cx = 0$$
has a general solution
$$x(t) = c_1 e^{-bt/2a} + c_2 t e^{-bt/2a}.$$

4.1.5 Exercises

1. Let $ax'' + bx' + cx = 0$, where $a \neq 0$ and $b^2 - 4ac = 0$.

(a) Show that $x_1(t) = e^{-bt/2a}$ is a solution to $ax'' + bx' + cx = 0$.

(b) Assume that

$$y = v(t)x_1(t) = v(t)e^{-bt/2a}$$

is a solution to $ax'' + bx' + cx = 0$ and show that $v(t) = c_1 + c_2 t$. Thus,

$$x(t) = c_1 e^{-bt/2a} + c_2 te^{-bt/2a}$$

is a general solution for $ax'' + bx' + cx = 0$.

2. *Reduction of Order.* Suppose that $x_1(t)$ is a solution (not identically zero) to the equation

$$x'' + p(t)x' + q(t)x = 0.$$

(a) Assume that $x(t) = v(t)x_1(t)$ is a solution to $x'' + p(t)x' + q(t)x = 0$ and derive the equation

$$x_1 v'' + (2x_1' + px_1)v' = 0. \tag{4.6}$$

(b) Let $u = v'$ and show that (4.6) is a first-order linear differential equation in u.

(c) Show that $x_1(t) = 1/t$ is a solution to

$$2t^2 x'' + 3tx' - x = 0 \tag{4.7}$$

for $t > 0$ and find a second linearly independent solution to (4.7).

4.1.6 Project

4.2 Forcing

Harmonic oscillators such as a spring-mass system (Subsection 1.1.3) or an RLC circuit (Section 4.1) can be modeled with second-order linear differential equations. Indeed, we can model a spring-mass system with the equation

$$mx''(t) + bx'(t) + kx(t) = g(t),$$

where m is the mass, b is the damping coefficient, k is the spring constant, and $F(t) = g(t)$ represents some external force applied to our system. RLC circuits can also be modeled with provide another example of forcing. If $I(t)$ is the rate at which a charge flows through a circuit (measured in amperes or amps), R is the resistance (measured in ohms), C is the capacitance (measured in farads), and the inductance, L, is (measured in henrys), then the derivative of the impressed voltage (measured in volts), $E(t)$, is the forcing term

$$LI'' + RI' + \frac{1}{C}I = E'(t).$$

What is different about these two equations are the terms on the righthand side, $g(t)$ and $E'(t)$. Such a term is called a ***forcing term***.

4.2.1 Nonhomogeneous Equations

A ***nonhomogeneous second-order linear differential equation*** is an equation of the form

$$x'' + p(t)x' + q(t)x = g(t).$$

We have already seen how examples of such equations arise when examining models of harmonic oscillators with forcing terms. Our goal is to be able to solve such equations. In general, these equations can be difficult to solve for an arbitrary function $g(t)$. Before we attempt to find solutions for some of the more common functions that might occur for $g(t)$, let us derive some fundamental facts about second-order linear differential equations.

Theorem 4.5. *Suppose that*

$$x'' + p(t)x' + q(t)x = g(t) \tag{4.8}$$

has solutions $x_1 = x_1(t)$ and $x_2 = x_2(t)$. Then $x_1(t) - x_2(t)$ is a solution of the homogeneous linear differential equation

$$x'' + p(t)x' + q(t)x = 0.$$

Proof. Since x_1 and x_2 are solutions of (4.8), we know that

$$\begin{aligned} x_1'' + p(t)x_1' + q(t)x_1 &= g(t) \\ x_2'' + p(t)x_2' + q(t)x_2 &= g(t). \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{d^2}{dt^2}(x_1 - x_2) + p(t)\frac{d}{dt}(x_1 - x_2) + q(t)(x_1 - x_2) \\ &= \left(\frac{d^2x_1}{dt^2} + p(t)\frac{dx_1}{dt} + q(t)x_1\right) - \left(\frac{d^2x_2}{dt^2} + p(t)\frac{dx_2}{dt} + q(t)x_2\right) \\ &= g(t) - g(t) = 0. \end{aligned}$$

□

We can use Theorem 4.5 to derive the fact that the general solution to

$$x'' + p(t)x' + q(t)x = g(t). \tag{4.9}$$

can be written in the form

$$x = x_h + x_p,$$

where y_h is the general solution of the homogeneous equation

$$x'' + p(t)x' + q(t)x = 0,$$

and x_p is any solution of (4.9). Indeed, suppose that x_q is another solution to (4.9). Then $x_q - x_p$ is a solution to the homogeneous equation

$$x'' + p(t)x' + q(t)x = 0.$$

Therefore,

$$x_q - x_p = x_h$$

or

$$x_q = x_h + x_p.$$

We state this fact in the following theorem.

Theorem 4.6. *Let x_p be a particular solution of the equation*

$$x'' + p(t)x' + q(t)x = g(t),$$

and x_h be the general solution of the corresponding homogeneous equation

$$x'' + p(t)x' + q(t)x = 0.$$

Then the general solution to $x'' + p(t)x' + q(t)x = g(t)$ is $x = x_h + x_p$.

4.2.2 Forcing Terms

The equation

$$mx'' + bx' + kx = g(t)$$

can be used to model a harmonic oscillator where forcing occurs. In general, we will not be able to solve this equation explicitly for a given $g(t)$; however, certain forcing functions often occur in practice. Some of the more important forcing functions are $g(t) = e^{-at}$, where the external force decreases exponentially over time; $g(t) = k$, where a constant force is applied; and $g(t) = \cos \omega t$ or $g(t) = \sin \omega t$, where a force is applied periodically.

In the case of the unforced damped harmonic oscillator,

$$mx'' + bx' + kx = 0,$$

we know that $m > 0$, $b > 0$, and $k > 0$. Thus, we can rewrite this equation as

$$x'' + px' + qx = 0,$$

where $p = b/m$ and $q = k/m$ are both positive. As a first-order linear system, the harmonic oscillator is

$$\begin{aligned} x' &= y \\ y' &= -qx - py. \end{aligned}$$

The matrix corresponding to this system,

$$A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix},$$

has trace of $-p$ and determinant q. Since $\operatorname{tr}(A) < 0$ and $\det(A) > 0$, we know that any solution of the unforced equation tends toward the origin as $t \to \infty$. That is, the solution is a sink. This leads us to the following conclusion.

Theorem 4.7. *Suppose*

$$x'' + px' + qx = g(t)$$

has solution $x = x_h + x_p$, where $p > 0$ and $q > 0$. If

$$x = x_h + x_p$$

is the general solution to the equation, then

$$x = x_h + x_p \to x_p$$

as $t \to \infty$.

In other words, all solutions of a damped harmonic oscillator with nonzero damping are essentially the same for large values of t.

4.2.3 The Method of Undetermined Coefficients

Example 4.8. Let us solve the differential equation

$$x'' + 5x' + 4x = 1, \tag{4.10}$$

a harmonic oscillator with a constant forcing function. It is easy to check that the general solution to the homogeneous equation

$$x'' + 5x' + 4x = 0$$

is

$$x_h = c_1 e^{-t} + c_2 e^{-4t}.$$

A particular solution to (4.10) is given by $x_p = 1/4$. Thus, the general solution is

$$x(t) = c_1 e^{-t} + c_2 e^{-4t} + \frac{1}{4}.$$

As $t \to \infty$, all solutions approach the constant solution $x = 1/4$ (Figure 4.9).

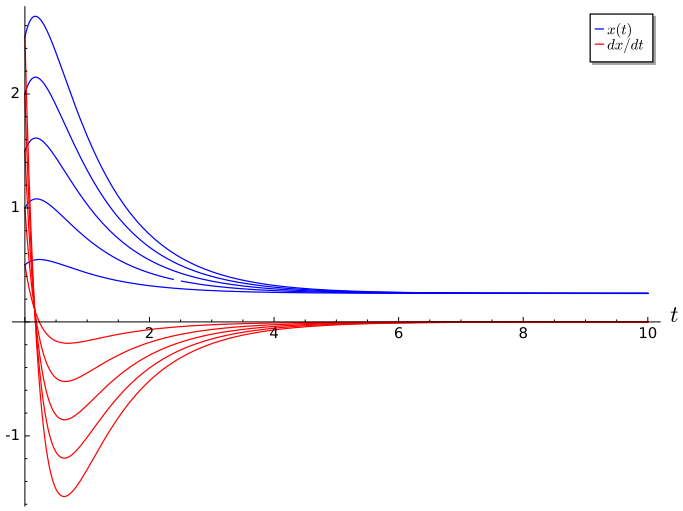

Figure 4.9: Solutions to $x'' + 5x' + 4x = 1$

Example 4.10. Now let us consider a more complicated example. Suppose that we wish to solve

$$x'' + 5x' + 4x = e^{-2t}. \tag{4.11}$$

This is the equation of a harmonic oscillator with a forcing function that decreases exponentially with time. We already know the solution to the homogeneous equation. We will use the ***method of undetermined coefficients*** to find a particular solution to (4.11). It is reasonable to guess that a particular solution will have the form

$$x_p = Ae^{-2t}.$$

Substituting this expression into (4.11), we find that

$$e^{-2t} = x_p'' + 5x_p' + 4x_p = 4Ae^{-2t} - 10Ae^{-2t} + 4Ae^{-2t} = -2Ae^{-2t}.$$

Hence, $A = -1/2$. Therefore, the solution that we seek is

$$x = c_1 e^{-t} + c_2 e^{-4t} - \frac{1}{2} e^{-2t}.$$

Again, all solutions approach zero as $t \to \infty$ (Figure 4.11).

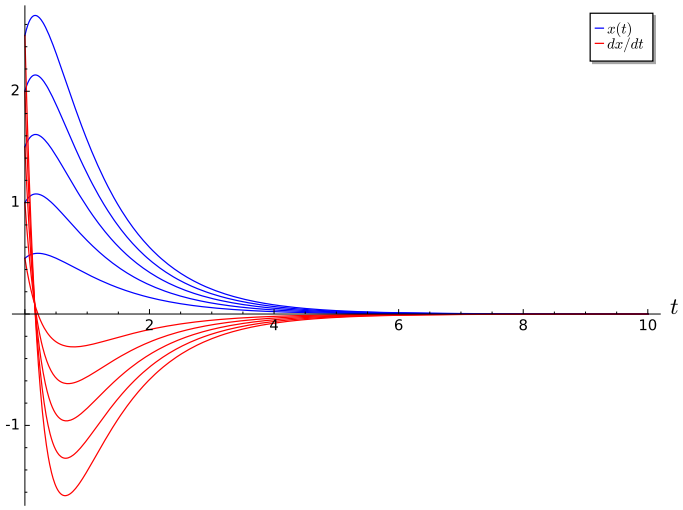

Figure 4.11: Solutions to $x'' + 5x' + 4x = e^{-2t}$

Example 4.12. Now let us examine what happens if we have a periodic forcing function. Let us assume that the particular solution to the equation

$$x'' + 5x' + 4x = 2\cos t.$$

takes the form

$$x_p = A\cos t + B\sin t.$$

Then

$$\begin{aligned} x_p' &= -A\sin t + B\cos t \\ x_p'' &= -A\cos t - B\sin t. \end{aligned}$$

Substituting these expressions into the differential equation, we see that

$$\begin{aligned} 2\cos t &= x_p'' + 5x_p' + 4x_p \\ &= (-A\cos t - B\sin t) + 5(-A\sin t + B\cos t) + 4(A\cos t + B\sin t) \\ &= (3A + 5B)\cos t + (-5A + 3B)\sin t. \end{aligned}$$

We must solve the following system of equations to find a particular solution:

$$\begin{aligned} 3A + 5B &= 2 \\ -5A + 3B &= 0. \end{aligned}$$

The solution of this system is $A = 3/17$ and $B = 5/17$. Consequently,

$$x_p = \frac{3}{17}\cos t + \frac{5}{17}\sin t$$

is a particular solution to $x'' + 5x' + 4x = 2\cos t$. The general solution to our equation is

$$x = c_1 e^{-t} + c_2 e^{-4t} + \frac{3}{17}\cos t + \frac{5}{17}\sin t.$$

The solutions to this equation are given in Figure 4.13.

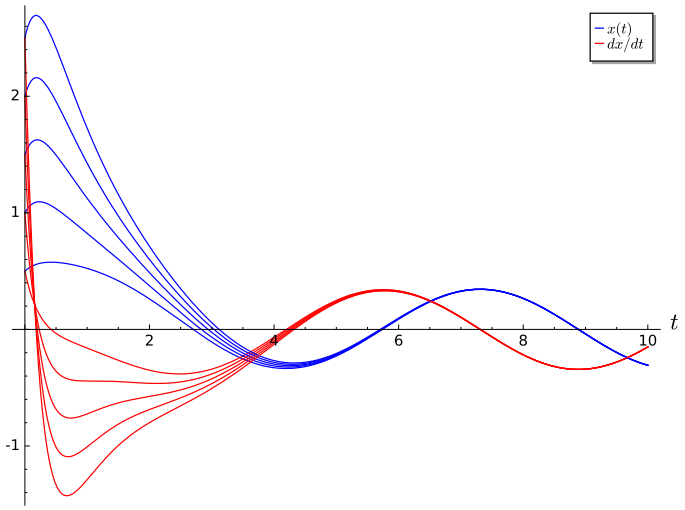

Figure 4.13: Solutions to $x'' + 5x' + 4x = 2\cos t$

Example 4.14. As a final example, consider the

$$x'' + 5x' + 4x = e^{-t}.$$

Recall that the solution to the homogeneous equation $x'' + 5x' + 4x = 0$ is

$$x_h = c_1 e^{-t} + c_2 e^{-4t}.$$

Our guess of $x_p(t) = Ae^{-t}$ for a particular solution will no longer work since e^{-t} is a solution to the homogeneous equation. We must therefore consider a function of a different form. Such a function must yield a multiple of e^{-t} when differentiated. The simplest such function is of the form

$$x_p = Ate^{-t}.$$

Using this guess,

$$\begin{aligned} e^{-t} &= \frac{d^2}{dt^2} Ate^{-t} + 5\frac{d}{dt} Ate^{-t} + 4Ate^{-t} \\ &= A(-e^{-t} - e^{-t} + te^{-t}) + 5A(e^{-t} - te^{-t}) + 4Ate^{-t} \\ &= 3Ate^{-t}. \end{aligned}$$

Thus, $A = 1/3$ and our general solution is

$$x = c_1 e^{-t} + c_2 e^{-4t} + \frac{1}{3} te^{-t}.$$

Solutions to the differential equation $x'' + 5x' + 4x = e^{-t}$ are given in Figure 4.15.

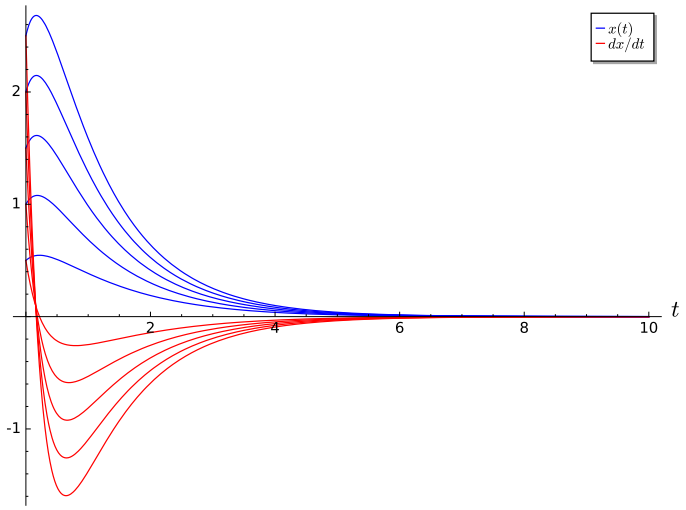

Figure 4.15: Solutions to $x'' + 5x' + 4x = e^{-t}$

4.2.4 A Strategy

We outline a general strategy for choosing x_p for the Method of Undetermined Coefficients in Table 4.16. Here $s = 0, 1, 2$ is the smallest integer that will ensure that no term in x_p is a solution of the corresponding homogeneous equation.

$g(t)$	x_p
$P_n(t) = a_n t^n + \cdots + a_1 t + a_0$	$t^s(A_n t^n + \cdots + A_1 t + A_0)$
$P_n(t)e^{\alpha t}$	$t^s(A_n t^n + \cdots + A_1 t + A_0)e^{\alpha t}$
$P_n(t)e^{\alpha t}\begin{cases}\sin\beta t\\ \cos\beta t\end{cases}$	$t^s e^{\alpha t}[(A_n t^n + \cdots + A_1 t + A_0)\cos\beta t$ $+(B_n t^n + \cdots + B_1 t + B_0)\sin\beta t]$

Table 4.16: Particular solutions of $x'' + px' + qx = g(t)$.

4.2.5 Important Lessons

- A ***nonhomogeneous second-order linear differential equation*** is an equation of the form
$$x'' + p(t)x' + q(t)x = g(t).$$
Forced harmonic oscillators and RLC circuits provide good examples of nonhomogeneous second-order linear differential equations.

- Suppose that
$$x'' + p(t)x' + q(t)x = g(t)$$
has solutions $x_1 = x_1(t)$ and $x_2 = x_2(t)$. Then $x_1(t) - x_2(t)$ is a solution of the homogeneous linear differential equation
$$x'' + p(t)x' + q(t)x = 0.$$

- Let x_p be a particular solution of the equation
$$x'' + p(t)x' + q(t)x = g(t),$$

and x_h be the general solution of the corresponding homogeneous equation

$$x'' + p(t)x' + q(t)x = 0.$$

Then the general solution to $x'' + p(t)x' + q(t)x = g(t)$ is $x = x_h + x_p$. In particular, if the solution to $x'' + p(t)x' + q(t)x = 0$ has a sink at the origin, all solutions of the equation $x'' + p(t)x' + q(t)x = g(t)$ approach $x_p(t)$ as $t \to \infty$.

- The Method of Undetermined Coefficients is useful for solving the equation $x'' + p(t)x' + q(t)x = g(t)$, when g is of the form

$$g(t) = P_n(t)e^{\alpha t} \begin{cases} \sin \beta t \\ \cos \beta t \end{cases}$$

4.2.6 Exercises

1. Find the solution of the initial-value problem

$$\begin{aligned} y'' + 6y' + 8y &= \cos 3t \\ y(0) &= 0 \\ y'(0) &= 0. \end{aligned}$$

Hint. The solution to the homogeneous equation $y'' + 7y' + 10y = 0$ is

$$y_h = c_1 e^{-4t} + c_2 e^{-2t}.$$

Using the method of undetermined coefficients, we can find a particular solution

$$y_p = \frac{2}{235}(18 \sin 3t - \cos 3t).$$

Thus, the general solution is

$$y = y_h + y_p = c_1 e^{-4t} + c_2 e^{-2t} + \frac{2}{235}(18 \sin 3t - \cos 3t).$$

Applying the initial conditions $y(0) = 0$ and $y'(0) = 0$, we have

$$y = -\frac{16}{235}e^{-4t} + \frac{18}{235}e^{-2t} + \frac{2}{235}(18 \sin 3t - \cos 3t).$$

2. Compute the solution of the initial value problem

$$\begin{aligned} y'' + 4y &= 3 \cos 2t \\ y(0) = y'(0) &= 0. \end{aligned}$$

Hint. The solution to the homogeneous equation, $y'' + 4y = 0$ is

$$y_h = c_1 \cos 2t + c_2 \sin 2t.$$

The complex version of the equation is

$$y'' + 4y = 3e^{2it}.$$

Assume that the particular solution to this equation is $y_c = ate^{2it}$. We include a factor of t, since e^{2it} is a solution to the homogeneous equation. Substituting y_c and y_c'' into the differential equation, we have

$$y_c'' + 4y_c = 4a(i-t)e^{2it} + 4ate^{2it} = 4aie^{2it} = 3e^{2it}.$$

Thus, $a = 3/(4i) = -3i/4$. Taking the real part of

$$y_c = -\frac{3}{4}it(\cos 2t + i\sin 2t),$$

we have

$$y_p = \frac{3}{4}t\sin 2t.$$

Therefore the general solution is

$$y = y_h + y_p = c_1\cos 2t + c_2\sin 2t + \frac{3}{4}t\sin 2t.$$

Applying the initial conditions $y(0) = y'(0) = 0$, we find that $c_1 = c_2 = 0$ and

$$y = \frac{3}{4}t\sin 2t.$$

3. Use the complex method to find a particular solution to

$$y'' + 7y' + 10y = -4\sin 3t.$$

Write your solution as a function of cosine. What is the amplitude, the period, and the frequency of your solution?

Hint. If we assume that the complex solution has form $y_c = Ae^{3it}$, then

$$y_c'' + 7y_c' + 10y_c = A(1+21i)e^{3it} = -4e^{3it},$$

and

$$A = \frac{-4}{1+21i} = \frac{-2+42i}{221}.$$

In polar form,

$$A = |A|e^{i\phi} = \frac{4}{\sqrt{442}}e^{i\phi},$$

where $\phi = \pi + \arctan(-21)$. Thus, $y_c = Ae^{3it} = 4e^{i(3t+\phi)}/\sqrt{442}$. Using the imaginary part of this solution, we obtain

$$y = \frac{4}{\sqrt{442}}\sin 3t + \phi = \frac{4}{\sqrt{442}}\cos(3t-\theta),$$

where $\theta = \phi - \pi/2 = \pi/2 + \arctan(-21)$. The amplitude is $4/\sqrt{442}$, the period is $2\pi/3$, and the frequency is $3/2\pi$.

4. We define two functions, $f(t)$ and $g(t)$, to be *linearly independent* on an open interval $I = (a, b)$ if there do not exist nonzero constants c_1 and c_2 such that

$$c_1 f(t) + c_2 g(t) = 0$$

for all $t \in I$. Otherwise, $f(t)$ and $f(t)$ are *linearly dependent*. Suppose that $f(t)$ and $g(t)$ are solutions to the homogeneous linear equation

$$y'' + py' + qy = 0.$$

Show that $f(t)$ and $g(t)$ are linearly dependent on an interval $I = (a, b)$ if and only if $W[f, g](t) \equiv 0$.

Hint. Suppose that that $f(t)$ and $g(t)$ are linearly dependent on an interval $I = (a, b)$. Then one function is a multiple of the other, say $f(t) = cg(t)$. Thus, $f'(t) = cg'(t)$.

$$W(f,g)(t) = \det \begin{pmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{pmatrix} = f(t)g'(t) - f'(t)g(t) = cg(t)g'(t) - cg'(t)g(t) = 0.$$

Conversely, suppose that

$$W(f,g)(t) = \det \begin{pmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{pmatrix} = 0,$$

for all t in (a, b). If $g = 0$, then $0f = g$ and the two functions are linearly dependent. Assume that $g(t_0) \neq 0$ for some t_0 in (a, b). Since g is differentiable, it must also be continuous and there is some interval (c, d) contained in (a, b) such that $t_0 \in (c, d)$ and g does not vanish on this interval. Therefore,

$$\frac{d}{dt}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2} = -\frac{W(f,g)}{g^2} = 0,$$

and f/g is constant on the interval (c, d). Thus, $f(t_0) = cg(t_0)$ and $f(t_0') = cg'(t_0)$. Since f and cg are both solutions to the differential equation $y'' + py' + qy = 0$ and have the same initial condition, $f(t) = cg(t)$ for all $t \in (a, b)$ by the existence and uniqueness theorem. Consequently, f and g are linearly dependent.

5. *Abel's Theorem.* If y_1 and y_2 are solutions of the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on an open interval $I = (a, b)$, show that

$$W[y_1, y_2](t) = c \exp\left(-\int p(t)\,dt\right),$$

for some constant c that depends on y_1 and y_2 but not on t.

(a) Use Abel's Theorem to find the Wronskian of $2t^2y'' + 3ty' - y = 0$ up to a constant multiple, where $t > 0$.

(b) Prove Abel's Theorem.

Hint.

(a) We can rewrite $2t^2y'' + 3ty' - y = 0$ as

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0.$$

Since $p(t) = 1/2t$, Abel's Theorem tells us that

$$W[y_1, y_2](t) = c \exp\left(-\int \frac{3}{2t}\,dt\right) = c \exp\left(-\frac{3}{2}\ln t\right) = ct^{-3/2}.$$

(b) Since y_1 and y_2 are solutions to our differential equation, we know that

$$y_1'' + p(t)y_1' + q(t)y_1 = 0$$

$$y_2'' + p(t)y_2' + q(t)y_2 = 0.$$

Multiplying the first equation by y_2 and the second equation by y_1 and subtracting, we obtain

$$(y_1y_2'' - y_1''y_2) + p(t)(y_1y_2' - y_1'y_2) = 0. \tag{4.12}$$

If

$$W(t) = W(y_1, y_2)(t) = y_1y_2' - y_1'y_2,$$

then

$$W' = y_1y_2'' - y_1''y_2,$$

and equation (A.12) becomes

$$W'' + p(t)W = 0.$$

This equation is separable with solution

$$W(t) = c\exp\left(-\int p(t)\,dt\right),$$

6. *The Method of Variation of Parameters.* In this problem we will describe another method of finding a particular solution to a nonhomogeneous equation,

$$y'' + p(t)y' + q(t)y = f(t), \tag{4.13}$$

if we know know that the general solution to the homogeneous equation $y''+p(t)y'+q(t)y = 0$ is

$$y_h = c_1y_1 + c_2y_2.$$

(a) Assume that a particular solution of (A.13) has the form

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

where

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0.$$

Substitute y_p into the left-hand side of (A.13) to show that

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = f(t).$$

(b) Show that

$$u_1'(t) = \frac{-y_2(t)f(t)}{W[y_1, y_2](t)}$$
$$u_2'(t) = \frac{y_1(t)f(t)}{W[y_1, y_2](t)},$$

where $W[y_1, y_2](t) = y_1(t)y_2'(t) - y_2(t)y_1'(t)$ is the *Wronskian* of y_1 and y_2.

(c) If p, q, and f are continuous on an interval I, show that

$$u_1(t) = -\int_{t_0}^{t} \frac{y_2(s)f(s)}{W[y_1, y_2](s)}\,ds$$
$$u_2(t) = \int_{t_0}^{t} \frac{y_1(s)f(s)}{W[y_1, y_2](s)}\,ds$$

for any point t_0 in I. Consequently, a particular solution to (A.13) is

$$y_p = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)f(s)}{W[y_1, y_2](s)}\,ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)f(s)}{W[y_1, y_2](s)}\,ds.$$

(d) Find the general solution of the differential equation

$$y'' + 4y = 3\csc t.$$

Hint.

(a) If $y_p = u_1y_1 + u_2y_2$, then

$$\begin{aligned} y_p' &= u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2' = u_1y_1' + u_2y_2' \\ y_p'' &= u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''. \end{aligned}$$

Substituting these expressions into equation (A.13), we have

$$\begin{aligned} y_p'' + py_p' + qy_p &= (u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'') + p(u_1y_1' + u_2y_2') \\ &\quad + q(u_1y_1 + u_2y_2) \\ &= u_1[y_1'' + py_1' + qy_1] + u_2[y_2'' + py_2' + qy_2] + u_1'y_1' + u_2'y_2' \\ &= u_1'y_1' + u_2'y_2' \\ &= f(t). \end{aligned}$$

(b) If we solve the system

$$\begin{aligned} u_1'(t)y_1(t) + u_2'(t)y_2(t) &= 0 \\ u_1'(t)y_1'(t) + u_2'(t)y_2'(t) &= f(t). \end{aligned}$$

for u_1' and u_2', we obtain

$$\begin{aligned} u_1'(t) &= \frac{-y_2(t)f(t)}{W[y_1, y_2](t)} \\ u_2'(t) &= \frac{y_1(t)f(t)}{W[y_1, y_2](t)}. \end{aligned}$$

(c) Integrate the two equations from part (2).

(d) The general solution to the homogeneous equation $y'' + 4y = 0$ is

$$y_h = c_1\cos 2t + c_2\sin 2t.$$

To find a particular solution, assume that the solution has the form

$$y_p = u_1(t)\cos 2t + u_2(t)\sin 2t.$$

By part (2)

$$\begin{aligned} u_1'(t) &= -3\cos t \\ u_2'(t) &= \frac{3}{2}\csc t - 3\sin t. \end{aligned}$$

Integrating, we obtain

$$\begin{aligned} u_1(t) &= -3\sin t \\ u_2(t) &= \frac{3}{2}\ln|\csc t - \cot t| + 3\cos t. \end{aligned}$$

Therefore,

$$\begin{aligned} y_p(t) &= u_1(t)y_1(t) + u_2(t)y_2(t) \\ &= -3\sin t\cos 2t + \left[\frac{3}{2}\ln|\csc t - \cot t| + 3\cos t\right]\sin 2t, \end{aligned}$$

and the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1\cos 2t + c_2\sin 2t - 3\sin t\cos 2t + \left[\frac{3}{2}\ln|\csc t - \cot t| + 3\cos t\right]\sin 2t. \end{aligned}$$

4.2.7 Project

4.3 Sinusoidal Forcing

If we consider different forcing functions $g(t)$ for the equation

$$x'' + px' + qx = g(t),$$

functions that are periodic are especially important. Recall that a function $g(t)$ is ***periodic*** if

$$g(t+T) = g(t)$$

for all t and some fixed constant T. The most familiar periodic functions are

$$g(t) = \sin\omega t \text{ and } g(t) = \cos\omega t.$$

The ***period*** for each of these two functions is $2\pi/\omega$ and the ***frequency*** is $\omega/2\pi$. These two functions share the additional property that their average value is zero. That is,

$$\frac{1}{T}\int_0^T g(t)\,dt = 0.$$

We say that ***sinusoidal forcing*** occurs in the differential equation

$$x'' + px' + qx = A\cos\omega t + B\sin\omega t.$$

4.3.1 Complexification

Given a second-order linear differential equation

$$ax'' + bx' + cx = A\cos\omega t + B\sin\omega t,$$

we can use Euler's formula, $e^{i\beta t} = \cos\beta t + i\sin\beta t$ to derive a particular solution. That is, we will assume that our particular solution has the form

$$x_c = x_{\text{Re}} + ix_{\text{Im}}$$

and use the properties of complex numbers.[1]

[1] If complex numbers make you uncomfortable, the alternative is to become an expert in trigonometric identities

Example 4.17. Let us consider the equation

$$x'' + 6x' + 5x = \sin 2t. \tag{4.14}$$

The solution to the corresponding homogeneous equation, $x'' + 6x' + 5x = 0$, is

$$x_h = c_1 e^{-5t} + c_2 e^{-t}.$$

To find a particular solution, we can use the method of undetermined coefficients and assume that the solution has the form

$$x_p = A \cos 2t + B \sin 2t.$$

If we carry out the appropriate calculations, we will obtain a particular solution

$$x_p = -\frac{12}{145} \cos 2t + \frac{1}{145} \sin 2t.$$

Thus, the general solution is

$$x = x_h + x_p = c_1 e^{-5t} + c_2 e^{-t} - \frac{12}{145} \cos 2t + \frac{1}{145} \sin 2t.$$

Notice that all solutions of (4.14) will approach the particular solution as $t \to \infty$.

Example 4.18. Now let us solve (4.14) using complex numbers. If we assume that the equation has a complex solution of the form $x_c = x_{\text{Re}} + i x_{\text{Im}}$, then

$$\begin{aligned} \frac{d^2}{dt^2} x_c + 6 \frac{d}{dt} x_c + 5 x_c &= \frac{d^2}{dt^2}(x_{\text{Re}} + i x_{\text{Im}}) + 6 \frac{d}{dt}(x_{\text{Re}} + i x_{\text{Im}}) + 5(x_{\text{Re}} + i x_{\text{Im}}) \\ &= e^{2it} \\ &= \cos 2t + i \sin 2t. \end{aligned}$$

Equating the real and imaginary parts of this equation, we obtain

$$\begin{aligned} x''_{\text{Re}} + 6x'_{\text{Re}} + 5x_{\text{Re}} &= \cos 2t \\ x''_{\text{Im}} + 6x'_{\text{Im}} + 5x_{\text{Im}} &= \sin 2t. \end{aligned}$$

Thus, if we can find a complex solution, we can find a solution to

$$x'' + 6x' + 5x = \sin 2t$$

simply by examining the imaginary part of the solution.

Now let us assume that our solution has the form $x_c = Ae^{2it}$. Then

$$\begin{aligned} x_c'' + 6x_c' + 5x_c &= -4Ae^{2it} + 12Aie^{2it} + 5Ae^{2it} \\ &= (1 + 12i)Ae^{2it} \\ &= (1 + 12i)A(\cos 2t + i \sin 2t). \end{aligned}$$

Equating the real and imaginary parts of this equation, we obtain

$$\begin{aligned} x''_{\text{Re}} + 6x'_{\text{Re}} + 5x_{\text{Re}} &= \cos 2t \\ x''_{\text{Im}} + 6x'_{\text{Im}} + 5x_{\text{Im}} &= \sin 2t \end{aligned}$$

and we immediately see that

$$A = \frac{1}{1+12i} = \frac{1}{145} - \frac{12}{145}i.$$

Therefore, the complex solution to

$$x'' + 6x' + 5x = e^{2it}$$

is

$$\begin{aligned} x_c &= Ae^{2it} \\ &= \left(\frac{1}{145} - \frac{12}{145}i\right) \cdot (\cos 2t + i \sin 2t) \\ &= \left(\frac{1}{145}\cos 2t + \frac{12}{145}\sin 2t\right) + i\left(-\frac{12}{145}\cos 2t + \frac{1}{145}\sin 2t\right). \end{aligned}$$

The imaginary part of this function is

$$x_{\text{Im}} = -\frac{12}{145}\cos 2t + \frac{1}{145}\sin 2t,$$

which is the particular solution that we have been seeking. Thus, our general solution agrees with what we found in Example 4.17.

4.3.2 Qualitative Analysis

We can use the complex solution of $ax'' + bx' + cx = A\cos\omega t + B\sin\omega t$ to analyze the qualitative behavior of solutions.

Example 4.19. We discovered that the complex solution of

$$x'' + 6x' + 5x = e^{2it}$$

to be $x_c = Ae^{2it}$, where $A = (1 - 12i)/145$. Let us rewrite A in polar form. Since

$$|A| = \frac{1}{\sqrt{145}},$$

we know that

$$A = \frac{1}{\sqrt{145}}e^{i\theta},$$

where $\theta = \arctan(-12) \approx -1.4877$. Therefore,

$$x_c = Ae^{2it} = \frac{1}{\sqrt{145}}e^{i\theta}e^{2it} = \frac{1}{\sqrt{145}}e^{i(2t+\theta)}.$$

Our particular solution is the imaginary part of x_c,

$$x_p(t) = \frac{1}{\sqrt{145}}\sin(2t+\theta) = \frac{1}{\sqrt{145}}\cos\left(2t + \theta - \frac{\pi}{2}\right) = \frac{1}{\sqrt{145}}\cos\left(2t - \phi\right),$$

where $\phi \approx 3.058451$. We say that ϕ is the ***phase angle*** of our solution. The amplitude of our solution is $1/\sqrt{145}$ and the period is π (Figure 4.20).

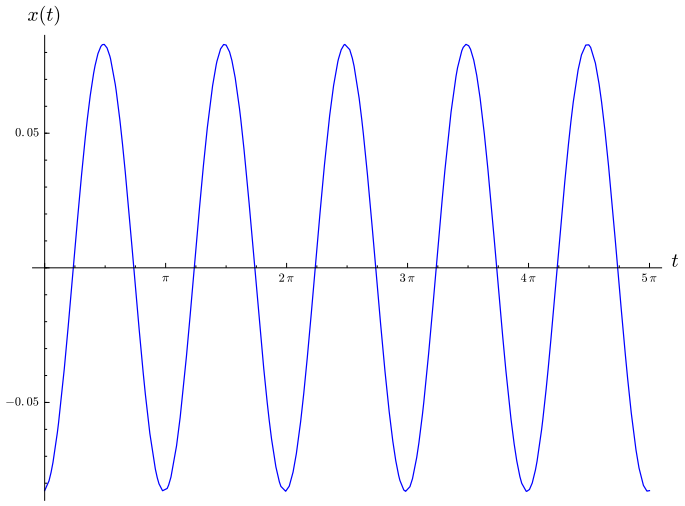

Figure 4.20: Steady state solution to $x'' + 6x' + 5x = \sin 2t$

The corresponding first order system for the differential equation

$$x'' + px' + qx = g(t),$$

is

$$\begin{aligned} x' &= y \\ y' &= -qx - py + g(t). \end{aligned}$$

This is a ***nonautonomous system***, and the tangent vector of a solution curve in the phase plane depends not only on the position (x, y), but also on the time t. In other words, the direction field changes with time. Since the direction field changes with time, two solutions with the same (x, y) value at different times can follow different paths. Consequently, solutions can cross each other in the xy-plane without violating the Existence and Uniqueness Theorem.

Example 4.21. Consider the harmonic oscillator that is modeled by the differential equation

$$x'' + 2x' + 17x = -2\sin 3t. \tag{4.15}$$

The solution to the homogeneous equation $x'' + 2x' + 17x = 0$ is

$$x_h = c_1 e^{-t}\cos 4t + c_2 e^{-t}\sin 4t$$

The complex version of this equation is

$$x'' + 2x' + 17x = -2e^{3it},$$

and we will use the Method of Undetermined Coefficients and assume that we can find a particular solution of the form $x_c = Ae^{3it}$. Substituting x_c into equation (4.15), we find that

$$(8 + 6i)Ae^{3it} = -2e^{3it}.$$

Thus, x_c is a solution if

$$A = \frac{-2}{8 + 6i} = -\frac{4}{25} + \frac{3}{25}i$$

We have

$$\begin{aligned} x_c(t) &= \left(-\frac{4}{25} + \frac{3}{25}i\right)(\cos 3t + i \sin 3t) \\ &= \left(-\frac{4}{25}\cos 3t - \frac{3}{25}\sin 3t\right) + i\left(\frac{3}{25}\cos 3t - \frac{4}{25}\sin 3t\right). \end{aligned}$$

The imaginary part of this function is the solution that we seek,

$$x_p = \frac{3}{25}\cos 3t - \frac{4}{25}\sin 3t.$$

Thus, the general solution to (4.15) is

$$x(t) = c_1 e^{-t}\cos 4t + c_2 e^{-t}\sin 4t + \frac{3}{25}\cos 3t - \frac{4}{25}\sin 3t.$$

Now suppose that $x(0) = 0$ and $x'(0) = 0$. We can quickly determine that

$$x'(t) = c_1 e^{-t}(4\cos 4t - \sin 4t) + c_2 e^{-t}(-\cos 4t - 4\sin 4t) - \frac{12}{25}\cos 3t - \frac{9}{25}\sin 3t$$

To solve this initial value problem, we must solve the linear system

$$\begin{aligned} c_1 + \frac{3}{25} &= 0 \\ -4c_1 - c_2 - \frac{9}{25} &= 0. \end{aligned}$$

We obtain $c_1 = -3/25$ and $c_2 = 3/25$, and the solution to our initial value problem is

$$x(t) = -\frac{3}{40}e^{-2t}\cos 4t + \frac{3}{80}e^{-2t}\sin 4t + \frac{3}{40}\cos 2t - \frac{3}{20}\sin 2t.$$

The graph of our solution is given in Figure 4.22.

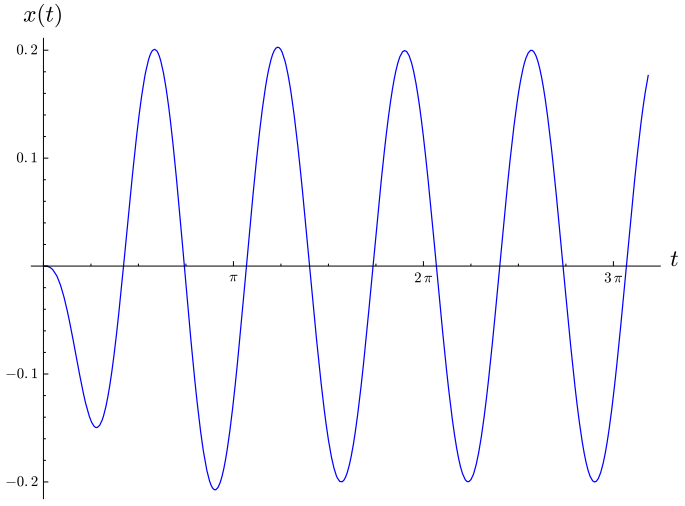

Figure 4.22: Solution to $x'' + 2x' + 17x = -2\sin 3t$, $x(0) = 0$, $x'(0) = 0$

Since $y = x'(t)$, we can now graph the solution curve in the phase plane (Figure 4.23). Notice how the solution curve can intersect itself. The restoring force and damping are proportional to x and $y = x'$, respectively. When x and y are close to the origin, the

external force is as large or larger than the restoring and damping forces. In this part of the xy-plane, the external force overcomes the damping and pushes the solution away from the origin.

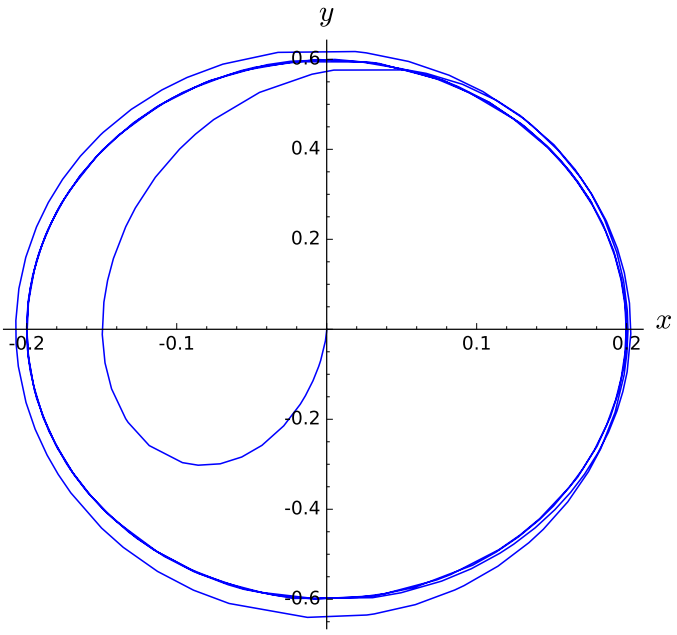

Figure 4.23: Phase Plane for $x'' + 2x' + 17x = -2\sin 3t$, $x(0) = 0$, $x'(0) = 0$

On the other hand, suppose we have initial conditions $x(0) = 2$ and $x'(0) = 2$, we can solve the linear system

$$\begin{aligned} c_1 + \frac{3}{25} &= 2 \\ -4c_1 - c_2 - \frac{9}{25} &= 2. \end{aligned}$$

to obtain $c_1 = 47/25$ and $c_2 = 109/100$. Thus, solution to our initial value problem is

$$x(t) = \frac{47}{25}e^{-t}\cos 4t + \frac{109}{100}e^{-t}\sin 4t + \frac{3}{40}\cos 2t - \frac{3}{20}\sin 2t.$$

The graph of our solution is given in Figure 4.24.

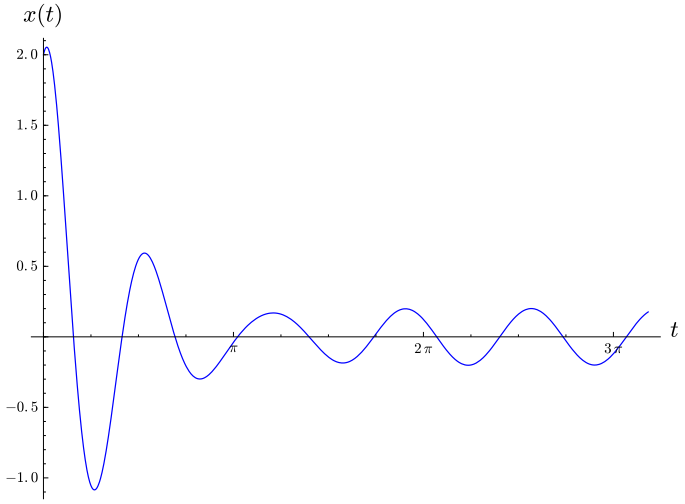

Figure 4.24: Solution to $x'' + 2x' + 17x = -2\sin 3t$, $x(0) = 2$, $x'(0) = 2$

If we examine the phase plane for this solution (Figure 4.25), we see that the initial damping and restoring forces are much larger than the external force. Thus, if we are far from the origin, the solutions in the xy-plane tend to spiral towards the origin and are similar to the solutions of the unforced equation.

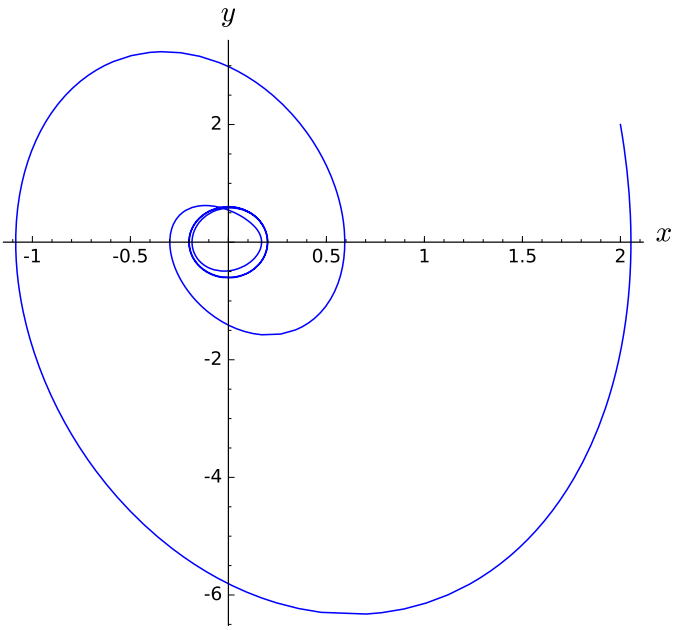

Figure 4.25: Phase Plane for $x'' + 2x' + 17x = -2\sin 3t$, $x(0) = 2$, $x'(0) = 2$

4.3.3 Important Lessons

- The functions $\sin \omega t$ and $g(t) = \cos \omega t$ are periodic with period $2\pi/\omega$ and frequency $\omega/2\pi$. These average value of each of these functions is zero.
- We can use Euler's formula and complexification to solve the equation

$$x'' + px' + qx = g(t),$$

where the forcing function $g(t)$ is $\sin \omega t$ or $\cos \omega t$. Furthermore, we can use complex numbers to express or solution in the form

$$x(t) = A\cos(\omega t - \phi),$$

where A is the amplitude of the solution, $\omega/2\pi$ is the frequency of the solution, and ϕ is the phase angle.

- If we write the equation

$$x'' + px' + qx = g(t),$$

as a first-order system,

$$\begin{aligned} x' &= y \\ y' &= -qx - py + g(t), \end{aligned}$$

we obtain a nonautonomous system. In this case the direction field changes with time, and two solutions with the same (x, y) value at different times can follow different paths. Therefore, solutions can cross each other without violating the Existence and Uniqueness Theorem.

- If we are far from the origin, the solutions in the xy-plane tend to spiral towards the origin and are similar to the solutions of the unforced equation. When x and y are close to the origin, the external force is as large or larger than the restoring and damping forces. In this part of the xy-plane, the external force overcomes the damping and pushes the solution away from the origin.

4.3.4 Exercises

1. Compute the solution of the initial value problem

$$\begin{aligned} y'' + 4y &= 3\cos 2t \\ y(0) = y'(0) &= 0. \end{aligned}$$

Hint.

4.3.5 Project

4.4 Forcing and Resonance

Resonance happens when the natural frequency and the forcing frequency of an undamped harmonic oscillator are the same. There are many real world examples of resonance. For example, it is possible for an trained singer shatter a champaign glass by simply singing the correct note (see https://www.youtube.com/watch?v=IZD8ffPwXRo). Shattering a glass by singing the correct note is can be modeled by an equation of an undamped harmonic oscillator with sinusoidal forcing. While a certain amount of damping occurs in all physical systems, the damping is often so small that it makes sense to use an undamped harmonic oscillator.

4.4.1 Resonance

Resonance was responsible for the collapse of the Broughton suspension bridge near Manchester, England in 1831. The collapse occurred when a column of soldiers marched in cadence over the bridge, setting up a periodic force of rather large amplitude. The frequency of the force was approximately equal to the natural frequency of the bridge. Thus, the bridge collapsed when large oscillations occurred. For this reason soldiers are ordered to break cadence whenever they cross a bridge.

The Millennium Bridge, the first new bridge to span the Thames River in London in over 100 years, is a modern example of how resonance can effect a bridge (Figure 4.26). This pedestrian bridge, which opened to the public in June 2000, was quickly closed after the bridge experienced high amplitude horizontal oscillations during periods of high traffic. Studies by designers found that the bridge experienced high amplitude horizontal oscillations in response to horizontal forcing at a rate of one cycle per second. Typically, people walk at a rate of two steps per second, so the time between two successive steps of the left foot is about one second. Thus, if people were to walk in cadence, they would could set up strong horizontal forcing that would place a destructive load on the bridge. The engineers did not envision this to be a problem since tourists do not generally march in time. However, a video of tourists crossing the bridge revealed the opposite. When the bridge began oscillating, people tended to walk in cadence in order to keep their balance (https://www.youtube.com/watch?v=gQK21572oSU).

Figure 4.26: The Millennium Bridge

Consider the case of an undamped harmonic oscillator with a sinusoidal forcing function.

We know that we can model this situation with the second-order linear differential equation

$$x'' + \omega_0^2 x = A\cos\omega t$$

Since the homogeneous solution to this equation is

$$x_h(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t,$$

we can see that the ***natural frequency*** of the harmonic oscillator is ω_0. This is the frequency of the oscillations if there is no forcing term. Depending on whether or not the frequency of the forcing term ω is equal to the natural frequency, we will have two very different situations. If the two frequencies are equal, then they will re-enforce one another and we will have ***resonance***. If the two frequencies are close but not equal, we will see the phenomenon of ***beats***. We wish to investigate what happens when the ***driving frequency*** ω is equal to the natural frequency and when it is not.

Let us examine the case where the forcing frequency and the natural frequency of the oscillator are the same,

$$x'' + \omega_0^2 x = A\cos\omega_0 t. \tag{4.16}$$

Since $A\cos\omega_0 t$ is a solution to the homogeneous equation $x'' + \omega_0^2 x = 0$, we cannot assume that a particular solution to equation (4.16) has the form $a\cos\omega_0 t + b\sin\omega_0 t$. Equivalently, if we use the complex method, we cannot assume that our solution has the form $ae^{i\omega_0 t}$. Therefore, we will look for a complex solution of the form

$$x_c = ate^{i\omega_0 t}$$

for the complex equation that corresponds to (4.16),

$$x'' + \omega_0^2 x = Ae^{i\omega_0 t}. \tag{4.17}$$

In this case,

$$\begin{aligned} x_c' &= a(1 + i\omega_0 t)e^{i\omega_0 t} \\ x_c'' &= a(2i\omega_0 - \omega_0^2 t)e^{i\omega_0 t}. \end{aligned}$$

Substituting x_c and x_c'' into the left-hand side of (4.17), we have

$$\begin{aligned} x_c'' + \omega_0^2 x_c &= a(2i\omega_0 - \omega_0^2 t)e^{i\omega_0 t} + \omega_0^2 ate^{i\omega_0 t} \\ &= 2ai\omega_0 e^{i\omega t}. \end{aligned}$$

In order for $x_c = ate^{i\omega_0 t}$ to be a solution to (4.17), we must have

$$a = \frac{A}{2i\omega_0}.$$

Thus, our solution to the complex form of the differential equation is

$$\begin{aligned} x_c &= \frac{A}{2i\omega_0} te^{i\omega_0 t} \\ &= -\frac{Ai}{2\omega_0} te^{i\omega_0 t} \\ &= -\frac{Ai}{2\omega_0} t(\cos\omega_0 t + i\sin\omega_0 t) \\ &= \frac{A}{2\omega_0} t\sin\omega_0 t - i\frac{A}{2\omega_0} t\cos\omega_0 t. \end{aligned}$$

The real part of x_c,

$$x_p(t) = \frac{A}{2\omega_0} t \sin \omega_0 t,$$

is a particular solution to (4.16). Thus, our general solution is

$$x(t) = x_h(t) + x_p(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{A}{2\omega_0} t \sin \omega_0 t.$$

Example 4.27. Now let us consider the initial-value problem

$$\begin{aligned} x'' + 144x &= 4 \cos 12t \\ x(0) &= 0 \\ x'(0) &= 0. \end{aligned}$$

where $A = 4$ and $\omega_0 = 12$. The solution to the homogeneous equation $x'' + 144x = 0$ is

$$x_h(t) = c_1 \cos 12t + c_2 \sin 12t.$$

To find a particular solution to $x'' + 144x = 4 \cos 12t$, we will use the complex method and try to find a particular solution to $x'' + 144x = 4e^{12it}$. We must assume that the solution has the form $x_c(t) = ate^{12it}$, since ae^{12it} is a solution to the homogeneous equation. As before, we have $x_c'' = a(24i - 144t)e^{12it}$. If we substitute x_c and x_c'' into the left-hand side of our differential equation, we have

$$x_c'' + 144x_c = a(24i - 144t)e^{12it} + 144ate^{12it} = 24aie^{12it}.$$

Therefore,

$$a = \frac{4}{24i} = -\frac{1}{6}i,$$

and our complex solution is

$$-\frac{1}{6}ie^{12it} = -\frac{1}{6}i(\cos 12t + i \sin 12t) = \frac{1}{6} \sin 12t - \frac{1}{6}i \cos 12t.$$

Taking the real part of our complex solution, we have a particular solution

$$x_p(t) = \frac{1}{6} t \sin 12t.$$

Thus, the general solution to $x'' + 144x = 4 \cos 12t$ is

$$x(t) = x_h(t) + x_p(t) = c_1 \cos 12t + c_2 \sin 12t + \frac{1}{6} t \sin 12t.$$

Applying the initial conditions, both $c_1 = 0$ and $c_2 = 0$. Consequently, the solution to the initial-value problem is

$$x(t) = \frac{1}{6} t \sin 12t.$$

The graph of this solution is given in Figure 4.28. Notice that our solution grows with time. This growth is due to the fact that the frequency of the forcing term is equal to the natural frequency of the oscillator. Since the force pulls and pushes at a frequency equal to the natural frequency of the oscillator, the amplitude increases with time. This type of behavior is called ***resonance***.

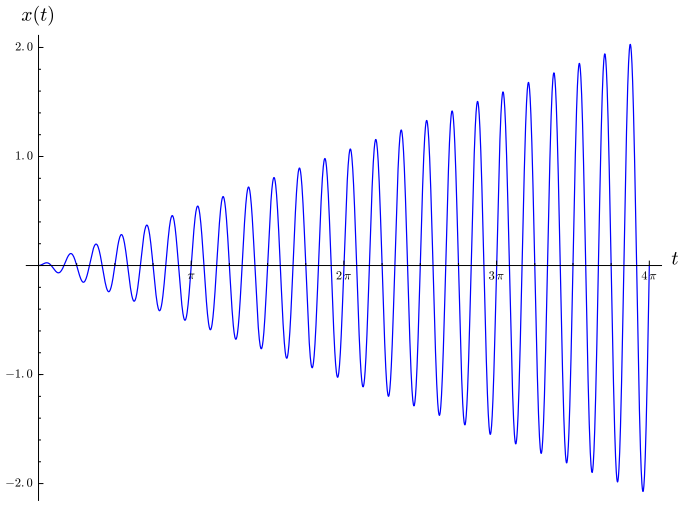

Figure 4.28: The solution to $x'' + 4x = 4\sin 2t$, $x(0) = 0$, $x'(0) = 0$.

4.4.2 Beats or the Case $\omega \neq \omega_0$

If the forcing frequency and natural frequency are not equal but close, then we have the phenomenon of beats. For example, a piano tuner uses this phenomenon to make certain that a particular string is correctly tuned. The tuner will strike a tuning fork which vibrates at the correct frequency. Next, the tuner hits the poorly tuned piano string. This modulation gives rise to beats in the tone that are readily audible. When the string is properly tuned, the beats will go away. To see another example of beats, watch the video https://www.youtube.com/watch?v=pRpN9uLiouI.

To understand the situation where the forcing and natural frequencies differ by a small amount, $\omega \neq \omega_0$, we will first consider the equation,

$$x'' + \omega_0^2 = A\cos\omega t. \tag{4.18}$$

The solution to the homogeneous equation $x'' + \omega_0^2 = 0$ is

$$x_h(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t.$$

Since the forcing frequency is not equal to the natural frequency, we can look for a particular solution to the complex of the form of equation (4.18),

$$x'' + \omega_0^2 = Ae^{i\omega t}. \tag{4.19}$$

If we assume out solution has the form $x_c = ae^{i\omega t}$ and substitute x_c into the left-hand side of (4.19), we find

$$\begin{aligned} x_c'' + \omega_0^2 x_c &= -a\omega^2 e^{i\omega t} + \omega_0^2 a e^{i\omega t} \\ &= a(\omega_0^2 - \omega^2)e^{i\omega t}. \end{aligned}$$

Therefore, for $x_c = ae^{i\omega t}$ to be a solution for

$$x'' + \omega_0^2 x = Ae^{i\omega t},$$

we must have

$$a = \frac{A}{\omega_0^2 - \omega^2}.$$

The real part of x_c is a particular solution to (4.18),

$$x_p(t) = \frac{A}{\omega_0^2 - \omega^2} \cos \omega t.$$

Thus, the general solution to (4.18) is

$$x(t) = x_h(t) + x_p(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{A}{\omega_0^2 - \omega^2} \cos \omega t.$$

Now let us examine the case where the motion starts at equilibrium. That is, we will investigate what happens when $x(0) = 0$ and $x'(0) = 0$. In this case, we can easily determine that $c_1 = -A/(\omega_0^2 - \omega^2)$ and $c_2 = 0$. Hence the solution to our initial value problem is

$$x(t) = \frac{A}{\omega_0^2 - \omega^2}(\cos \omega t - \cos \omega_0 t). \tag{4.20}$$

Thus, we have a superposition to two oscillations of different frequencies.

To understand how the superposition of the two frequencies works, we will let

$$\overline{\omega} = \frac{\omega_0 + \omega}{2}$$

be the ***mean frequency*** and

$$\delta = \frac{\omega_0 - \omega}{2}$$

be the ***half difference***. Since

$$\begin{aligned} \omega &= \overline{\omega} - \delta, \\ \omega_0 &= \overline{\omega} + \delta. \end{aligned}$$

we know that

$$\begin{aligned} x(t) &= \frac{A}{\omega_0^2 - \omega^2}(\cos \omega t - \cos \omega_0 t) \\ &= \frac{A}{(\overline{\omega} + \delta)^2 - (\overline{\omega} - \delta)^2}(\cos[(\overline{\omega} - \delta)t] - \cos[(\overline{\omega} + \delta)t]) \\ &= \frac{A}{4\overline{\omega}\delta}[\cos(\overline{\omega}t - \delta t) - \cos(\overline{\omega}t + \delta t)] \\ &= \frac{A}{4\overline{\omega}\delta}[\cos(\overline{\omega}t)\cos(\delta t) + \sin(\overline{\omega}t)\sin(\delta t) - \cos(\overline{\omega}t)\cos(\delta t) + \sin(\overline{\omega}t)\sin(\delta t)] \\ &= \frac{A}{4\overline{\omega}\delta}[2\sin(\overline{\omega}t)\sin(\delta t)] \\ &= \frac{A \sin \delta t}{2\overline{\omega}\delta} \sin \overline{\omega} t. \end{aligned}$$

Thus, we have a sine function, $\sin \overline{\omega} t$ whose amplitude varies according to $|(A/2\overline{\omega}\delta) \sin \delta t|$, and

$$x(t) = \frac{A \sin \delta t}{2\overline{\omega}\delta} \sin \overline{\omega} t. \tag{4.21}$$

Example 4.29. In the initial-value problem

$$x'' + 144x = 4\cos(13t) \tag{4.22}$$

$$x(0) = 0 \tag{4.23}$$

$$x'(0) = 0, \tag{4.24}$$

we have $A = 4$, $\omega_0 = 12$, and $\omega = 13$. The solution to this equation is

$$x(t) = \frac{4}{25}(\cos 12t - \cos 13t).$$

The graph of this solution is given in Figure 4.30 and illustrate the phenomenon of ***beats***. The two frequencies are almost equal and interfere with each other.

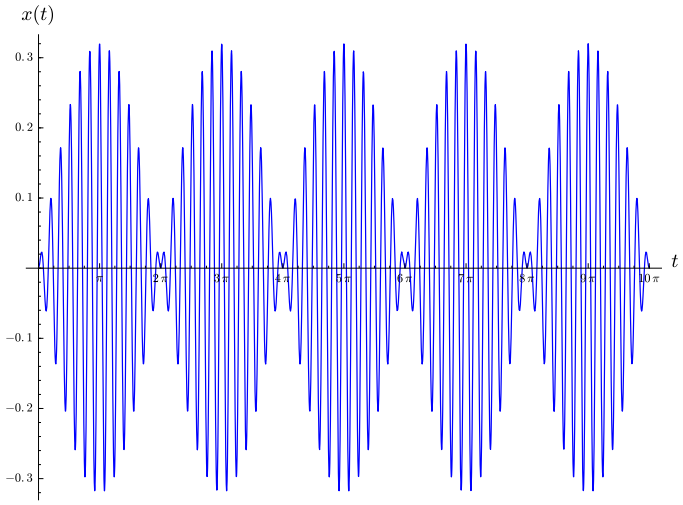

Figure 4.30: Solution to the initial-value problem (4.22)

In our example, $\delta = -1/2$ and $\overline{\omega} = 25/2$. The factor $\sin \delta t = \sin(-t/2)$ in (4.21) oscillates very slowly in comparison to $\sin(\overline{\omega}) = \sin(25t/2)$. Thus, we can view the solution (4.21) as a fast oscillation with a frequency $\overline{\omega}$ and amplitude

$$\left| \frac{A \sin \delta t}{2\overline{\omega}\delta} \right| = \left| \frac{8}{25} \sin\left(\frac{t}{2}\right) \right|, \tag{4.25}$$

which oscillates much more slowly. If we superimpose the graph of (4.25) onto the graph in Figure 4.30, we obtain an ***envelope*** of the faster oscillation.

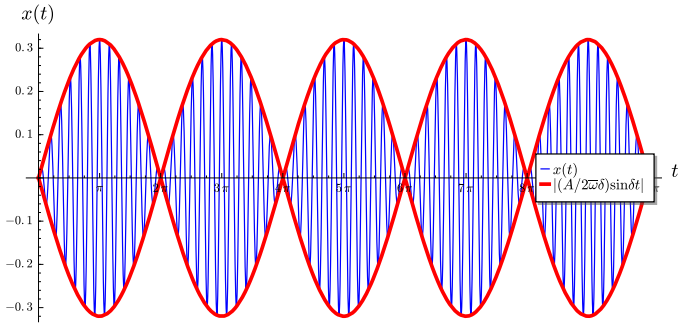

Figure 4.31: Envelope for the initial-value problem (4.22)

4.4.3 Forced Damped Harmonic Motion

Since perpetual motion does not really exist, we should consider what happens when we add some damping to our harmonic oscillator,

$$x'' + 2cx' + \omega_0^2 x = A \cos \omega t. \tag{4.26}$$

Example 4.32. Let us consider the differential equation

$$x'' + 2x' + 2x = 3\cos 4t. \tag{4.27}$$

The corresponding homogeneous equation is

$$x'' + 2x' + 2x = 0,$$

which has solution

$$x_h(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t.$$

To find a particular solution for the equation (4.27), we will look for a solution of the form $x_c = ae^{4it}$ to the equation

$$x'' + 2x' + 2x = 3e^{4it} \tag{4.28}$$

and then set $x_p(t)$ to the real part of the solution. Substituting x_c into the lefthand side of (4.28), we obtain

$$x_c'' + 2x_c' + 2x_c = (-14 + 8i)ae^{4it} = P(4i)ae^{4it} = 3e^{4it},$$

where $P(\lambda) = \lambda^2 + 2\lambda + 2$ is the characteristic polynomial of $x'' + 2x' + 2x = 0$. Therefore,

$$a = \frac{3}{P(4i)} = \frac{3}{-14 + 8i} = -\frac{3}{130}(7 + 4i),$$

and the solution to the complex equation is

$$\begin{aligned} x_c &= -\frac{3}{130}(7 + 4i)e^{4it} \\ &= -\frac{3}{130}(7 + 4i)(\cos 4t + i\sin 4t) \\ &= \frac{3}{130}[(-7\cos 4t + 4\sin 4t) + i(-4\cos 4t - 7\sin 4t)]. \end{aligned}$$

Thus, we can determine a particular solution by taking the real part of x_c,

$$x_p(t) = \frac{3}{130}(-7\cos 4t + 4\sin 4t).$$

The general solution for (4.27) is

$$x(t) = x_h(t) + x_p(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t + \frac{3}{130}(-7\cos 4t + 4\sin 4t).$$

It is easy to see that all solutions approach $x_p(t)$ as $t \to \infty$ no matter what the initial conditions may be. However, it would be much more useful if we would write the particular solution in the form

$$x_p(t) = K\cos(4t - \phi)$$

for some constants K and ϕ rather than a linear combination of $\cos 4t$ and $\sin 4t$.

Returning to the general case, the associated homogeneous equation for (4.26) is

$$x'' + 2cx' + \omega_0^2 x = 0. \tag{4.29}$$

The characteristic polynomial of (4.29),

$$P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2, \tag{4.30}$$

has roots

$$\lambda = -c \pm \sqrt{c^2 - \omega_0^2}.$$

If our harmonic oscillator is undedamped ($c < \omega_0$), then the solution to the homogeneous equation (4.29) is

$$x_h = e^{-ct}(c_1 \cos \beta t + c_2 \sin \beta t),$$

where $\beta = \sqrt{\omega_0^2 - c^2}$.

As in Example 4.32, we will look for a particular solution to (4.26) by examining the equivalent complex differential equation. That is, we shall look for a solution of the form $x_c = ae^{i\omega t}$ to the equation

$$x_c'' + 2cx_c' + \omega_0^2 x_c = Ae^{i\omega t}, \tag{4.31}$$

and then set x_p equal to the real part of our solution. Substituting x_c into the right-hand side of (4.31), we obtain

$$x_c'' + 2cx_c' + \omega_0^2 x_c = [(i\omega)^2 + 2c(i\omega) + \omega_0^2]ae^{i\omega t} = P(i\omega)ae^{i\omega t},$$

where P is the characteristic polynomial (4.30). Thus, equation (4.31) becomes

$$P(i\omega)ae^{i\omega t} = Ae^{i\omega t},$$

and

$$x_c(t) = ae^{i\omega t} = \frac{A}{P(i\omega)}e^{i\omega t} = H(i\omega)Ae^{i\omega t},$$

where

$$H(i\omega) = \frac{1}{P(i\omega)}.$$

We say that $H(\lambda) = 1/P(\lambda)$ is the ***transfer function***.

Let us examine the transfer function more closely to see if we can use it to write our particular solution in a more useable form. First, let us write

$$P(i\omega) = (i\omega)^2 + 2c(i\omega) + \omega_0^2 = (\omega_0^2 - \omega^2) + 2ic\omega.$$

in polar form,

$$P(i\omega) = Re^{i\phi} = R(\cos\phi + i \sin\phi),$$

where

$$R = \sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}$$

and ϕ is the angle defined by the equations

$$\cos\phi = \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 - 4c^2\omega^2}},$$
$$\sin\phi = \frac{2c\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 - 4c^2\omega^2}}.$$

Since $2c\omega > 0$, we know that $\sin\phi > 0$. Equivalently, $0 < \phi < \pi$. Thus,

$$\phi = \phi(\omega) = \cot^{-1}\left(\frac{\omega_0^2 - \omega^2}{2c\omega}\right).$$

Therefore, we can write the transfer function as

$$H(i\omega) = \frac{1}{P(i\omega)} = \frac{1}{R}e^{-i\phi}.$$

We define the ***gain*** to be

$$G(\omega) = \frac{1}{R} = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}},$$

and we will rewrite the transfer function as

$$H(i\omega) = G(\omega)e^{-i\phi(\omega)}.$$

Thus, the solution to

$$x_c'' + 2cx_c' + \omega_0^2 x_c = Ae^{i\omega t}$$

is

$$x_c(t) = H(i\omega)Ae^{i\omega t} = G(\omega)Ae^{i(\omega t - \phi)}. \tag{4.32}$$

Taking the real part of (4.32), our particular solution is

$$x_p(t) = \mathrm{Re}(x_c(t)) = G(\omega)A\cos(\omega t - \phi).$$

It is now clear that particular solution has the same frequency as the forcing term. In addition, x_p is out of phase with the driving force by the amount

$$\phi = \phi(\omega) = \cot^{-1}\left(\frac{\omega_0^2 - \omega^2}{2c\omega}\right).$$

The general solution to

$$x'' + 2cx' + \omega_0^2 x = A\cos\omega t$$

is

$$x(t) = x_h(t) + x_p(t) = e^{-ct}(c_1\cos(\eta t) + c_2\sin(\eta t)) + G(\omega)A\cos(\omega t - \phi).$$

Since x_h has the factor e^{-ct}, the homogeneous part of the solution quickly decays to zero as $t \to \infty$. For this reason, x_h is called the ***transient term*** while x_p is called the ***steady-state term***.

Example 4.33. Let us examine the steady-state solution of the harmonic oscillator in Example 4.32. We have already determined that the second-order linear differential equation

$$x'' + 2x' + 2x = 3\cos 4t$$

has solution

$$x(t) = x_h(t) + x_p(t) = c_1e^{-t}\cos t + c_2e^{-t}\sin t + \frac{3}{130}(-7\cos 4t + 4\sin 4t).$$

The natural frequency is $\omega_0 = \sqrt{2} \approx 1.4142$ If we choose the initial conditions so that $c_1 = 0$ and $c_2 = 1$, our transient term is

$$x_h(t) = e^{-ct}\sin\beta t = e^{-t}\sin t.$$

Since $c = 1$, $\omega = 4$, and $A = 3$, the gain is $G(4) \approx 0.0620$, the amplitude of the steady state solution is $G(4) \cdot 4 \approx 0.2481$. The phase is $\phi = \phi(4) \approx 2.6224$. The solution and the steady-state solution are given in Figure 4.34. As we can see, transient solutions can be quite large when compared to steady-state solutions.

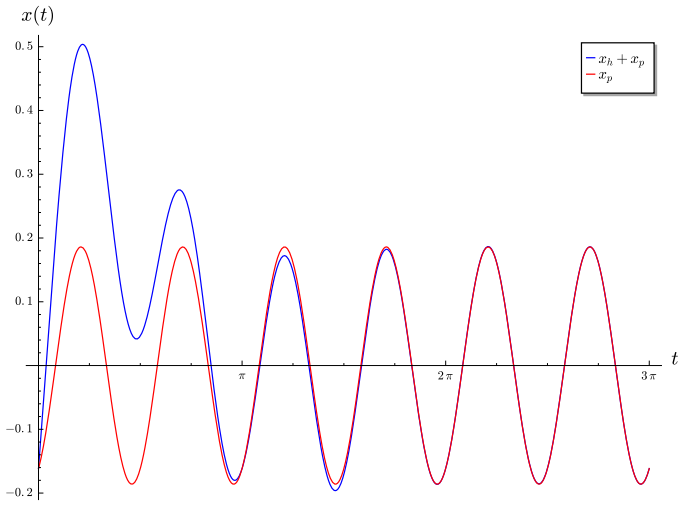

Figure 4.34: A forced damped harmonic oscillator and the steady-state solution.

Large transient currents can be destructive in electrical circuits. For example, a light bulb usually burns out when a large transient flows through an already weakened bulb. This, usually occurs at the time the light bulb is turned out or turned off. Large transient currents are particularly harmful to the hard drive in a computer. For this reason, computers should be left on except in the case that they are not being used for a period of several days.

4.4.4 Important Lessons

- A forced damped harmonic oscillator,

  $$x'' + 2cx' + \omega_0^2 x = A\cos\omega t$$

  has homogeneous solution

  $$x_h = e^{-ct}(c_1\cos\eta t + c_2\sin\eta t),$$

  where $\eta = \sqrt{\omega_0^2 - c^2}$. A particular solution is given by

  $$x_p(t) = \mathrm{Re}(x_c(t)) = G(\omega)A\cos(\omega t - \phi),$$

  where the phase angle ϕ is

  $$\phi = \phi(\omega) = \cot^{-1}\left(\frac{\omega_0^2 - \omega^2}{2c\omega}\right)$$

  and the gain G is given by

  $$G(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}.$$

- The amplitude and phase of the steady-state solution,

  $$x_p(t) = G(\omega)A\cos(\omega t - \phi),$$

  are determined by

  $$\omega_0^2 G = \frac{1}{\sqrt{(1-s^2)^2 + D^2 s^2}},$$

  where $\omega = s\omega_0$ and $D = 2c/\omega_0$.

- In an undamped harmonic oscillators with a sinusoidal forcing term, the natural frequency of the solution interacts with the frequency of the forcing term. As the frequency of the forcing term approaches the natural frequency of the equation, we can observe a phenomenon called ***resonance***. We can use the equation

$$x'' + \omega_0^2 x = A\cos\omega t$$

to model an undamped harmonic oscillator with sinusoidal forcing.

- If the driving frequency is not equal to the natural frequency ($\omega \neq \omega_0$), we have a particular solution

$$x_p(t) = \frac{A}{\omega_0^2 - \omega^2}(\cos\omega t - \cos\omega_0 t).$$

If the two frequencies are close, we can see a phenomenon called ***beats***.

- If the driving frequency is equal to the natural frequency, we have a particular solution

$$x_p(t) = \frac{A}{2\omega_0} t\sin\omega_0 t$$

In this case, we can see the ***resonance*** phenomenon.

- A forced damped harmonic oscillator,

$$x'' + 2cx' + \omega_0^2 x = A\cos\omega t$$

has homogeneous solution

$$x_h = e^{-ct}(c_1\cos\eta t + c_2\sin\eta t),$$

where $\eta = \sqrt{\omega_0^2 - c^2}$. A particular solution is given by

$$x_p(t) = \mathrm{Re}(x_c(t)) = G(\omega)A\cos(\omega t - \phi),$$

where the phase angle ϕ is

$$\phi = \phi(\omega) = \cot^{-1}\left(\frac{\omega_0^2 - \omega^2}{2c\omega}\right)$$

and the gain G is given by

$$G(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}.$$

4.4.5 Exercises

1. Given the equation

$$y'' + 10y = \sin 3t,$$

determine the following.

(a) The frequency of the beats.

(b) The frequency of the rapid oscillations.

(c) Use parts (1) and (2) to sketch a graph of a typical solution.

Hint.

2.

(a) Suppose that an opera singer can break a glass by singing a particular note. Will the singer have to sing a higher note or a lower note to break an identical glass that is half full of water?

(b) Suppose that both notes are within the opera singer's range. Will it be harder or easier to break the glass when it is half full of water?

(c) Commercials for a maker of recording tape show a crystal glass being broken by the sound of a recording of an opera singer's voice. Is this a good test of the quality of the tape? Why or why not?

Hint.

3. For large t, every solution of

$$y'' + py' + qy = \cos \omega t$$

oscillates with angular frequency ω and amplitude A given by

$$A(\omega, p, q) = \frac{1}{\sqrt{(q - \omega^2)^2 + p^2\omega^2}}.$$

That is, the amplitude of A is a function of the parameters ω, p, and q.

(a) Compute $\dfrac{\partial A}{\partial \omega}$.

(b) For fixed p and q, let $M(p, q)$ denote the maximum value of $A(\omega, p, q)$ as a function of ω. Compute an expression for $M(p, q)$.

(c) Set $q = 1$ and plot $M(p, q)$ as a function of p.

(d) Explain why $M(p, q)$ is proportional to $1/p$ as $p \to 0$.

Hint.

4. Let us examine the amplitude and phase of the steady-state solution,

$$x_p(t) = G(\omega)A \cos(\omega t - \phi),$$

where

$$G(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}.$$

Now let $s = \omega/\omega_0$ and $D = 2c/\omega_0$. These new constants, s and D, measure the ratio of the driving frequency to the natural frequency and the effect of the damping force, respectively. Thus,

$$G = \frac{1}{\omega_0^2\sqrt{(1 - s^2)^2 + D^2s^2}}$$

or

$$\omega_0^2 G = \frac{1}{\sqrt{(1 - s^2)^2 + D^2s^2}}.$$

This expression shows us how the gain varies as $s = \omega/\omega_0$ varies. The natural frequency is fixed in

$$x'' + 2cx' + \omega_0^2 x = A \cos \omega t$$

and $D = 2c/\omega_0$ is proportional to the damping constant.

Hint.

4.4.6 Project—The Tacoma Narrows Bridge

The History of the Tacoma Narrows Bridge

On July 1, 1940, the Tacoma Narrows Bridge was completed and opened to traffic (Figure 4.35). From the first day of operation, the bridge experienced vertical oscillations of several feet. The bridge received the nickname "Galloping Gertie" for its wild behavior. On the morning of November 7, the bridge began undulating up and down with motions as large as three feet. Later, the bridge began to experience torsional oscillations as large as 23 degrees. The bridge finally came crashing down at 11:10 a.m. For a long time, the collapse of the Tacoma Narrows Bridge was attributed to resonance. However, we shall soon see the the story is more complicated.

Figure 4.35: The Tacoma Narrows Bridge

Construction on first Tacoma Narrows Bridge, A.K.A. "Galloping Gertie", was completed on July 1, 1940, the bridge was opened to traffic (Figure 4.36). Even on the first day of operation, people driving bridge experienced vertical oscillations of several feet. On the morning of November 7, these vertical oscillations were as large as three feet. Later that morning, the bridge began to experience torsional oscillations as great as 23 degrees. The bridge failed castrophically at 11:10 a.m. with the center span falling into Puget Sound below.

Figure 4.36: The Tacoma Narrows Bridge Video

There are many interesting stories connected with the first Tacoma Narrows bridge. One of the insurance policies had been written by a local travel agent who pocketed the premium and failed to report the $800,000 policy to his company. During sentencing at his trial, he pointed out that bridge officials had planned to cancel all policies in one more week after which his crime would have never been discovered. After the collapse of the Tacoma Narrows bridge, the governor of the State of Washington announced, "We are going to build the exact same bridge, exactly as before." Theodore Von Kármán, director of the Guggenheim Aeronautical Laboratory at the California Institute of Technology and a member of the board of inquiry into the collapse,[2] sent a telegram to the governor stating, "If you build the exact same bridge exactly as before, it will fall into the exact same river exactly as before."

The Design of the Narrows Bridge

The Tacoma Narrows bridge is a suspension bridge like the Golden Gate bridge. The road bed of the suspension bridge was hung from vertical cables attached to other cables strung between the two towers. If we think of the cables as long springs, it is tempting to model the oscillations of the road bed with an harmonic oscillator, and one's first guess as to the reason for the collapse of the bridge would be resonance.

However, the answer may not be so simple. If the collapse of the bridge was due to resonance, the forcing frequency of a forced harmonic oscillator must be close to its natural frequency, but the wind (the forcing) simply does not behave like this. One explanation might be the fact that cables are not true springs. They act more like rubber bands. Imagine a mass suspended by both a spring and a rubber band. The rubber band acts like a spring when it is stretched, but there is no restoring force if the oscillator is in a compressed position (Figure 4.37).[3]

[2]The Jet Propulsion Laboratory, managed by the California Institute of Technology for the National Aeronautics and Space Administration grew out of the Guggenheim Aeronautical Laboratory.

[3]For the arguments for and against resonance, consult the references at the end of this section.

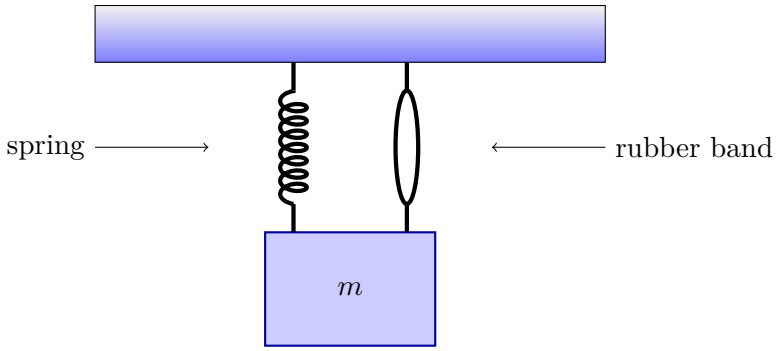

Figure 4.37: A spring-rubber band system

The Spring-Rubber Band Model

Suppose that $y(t)$ is vertical displacement of the mass the spring-rubber band model (Figure 4.37), where the displacement is positive in the downward direction and negative in the upward direction. We will also assume that we have a damped system. If this is the case, we can model the motion of the mass with the differential equation

$$my'' = -by' - k_1 y - k_2 y^+ + F(t), \tag{4.33}$$

where $y^+ = \max(y, 0)$. Here, m is the mass, b is the damping coefficient, k_1 is the spring constant, and $F(t)$ is the forcing term. The term $k_2 y^+$ corresponds to the rubber band in our system and can be explained as follows:

$$y^+(t) = \begin{cases} y(t), & \text{the rubber band is stretched,} \\ 0, & \text{the rubber band is slack or tight but not stretched.} \end{cases}$$

Thus, the rubber band acts as a spring with spring coefficient k_2, if the rubber band is stretched, but has no effect on the system if the rubber band is slack. Of course, the term my'' is just an application of Newton's Second law of Motion.

We can rewrite equation (4.33) in the more familiar form

$$my'' + by' + k_1 y + k_2 y^+ = F(t), \tag{4.34}$$

Our first observation is that this equation is a nonlinear due to the y^+ term. This may effect how sensitive the system is to initial conditions. A linear harmonic oscillator is not sensitive to initial conditions. For example, suppose we consider the harmonic oscillator

$$y'' + 2y' + 2y = \sin t$$

and examine the behavior of the solutions for different initial conditions. In Figure 4.38, we give the solutions for $y(0) = 0$ with $y'(0) = -4$, $y'(0) = 1$, and $y'(0) = 3$. Even though the transient part of the solution can differ a great deal depending on the initial conditions, all of the solutions eventually approach the steady-state solution.

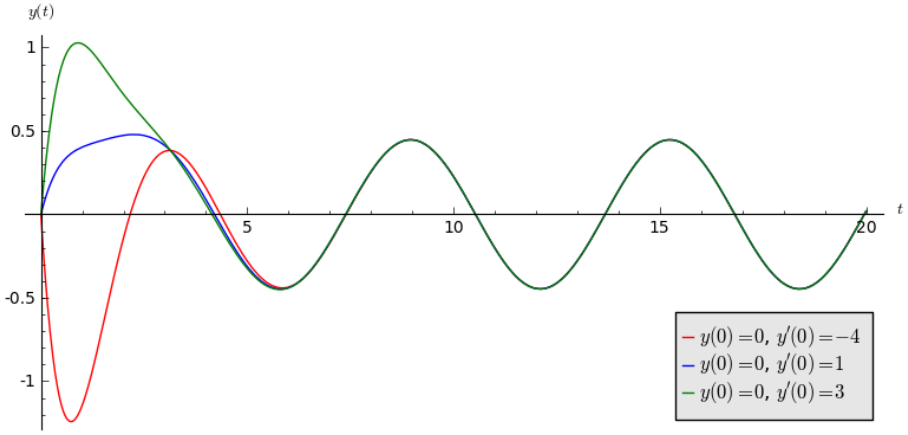

Figure 4.38: Solutions of $y'' + 2y' + 2y = \sin t$

Let us return to our spring-rubber band model and let us assume that $m = 1$. We can rewrite equation (4.34) as a system of first-order equations,

$$\begin{aligned} y' &= v \\ v' &= -k_1 y - k_2 y^+ - bv + F(t) \end{aligned}$$

or

$$\begin{aligned} x' &= v \\ v' &= -h(y) - bv + F(t), \end{aligned}$$

where

$$h(y) = \begin{cases} k_1 y & \text{if } y < 0 \\ (k_1 + k_2)y & \text{if } y \geq 0. \end{cases}$$

Let us assume that the forcing term is $F(t) = 10 + \lambda \sin \mu t$. If we keep λ small and assume that both $y(0)$ and $y'(0)$ are small, there will always be tension on the rubber band. Or in the case of the Tacoma Narrows Bridge, there will always be tension on the cables of the suspension bridge. However, a large initial velocity such as a gust of wind on the bridge, might cause the cables to go slack.

To see how this works in practice, we choose constants $\lambda = 0.1$, $\mu = 4$, $k_1 = 13$, $k_2 = 4$, and $b = 0.01$, the system becomes

$$\begin{aligned} y' &= v \\ v' &= -h(y) - 0.01v + 10 + 4 \sin 0.1t \end{aligned}$$

where

$$h(y) = \begin{cases} 13x & \text{if } x < 0 \\ 17x & \text{if } x \geq 0. \end{cases}$$

This is a nonautonomous (and nonlinear) system and can be very sensitive to initial conditions. If we find a numerical solution to this system with $y(0) = 0$ and $v(0) = 4.764$, we obtain low amplitude solutions (Figure 4.39). However, we only have to change the initial velocity of the system slightly to $v(0) = 4.765$ in order to obtain large amplitude solutions that do not die out over time.

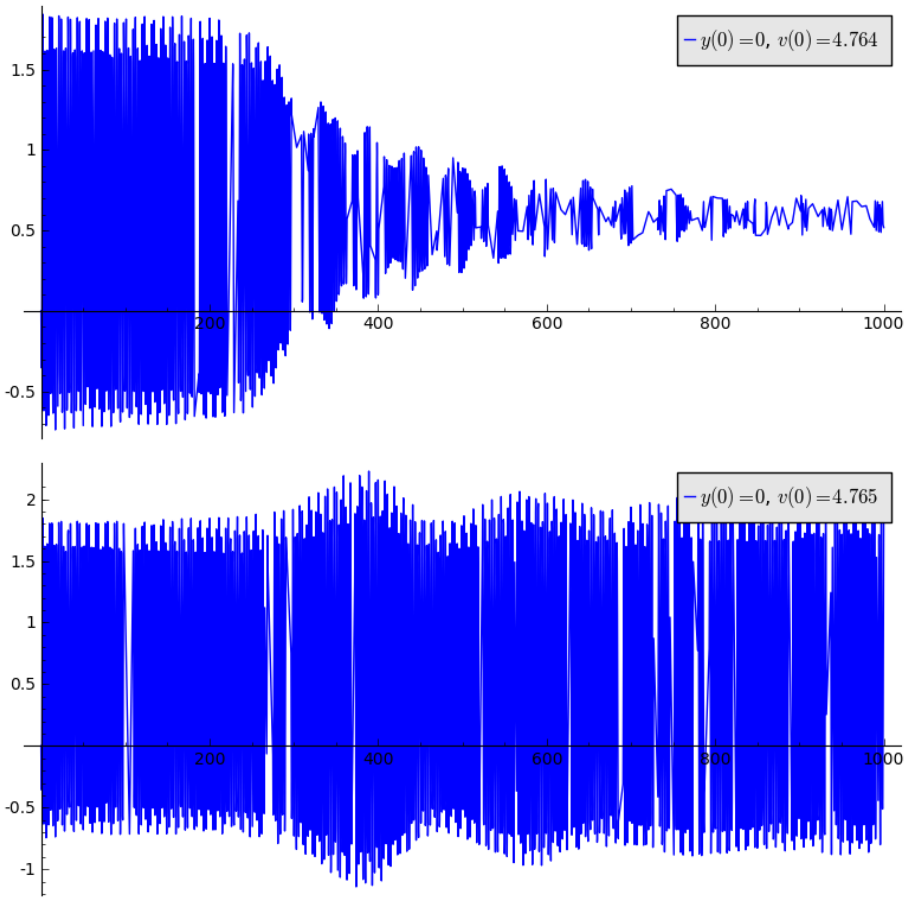

Figure 4.39: Initial conditions $y(0) = 0$ with $y'(0) = 4.764$ and $y'(0) = 4.765$

The Bridge Model

We can now model the bridge by

$$my'' + \alpha y' + \beta y + \gamma y^+ = -gm + F(t),$$

where

- y is the height of the road bed at the center of the bridge.
- m is the mass of the bridge.
- $\alpha y'$ is the damping term. We assume that α is small since suspension bridges are relatively flexible structures.
- βy is the force provided by the material of the bridge pulling the bridge back to $y = 0$.
- γy^+ is the force provided by the pull of the cables.
- $-gm$ is the force provided by gravity.
- $F(t)$ is the periodic force provided by the wind. Yes, wind does provide a periodic force. Just think of how a flag flutters in a wind.

The Model for Torsion

We have still not accounted for the large torsional oscillations that occurred on November 7, 1940, the day that the Tacoma Narrows bridge collapsed. If we look down the bridge (Figure 4.36), we see the road bed suspended from two cables—one on each side of the road. Thus, we can view a cross-section of the bridge as a rod suspended by cables or springs on each end. The bar is free to move vertically and rotate about its center of mass.

If a spring with spring constant k is extended by a distance y, the potential energy is $ky^2/2$. Let θ be the angle of the rod from the horizontal. If a rod of mass m and length $2l$ rotates about its center of gravity with angular velocity $d\theta/dt$, then its kinetic energy is given by[4]

$$\frac{1}{6}ml^2\left(\frac{d\theta}{dt}\right)^2.$$

The potential due to gravity is $-mgy$.

The extension of one spring is given by $(y-l\sin\theta)^+$ and $(y+l\sin\theta)^+$ in the other, where $y^+ = \max(y,0)$. We are making the assumption that the cables remain vertical. Since the cables are relatively long compared to the length of the roadbed, this is a reasonable assumption.

The total potential energy is given by

$$V = \left(\frac{k}{2}\right)\left(\left((y-l\sin\theta)^+\right)^2 + (y+l\sin\theta)^+\right)^2 - mgy,$$

and the total kinetic energy is given by

$$T = \frac{m}{2}\left(\frac{dy}{dt}\right)^2 + \frac{1}{6}ml^2\left(\frac{d\theta}{dt}\right)^2.$$

Let $L = T - V$ and put

$$\frac{d}{dt}\left(\frac{\delta L}{\delta\theta'}\right) = \frac{\delta L}{\delta\theta}$$

and

$$\frac{d}{dt}\left(\frac{\delta L}{\delta y'}\right) = \frac{\delta L}{\delta y}$$

where $\delta\theta'$ and $\delta y'$ are small damping terms. Thus, we obtain the equations

$$\begin{aligned}\frac{1}{3}ml^2\theta'' &= kl\cos\theta[(y-l\sin\theta)^+ - (y+l\sin\theta)^+]\\ y'' &= -k[(y-l\sin\theta)^+ + (y+l\sin\theta)^+] + mg\end{aligned}$$

Since the force exerted by the springs (cables) is not perpendicular to the rod but at an angle θ, $\cos\theta$ must appear in the first equation. Adding forcing and damping terms, we have

$$\begin{aligned}\theta'' &= -\delta\theta' + \frac{3k}{ml}\cos\theta[(y-l\sin\theta)^+ - (y+l\sin\theta)^+] + F(t)\\ y'' &= -\delta y' - \frac{k}{m}[(y-l\sin\theta)^+ + (y+l\sin\theta)^+] + g.\end{aligned}$$

If we assume that the cables never lose tension, then

$$(y-l\sin\theta)^+ = y - l\sin\theta$$

[4]Consult your local physicist.

$$(y + l\sin\theta)^+ = y + l\sin\theta$$

and we end up with the uncoupled equations

$$\theta'' = -\delta\theta' - \frac{6k}{m}\cos\theta\sin\theta + F(t)$$
$$y'' = -\delta y' - \frac{2k}{m}y + g.$$

The first equation models the torsional motion while the second models the vertical motion.

If the torsional oscillations are very small, then we can assume that

$$\sin\theta \approx \theta \qquad \text{and} \qquad \cos\theta \approx 1,$$

or

$$\theta'' = -\delta\theta' - \frac{6k}{m}\theta + F(t)$$
$$y'' = -\delta y' - \frac{2k}{m}y + g,$$

each of which is which is linear. However, the torsional oscillations for the Narrow bridge were as large as 23 degrees. The linear model

$$\theta'' = -0.01\theta' - (2.4)\theta + (1.2)\sin(0.06t)$$

and the nonlinear model

$$\theta'' = -0.01\theta' - (2.4)\cos\theta\sin\theta + (1.2)\sin(0.06t)$$

behave very differently under the initial conditions $\theta(0) = 1.2$ and $\theta'(0) = 0$. See Figures 4.40 and 4.41

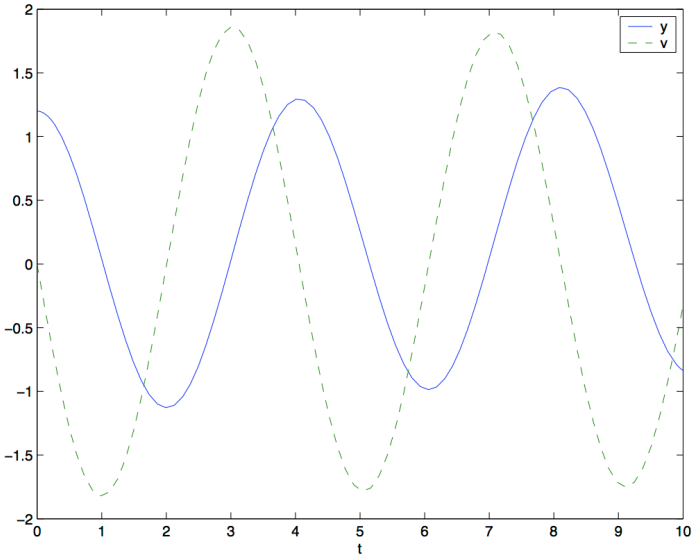

Figure 4.40: Solution of the linear model for torsion

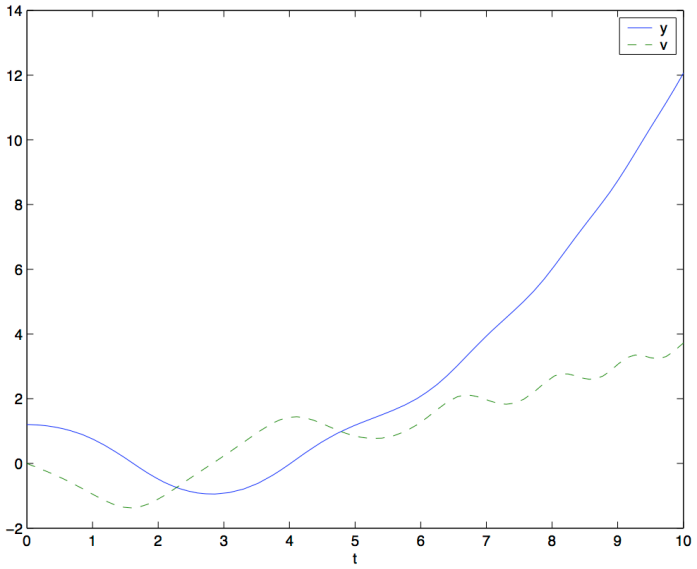

Figure 4.41: Solution of the nonlinear model for torsion

5

Nonlinear Systems

5.1 Linearization

In Chapter 3, we explored how to solve linear systems. However, if we are asked to solve a nonlinear system such as

$$\frac{dx}{dt} = x - 3y + xy^2$$
$$\frac{dy}{dt} = 2x - 4y - x^2y,$$

we are faced with a much more difficult task. In general, it may not be possible to find solutions for a nonlinear system in terms of elementary functions. However, for a given modeling problem, we can ask many questions that may be answered without finding an explicit solution for the associated system of differential equations. For example, what are the equilibrium points? Are the equilibrium points stable? Do closed solution curves exist in the phase plane?

5.1.1 Equilibrium Solutions

One of the most useful methods of determining the nature of an equilibrium solution for a given nonlinear system is to approximate the nonlinear system with a linear system. More specifically, an equilibrium solution occurs for the linear system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

when an x-nullcline intersects a y-nullcline. That is, when a curve defined by $dx/dt = f(x, y) = 0$ intersects a curve defined $dy/dt = g(x, y) = 0$, we have an equilibrium solution. Since the x and y-nullclines are simply curves in the xy-plane, we can approximate them locally by intersecting straight lines. Translating the pair of intersection lines to the origin, we obtain a linear system, and we can apply everything that we learned about such systems in Chapter 3.

An ***equilibrium solution*** for a first-order system of differential equations

$$x'(t) = f(x, y)$$
$$y'(t) = g(x, y)$$

is a point (x_0, y_0) such that

$$f(x_0, y_0) = g(x_0, y_0) = 0.$$

Notice that neither x nor y is changing at this point. If we have the initial conditions, $x(0) = x_0$ and $y(0) = y_0$, then the solution to the system is $x(t) = x_0$ and $y(t) = y_0$. Of course the interesting question is what happens if our initial conditions are close to an equilibrium solution. Do solutions tend toward the equilibrium solution, away from the equilibrium solution, or is there a combination of the two?

One of the most useful methods of determining the nature of an equilibrium solution for a given nonlinear system is to approximate the nonlinear system with a linear system. More specifically, an equilibrium solution occurs for the linear system

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned}$$

when an x-nullcline intersects a y-nullcline. That is, when a curve defined by $dx/dt = f(x, y) = 0$ intersects a curve defined $dy/dt = g(x, y) = 0$, we have an equilibrium solution. Since the x and y-nullclines are simply curves in the xy-plane, we can approximate them locally by intersecting straight lines. Translating the pair of intersection lines to the origin, we obtain a linear system, and we can apply everything that we learned about such systems in Chapter 3.

Example 5.1. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= x - 3y + xy^2 \\ \frac{dy}{dt} &= 2x - 4y - x^2 y. \end{aligned}$$

From

$$\begin{aligned} \frac{dx}{dt} &= x - 3y + xy^2 = 0 \\ \frac{dy}{dt} &= 2x - 4y - x^2 y = 0, \end{aligned}$$

we can quickly conclude that the only equilibrium solution to the system is $(0, 0)$. The phase portrait for this system is given in Figure 5.2. If we have the initial conditions $x(0) = 1$ and $y(0) = 1$, we can see that the solution tends toward the origin as $t \to \infty$. However, it is unclear from the phase portrait if the solution curves of all initial value problems with initial conditions near the origin tend towards the equilibrium solution as $t \to \infty$.

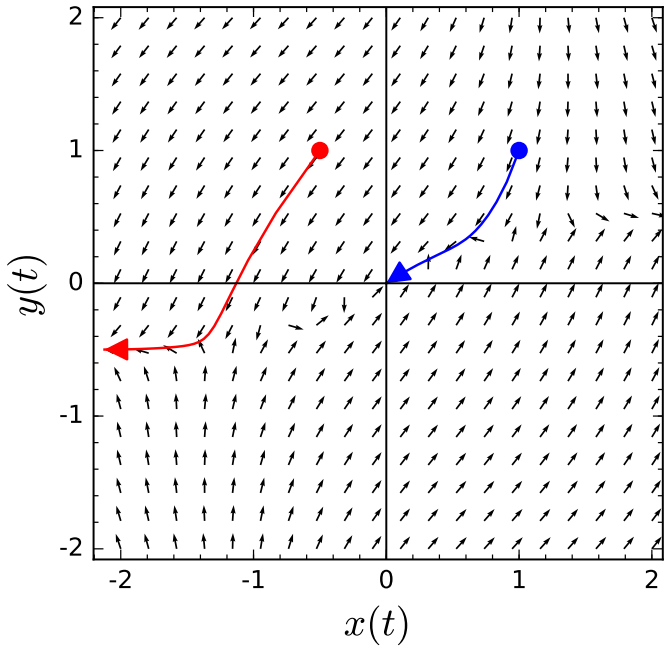

Figure 5.2: Phase Portrait for $x' = x - 3y + xy^2$ and $2x - 4y - x^2y$

Since the nonlinear terms of the system do not contribute much towards dx/dt and dy/dt for values of x and y near zero, we can determining the nature of the equilibrium solution by examining the system consisting of only linear terms on the right-hand side of the equation,

$$\frac{dx}{dt} = x - 3y$$
$$\frac{dy}{dt} = 2x - 4y.$$

The matrix for this linear system,

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -4 \end{pmatrix},$$

has eigenvalues $\lambda = -1$ and $\mu = -2$ with eigenvectors $\mathbf{u} = (3, 2)$ and $\mathbf{v} = (1, 1)$, respectively. The general solution for this linear system is

$$x(t) = 3c_1e^{-t} + c_2e^{-2t}$$
$$y(t) = 2c_1e^{-t} + c_2e^{-2t}.$$

This indeed suggests that solutions near the origin tend towards the origin as $t \to \infty$. In this case, we say that the equilibrium solution is ***stable***. Of course, if we are given an initial condition such as $x(0) = -0.5$ and $y(0) = 1$.

Example 5.3 (A Competing Species Model)**.** Suppose that x and y are the population of two distinct species that compete for the same resources. For example, two species of fish may compete for the same food in a lake or sheep and cattle competing for the same grazing land. Recall from Section 2.2 that we can model two competing species using the following system of first-order differential equations,

$$\frac{dx}{dt} = \alpha x \left(1 - \frac{x}{M}\right) - \beta xy$$

$$\frac{dy}{dt} = \gamma y\left(1 - \frac{y}{N}\right) - \delta xy.$$

The first term in each equation is the logistic growth of each species. The second term in each equation tells how each species is affected by interacting with the competing species.

More specifically, let us consider the following system,

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy \tag{5.1}$$

$$\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy. \tag{5.2}$$

It is easy to see that the four equilibrium solutions: $(0,0)$, $(0,3)$, $(2,0)$, and $(1,1)$. We can view the direction field, the phase plane, and some solution curves for this system in Figure 5.4.

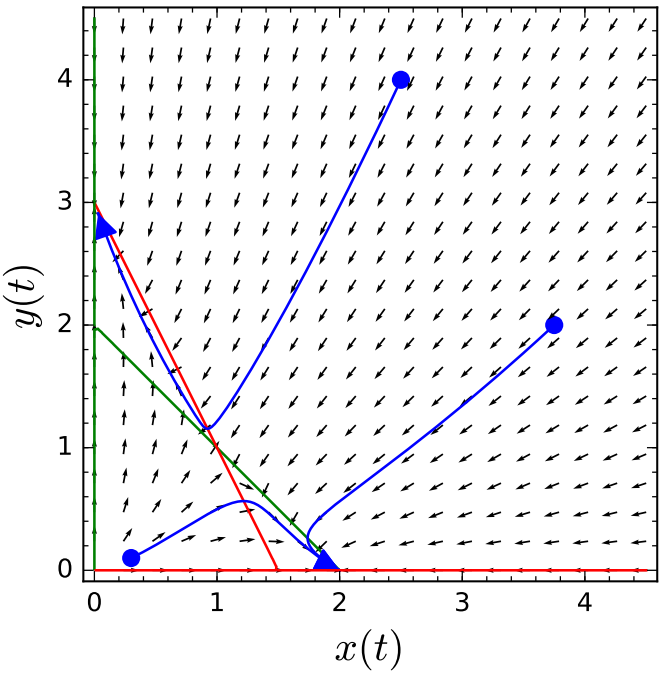

Figure 5.4: A Competing Species Model

Let us analyze what happens at the equilibrium solution $(1,1)$. If we decide on an appropriate change of variables, we can translate the entire system so that this equilibrium solution is at the origin. If we let

$$u = x - 1$$
$$v = y - 1,$$

then

$$\frac{du}{dt} = \frac{d}{dt}(x-1) = \frac{dx}{dt},$$
$$\frac{dv}{dt} = \frac{d}{dt}(y-1) = \frac{dy}{dt}.$$

Equations (5.1) and (5.2) now become

$$\frac{du}{dt} = -u - v - u^2 - uv, \tag{5.3}$$

$$\frac{dv}{dt} = -2u - v - 2uv - v^2. \tag{5.4}$$

As before we consider only the linear part of equations (5.4) and (5.4) are

$$\frac{du}{dt} = -u - v, \tag{5.5}$$
$$\frac{dv}{dt} = -2u - v. \tag{5.6}$$

The idea is that the linear part is a good ***local*** approximation to the original equations much like a tangent line is a good local approximation to a smooth function in calculus. We can determine the local nature of the equilibrium solution by examining the eigenvalues of the matrix

$$A = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix}.$$

The eigenvalues of A are $\lambda = -1 + \sqrt{2} = 0.4142$ and $\mu = -1 - \sqrt{2} \approx -2.4142$ with eigenvectors $\mathbf{u} = (1, \sqrt{2}\,) \approx (1, 1.4142)$ and $\mathbf{v} = (1, -\sqrt{2}\,) \approx (1, -1.4142)$, respectively. The solution to the linear system is now

$$\mathbf{x}(t) = c_1 e^{\lambda t} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + c_2 e^{\mu t} \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}.$$

or

$$x(t) = c_1 e^{\lambda t} + c_2 e^{\mu t} \tag{5.7}$$
$$y(t) = c_1 \sqrt{2} e^{\lambda t} - c_2 \sqrt{2} e^{\mu t}. \tag{5.8}$$

If we have initial conditions $x(0) = 3$ and $y(0) = 4$, one can quickly determine that $c_1 = 3/2 + \sqrt{2} \approx 2.9142$ and $c_2 = 3/2 - \sqrt{2} \approx 0.0859$, and equations (5.7) and (5.8) become

$$x(t) = 2.9142e^{0.4142t} + 0.0859e^{-2.4142t} \tag{5.9}$$
$$y(t) = 4.1213e^{0.4142t} - 0.1213e^{-2.4142t}. \tag{5.10}$$

As $t \to \infty$ notice that both $x(t)$ and $y(t)$ become very large and tend away from the origin.

We can conclude that the equilibrium solution $(1, 1)$ is ***unstable***. That is, tend away from the equilibrium solution as $t \to \infty$ if one population begins with a slight advantage over the other. If neither popluation has an initial advantage over the other, then the solution curve will approach the equilibrium solution as $t \to \infty$.

Example 5.5 (A Nonpolynomial Example)**.** If systems such as the following can be approximated by linear systems,

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -y - \sin x.$$

Certainly, this sytems has an equilibrium solution at $(0, 0)$. We can expand $\sin x$ into a power series,

$$\frac{dy}{dt} = -y - \sin x = -y - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right).$$

Thus, this system can be approximated by the linear system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -y - x.$$

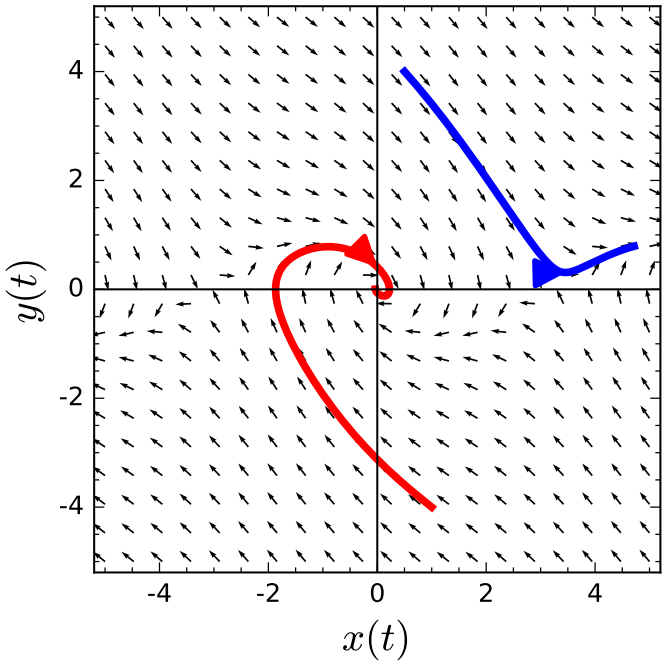

Figure 5.6: Phase portrait of $x' = y$ and $y' = -y - \sin x$

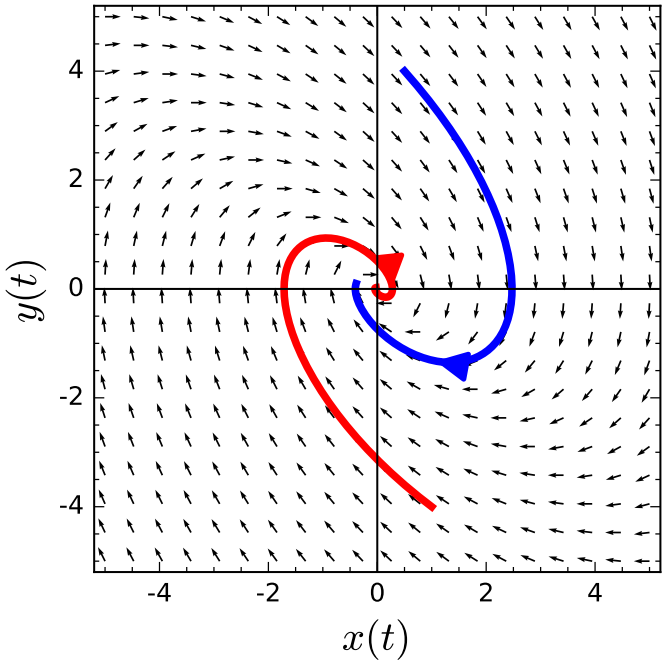

Figure 5.7: Phase portrait of $x' = y$ and $y' = -y - x$

Example 5.8. For example, consider the system

$$\frac{dx}{dt} = x^2 - 4x - y + 4 \tag{5.11}$$

$$\frac{dy}{dt} = x^3 - y. \tag{5.12}$$

The x and y-nullclines of this system are the curves $y = x^2 - 4x + 4$ and $y = x^3$, respectively. Since the two nullclines intersect only at $(1, 1)$, we have a single equilibrium solution. From the phase plane, it appears that $(1, 1)$ is a stable equilibrium solution. That is, all solution curves starting sufficiently close to $(1, 1)$ will approach the equilibrium solution as $t \to \infty$ (Figure 5.9).

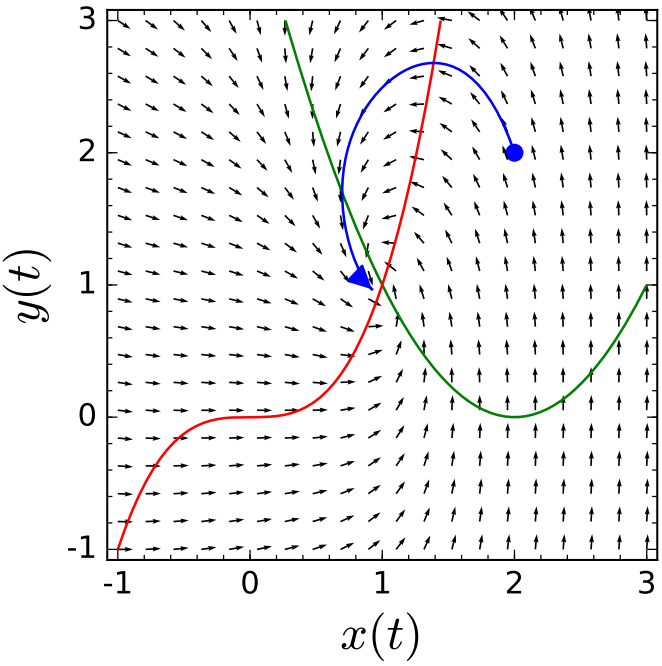

Figure 5.9: Phase portrait for the system (5.11)–(5.12)

Making the substitution $u = x - 1$ and $v = y - 1$, simply translates the entire system to the origin, and we obtain the system

$$\frac{du}{dt} = u(u - 2) - v \tag{5.13}$$

$$\frac{dv}{dt} u(u^2 + 3u + 3) - v. \tag{5.14}$$

Our new system has u and v-nullclines $v = u(u - 2)$ and $v = u(u^2 + 3u + 3)$, respectively. Notice that we have simply moved the phase portrait of the original system so that our equilibrium solution is now at the origin. Furthermore, we can approximate the u and v-nullclines by their tangent lines $v = -2u$ and $v = 3u$, respectively. From Figure 5.9, it appears that we are approximating our original system with a linear system.

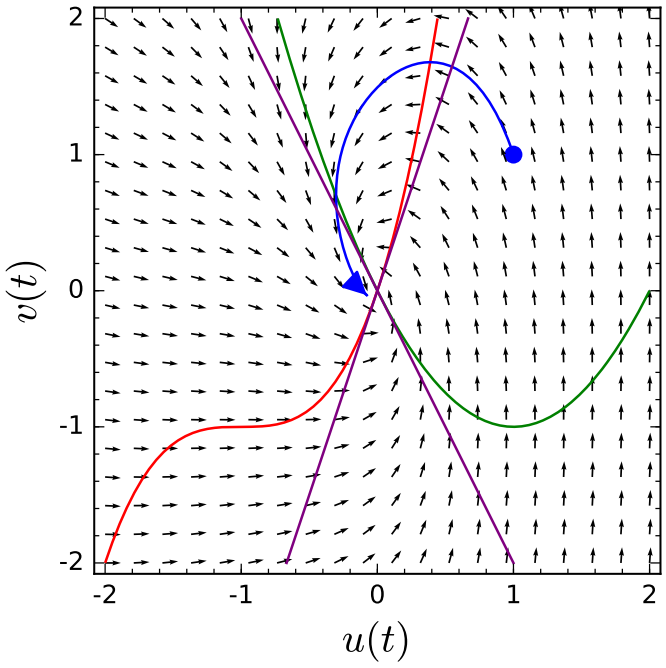

Figure 5.10: Phase portrait for the system (5.13)–(5.14)

To determine the general case, we must approximate a nonlinear system

$$x'(t) = f(x, y) \tag{5.15}$$
$$y'(t) = g(x, y) \tag{5.16}$$

near each equilibrium point (x_0, y_0) with a linear system. The idea is to move the equilibrium solution to the origin with a change of coordinates and then approximate the nonlinear system with a linear system. In order to move the equilibrium solution to the origin, we will introduce new variables

$$u = x - x_0$$
$$v = y - y_0.$$

If (x, y) is close to the equilibrium solution (x_0, y_0), then (u, v) will be close to the origin. Under this change of coordinates,

$$\frac{du}{dt} = \frac{d}{dt}(x - x_0) = \frac{dx}{dt} = f(x, y) = f(x_0 + u, y_0 + v).$$

Similarly, $dv/dt = dy/dt = g(x_0 + u, y_0 + v)$.

In order to approximate the nonlinear system with a linear system, we will use a Taylor series in two variables. That is, we can write

$$\begin{aligned} f(x_0 + u, y_0 + v) = f(x_0, y_0) &+ f_x(x_0, y_0)u + f_y(x_0, y_0)v \\ &+ \frac{1}{2}f_{xx}(x_0, y_0)u^2 + f_{xy}(x_0, y_0)uv + \frac{1}{2}f_{yy}(x_0, y_0)v^2 + \cdots . \end{aligned}$$

If we only use the linear terms of the Taylor series, we obtain a fairly accurate approximation of f provided we are near the equilibrium solution. Of course, f and its linear approximation

may be quite different for values far away from the equilibrium solution. The linearization of the system of equations (5.15)–(5.16) now becomes

$$\begin{aligned}\frac{du}{dt} &= f(x_0, y_0) + f_x(x_0, y_0)u + f_y(x_0, y_0)v \\ \frac{dv}{dt} &= g(x_0, y_0) + g_x(x_0, y_0)u + g_y(x_0, y_0)v.\end{aligned}$$

Since (x_0, y_0) is an equilibrium solution, the constant terms vanish in each equation and

$$\begin{aligned}\frac{du}{dt} &= f_x(x_0, y_0)u + f_y(x_0, y_0)v \\ \frac{dv}{dt} &= g_x(x_0, y_0)u + g_y(x_0, y_0)v.\end{aligned}$$

If we write our linearization in matrix form, we have

$$\begin{pmatrix} du/dt \\ dv/dt \end{pmatrix} = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

The matrix

$$J = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix}$$

is the ***Jacobian matrix*** of the system. We can now classify equilibrium solutions of nonlinear systems by examining the eigenvalues of the Jacobian matrix of the system or by using the trace-determinant plane. For example, if $D = \det(J) > 0$ and $T = \operatorname{tr}(J) < 0$, we have a sink. We can determine the sink to be spiral or nodal by examining whether (T, D) lies above or below the parabola $T^2 = 4D$ in the trace-determinant plane.

It is important to note that linearization only tells us the local story. A solution curve might behave quite differently if it is far away from the equilibrium solution.

Example 5.11. In Example 5.8, we considered the system

$$\begin{aligned}\frac{dx}{dt} &= x^2 - 4x - y + 4 \\ \frac{dy}{dt} &= x^3 - y.\end{aligned}$$

The Jacobian matrix of this system is

$$J = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix} = \begin{pmatrix} 2x_0 - 4 & -1 \\ 3x_0^2 & -1 \end{pmatrix}$$

For the equilibrium solution $(1, 1)$,

$$J = \begin{pmatrix} -2 & -1 \\ 3 & -1 \end{pmatrix}.$$

Since J has eigenvalues $\lambda = (-3 \pm i\sqrt{11})/2$, the equilibrium solution will act as a spiral sink for initial values close to $(1, 1)$.

5.1.2 When Linearization Fails

There are at least two cases when linearization does not give us the information that we seek. First, it might well be the case that the linear terms vanish in the nonlinear system. For example, the system

$$\begin{aligned}\frac{dx}{dt} &= xy \\ \frac{dy}{dt} &= -x^2 + xy\end{aligned}$$

has an equilibrium solution at the origin, but this linearization of this system vanishes.

A more subtle example is the system

$$\begin{aligned}\frac{dx}{dt} &= y - (x^2 + y^2)x \\ \frac{dy}{dt} &= -x - (x^2 + y^2)y.\end{aligned}$$

The origin is an equilibrium solution, and the linearization of this system is

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x.\end{aligned}$$

The eigenvalue of the matrix corresponding to the linear system,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

are $\lambda = \pm i$. Thus, the solution curves of the linear system are simply circles about the origin. However, the nonlinear system acts quite differently. In the nonlinear system, solutions spiral out slowly from the origin (Figure 5.12). This system has no periodic solutions.

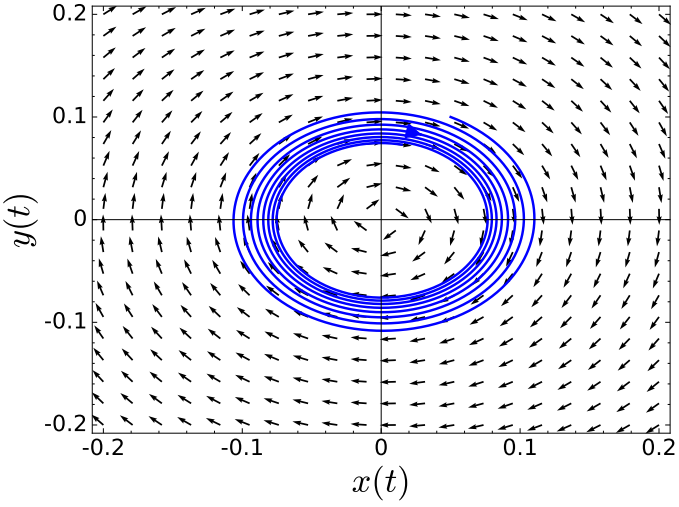

Figure 5.12: When linearization fails

5.1.3 Important Lessons

- We can approximate a nonlinear system

$$x'(t) = f(x, y)$$
$$y'(t) = g(x, y)$$

near each equilibrium point (x_0, y_0) with a linear system by using a Taylor series approximation for f and g. The matrix of our linear approximation,

$$J = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix},$$

is the ***Jacobian matrix*** of the system. We can classify nonlinear systems by examining the Jacobian matrix of the system and using the trace-determinant plane.

- Linearization only tells us how solutions behave near the equilibrium point. A solution curve might behave quite differently if it is far away from the equilibrium solution.

- In some cases, linearization can fail.

5.1.4 Exercises

1. Consider the following three systems

(a)

$$x' = 3\sin x + y$$
$$y' = 4x + \cos y - 1$$

(b)

$$x' = -3\sin x + y$$
$$y' = 4x + \cos y - 1$$

(c)

$$x' = -3\sin x + y$$
$$y' = 4x + 3\cos y - 3$$

All three systems have an equilibrium solution at $(0, 0)$. Which two systems have phase portraits with the same "local picture" near $(0, 0)$? Justify your answer.

Hint.

2. Let us consider an epidemic model for a city. We make the following additional assumptions about our model.

- The city has a constant birth rate rate of α persons per day. All new born babies are susceptible to the disease.
- Infected people either recover or die after a certain number of days. If an individual recovers, he or she is immune.

If we let $S(t)$ be the number of susceptible individuals at time t and $I(t)$ be the number of infected individuals at time t, our assumptions give rise to the following system of differential equations,

$$\frac{dS}{dt} = -\alpha SI + \beta$$
$$\frac{dI}{dt} = -\gamma I + \alpha SI.$$

The constant α is determined by the probability of an interaction between a susceptible individual and an infected individual, and γ is the rate at which infected individuals are removed from the population. If

$$\frac{dS}{dt} = -\alpha SI + \beta = 0$$
$$\frac{dI}{dt} = -\gamma I + \alpha SI = 0,$$

then both the susceptible and infected populations do not change. This will occur at

$$S_0 = \frac{\gamma}{\alpha}$$
$$I_0 = \frac{\beta}{\gamma}.$$

We are interested in the behavior of solutions near (S_0, I_0). If solutions approach this equilbrium point, then the disease will become endemic to the population.

Hint.

5.1.5 Project—Modeling the Nervous System

Hodgkin and Huxley developed a model for the firing of nerve cells in the 1950s. The model was a system of four differential equations that described the electrochemical transmission of neuronal signals along cell membranes in giant squid. The system itself was very difficult to analyze due to its highly nonlinear nature, but it did earn the two scientists a Nobel Prize for medicine in 1963 (the year after Watson and Crick won their prize).

Fitzhugh and Nagumo managed to simplify the Hodgkin-Huxley model to two equations

$$x' = y + x - x^3/3 + I$$
$$y' = -x + a - by,$$

where a and b are constants satisfying a certain condition and I is a parameter. In these equations, x represents the excitability of the system and y represents a combination of other forces that return the system to rest. Similar models have been constructed to model the systolic and diastolic movements of the heart.

We will now consider the system

$$\frac{dx}{dt} = x + y - x^3$$
$$\frac{dy}{dt} = -0.5x,$$

which has been used to study nerve cells and is similar to the Fitzhugh-Nagumo model. The x-nullcline for this system is $y = -x + x^3$. The x-component of the direction field

above this nullcline is positive. Below the nullcline the x-component of the direction field is negative.

The y-nullcline of the system is the y-axis. To the right of the y-nullcline, we have $dy/dt < 0$, and $dy/dt > 0$ to the left of the y-nullcline. Thus, the x and y-nullclines divide the plane into four regions and all solutions must circulate in a clockwise manner about the origin (Figure 5.13).

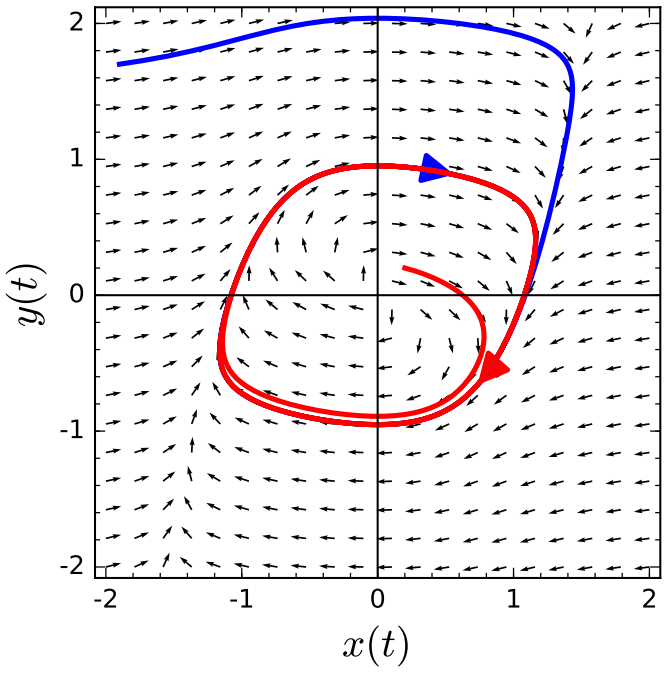

Figure 5.13: Modeling nerve cells

The origin is the only equilibrium solution of the system; hence, we can examine the linearization

$$\begin{aligned} \frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= -0.5x, \end{aligned}$$

to determine the nature of the equilibrium solution. Since the eigenvalues of this linear system are $(1 \pm i)/2$, we can conclude that the origin is a spiral source. A solution that passes close to the origin tends to spiral outward; however, solutions that are far from the origin spiral inward. We shall later see that we must have a closed periodic solution to this system by the Poincaré-Bendixson Theorem.

5.2 Hamiltonian Systems

An undamped harmonic oscillator, $my'' + ky = 0$, can be written as the system

$$\begin{aligned} y' &= v \\ v' &= -\frac{k}{m}y. \end{aligned}$$

Now suppose that $(y(t), v(t))$ is a solution curve in the yv-plane. We will calculate the slope of the solution curve, dv/dy. Using the fact from calculus that the derivative of an inverse function is

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(x)},$$

we have

$$\frac{dv}{dy} = \frac{dv}{dt}\cdot\frac{dt}{dy} = \frac{dv/dt}{dy/dt} = -\frac{ky}{mv}.$$

In general for the system

$$\begin{aligned}\frac{dx}{dt} &= f(x,y)\\ \frac{dy}{dt} &= g(x,y),\end{aligned}$$

we have

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}.$$

Using separation of variables to solve the equation

$$\frac{dv}{dy} = -\frac{ky}{mv},$$

we obtain the solution

$$\frac{1}{2}mv^2 + \frac{1}{2}ky^2 = C, \tag{5.17}$$

where C is a constant. The first term of (5.17) is the ***kinetic energy function*** of the harmonic oscillator

$$K = \frac{1}{2}mv^2,$$

while the second term is the ***potential energy function***

$$U = \int_0^y ks\,ds = \frac{1}{2}ky^2.$$

For this reason, we call the function

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}ky^2$$

the ***total energy function*** of the harmonic oscillator. Equation (5.17) tells us that energy is conserved. That is, the sum of the potential energy and the kinetic energy is constant.

5.2.1 The Nonlinear Pendulum

While pendulums have long been used in clocks to keep time, they have also been used to measure gravity as well as used in early seismometers to measure the effect of earthquakes. One of the more interesting uses of a pendulum has been to measure the rotation of the earth. In 1851, the French physicist, Léon Foucault, used a pendulum (Figure 5.14) to demonstrate that the earth actually rotated on its axis. The fact that the earth rotates had been known for a long time, but Foucault's experiment give the first simple proof of this phenomena.

Figure 5.14: Foucault Pendulum Clock—California Academy of Sciences

Let us consider a pendulum made of a light rod of length L called the ***arm*** of the pendulum with a mass $m$on the end of the rod called the ***bob*** (Figure 5.15). We will ignore the mass of the arm in our system. The position of the bob is given by $\theta(t)$, which we will measure in a counterclockwise direction. We will assume that $\theta(0)$ is when the bob is in the vertical position.

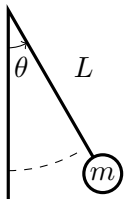

Figure 5.15: The nonlinear pendulum

There are two forces acting on the pendulum—gravity and friction. The position of the bob at time t is

$$(L\sin\theta(t), -L\cos\theta(t)),$$

and the velocity of the bob is $L(d\theta/dt)$, the length of the velocity vector (Figure 5.16) The component of the acceleration that points along the direction of motion of the bob is

$$L\frac{d^2\theta}{dt^2}.$$

We can take the force due to friction to be proportional to the velocity,

$$-bL\frac{d\theta}{dt},$$

where $b \geq 0$. Thus, Newton's second law tells us that

$$mL\frac{d^2\theta}{dt^2} = -bL\frac{d\theta}{dt} - mg\sin\theta$$

or

$$\frac{d^2\theta}{dt^2} + \frac{b}{m}\frac{d\theta}{dt} + \frac{g}{L}\sin\theta = 0.$$

As a system, we can model the pendulum as

$$\begin{aligned}\frac{d\theta}{dt} &= v\\ \frac{dv}{dt} &= -\frac{b}{m}v - \frac{g}{L}\sin\theta.\end{aligned}$$

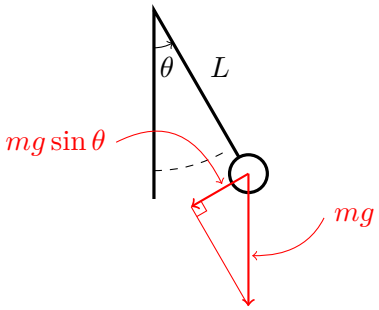

Figure 5.16: Forces on the nonlinear pendulum

If there is no damping for our pendulum, we say that we have an ***ideal pendulum***. In this case,

$$\begin{aligned}\frac{d\theta}{dt} &= v\\ \frac{dv}{dt} &= -\frac{g}{L}\sin\theta.\end{aligned}$$

Since

$$\frac{dv}{d\theta} = -\frac{g\sin\theta}{Lv},$$

we know that the energy function for the pendulum is

$$E(\theta, v) = \frac{1}{2}v^2 - \frac{g}{L}\cos\theta.$$

Of course, $E(\theta, v)$ is constant on the ideal pendulum.

5.2.2 Hamiltonian Systems

The ideal pendulum and the undamped harmonic oscillator are examples of Hamiltonian systems. More specifically, a ***Hamiltonian system*** is a system of the form

$$\begin{aligned}\frac{dx}{dt} &= \frac{\partial H}{\partial y}(x, y)\\ \frac{dy}{dt} &= -\frac{\partial H}{\partial x}(x, y),\end{aligned}$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function.

Example 5.17. The system

$$\begin{aligned}\frac{dx}{dt} &= -2x - 3y^2\\ \frac{dy}{dt} &= -3x^2 + 2y\end{aligned}$$

is Hamiltonian since for $H(x,y) = x^3 - 2xy - y^3$

$$\frac{\partial H}{\partial y} = -2x - 3y^2$$
$$\frac{\partial H}{\partial x} = -(-3x^2 + 2y).$$

Example 5.18. In the case of the harmonic oscillator and ideal pendulum, H is just the energy function. For the harmonic oscillator, the Hamiltonian function is

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}ky^2.$$

In this case,

$$\frac{\partial E}{\partial v} = y' = mv$$
$$-\frac{\partial E}{\partial y} = v' = -ky.$$

A change of variables ($y = my$) tells us that this system is equivalent to the system

$$y' = v$$
$$v' = -\frac{k}{m}y.$$

For the ideal pendulum

$$\frac{\partial E}{\partial v} = \frac{d\theta}{dt} = v$$
$$-\frac{\partial E}{\partial \theta} = \frac{dv}{dt} = -\frac{g}{L}\sin\theta,$$

where

$$E(\theta, v) = \frac{1}{2}v^2 - \frac{g}{L}\cos\theta.$$

The following theorem tells the the importance of Hamiltonian systems. That is, the solution curves of the system are simply the level sets of the Hamiltonian function.

Theorem 5.19. *Let*

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}(x,y)$$
$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}(x,y),$$

be a Hamiltonian system, where $H : \mathbb{R}^2 \to \mathbb{R}$. Then H is constant along any solution curve.

Proof. Let $(x(t), y(t))$ be a solution curve for the system. Using the chain rule

$$\begin{aligned}\frac{dH}{dt} &= \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} \\ &= \frac{\partial H}{\partial x}\left(\frac{\partial H}{\partial y}\right) + \frac{\partial H}{\partial y}\left(-\frac{\partial H}{\partial x}\right) \\ &= 0.\end{aligned}$$

□

Theorem 5.19 tells us how to draw the solution curves in the phase plane without solving the system. Assuming that the Hamiltonian function H is not constant on any open set in $\mathbb{R}^2$, we simply need to plot the level curves, $H(x, y) = C$. The solutions of the system live on these level sets, and all we need to do is find the direction of the solution curve. However, this is quite easy since we know the vector field of the system. Furthermore, the equilibrium points of the Hamiltonian system occur at the critical points of H (where the partials of H vanish). For example, we can see the solution curves of

$$\begin{aligned} \frac{\partial H}{\partial y} &= -2x - 3y^2 \\ \frac{\partial H}{\partial x} &= -(-3x^2 + 2y). \end{aligned}$$

in Figure 5.20.

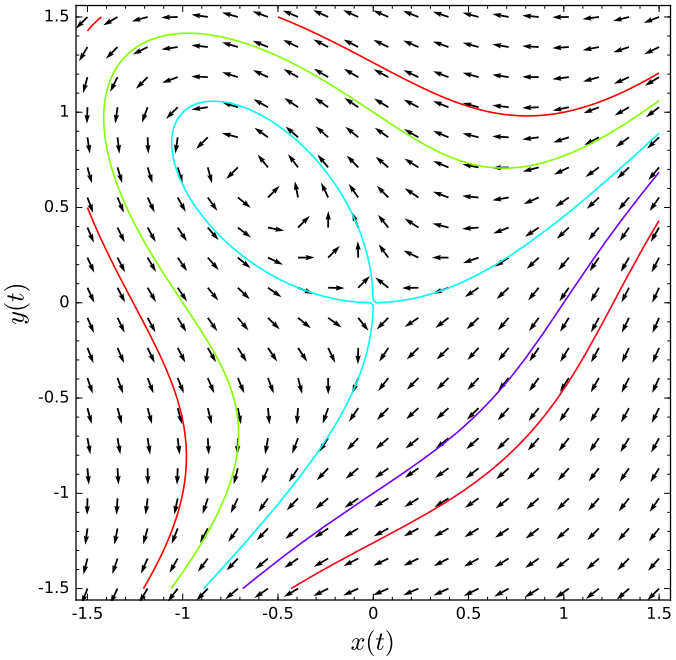

Figure 5.20: Solution curves of $x' = x - 3y^2$ and $y' = -y$

5.2.3 The Ideal Pendulum Revisited

The ideal pendulum

$$\begin{aligned} \frac{d\theta}{dt} &= v \\ \frac{dv}{dt} &= -\frac{g}{L}\sin\theta. \end{aligned}$$

is Hamiltonian, since the total energy function

$$H(\theta, v) = E(\theta, v) = \frac{1}{2}v^2 - \frac{g}{L}\cos\theta. \tag{5.18}$$

is the Hamiltonian function for the system. If we set $g/L = 1$, then the solution curves of the system are just the level curves of (5.18). In Figure 5.21, the closed ellipses correspond to the normal motion of a pendulum, while cosine curves correspond to a pendulum that

always rotates in the same direction. The curves that join the equilibrium points correspond to the pendulum that rotates exactly to the top of the arc and then rotates back in the other direction.

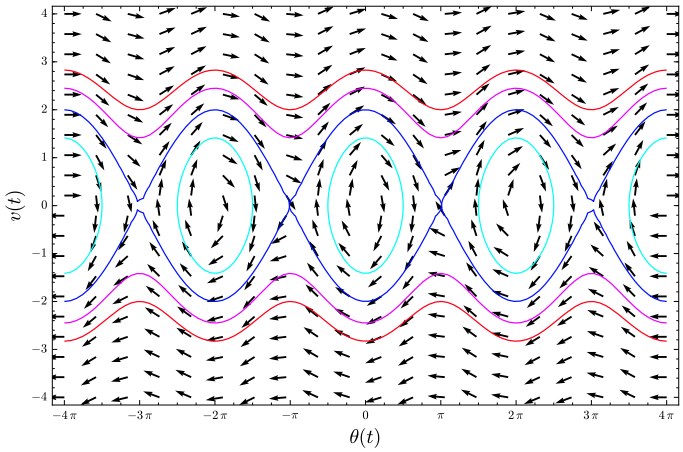

Figure 5.21: The ideal pendulum

Hamiltonian systems are rather rare. Given a system, we need a test to see if it is indeed Hamiltonian. For the system

$$\begin{aligned}\frac{dx}{dt} &= f(x,y)\\ \frac{dy}{dt} &= g(x,y),\end{aligned}$$

we wish to find a function $H(x,y)$ such that

$$\begin{aligned}\frac{\partial H}{\partial y} &= f(x,y)\\ \frac{\partial H}{\partial x} &= -g(x,y).\end{aligned}$$

If such a function exists, then

$$\frac{\partial f}{\partial x} = \frac{\partial^2 H}{\partial x \partial y} = \frac{\partial^2 H}{\partial y \partial x} = -\frac{\partial g}{\partial y}.$$

Thus,

$$\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y},$$

if our system is Hamiltonian.

Example 5.22. We already know that the system

$$\begin{aligned}\frac{dx}{dt} &= -2x - 3y^2\\ \frac{dy}{dt} &= -3x^2 + 2y\end{aligned}$$

is Hamiltonian. Indeed,

$$\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y} = -2.$$

The condition that

$$\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y},$$

is also sufficient for

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y), \end{aligned}$$

to be a Hamiltonian system. That is, we can construct a Hamiltonian function if $\partial f/\partial x = -\partial g/\partial y$. If

$$f(x,y) = \frac{\partial H}{\partial y}(x,y),$$

then

$$H(x,y) = \int f(x,y)\,dy + \phi(x).$$

Differentiating H with respect to x, tells us that

$$\frac{\partial H}{\partial x}(x,y) = \frac{\partial}{\partial x}\int f(x,y)\,dy + \phi'(x) = -g(x,y).$$

Now we can determine $\phi(x)$ by solving the first-order differential equation

$$\phi'(x) = -g(x,y) - \frac{\partial}{\partial x}\int f(x,y)\,dy.$$

We summarize our results in the following theorem.

Theorem 5.23. *Let*

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y), \end{aligned}$$

where f and g are continuously differentiable.

1. *If the system is Hamiltonian, then*

$$\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}. \tag{5.19}$$

2. *If equation (5.19) is true on a rectangle R, then the system is Hamiltonian on R and*

$$H(x,y) = \int f(x,y)\,dy + \phi(x),$$

where

$$\phi'(x) = -\frac{\partial}{\partial x}\left(\int f(x,y)\,dy\right) - g(x,y).$$

Example 5.24. As an example, we will show how we constructed the Hamiltonian function given in Example 5.17 for the system

$$\frac{dx}{dt} = -2x - 3y^2$$

$$\frac{dy}{dt} = -3x^2 + 2y$$

First,

$$H(x,y) = \int f(x,y)\,dy + \phi(x) = \int (-2x - 3y^2)\,dy + \phi(x) = -2xy - y^3 + \phi(x).$$

Consequently,

$$\begin{aligned}\phi'(x) &= -g(x,y) - \frac{\partial}{\partial x}\int f(x,y)\,dy \\ &= -(-3x^2 + 2y) - \frac{\partial}{\partial x}(-2xy - y^3) \\ &= 3x^2 - 2y + 2y \\ &= 3x^2,\end{aligned}$$

and $\phi(x) = x^3 + C$, where C is any constant. If we choose $C = 0$, our Hamiltonian function is

$$H(x,y) = -2xy - y^3 + \phi(x) = -2xy - y^3 + x^3.$$

Example 5.25. As a second example, suppose

$$\begin{aligned}\frac{dx}{dt} &= -x^2 \cos y \sin x \\ \frac{dy}{dt} &= x^2 \cos x \sin y + 2x(\sin x \sin y - 1).\end{aligned}$$

Since

$$\frac{\partial f}{\partial x} = x^2 \cos x \cos y - 2x \cos y \sin x = -\frac{\partial g}{\partial y}$$

our system is Hamiltonian. To find a Hamiltonian, we compute

$$H(x,y) = \int f(x,y)\,dy + \phi(x) = \int (-x^2 \cos y \sin x)\,dy + \phi(x) = -x^2 \sin x \sin y + \phi(x).$$

Thus,

$$\begin{aligned}\phi'(x) &= -g(x,y) - \frac{\partial}{\partial x}\int f(x,y)\,dy \\ &= -(x^2 \cos x \sin y + 2x(\sin x \sin y - 1)) - (2x \sin x \sin y - x^2 \cos x \sin y) \\ &= 2x,\end{aligned}$$

and $\phi(x) = x^2 + C$. Letting $C = 0$, our Hamiltonian function is

$$H(x,y) = x^2 - x^2 \sin x \sin y = x^2(1 - \sin x \sin y).$$

Solution curves for this system are given in Figure 5.26.

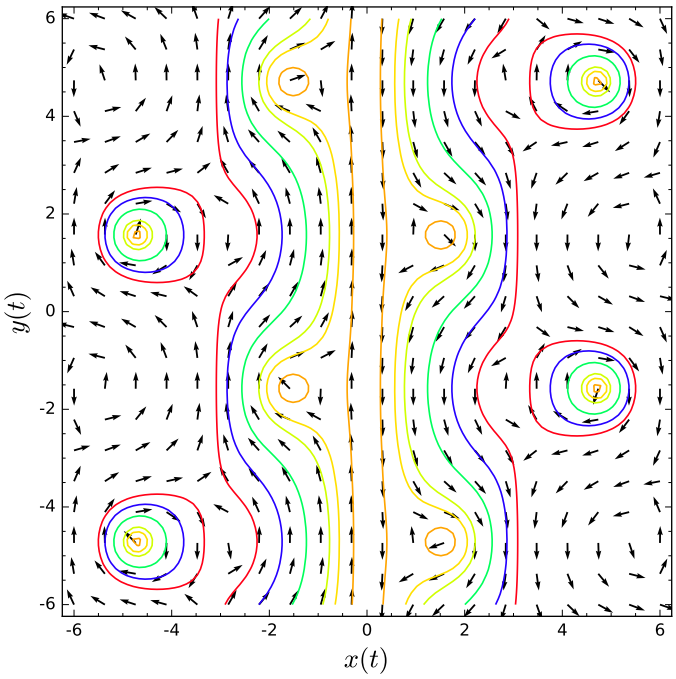

Figure 5.26: Solution curves for Example 5.25

5.2.4 Equilibrium Solutions of Hamiltonian Systems

The system that we studied in Example 5.17,

$$\frac{dx}{dt} = -2x - 3y^2$$
$$\frac{dy}{dt} = -3x^2 + 2y$$

has equilibrium solutions at $(0, 0)$ and $(-2/3, 2/3)$. Referring to the phase portrait for the system (Figure 5.20), we might guess that $(0, 0)$ is a saddle and $(-2/3.2/3)$ is either a center or a spiral sink. Since solution curves must follow contours for the Hamiltonian function $H(x, y) = x^3 - 2xy - y^3$, spiral sinks do not make sense.

Let us examine the possible types of equilibrium solutions for a Hamiltonian system. Suppose that (x_0, y_0) is an equilibrium solution for the system

$$\frac{dx}{dt} = f(x, y) = \frac{\partial H}{\partial y}(x, y)$$
$$\frac{dy}{dt} = g(x, y) = -\frac{\partial H}{\partial x}(x, y).$$

In order to determine the nature of the equilibrium solution, we will compute the Jacobian matrix of the system at (x_0, y_0),

$$J = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix} = \begin{pmatrix} H_{yx}(x_0, y_0) & H_{yy}(x_0, y_0) \\ -H_{xx}(x_0, y_0) & -H_{xy}(x_0, y_0) \end{pmatrix}.$$

If we let

$$\alpha = H_{xy}(x_0, y_0) = H_{yx}(x_0, y_0)$$
$$\beta = H_{yy}(x_0, y_0)$$

$$\gamma = -H_{xx}(x_0, y_0),$$

then J becomes

$$\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$\lambda^2 - \alpha^2 - \beta\gamma.$$

Therefore, our matrix has eigenvalues

$$\lambda = \pm\sqrt{\alpha^2 + \beta\gamma},$$

and we have the following possibilities.

- If $\alpha^2 + \beta\gamma > 0$, we have two real eigenvalues of opposite signs. Therefore, our equilibrium solution is a saddle.
- If $\alpha^2 + \beta\gamma = 0$, the only eigenvalue is zero.
- If $\alpha^2 + \beta\gamma < 0$, we have two purely imaginary eigenvalues.

In particular, a Hamiltonian system has no spiral sinks or sources.

Returning to Example 5.17, the Jacobian matrix is

$$J = \begin{pmatrix} H_{yx}(x_0, y_0) & H_{yy}(x_0, y_0) \\ -H_{xx}(x_0, y_0) & -H_{xy}(x_0, y_0) \end{pmatrix}. = \begin{pmatrix} -2 & -6y_0 \\ -6x_0 & 2 \end{pmatrix}.$$

For the equilibrium solution $(0, 0)$, we have eigenvalues $\lambda = \pm 2$, which tells us that the origin is a nodal saddle. The Jacobian matrix corresponding to $(x_0, y_0) = (-2/3, 2/3)$ is

$$J = \begin{pmatrix} -2 & -4 \\ -4 & 2 \end{pmatrix}.$$

Since J has eigenvalues $\lambda = \pm 2\sqrt{5}\, i$. Therefore, solution curves near $(-2/3, 2/3)$ should look like closed orbits.

5.2.5 Important Lessons

- A pendulum can be modeled by the equation

$$\frac{d^2\theta}{dt^2} + \frac{b}{m}\frac{d\theta}{dt} - \frac{g}{L}\sin\theta = 0.$$

  or the system

$$\begin{aligned} \frac{d\theta}{dt} &= v \\ \frac{dv}{dt} &= -\frac{b}{m}v - \frac{g}{L}\sin\theta. \end{aligned}$$

- A ***Hamiltonian system*** is a system of the form

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial y}(x, y) \\ \frac{dy}{dt} &= -\frac{\partial H}{\partial x}(x, y), \end{aligned}$$

  where $H : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth function.

- If

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}(x, y)$$
$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}(x, y),$$

  is a Hamiltonian system, then H is constant along any solution curve. In particular, the solution curves of a Hamiltonian system are the level sets of H.

- If the system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y),$$

  is Hamiltonian, then

$$\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}.$$

- If $f_x = -g_y$ is true on a rectangle R, then the system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y),$$

  is Hamiltonian on R and

$$H(x, y) = \int f(x, y)\, dy + \phi(x),$$

  where

$$\phi'(x) = -\frac{\partial}{\partial x}\left(\int f(x, y)\, dy\right) - g(x, y).$$

- A Hamiltonian system has no spiral sinks or sources.

5.2.6 Projects

Possible project

5.3 More Nonlinear Mechanics

The equation for the nonlinear pendulum with damping is

$$\frac{d^2\theta}{dt^2} + \frac{b}{m}\frac{d\theta}{dt} + \frac{g}{L}\sin\theta = 0.$$

As a system, we can model the pendulum as

$$\frac{d\theta}{dt} = v \tag{5.20}$$
$$\frac{dv}{dt} = -\frac{b}{m}v - \frac{g}{L}\sin\theta. \tag{5.21}$$

This system is not Hamiltonian since

$$\frac{\partial v}{\partial \theta} = 0,$$

but

$$-\frac{\partial}{\partial v}\left(-\frac{b}{m}v - \frac{g}{L}\sin\theta\right) = \frac{b}{m} \neq 0.$$

We shall have to invent new techniques to analyze such systems.

5.3.1 The Nonlinear Pendulum and Damping

To find the nullclines of the nonlinear pendulum, let

$$\begin{aligned}\frac{d\theta}{dt} &= v = 0\\ \frac{dv}{dt} &= -\frac{b}{m}v - \frac{g}{L}\sin\theta = 0.\end{aligned}$$

The θ-nullcline is $v = 0$, while the v-nullcline is

$$v = -\frac{mg}{bL}\sin\theta.$$

It follows that the equilibrium solutions for the nonlinear pendulum occur at

$$(\theta, v) = (0,0), (\pm\pi, 0), (\pm 2\pi, 0), \ldots.$$

This makes sense since the pendulum should not move if the bob is initially hanging downward ($\theta = 2\pi n$) or is at the very top or the very bottom of a swing ($\theta = (2n+1)\pi$). Since our first goal is to determine the nature of each equilibrium solution, we will compute the Jacobian of the system (5.20)–(5.21). This is just

$$\begin{pmatrix} 0 & 1 \\ -(g/L)\cos\theta & -b/m \end{pmatrix}.$$

At the equilibrium solutions $(\theta, v) = (0,0), (\pm 2\pi, 0), (\pm 4\pi, 0), \ldots$, the pendulum is hanging downward, and the Jacobian matrix becomes

$$J_1 = \begin{pmatrix} 0 & 1 \\ -g/L & -b/m \end{pmatrix}.$$

On the other hand, if $(\theta, v) = (\pm\pi, 0), (\pm 3\pi, 0), (\pm 5\pi, 0), \ldots$, the pendulum is at the top of its swing, and the Jacobian matrix is

$$J_2 = \begin{pmatrix} 0 & 1 \\ g/L & -b/m \end{pmatrix}.$$

We will first examine the case where the pendulum is hanging downward. The characteristic polynomial of J_1 is

$$\lambda^2 + \frac{b}{m}\lambda + \frac{g}{L}.$$

Thus, the eigenvalues of J_1 are

$$\lambda = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{g}{L}}. \tag{5.22}$$

We can analyze the nature of these eigenvalues by examining the sign of

$$\Delta = \left(\frac{b}{2m}\right)^2 - \frac{g}{L}.$$

- If $\Delta < 0$, the eigenvalues of the Jacobian are complex. Since the real part of (5.22) is negative, these equilibrium solutions are spiral sinks.
- If $\Delta > 0$, we have two distinct real eigenvalues. Since

$$\left(\frac{b}{2m}\right)^2 - \frac{g}{L} < \left(\frac{b}{2m}\right)^2,$$

we know that

$$\left(\frac{b}{2m}\right) > \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{g}{L}}.$$

Thus, both of our eigenvalues must be negative. Therefore, we have a nodal sink.

- If $\Delta = 0$, we have a single real negative eigenvalue. Thus, we also have a sink.

Consequently, if we assume that b is small, then $\Delta < 0$ and we will only have spiral sinks (Figure 5.27).

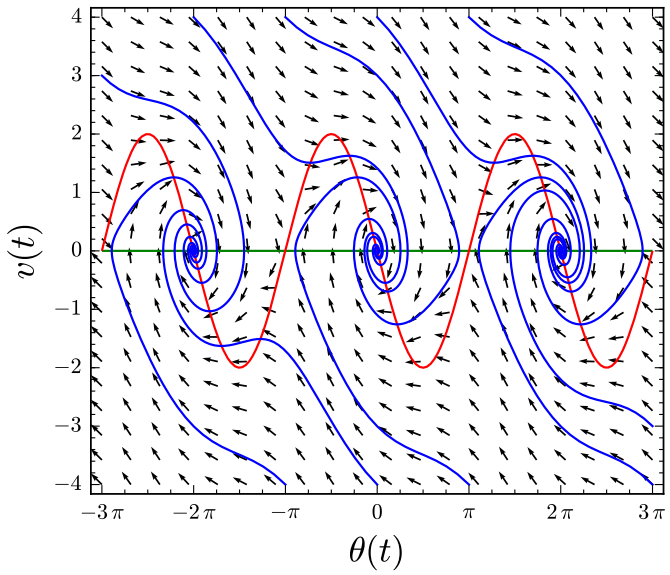

Figure 5.27: Nullclines of the damped pendulum

Now let us consider the type of equilibrium solutions that we will obtain when the pendulum is standing upright. These solutions will occur at $(\theta, v) = (\pm\pi, 0), (\pm 3\pi, 0), (\pm 5\pi, 0), \ldots$. The characteristic polynomial of the Jacobian matrix J_2 at these points is

$$\lambda^2 + \frac{b}{m}\lambda - \frac{g}{L};$$

hence, the eigenvalues of J_2 are

$$\lambda = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 + \frac{g}{L}}. \tag{5.23}$$

Furthermore, we have distinct real eigenvalues since

$$\left(\frac{b}{2m}\right)^2 + \frac{g}{L} > 0.$$

In fact, the eigenvalues will have opposite sign

$$\lambda_1 = -\frac{b}{2m} + \sqrt{\left(\frac{b}{2m}\right)^2 + \frac{g}{L}} > 0$$
$$\lambda_2 = -\frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 + \frac{g}{L}} < 0.$$

Thus, the equilibrium solutions are saddles.

5.3.2 Lyapunov Functions

The function

$$H(\theta, v) = \frac{1}{2}v^2 - \frac{g}{L}\cos\theta$$

is a Hamiltonian function for the ideal pendulum

$$\frac{d\theta}{dt} = v$$
$$\frac{dv}{dt} = -\frac{g}{L}\sin\theta,$$

since $dH/dt = 0$. However, if $b > 0$, then our system

$$\frac{d\theta}{dt} = v$$
$$\frac{dv}{dt} = -\frac{b}{m}v - \frac{g}{L}\sin\theta.$$

has damping and

$$\begin{aligned}\frac{dH}{dt} &= \frac{\partial H}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial H}{\partial v}\frac{dv}{dt} \\ &= \left(\frac{g}{L}\sin\theta\right)v + v\left(-\frac{b}{m}v - \frac{g}{L}\sin\theta\right) \\ &= -\frac{b}{m}v^2 \leq 0.\end{aligned}$$

Thus, H is decreasing whenever $v \neq 0$. Hence solution curves in the θv-plane cross the level sets of H moving from larger to smaller H values.

We can now devise a strategy for sketching the phase plane of the damped pendulum. If b/m and v are both small, the value of H decreases slowly along the solutions (Figure 5.27).

The function H in the case of the damped pendulum is an example of a Lyapunov function. Specifically, a function $L(x, y)$ is called a ***Lyapunov function*** for the system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

if for every solution of the system, $(x(t), y(t))$, that is not an equilibrium solution of the system,

$$\frac{dL}{dt} \leq 0,$$

with strict inequality except possible for a discrete set of ts.

As an example, let us return to the damped harmonic oscillator

$$\begin{aligned}\frac{dy}{dt} &= v\\ \frac{dv}{dt} &= -qy - pv.\end{aligned}$$

If $p = 0$, then

$$H(y, v) = \frac{1}{2}v^2 + \frac{q}{2}y^2$$

is a Hamiltonian function for our system. Recall that we also call H the energy function of the system. However, if $p > 0$ and $(y(t), v(t))$ is a solution for our system, we have

$$\begin{aligned}\frac{dH}{dt}(y, v) &= \frac{\partial H}{\partial y}\frac{dy}{dt} + \frac{\partial H}{\partial v}\frac{dy}{dt}\\ &= (qy)v + v(-qy - pv)\\ &= -pv^2 \le 0.\end{aligned}$$

Consequently, $H(y(t), v(t))$ decreases at a nonzero rate (except when $v = 0$), and H is a Lyapunov function. The level sets of H are ellipses in the yv-plane. As H decreases, the energy dissipates and the ellipses become spiral sinks.

It is easy to see that the system

$$\frac{dx}{dt} = y + \alpha x(x^2 + y^2) \tag{5.24}$$

$$\frac{dy}{dt} = -x + \alpha y(x^2 + y^2) \tag{5.25}$$

has an equilibrium solution at the origin no matter what the value of α is. The Jacobian of this system is

$$J = \begin{pmatrix} 3\alpha x^2 + \alpha y^2 & 1 + 2\alpha xy \\ -1 + 2\alpha xy & \alpha x^2 + 3\alpha y^2 \end{pmatrix}.$$

Since our equilibrium solution is the origin,

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and the linearization of our system at the origin is

$$\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= -x.\end{aligned}$$

Since the eigenvalues of

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

are $\pm i$, the linearization has a center at the origin. The phase plane consists of circles about the origin (Figure 5.28). Notice that the linearization does not depend on α.

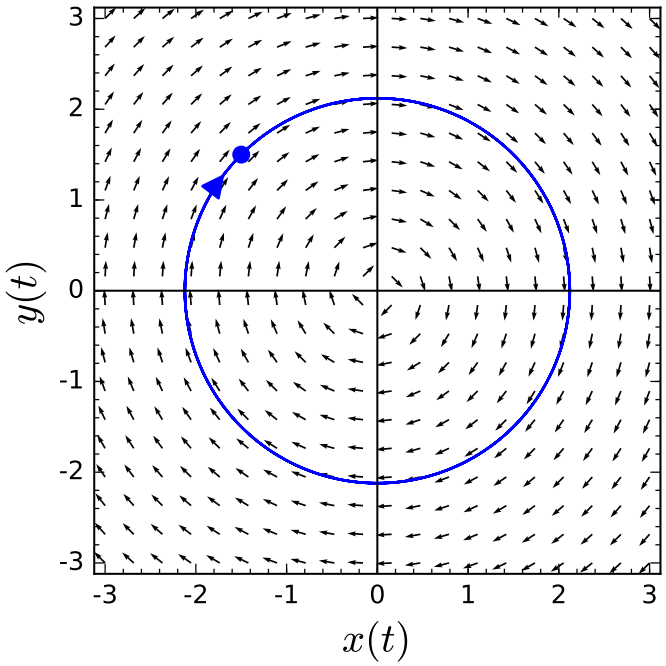

Figure 5.28: Solution curves for the linearization.

Now let us consider what happens to system (5.24)–(5.25) if we consider different values of α. If $\alpha = 5$, the situation is quite different that the linearization of our system. A solution curve spirals out from the origin as $t \to \infty$ (Figure 5.29). As $t \to -\infty$, the solution curve spirals back into the origin, but it seems to stop before actually reaching the origin. If $\alpha = -5$ on the other hand, we seem to have the opposite behavior with the solution curves spiraling into the origin as $t \to \infty$. As before, the solutions do not seem to reach the origin (Figure 5.30).

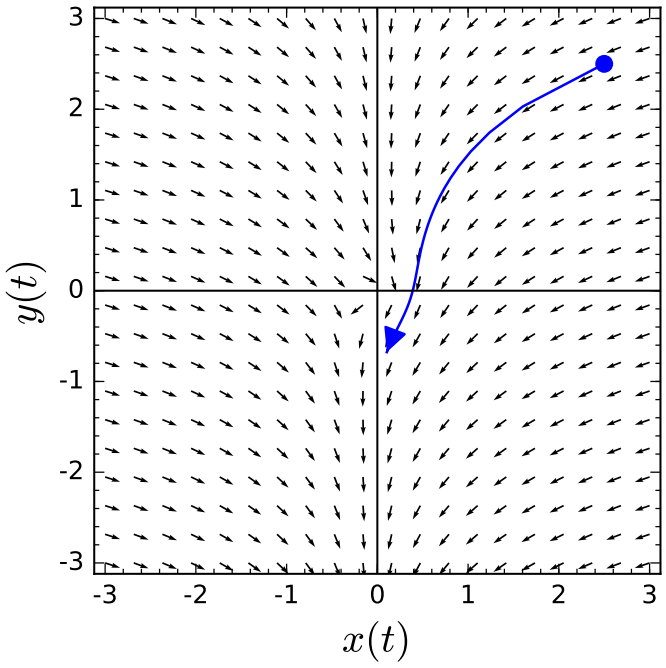

Figure 5.29: Solution curves $\alpha = 5$.

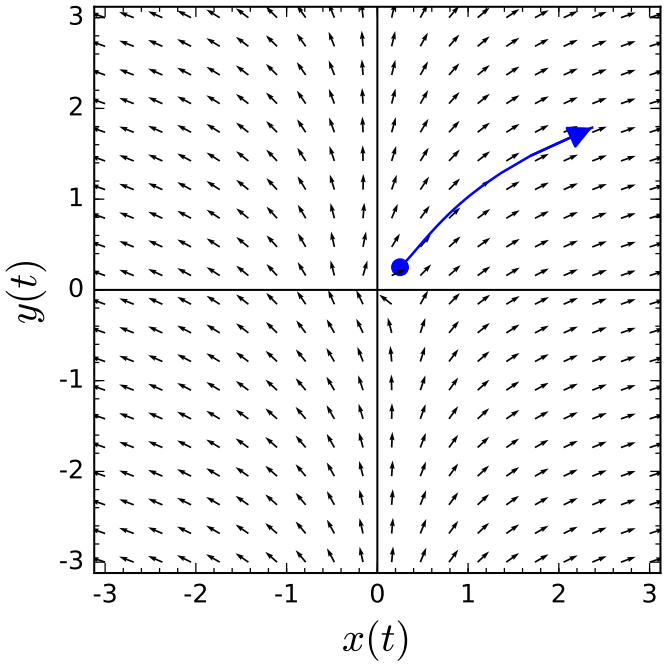

Figure 5.30: Solution curves for $\alpha = -5$.

Suppose that $(x(t), y(t))$ is a solution to the nonlinear system. The function

$$r(t) = \sqrt{x(t)^2 + y(t)^2}$$

is the distance of a point on the solution curve to the origin in the xy-plane. To see how r changes as $t \to \pm\infty$, we can compute the derivative of r. Actually, it is easier to work with

the equation $r(t)^2 = x(t)^2 + y(t)^2$. Thus,

$$\begin{aligned}\frac{d}{dt}r^2 &= \frac{d}{dt}(x(t)^2 + y(t)^2)\\ &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt}\\ &= 2x(y + \alpha x(x^2+y^2)) + 2y(-x + \alpha y(x^2+y^2))\\ &= 2\alpha(x^2+y^2)^2\\ &= 2\alpha r^4.\end{aligned}$$

Since $(r^2)' = 2rr'$, we have

$$r' = \alpha r^3 \tag{5.26}$$

for $r \neq 0$.

Equation (5.26) is separable, and it is quite easy to determine the solution as

$$r(t) = \frac{1}{\sqrt{C - 2\alpha t}}.$$

However, we do not need to know this solution to determine the nature of the equilibrium solution at the origin. If $\alpha > 0$ and $t \to -\infty$, equation (5.26) tells us that $r(t) \to 0$. Thus, any solution to the system (5.24)–(5.25) we have a spiral sink at the origin if $\alpha = -5$. Even though linearization fails to tell us the nature of the equilibrium solution at the origin, we were able to determine the nature of the equilibrium solution with further analysis.

We will now try to exploit what we have learned from our last example and from Hamiltonian systems to see if it is possible to analyze more general systems. If we consider solutions, $(x(t), y(t))$, of the system

$$\begin{aligned}\frac{dx}{dt} &= f(x,y)\\ \frac{dy}{dt} &= g(x,y),\end{aligned}$$

we might ask how a function $V(x,y)$ varies along the solution curve. We already have an answer if our system is Hamiltonian, and V is the corresponding Hamiltonian function. In this case $dV/dt = 0$. In general, we know that

$$\begin{aligned}\frac{dV}{dt}(x(t),y(t)) &= \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt}\\ &= \frac{\partial V}{\partial x}f(x,y) + \frac{\partial V}{\partial y}g(x,y).\end{aligned}$$

Thus, if we let

$$\dot{V} = \frac{\partial V}{\partial x}f(x,y) + \frac{\partial V}{\partial y}g(x,y),$$

we know that

$$\frac{dV}{dt}(x(t),y(t)) = \dot{V}(x(t),y(t)).$$

Thus, V is increasing along a solution curve if $\dot{V}(x,y) > 0$ and decreasing along a solution curve if $\dot{V}(x,y) < 0$. Our example suggests that we can determine this information without finding the solution.

Recall that the gradient of a function $V : \mathbb{R}^2 \to \mathbb{R}$ is

$$\nabla V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}\right).$$

If

$$\mathbf{F}(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix},$$

then

$$\dot{V} = \nabla V \cdot \mathbf{F},$$

is the directional derivative of V in the direction of $\mathbf{F}$.

Let us use this new information about V to obtain information about equilibrium solutions of our system. We do know that $V(x,y)$ graphs as a surface in $\mathbb{R}^3$ and

$$V(x,y) = \text{constant}$$

gives the contour lines or level curves of the surface in the xy-plane.[1] We also know that the gradient of V points in the direction that V is increasing the fastest and that the gradient is orthogonal to the level curves of V. Thus, if $\dot{V}(x,y) > 0$, we know that V is increasing in the direction of the direction of the vector field $\mathbf{F}$ and the elevation of the solution curve through (x,y) in $\mathbb{R}^3$ is increasing. That is, the solution curve is traveling uphill. Similarly, if $\dot{V}(x,y) < 0$, we know that the solution curve at (x,y) is going downhill.[2]

Now suppose that V is a real-valued function defined on a set S in the xy-plane, where the point $\mathbf{x}_0 = (x_0, y_0)$ is in S. We say that V is ***positive definite*** if $V(\mathbf{x}) > 0$ for all $\mathbf{x}$ in S, where $\mathbf{x} \neq \mathbf{x}_0$, and V is ***positive semidefinite*** if $V(\mathbf{x}) \geq 0$. Similarly, we say that V is ***negative definite*** and ***negative semidefinite*** if $V(\mathbf{x}) < 0$ and $V(\mathbf{x}) \leq 0$, respectively.

For example, if we consider the system for a harmonic oscillator

$$\begin{aligned} y' &= v \\ v' &= -\frac{k}{m}y - \frac{b}{m}v, \end{aligned}$$

then the energy function,

$$E(y,v) = \frac{1}{2}mv^2 + \frac{1}{2}ky^2,$$

is positive definite.

Theorem 5.31. *Suppose that the system*

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y) \end{aligned}$$

has an equilibrium solution at (x_0, y_0). Let V be a continuously differentiable function defined on a neighborhood U of (x_0, y_0) that is positive definite with minimum at (x_0, y_0).

1. *If $\dot{V}$ is negative semidefinite on U, then (x_0, y_0) is a stable equilibrium solution. That is, any solution that starts near the equilibrium solution will stay near the equilibrium solution.*

2. *If $\dot{V}$ is negative definite on U, then (x_0, y_0) is a asymptotically stable equilibrium solution or a sink.*

[1] See Figures 1 and 2 in John Polking, Albert Boggess, and David Arnold. *Differential Equations.* Prentice Hall, Upper Saddle River, NJ, 2001, p. 611.

[2] The argument that we have made here also works in higher dimensions.

Recall our example,

$$\begin{aligned}\frac{dx}{dt} &= y + \alpha x(x^2+y^2)\\ \frac{dy}{dt} &= -x + \alpha y(x^2+y^2).\end{aligned}$$

The function $V(x,y) = x^2+y^2$ is positive definite on $\mathbb{R}^2$, with an isolated minimum at the origin. We can compute

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x}(x,y)f(x,y) + \frac{\partial V}{\partial y}(x,y)g(x,y)\\ &= 2x(y+\alpha x(x^2+y^2)) + 2y(-x+\alpha y(x^2+y^2))\\ &= 2\alpha(x^2+y^2)^2.\end{aligned}$$

If $\alpha < 0$, then $\dot{V}$ is negative definite on $\mathbb{R}^2$. Theorem 5.31 tells us that the origin is a stable equilibrium point.

The function V in Theorem 5.31 is called a ***Lyapunov function***. If we compare Theorem 1 to using linearization to determine stability of an equilibrium solution, we will find that we can apply this result where linearization fails. Also, Lyapunov functions are defined on a domain U, where linearization only tells us what happens on a small neighborhood around the equilibrium solution. Unfortunately, there are no general ways of finding Lyapunov functions.

5.3.3 Gradient Systems

Let S be a real-valued function on the xy-plane. The gradient of S is

$$\nabla S = \left(\frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}\right).$$

The system

$$\begin{aligned}\frac{dx}{dt} &= f(x,y)\\ \frac{dy}{dt} &= g(x,y)\end{aligned}$$

is a ***gradient system*** if

$$\begin{aligned}f(x,y) &= \frac{\partial S}{\partial x}\\ g(x,y) &= \frac{\partial S}{\partial y}.\end{aligned}$$

For example, the system

$$\begin{aligned}\frac{dx}{dt} &= x - x^3\\ \frac{dy}{dt} &= -y\end{aligned}$$

is a gradient system, where

$$S(x,y) = \frac{x^2}{2} - \frac{x^4}{4} - \frac{y^2}{2} + 8.$$

Now let us see what happens on the solution curves of this gradient system. If $(x(t), y(t))$ is a solution curve,

$$\frac{dS}{dt} = \frac{\partial S}{\partial x}\frac{dx}{dt} + \frac{\partial S}{\partial y}\frac{dy}{dt} = (x - x^3)^2 + y^2 \geq 0.$$

Thus, S increases at point on the solution curve where the gradient of S is nonzero. That is, S increases at every point on the solution curves except at the equilibrium points.

In general, suppose that

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial S}{\partial x} \\ \frac{dy}{dt} &= \frac{\partial S}{\partial y} \end{aligned}$$

is a gradient system. Since

$$\frac{dS}{dt}(x(t), y(t)) = \frac{\partial S}{\partial x}\frac{dx}{dt} + \frac{\partial S}{\partial y}\frac{dy}{dt} = \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 \geq 0,$$

we know that S increases on every solution to the system except at the critical points of $S(x(t), y(t))$.

Let us see what this means in terms of the linearization of the system. The Jacobian matrix of (f, g) is

$$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{pmatrix}.$$

Since $S_{xy} = S_{yx}$, the matrix J must have the form

$$J = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix},$$

where $\alpha = S_{xx}$, $\beta = S_{xy}$, and $\gamma = S_{yy}$. The characteristic polynomial of J is

$$\lambda^2 - (\alpha + \gamma)\lambda + \alpha\gamma - \beta^2.$$

Therefore, the eigenvalues are

$$\lambda = \frac{1}{2}(\alpha + \beta) \pm \frac{1}{2}\sqrt{(\alpha - \gamma)^2 + 4\beta^2}.$$

Since we have real eigenvalues, a gradient system can have no spiral sources, spiral sinks, or centers.

For example, $x^2 + y^2$ is a Lyapunov function for the system

$$\begin{aligned} \frac{dx}{dt} &= -x + y \\ \frac{dy}{dt} &= -x - y. \end{aligned}$$

However, the origin is a spiral sink, so this system cannot be a gradient system.

5.3.4 Important Lessons

- The equation for the nonlinear pendulum with damping

$$\frac{d\theta}{dt} = v$$
$$\frac{dv}{dt} = -\frac{b}{m}v - \frac{g}{L}\sin\theta.$$

  can be analyzed by examining dH/dt, where H is the Hamiltonian function for the ideal pendulum. The function H for this system is an example of a ***Lyapunov function***.

- Let V be a real-valued function defined on a set S in the xy-plane such that the point $\mathbf{x}_0 = (x_0, y_0)$ is in S and $V(\mathbf{x}_0) = 0$.
  - We say that V is ***positive definite*** if $V(\mathbf{x}) > 0$ for all $\mathbf{x}$ in S, where $\mathbf{x} \neq \mathbf{x}_0$.
  - We say that V is ***positive semidefinite*** if $V(\mathbf{x}) \geq 0$ for all $\mathbf{x}$ in S.
  - We say that V is ***negative definite*** if $V(\mathbf{x}) < 0$ for all $\mathbf{x}$ in S, where $\mathbf{x} \neq \mathbf{x}_0$.
  - We say that V is ***negative semidefinite*** if $V(\mathbf{x}) \leq 0$ for all $\mathbf{x}$ in S.

- Suppose that the system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

  has an equilibrium solution at (x_0, y_0). Let V be a continuously differentiable function defined on a neighborhood U of (x_0, y_0) that is positive definite with minimum at (x_0, y_0).

  1. If $\dot{V}$ is negative semidefinite on U, then (x_0, y_0) is a stable equilibrium solution. That is, any solution that starts near the equilibrium solution will stay near the equilibrium solution.
  2. If $\dot{V}$ is negative definite on U, then (x_0, y_0) is a asymptotically stable equilibrium solution or a sink.

  We can use these results to analyze the behavior of equilibrium solutions where linearization fails. The function V is called a ***Lyapunov function***. We have no general methods for finding Lyapunov functions.

- The system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

  is a ***gradient system*** if

$$f(x, y) = \frac{\partial S}{\partial x}$$
$$g(x, y) = \frac{\partial S}{\partial y},$$

where S be a real-valued function on the xy-plane. Since

$$\frac{dS}{dt}(x(t), y(t)) = \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 \geq 0,$$

S increases on every solution to the system except at the critical points of $S(x(t), y(t))$. Since the eigenvalues of a gradient system are real, a gradient system has no spiral sources, spiral sinks, or centers.

5.3.5 Exercises

1. This is an exercise.

Hint.

5.3.6 Project—Tuned-Mass Dampers

The 60-story, 790-foot mirror-glass John Hancock Tower, New England's tallest building, was completed in 1976 and was designed by Henry N. Cobb, who founded the firm Pei Cobb Freed & Partners with famed architect I.M. Pei. (Figure 5.32).[3] The building suffered many problems during construction, the most notorious of which was how the original glass windows were attached. In early 1972 when construction was still underway, one of the 500-pound windows popped out of the building and committed suicide on the sidewalk below. In all more than 100 of the building's windows suffered the same fate. Fortunately, no one was injured. Initially, the architects and engineers thought thought that the problem was caused by the building's tendancy to sway excessively in high winds. However, they later determined that the falling-window problem was caused by the air space between the double-paned windows and pressure differentials between the interior and exterior of the building. The problem was solved by replacing all of the windows with single sheets of fully tempered glass. During the repairs, the windows were replaced with plywood, and the building was nicknamed the "Plywood Palace."

[3]The John Hancock Tower is officially known by its street address, 200 Clarendon.

Figure 5.32: The John Hancock Tower

Another flaw in the design of the building resulted in an extreme amount of swaying. In fact, office workers in the upper stories of the building complained of motion sickness. The building actually twisted as it was swaying back and forth. Engineers determined that the building's natural sway period was dangerously close to the period of its torsion. To remedy than problem, the engineers installed a pair of tuned mass dampeners on the 58th floor of the building. This is the floor of concrete that you see near the top of the structure (Figure 5.32). In addition, 1,500 tons of steel braces were installed to keep the building from falling over in a high wind.

To construct a simple model of a tuned-mass damper, we will consider a mass-spring system connected to a fixed wall. The mass is allowed to slide freely with no damping. We

can think of this mass as the top of the building moving back and forth. To this first mass, we attach a second mass by using a spring and a dashpot. This second mass is called the ***tuned mass-damper***. If we "tune" the mass-damper by choosing the appropriate spring, we can remove energy from the system and the motion of the first mass (the building) will decrease.

We will first model our system without a dashpot. If we let L_1 and L_2 be the natural lengths of the springs, the first spring will exert a force $-k_1(x_1 - L_1)$. The second spring pushes the two masses in opposite directions according to $\pm k_2(x_2 - x_1 - L_2)$. Thus, we can use Newton's second law to derive two equations

$$m_1 \frac{d^2x_1}{dt^2} = -k_1(x_1 - L_1) + k_2(x_2 - x_1 - L_2) \tag{5.27}$$

$$m_2 \frac{d^2x_2}{dt^2} = -k_2(x_2 - x_1 - L_2). \tag{5.28}$$

If we let

$$p_1 = m_1 \frac{dx_1}{dt}$$
$$p_2 = m_2 \frac{dx_2}{dt}$$

be the momenta of the two masses, we can rewrite this system as a system of four first-order linear equations,

$$\begin{aligned}
\frac{dx_1}{dt} &= \frac{p_1}{m_1} \\
\frac{dx_2}{dt} &= \frac{p_2}{m_2} \\
\frac{dp_1}{dt} &= -k_1(x_1 - L_1) + k_2(x_2 - x_1 - L_2) \\
\frac{dp_2}{dt} &= -k_2(x_2 - x_1 - L_2).
\end{aligned}$$

If we consider the kinetic and potential energy of both masses and springs, we can derive the Hamiltonian function

$$H(x_1, x_2, p_1, p_2) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{k_1}{2}(x_1 - L_1)^2 + \frac{k_2}{L}(x_2 - x_1 - L_2)^2.$$

Indeed, it is easy to check that

$$\begin{aligned}
\frac{dx_1}{dt} &= \frac{\partial H}{\partial p_1} \\
\frac{dx_2}{dt} &= \frac{\partial H}{\partial p_2} \\
\frac{dp_1}{dt} &= -\frac{\partial H}{\partial x_1} \\
\frac{dp_2}{dt} &= -\frac{\partial H}{\partial x_2}.
\end{aligned}$$

We can show that a solution to the system, $(x_1(t), x_2(t), p_1(t), p_2(t))$, is constant on H.[4]

[4]In general, a Hamiltonian system is a system of $2n$ equations of the form

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}$$

Now suppose that we add a damping term to our system,

$$\pm b\left(\frac{dx_2}{dt}-\frac{dx_1}{dt}\right)=\pm b\left(\frac{p_2}{m_2}-\frac{p_1}{m_1}\right).$$

Then equations (5.27) and (5.28) become

$$\begin{aligned}
m_1\frac{d^2x_1}{dt^2}&=-k_1(x_1-L_1)+k_2(x_2-x_1-L_2)+b\left(\frac{p_2}{m_2}-\frac{p_1}{m_1}\right)\\
m_2\frac{d^2x_2}{dt^2}&=-k_2(x_2-x_1-L_2)-b\left(\frac{p_2}{m_2}-\frac{p_1}{m_1}\right),
\end{aligned}$$

and our new first-order system is

$$\begin{aligned}
\frac{dx_1}{dt}&=\frac{p_1}{m_1}\\
\frac{dx_2}{dt}&=\frac{p_2}{m_2}\\
\frac{dp_1}{dt}&=-k_1(x_1-L_1)+k_2(x_2-x_1-L_2)+b\left(\frac{p_2}{m_2}-\frac{p_1}{m_1}\right)\\
\frac{dp_2}{dt}&=-k_2(x_2-x_1-L_2)-b\left(\frac{p_2}{m_2}-\frac{p_1}{m_1}\right).
\end{aligned}$$

Hamiltonian. Indeed,

$$\frac{dH}{dt}=-b\left(\frac{p_2}{m_2}-\frac{p_1}{m_1}\right)^2=-b\left(\frac{dx_2}{dt}-\frac{dx_1}{dt}\right)^2. \tag{5.29}$$

Equation (5.29) tells us that $dH/dt\le 0$. Moreover, $dH/dt<0$ whenever the distance between the two masses is changing. Thus, energy decreases whenever the second mass is moving relative to the first. Thus, if the wind or an earthquake starts our building (m_1) swaying back and forth, then the tuned mass-damper (m_2) located on one of the top floors of the building will start to move relative to the building and energy will be removed from the system by the dampers.

Of course, m_1 and k_1 involve the building and are set by the architects and engineers. We, however, are free to choose m_2, k_2 and b. We want to choose b fairly large so that there is a rapid loss of energy; i.e., the magnitude of dH/dt is large. We should choose m_2 large enough so that this mass oscillates with respect to m_1. If we choose m_2 two small, then the strong damper will almost serve as a rigid connection between m_1 and m_2. Therefore, we wish to choose m_2 to be large so that we guaranteed that this mass will oscillate with respect to m_1. We should, however, remember that m_2 is sitting on top of a very tall building. Finally, we should choose k_2 to maximize the rate at which the second mass oscillates with respect to the first and to maximize the oscillations of m_2. If we choose k_2 so that the second spring is in resonance with the first, the oscillations of the first mass force the second at its resonance frequency. Thus, we will have relatively large oscillations of the second mass and a large loss of energy.

Let us consider example. Suppose that $m_1=1$, $k_1=1$, We will also choose m_2 to be 0.05 and $b=0.1$. If the initial conditions for our system are

$$x_1(0)=10$$

$$\frac{dy_i}{dt}=-\frac{\partial H}{\partial x_i}$$

for $i=1,2,\dots,n$, where $H(x_1,\dots,x_n,y_1,\dots y_n$ is a real-valued differentiable function on $\mathbb{R}^{2n}$ such that H is nonconstant on every open ball in $\mathbb{R}^{2n}$. We can show that H is constant on solution curves of the system.

$$x_2(0) = 0$$
$$p_1(0) = 0$$
$$p_2(0) = 0,$$

we will choose k_2 to tune our system. To do this we will plot the time t that it takes for the amplitude of the oscillations of m_1 to reach and stay below 2.5 for various values of k_2.[5] If we choose $k_2 \approx 0.05$, then we can minimize the time to be $t \approx 120$.

5.4 The Hopf Bifurcation

If we consider the phase portrait of a system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y),$$

we may see closed orbits and various types of equilibrium solutions. If we make a slight change in the system, then we might reasonably expect the phase portrait to change slightly. For example, a closed orbit might expand or contract, an equilibrium solution might shift, or a spiral source might tighten up, but we would not expect new equilibrium solutions to appear or a spiral source to turn into a spiral sink. However, this may be exactly what happens. If such a change occurs in the system through an adjustment of a parameter, we say a ***bifurcation*** has occurred.

5.4.1 Bifurcations

The simplest bifurcations occur when an equilibrium solution appears, disappears, or splits into two or more equilibrium solutions. For example, the system

$$\frac{dx}{dt} = \alpha + x^2$$
$$\frac{dy}{dt} = -y$$

has no equilibrium solution if $\alpha > 0$ (Figure 5.33). However, we have a single equilibrium solution at the origin if $\alpha = 0$ and two equilibrium solutions, $(\pm\sqrt{-\alpha}, 0)$, if $\alpha < 0$.

[5] See Figure 5.42 in Paul Blanchard, Robert L. Devaney, and Glen R. Hall. *Differential Equations*, Second edition. Brooks/Cole, Pacific Grove, CA, 2002, p. 499.

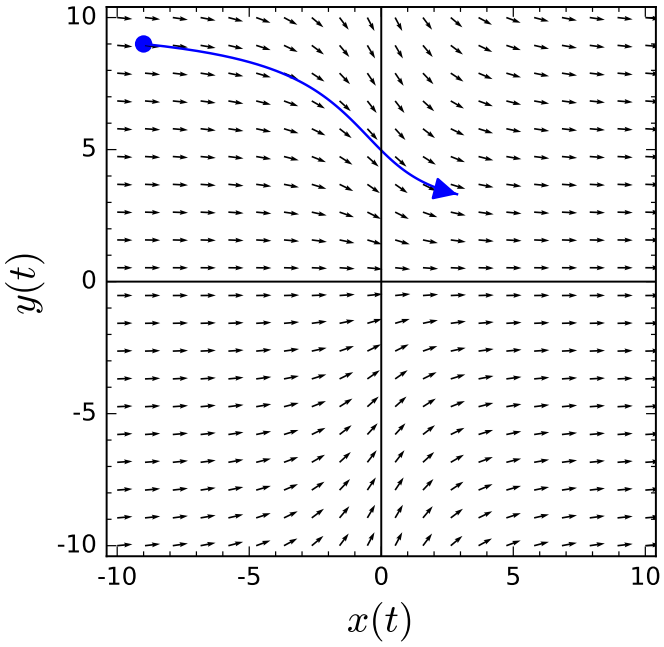

Figure 5.33: Solutions for $\alpha > 0$

The Jacobian of the system

$$\begin{aligned} \frac{dx}{dt} &= \alpha + x^2 \\ \frac{dy}{dt} &= -y \end{aligned}$$

is

$$J(x, y) = \begin{pmatrix} 2x & 0 \\ 0 & -1 \end{pmatrix}.$$

If $\alpha = 0$, then J has eigenvalues 0 and -1. This is not an elementary equilibrium point such a saddle, a sink, or a spiral source. We call this type of equilibrium solution a ***saddle-node***. On the right half of the xy-plane, the equilibrium solution resembles a saddle, while on the left half it resembles a sink (Figure 5.34).

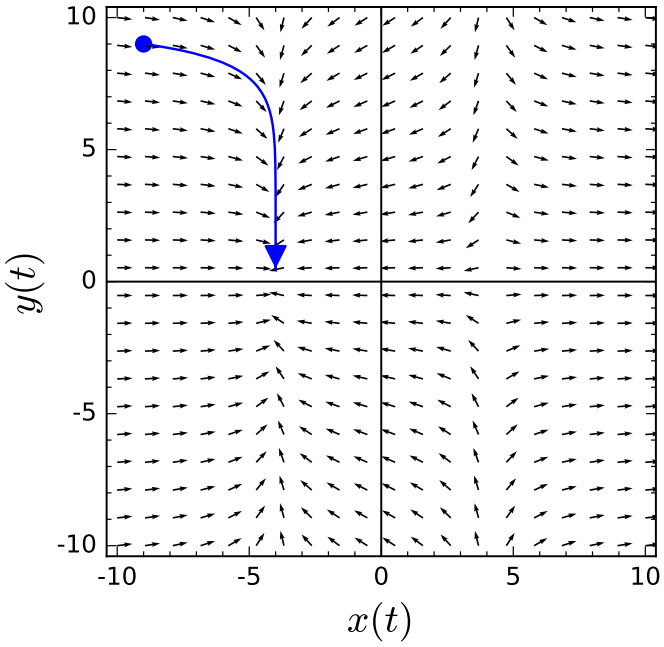

Figure 5.34: Solutions for $\alpha < 0$

If $\alpha < 0$, then

$$J = \begin{pmatrix} 2\sqrt{\alpha} & 0 \\ 0 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} -2\sqrt{\alpha} & 0 \\ 0 & -1 \end{pmatrix}.$$

In the first case, we have a saddle since the eigenvalues are real and of opposite sign. In the second case, we have a nodal sink, since both eigenvalues are negative. The two equilibrium solutions move in opposite directions as α decreases. We can summarize what happens for various values of α with a bifurcation diagram such as the one in Figure 5.35.

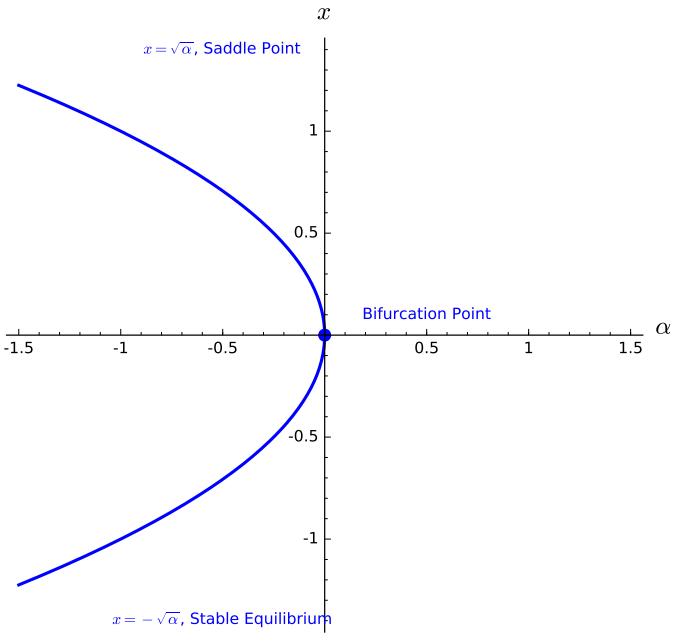

Figure 5.35: Bifurcation diagram

5.4.2 The Hopf Bifurcation

Now let us consider an entirely different type of bifurcation by examining the system

$$\begin{aligned}\frac{dx}{dt} &= \alpha x + 5y - x(x^2+y^2)\\ \frac{dy}{dt} &= -5x + \alpha y - y(x^2+y^2).\end{aligned}$$

The origin is an equilibrium solution for all values of α. The linearization of our system is

$$\begin{aligned}\frac{dx}{dt} &= \alpha x + 5y\\ \frac{dy}{dt} &= -5x + \alpha y.\end{aligned}$$

Since the eigenvalues of this system are $\alpha \pm 5i$, we can easily determine the nature of of our linearization. If $\alpha < 0$, we have a spiral sink. As α moves from negative values to positive values, the origin changes to a center ($\alpha = 0$), and then to a spiral source ($\alpha > 0$).

In the case of the nonlinear system, the origin is still a spiral sink for $\alpha > 0$ (Figure 5.36). If $\alpha > 0$, the origin is still a spiral source (Figure 5.37). However, something quite different happens at $\alpha = 0$. The origin is stable equilibrium solution with solutions spiraling into the origin very slowly. As α increases past zero, the equilibrium solution destabilizes and becomes a source. In addition, a closed orbit of radius $\sqrt{\alpha}$ develops. Solutions inside of this closed orbit spiral out towards the orbit, while solutions outside of the orbit spiral inward (Figure 5.38).

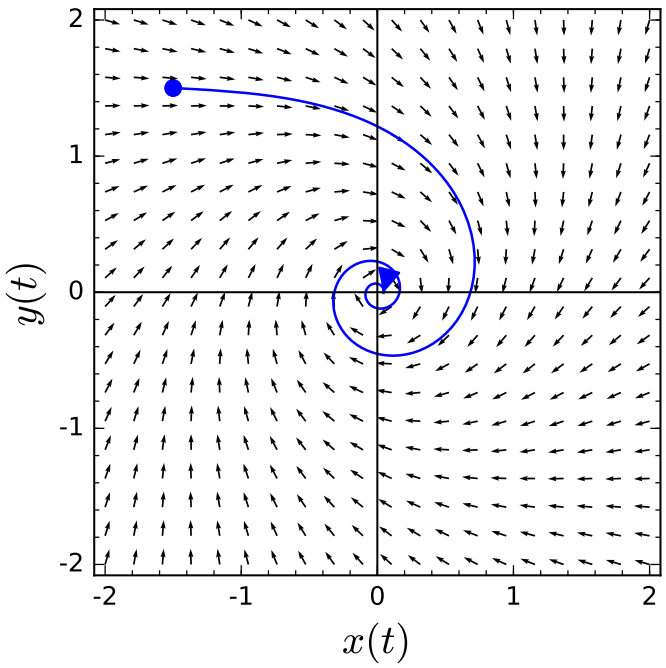

Figure 5.36: Solutions for $\alpha < 0$

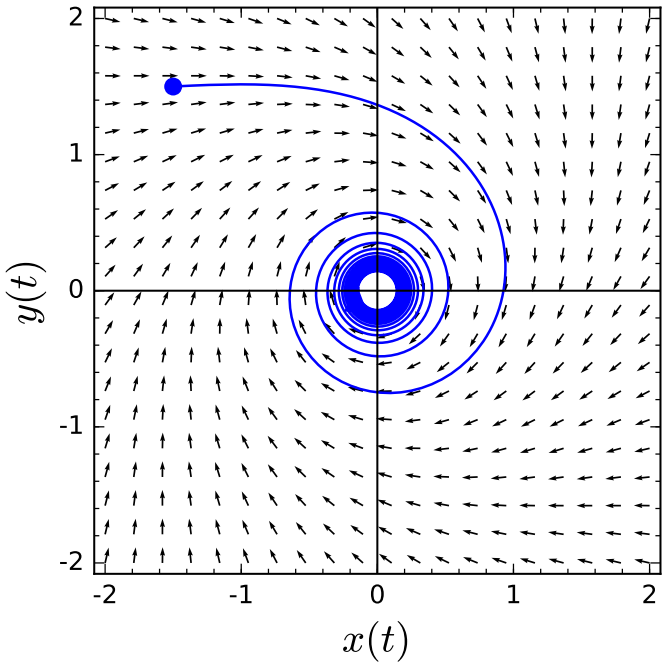

Figure 5.37: Solutions for $\alpha = 0$

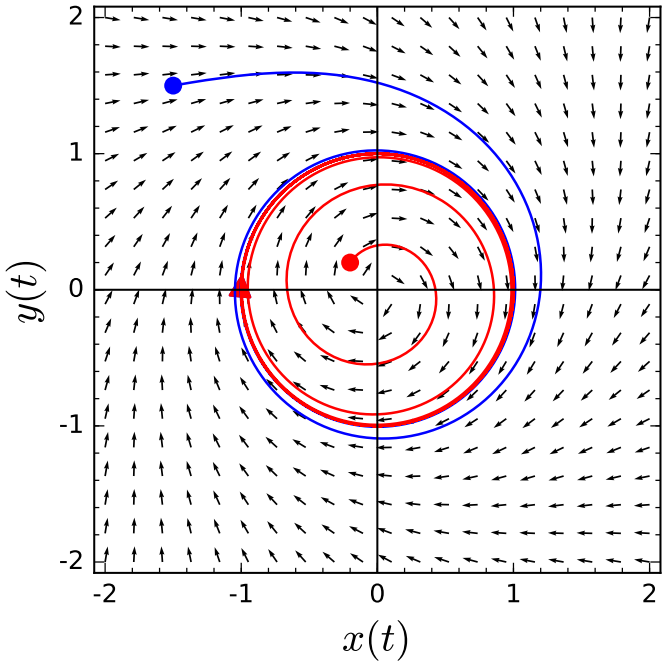

Figure 5.38: Solutions for $\alpha > 0$

To see exactly what happens as α passes zero and becomes positive, we will rewrite our system in polar coordinates. If we make the substitution $x = r\cos\theta$ and $y = r\sin\theta$, our nonlinear system can be rewritten as

$$\frac{dr}{dt} = \alpha r - r^3$$

$$\frac{d\theta}{dt} = -5.$$

If $\alpha < 0$, the origin is a sink since $\alpha r - r^3 < 0$ for all $r > 0$. In this case all solutions tend towards the origin as $t \to \infty$. When $\alpha > 0$, the origin is still an equilibrium solution. Moreover, $r' = 0$, when $r = \sqrt{\alpha}$. We also know that $r' > 0$ for $0 < r < \sqrt{\alpha}$ and $r' < 0$ if $r > \sqrt{\alpha}$. So the circle of radius $\sqrt{\alpha}$ is a periodic solution with the trajectory moving clockwise since $\theta' = -5 < 0$. All nonzero solutions spiral towards this orbit as $t \to \infty$. This type of bifurcation is called a ***Hopf bifurcation***. No new equilibrium solutions arise, but a periodic solution develops as α passes through the bifurcation value.

Theorem 5.39 (Hopf Bifurcation)**.** *Suppose that $(x_0(\alpha), y_0(\alpha))$ is a equilibrium solution for the family of systems*

$$\frac{dx}{dt} = f_\alpha(x, y)$$
$$\frac{dy}{dt} = g_\alpha(x, y)$$

parameterized by α, and the Jacobian matrix for the system evaluated at this equilibrium has eigenvalues $a(\alpha) \pm ib(\alpha)$. Assume that at some $\alpha = \alpha_0$, we have $a(\alpha_0) = 0$, $a'(\alpha_0) > 0$, $b(\alpha_0) \neq 0$, and that the equilibrium point is asymptotically stable. Then there exists an α_1 such that the system has a periodic solution encircling the equilibrium solution for $\alpha_0 < \alpha < \alpha_1$.[6]

Recall that our example system

$$\frac{dx}{dt} = \alpha x + 5y - x(x^2 + y^2)$$
$$\frac{dy}{dt} = -5x + \alpha y - y(x^2 + y^2)$$

has an equilibrium at $(0, 0)$ with eigenvalues $\alpha \pm 5i$. If we let $\alpha_0 = 0$, then $a(0) = 0$, $a'(0) = 1$, and $b(0) = 5$. Thus, the hypothesis of the Hopf Bifurcation Theorem are satisfied and we are guaranteed a period solution surrounding our equilibrium solution.

Example 5.40 (Van der Pol's equation)**.** Van der Pol's equation, a simple nonlinear equation having applications in electrical engineering and mathematical biology, is $x'' - x(1 - x^2)x' + x = 0$. This equation can be written as the system

$$\frac{dx}{dt} = y \tag{5.30}$$
$$\frac{dy}{dt} = -x + x(1 - x^2)y. \tag{5.31}$$

The phase portrait for Van der Pol's equation is given in Figure 5.41. The origin is the only equilibrium solution to Van der Pol's equation, and one might guess that this solution acts like a spiral source by examining the phase portrait. If we examine the system consisting of only linear terms on the right-hand side of the equation,

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x + y$$

[6]For a proof and description of the Hopf Bifurcation Theorem see C. Chicone. *Ordinary Differential Equations with Applications*. Springer-Verlag, New York, 1999.

we might get a better sense of what is happening. The matrix for this linear system,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},$$

has eigenvalues $\lambda = (1 \pm i\sqrt{3})/2$. This suggests that the origin looks like a spiral source at least locally. Indeed, the Jacobian matrix for (5.30)–(5.30) is

$$J = \begin{pmatrix} 0 & 1 \\ -1 - 2x_0y_0 & 1 - x_0^2 \end{pmatrix},$$

which agrees with A for $(x_0, y_0) = (0, 0)$.

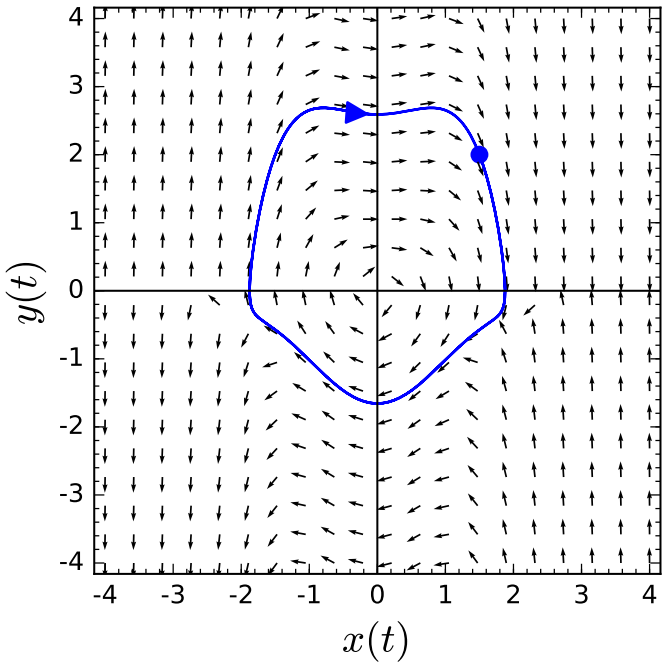

Figure 5.41: Van der Pol's equation

Let us examine how a bifurcation might occur in a predator-prey model. In our model, we will assume that a predator's appetite is satiated when food is abundant. If this is the case, an increase in the prey population has little effect on the interaction terms in our model. We might model this satiable predator-prey scenario by the equations

$$\begin{aligned} \frac{dx}{dt} &= (a - bx)x - \frac{cxy}{d + \alpha x} \\ \frac{dy}{dt} &= -ey + \frac{fxy}{d + \alpha x}, \end{aligned}$$

where $x(t)$ is the prey population at time t and $y(t)$ is the predator population at time t. Here, $a, \ldots, f$ and α are constants. Let us choose the constants $a, \ldots, f$ so that our system becomes

$$\begin{aligned} \frac{dx}{dt} &= (1 - x)x - \frac{xy}{0.3 + \alpha x} \\ \frac{dy}{dt} &= -0.5y + \frac{xy}{0.3 + \alpha x}. \end{aligned}$$

The constant α is know as the ***satiation constant***. The larger the value of α, the more quickly a predator's appetite is satiated as the prey population increases.

Let us see examine the phase portrait for several values of α. If $\alpha = 0$, then there is no satiation effect. In this case, all trajectories inside the population quadrant spiral asymptotically towards a single stable equilibrium point (Figure 5.42). If $\alpha = 1.35$, we have a very similar phase portrait with solutions spiraling in towards a stable equilibrium solution (Figure 5.43). However, if $\alpha = 0.9$, we have a periodic trajectory. All other trajectories are pulled toward this attracting periodic orbit (Figure 5.44).

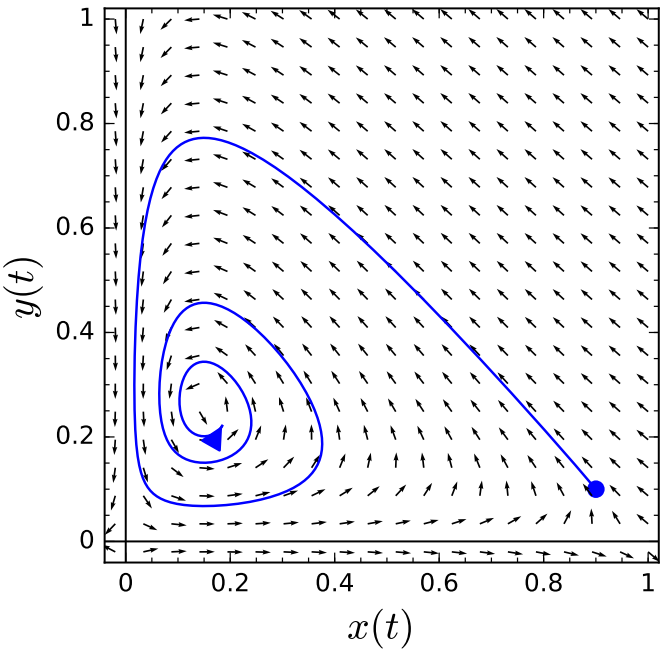

Figure 5.42: Predator-prey model with $\alpha = 0$

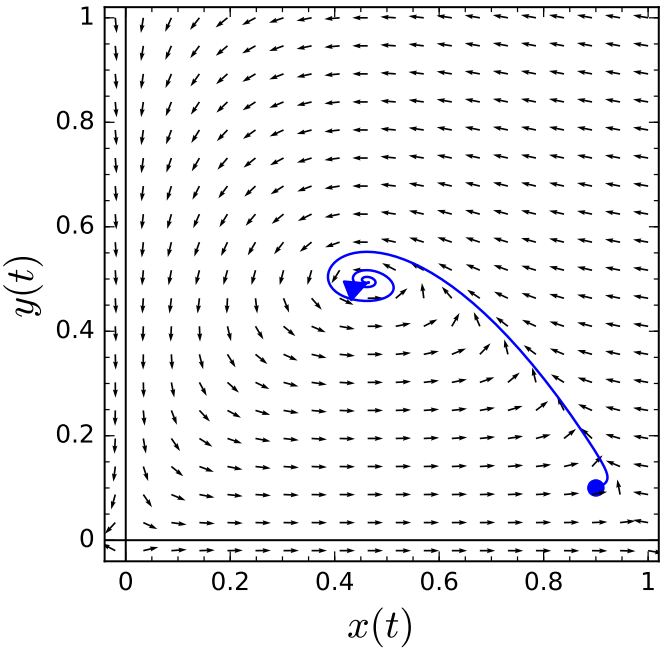

Figure 5.43: Predator-prey model with $\alpha = 1.35$

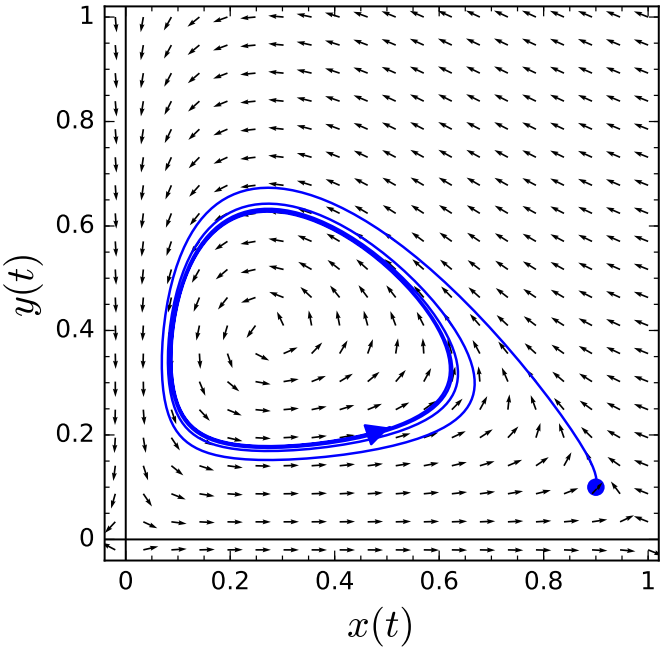

Figure 5.44: Predator-prey model with $\alpha = 0.9$

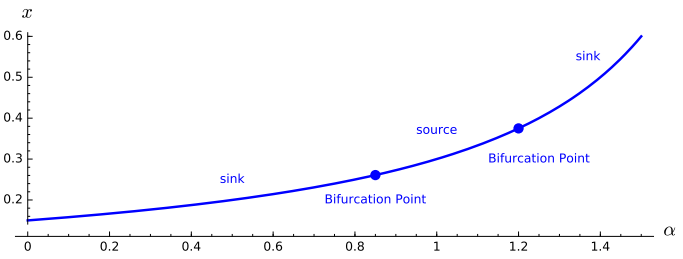

Figure 5.45: Bifurcation diagram for the satiable predator-prey model

As α continues to increase a Hopf bifurcation occurs at $\alpha \approx 0.5$. At this point the equilibrium solution destabilizes and spawns an attracting periodic orbit. As α continues to increase, the amplitude of this periodic orbit increases. However, at $\alpha \approx 0.85$, the amplitude of the periodic solution begins to decrease. At $\alpha \approx 1.2$, the equilibrium point re-stabilizes and the periodic solution is absorbed. For $\alpha > 1.2$, we only have a stable equilibrium solution in the first quadrant. The x-coordinate of our equilibrium solution is

$$x = \frac{0.15}{1 - 0.5\alpha},$$

and we can summarize our findings with a bifurcation diagram (Figure 5.45).

5.4.3 Important Lessons

- If we make a slight change in the system, then we might reasonably expect the phase portrait to change slightly. For example, a closed orbit might expand or contract, an equilibrium solution might shift, or a spiral source might tighten up, but we would not expect new equilibrium solutions to appear or a spiral source to turn into a spiral sink. However, this may be exactly what happens. If such a change occurs in the system through an adjustment of a parameter, we say a ***bifurcation*** has occurred.

- A ***Hopf bifurcation*** if an attracting periodic solution encircling an equilibrium solution develops as α passes through the bifurcation value.

- Suppose that $(x_0(\alpha), y_0(\alpha))$ is a equilibrium solution for the family of systems

$$\frac{dx}{dt} = f_\alpha(x, y)$$
$$\frac{dy}{dt} = g_\alpha(x, y)$$

  parameterized by α, and the Jacobian matrix for the system evaluated at this equilibrium has eigenvalues $a(\alpha) \pm ib(\alpha)$. Assume that at some $\alpha = \alpha_0$, we have $a(\alpha_0) = 0$, $a'(\alpha_0) > 0$, $b(\alpha_0) \neq 0$, and that the equilibrium point is asymptotically stable. Then there exists an α_1 such that the system has a periodic solution encircling the equilibrium solution for $\alpha_0 < \alpha < \alpha_1$.

- A satiable predator-prey scenario can be modeled by the equations

$$\frac{dx}{dt} = (a - bx)x - \frac{cxy}{d + \alpha y}$$
$$\frac{dy}{dt} = -ey + \frac{fxy}{d + \alpha y},$$

where $x(t)$ is the prey population at time t and $y(t)$ is the predator population at time t. Here, $a, \ldots, f$ and α are constants. The constant α is know as the ***satiation constant***. The larger the value of α, the more quickly a predator's appetite is satiated as the prey population increases.

5.4.4 Exercises

1. Consider the system

$$\begin{aligned} x' &= y \\ y' &= -x - \frac{y}{4} + x^2. \end{aligned}$$

(a) Verify that the function

$$L(x, y) = \frac{x^2}{2} + \frac{y^2}{2} - \frac{x^3}{3}$$

is a Lyapunov function for the system.

(b) Sketch the level sets of L.

(c) What can you conclude about the phase portrait of the system from (1) and (2)?

Hint.

2. Let $G(x, y) = x^3 - 3xy^2$.

(a) What is the gradient system with vector field given by the gradient of G?

(b) Sketch the graph of G and the level sets of G.

(c) Sketch the phase portrait of the gradient system in (1).

Hint.

3. Suppose that the smell of a bunch of dead fish in the region $-2 \leq x \leq 2$, $-2 \leq y \leq 2$ is given by the function

$$S(x, y) = x^2 + y^2 - \frac{x^4 + y^4}{4} - 3x^2y^2 + 100.$$

(a) What is the gradient system who vector field is the gradient of S?

(b) Use Sage to graph the phase portrait of the system.

(c) How many dead fish are there and where are they?

(d) Using the results from part (2), sketch the level sets of S.

(e) Why is this model not realistic for large values of x or y?

Hint.

4. Consider the system of differential equations

$$\begin{aligned} x' &= x(-x - y + 1) \\ y' &= y(-ax - y + b), \end{aligned}$$

where a and b are parameters with $a, b > 0$. Suppose that this system is only defined for $x, y \geq 0$.

(a) Use nullclines to sketch the phase portrait for this system for several values of a and b.

(b) Determine the values of a and b at which a bifurcation occurs.

Hint.

5. Show that the quadratic polynomial

$$V(x, y) = ax^2 + 2bxy + cy^2$$

is positive definite with minimum at $(x, y) = (0, 0)$ if and only if $a > 0$ and $ac - b^2 > 0$.

Hint.

6. Use the positive definite function $V(x, y) = x^2 + y^2$ to argue that the system

$$x' = -x - 2y^2$$
$$y' = 2xy - y^3$$

has an asymptotically stable equilibrium solution at $(x, y) = (0, 0)$.

Hint.

5.4.5 Project—The Lorenz Equations

The Lorenz Equations

In 1963, an MIT professor, Edward N. Lorenz, published a paper on his research in meteorology. Using differential equations, Lorenz had developed a simplified system to model certain weather-related phenomena. When he analyzed the system, however, he found that the trajectories of the solutions were incredibly convoluted and effectively unpredictable for certain parameters. The ***Lorenz equations*** can be written as

$$\frac{dx}{dt} = -\sigma x + \sigma y$$
$$\frac{dy}{dt} = \rho x - y - xz$$
$$\frac{dz}{dt} = -\beta z + xy,$$

where σ, ρ, and β are constants. The Existence and Uniqueness Theorem for systems of differential equations guarantees a unique solution for each set of initial conditions,

$$x(0) = x_0$$
$$y(0) = y_0$$
$$z(0) = z_0.$$

However, Lorenz discovered that the trajectories of the solutions were incredibly convoluted and effectively unpredictable for certain parameters. For certain values of σ, ρ, and β, the trajectories are extremely sensitive to initial conditions. Since real data always has some inherent uncertainty, initial values are never precisely known, and we may have little success modeling real world phenomena. In addition, solutions can stay in a bounded region of the three dimensional version of the phase plane and wind through the region along an incredibly convoluted path. There is much more freedom to move around in three dimensions than there is in two.

Lorenz noticed that the system behaved strangely, when he let $\sigma = 10$, $\rho = 28$, and $\beta = 8/3$. Thus, our system

$$\frac{dx}{dt} = -10x + 10y$$

$$\frac{dy}{dt} = 28x - y - xz$$
$$\frac{dz}{dt} = -\frac{8}{3}z + xy,$$

defines a vector field in $\mathbb{R}^3$,

$$\mathbf{F}(x, y, z) = \left(-10x + 10y, 28x - y - xz, -\frac{8}{3}z + xy\right),$$

and the equilibrium solutions occur exactly when this vector field is zero. That is, (x, y, z) is an equilibrium solution if

$$\mathbf{F}(x, y, z) = \left(-10x + 10y, 28x - y - xz, -\frac{8}{3}z + xy\right) = (0, 0, 0).$$

It is easy to check that $(0, 0, 0)$, $(6\sqrt{2}, 6\sqrt{2}, 27)$, and $(-6\sqrt{2}, -6\sqrt{2}, 27)$ are the equilibrium solutions to the Lorenz system for the parameters that we have chosen. With the exception of these equilibrium solutions it is probably impossible to find solutions for the system, so we will have to use a numerical algorithm to approximate solutions (Figure 5.46).

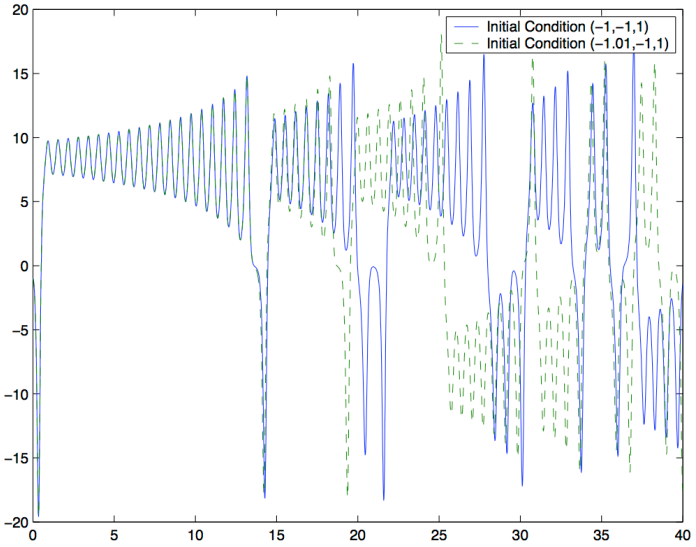

Figure 5.46: The solution curves for two systems with nearly equal initial conditions.

The Equilibrium Solutions

If we wish to understand the nature of the equilibrium solutions of the Lorenz system, it makes sense to linearize the system. The Jacobian matrix of the Lorenz system is

$$J(x, y, z) = \begin{pmatrix} -10 & 10 & 0 \\ 28 - z & -1 & -x \\ y & x & -8/3 \end{pmatrix}.$$

At the origin, the Jacobian is

$$J = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix}.$$

Thus, the linear system that approximates the Lorenz system at the origin is

$$\begin{aligned}\frac{dx}{dt} &= -10x + 10y\\ \frac{dy}{dt} &= 28x - y\\ \frac{dz}{dt} &= -\frac{8}{3}z.\end{aligned}$$

This system decouples into two systems,

$$\begin{aligned}\frac{dx}{dt} &= -10x + 10y\\ \frac{dy}{dt} &= 28x - y,\end{aligned}$$

and

$$\frac{dz}{dt} = -\frac{8}{3}z.$$

Consequently, the origin is a sink in the z-direction and a saddle in the xy-plane.

Now let us consider what happens at the other two equilibrium solutions. For the equilibrium point $(6\sqrt{2}, 6\sqrt{2}, 27)$, we have the Jacobian matrix

$$J = \begin{pmatrix} -10 & 10 & 0\\ 1 & -1 & -6\sqrt{2}\\ 6\sqrt{2} & 6\sqrt{2} & -8/3\end{pmatrix}.$$

The eigenvalues of this matrix are

$$\begin{aligned}\lambda_1 &\approx -13.8\\ \lambda_2 &\approx 0.094 + 10.2i\\ \lambda_3 &\approx 0.094 - 10.2i.\end{aligned}$$

Therefore, this equilibrium point is a spiral saddle that has a line of solutions that tend toward the equilibrium point as t increases and a plane of solutions that spiral towards the equilibrium point as t decreases. That is, solutions approach this equilibrium point along the direction of the eigenvalue of the negative eigenvector, then spiral slowly away along the plane corresponding to the complex eigenvalues. Similarly, the equilibrium solution $(-6\sqrt{2}, -6\sqrt{2}, 27)$ is a spiral saddle.

Now that we have some idea of how the solutions of the Lorenz system behave near the equilibrium solutions, we can start to understand global behavior of the system. A trajectory approaches one of the two equilibrium solutions $(\pm 6\sqrt{2}, \pm 6\sqrt{2}, 27)$ along the straight line corresponding to the negative eigenvalue. When the trajectory gets close enough to the equilibrium solutions, it begins to spiral away. When the spiral is large enough, the trajectory come under the influence of the saddle at the origin and either returns to repeat it previous pattern or moves over to the other equilibrium solution (Figure 5.47).

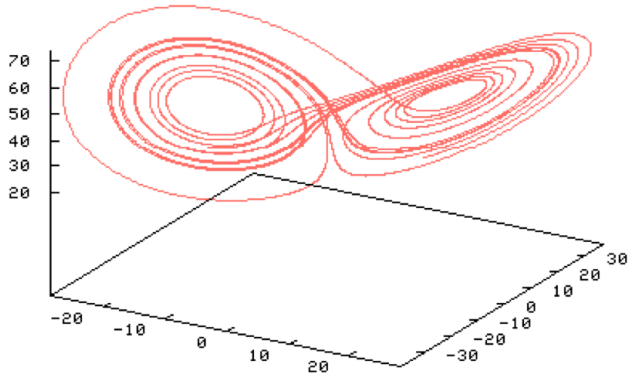

Figure 5.47: Solutions in $\mathbb{R}^3$.

Exercise 5.48. Consider the Lorenz system

$$\begin{aligned} x' &= -ax + ay \\ y' &= rx - y - xz \\ z' &= -bz + xy. \end{aligned}$$

In this exercise, we shall show that the solutions of the system are bounded.

1. If
$$V(x, y, z) = rx^2 + ay^2 + a(z - 2r)^2,$$
Show that
$$\dot{V} = -2a[rx^2 + y^2 + b(z - r)^2 - br^2].$$

2. It is true that $(z - r)^2 \geq (z - 2r)^2 - r^2$. Assuming this, show that
$$rx^2 + y^2 + b(z - r)^2 \geq mV - br^2,$$
where m is the smallest of the three numbers 1, $1/a$, and $b/2a$.

3. Use parts (1) and (2) to show that
$$\dot{V} \leq -2maV + 4abr^2.$$

4. Use part (3) to show that $\dot{V} \leq -2abr^2 < 0$ everywhere outside of the ellipsoid
$$R = \left\{(x, y, z) | V(x, y, z) \leq 3br^2/m\right\}.$$

5. Use part (4) to show that the ellipsoid is invariant, and that every solution curve ends up inside of R.

Hint.

Solution.

6

The Laplace Transform

6.1 The Laplace Transform

Consider the electrical circuit governed by the differential equation

$$LI'' + RI' + \frac{1}{C}I = E'(t).$$

The function $E'(t)$ can have discontinuities. For example, the voltage in the circuit can be periodically turned on and off. The previous methods that we have used to solve second order linear differential equations may not apply here. However, the ***Laplace transform***, an integral transform, allows us to change a differential equation to an algebraic equation.

As a second example, let us consider a population of fish that is governed by exponential growth,

$$\begin{aligned} \frac{dP}{dt} &= kP \\ P(0) &= P_0, \end{aligned}$$

and suppose that we wish to determine the effects of seasonal fishing. In other words, harvesting will not be continuous. For example, we might only allow harvesting during the first half of the year,

$$H(t) = \begin{cases} r & 0 \le t \le 1/2, \\ 0 & 1/2 < t < 1, \\ r & 1 \le t \le 3/2, \\ 0 & 3/2 < t < 2, \\ \quad \vdots \end{cases}$$

We do not yet have the tools to analyze differential equations possessing discontinuous terms.

Given an initial-value problem

$$\begin{aligned} ay'' + by' + cy &= g(t) \\ y(0) &= y_0 \\ y'(0) &= y_0', \end{aligned}$$

the idea is to use the Laplace transform to change the differential equation into an equation that can be solved algebraically and then transform the algebraic solution back into a solution of the differential equation. Surprisingly, this method will even work when g is a discontinuous function, provided the discontinuities are not too bad.

6.1.1 Definition of the Laplace Transform

We shall define the ***Laplace transform*** of a function $f(t)$ by

$$\mathcal{L}(f)(s) = F(s) = \int_0^\infty e^{-st} f(t)\, dt,$$

provided the integral converges.

The Laplace transform has many nice properties, especially with respect to the derivative of f. However, before we investigate these properties, let us compute several Laplace transforms.

- Let $f(t) = k$, where k is a constant. Then

  $$\mathcal{L}(f)(s) = \int_0^\infty ke^{-st}\, dt = \frac{k}{s},$$

  for $s > 0$.

- If $f(t) = e^{at}$, then the Laplace transform of f is

  $$\mathcal{L}(f)(s) = \int_0^\infty e^{-st}e^{at}\, dt = \frac{1}{s-a},$$

  for $s > a$.

- For $f(t) = t$, the Laplace transform is

  $$\mathcal{L}(f)(s) = \int_0^\infty te^{-st}\, dt = \frac{1}{s^2},$$

  for $s > 0$.

The Laplace transform of does not always exist, even for functions that are infinitely differentiable. For example, let $f(t) = e^{t^2}$. Then for any real number s,

$$\int_0^b e^{t^2}e^{-st}\, dt = \int_0^b e^{t(t-s)}\, dt,$$

and the integrand is greater then one for $t > s$. Thus, the integral

$$\int_0^\infty e^{t^2}e^{-st}\, dt$$

does not converge.

The Laplace transform, however, does exist in many cases. We will show that the Laplace transform of a function exists if the function does not grow too quickly and has no bad discontinuities. We say that a function f is ***piecewise continuous*** on an interval $[a, b]$ if f satisfies the following conditions.

1. There are a finite number of discontinuities of f on $[a, b]$.

2. If $c \in [a, b]$ is a discontinuity, then the one-sided limits

   $$\lim_{t\to c^-} f(t) \qquad \text{and} \qquad \lim_{t\to c^+} f(t)$$

   both exist. We also assume that $\lim_{t\to a^+} f(t)$ and $\lim_{t\to b^-} f(t)$ exist.

We say that a function is ***piece-wise continuous*** on an interval $[0, \infty)$ if it is piecewise continuous on the interval $[0, b]$ for all $b > 0$. The types of discontinuities that we are describing are sometimes called ***jump discontinuities***. If a function is piecewise continuous, then the continuities are not too bad.

The square wave function

$$f(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & t > 1, \end{cases}$$

where $f(t+2) = f(t)$ is an example of a piecewise continuous function defined on $[0, \infty)$.

We also wish to be certain that a function $f(t)$ defined on $[0, \infty)$ does not grow too quickly as $t \to \infty$. A function $f(t)$ is ***exponentially bounded*** on $[0, \infty)$ if there exist constants $M \ge 0$ and a such that

$$|f(t)| \le Me^{at},$$

for all t in $[0, \infty)$. In other words, the graph of f must lie between the curves $y = Me^{at}$ and $y = -Me^{at}$.

Theorem 6.1. *If f is a piecewise continuous, exponentially bounded function defined $[0, \infty)$, then the Laplace transform of f,*

$$\mathcal{L}(f)(s) = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

exists.

6.1.2 Properties of the Laplace Transform

One of the most important properties of the Laplace transform is linearity. That is,

$$\mathcal{L}(\alpha f + \beta g)(s) = \alpha \mathcal{L}(f)(s) + \beta \mathcal{L}(g)(s),$$

where $\alpha, \beta \in \mathbb{R}$ provided the Laplace transforms of f and g exist. The proof of this statement follows directly from the definition of the Laplace transform and the properties of integrations. Transforms of functions having the linearity property are called *linear operators*.

Laplace transforms forms behave very nicely with respect to derivatives. In fact, this is the reason why Laplace transforms are so useful when trying to solve initial value problems.

Theorem 6.2. *Let $y = y(t)$ be a piecewise continuous, exponentially bounded function and assume that y' is also exponentially bounded. Then for large values of s*

$$\mathcal{L}(y')(s) = s\mathcal{L}(y)(s) - y(0) = sY(s) - y(0),$$

where $Y(s)$ is the Laplace transform of y.

Proof. We can evaluate the Laplace transform of y' by using integration by parts,

$$\begin{aligned} \mathcal{L}(y')(s) &= \int_0^\infty y'(t) e^{-st}\, dt \\ &= \lim_{b \to \infty} \left[e^{-st} y(t) \Big|_0^b + s \int_0^b y(t) e^{-st}\, dt \right] \\ &= \lim_{b \to \infty} \left[e^{-sb} y(b) - y(0) + s\mathcal{L}(y)(s) \right]. \end{aligned}$$

We claim that $\lim_{b\to\infty} e^{-sb}y(b) = 0$. Since, y is exponentially bounded, there exist constants $M \geq 0$ and a such that $|y(t)| \leq Me^{at}$, for all t in $[0,\infty)$. Thus,

$$|e^{-sb}y(b)| \leq Me^{-(s-a)b}.$$

The right-hand side of this inequality as $b \to \infty$ for $s > a$. Thus,

$$\mathcal{L}(y')(s) = s\mathcal{L}(y)(s) - y(0). \qquad \square$$

The Laplace transform also behave nicely with respect to higher order derivatives.

Theorem 6.3. *Let $y = y(t)$ and $y'(t)$ be piecewise continuous, exponentially bounded functions and assume that y'' is exponentially bounded. Then for large values of s*

$$\mathcal{L}(y'')(s) = s^2\mathcal{L}(y)(s) - sy(0) - y'(0) = s^2Y(s) - sy(0) - y'(0),$$

where $Y(s)$ is the Laplace transform of y. In general, if $y = y(t)$ and all of its derivatives up to order $k-1$ are piecewise continuous, exponentially bounded functions and $y^{(k)}$ is piecewise continuous, then

$$\begin{aligned}\mathcal{L}(y^{(k)})(s) &= s^k\mathcal{L}(y)(s) - s^{k-1}y(0) - \cdots - sy^{(k-2)}(0) - y^{(k-1)}(0)\\ &= s^kY(s) - s^{k-1}y(0) - s^{k-1}y(0) - \cdots - sy^{(k-2)}(0) - y^{(k-1)}(0).\end{aligned}$$

6.1.3 Important Lessons

- Many initial value problems have discontinuous forcing terms.
- The ***Laplace transform*** of a function $f(t)$ by

  $$\mathcal{L}(f)(s) = F(s) = \int_0^\infty e^{-st}f(t)\,dt,$$

  provided the integral converges.
- If f is a piecewise continuous, exponentially bounded function defined $[0,\infty)$, then the Laplace transform of f,

  $$\mathcal{L}(f)(s) = F(s) = \int_0^\infty e^{-st}f(t)\,dt$$

  exists.
- The Laplace transform is a linear operator; i.e.,

  $$\mathcal{L}(\alpha f + \beta g)(s) = \alpha\mathcal{L}(f)(s) + \beta\mathcal{L}(g)(s),$$

  where $\alpha, \beta \in \mathbb{R}$ provided the Laplace transforms of f and g exist.
- Let $y = y(t)$ be a piecewise continuous, exponentially bounded function and assume that y' is also exponentially bounded. Then for large values of s

  $$\mathcal{L}(y')(s) = s\mathcal{L}(y)(s) - y(0) = sY(s) - y(0),$$

  where $Y(s)$ is the Laplace transform of y.

- Let $y = y(t)$ and $y'(t)$ be piecewise continuous, exponentially bounded functions and assume that y'' is exponentially bounded. Then for large values of s

$$\mathcal{L}(y'')(s) = s^2\mathcal{L}(y)(s) - sy(0) - y'(0) = s^2Y(s) - sy(0) - y'(0),$$

where $Y(s)$ is the Laplace transform of y. In general, if $y = y(t)$ and all of its derivatives up to order $k-1$ are piecewise continuous, exponentially bounded functions and $y^{(k)}$ is piecewise continuous, then

$$\begin{aligned}\mathcal{L}(y^{(k)})(s) &= s^k\mathcal{L}(y)(s) - s^{k-1}y(0) - \cdots - sy^{(k-2)}(0) - y^{(k-1)}(0)\\ &= s^kY(s) - s^{k-1}y(0) - s^{k-1}y(0) - \cdots - sy^{(k-2)}(0) - y^{(k-1)}(0).\end{aligned}$$

6.1.4 Exercises

1. Find the Laplace transform of each of the following functions.

(a) te^{-2t}

(b) $\sin at$

(c) $\cos at$

(d) $e^{at}\sin bt$

2. If f is a piecewise continuous, exponentially bounded function defined $[0, \infty)$, prove that the Laplace transform of f,

$$\mathcal{L}(f)(s) = F(s) = \int_0^\infty e^{-st}f(t)\,dt$$

exists.

3.

(a) Let $y = y(t)$ and $y'(t)$ be piecewise continuous, exponentially bounded functions and assume that y'' is exponentially bounded. Prove that

$$\mathcal{L}(y'')(s) = s^2\mathcal{L}(y)(s) - sy(0) - y'(0) = s^2Y(s) - sy(0) - y'(0)$$

for large values of s, where $Y(s)$ is the Laplace transform of y.

(b) If $y = y(t)$ and all of its derivatives up to order $k-1$ are piecewise continuous, exponentially bounded functions and $y^{(k)}$ is piecewise continuous, prove that

$$\begin{aligned}\mathcal{L}(y^{(k)})(s) &= s^k\mathcal{L}(y)(s) - s^{k-1}y(0) - \cdots - sy^{(k-2)}(0) - y^{(k-1)}(0)\\ &= s^kY(s) - s^{k-1}y(0) - s^{k-1}y(0) - \cdots - sy^{(k-2)}(0) - y^{(k-1)}(0).\end{aligned}$$

4. Define the *gamma function* to be

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt.$$

(a) Show that $\Gamma(1) = 1$.

(b) Prove that $\Gamma(x+1) = x\Gamma(x)$ and deduce that $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$.

(c) Show that the Laplace transform of $f(t) = t^\alpha$ is

$$\mathcal{L}(f)(s) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}.$$

If n is a positive integer, then $\mathcal{L}(t^n)(s) = n!/s^{n+1}$.

6.2 Solving Initial Value Problems

6.2.1 The Inverse Laplace Transform

As we mentioned in the previous section, we can use Laplace transforms to help us solve initial value problems. The idea is to convert an initial value problem into an algebraic equation. Consider

$$\begin{aligned} y'' - y' - 6y &= 0 \\ y(0) &= 2 \\ y'(0) &= -1, \end{aligned}$$

and recall that

$$\begin{aligned} \mathcal{L}(y')(s) &= s\mathcal{L}(y)(s) - y(0) = sY(s) - y(0) \\ \mathcal{L}(y'')(s) &= s^2\mathcal{L}(y)(s) - sy(0) - y'(0) = s^2Y(s) - sy(0) - y'(0). \end{aligned}$$

where $Y(s)$ is the Laplace transform of y. If we take the Laplace transform of both sides of the equation $y'' - y' - 6y = 0$, we have

$$\begin{aligned} \mathcal{L}(y'' - y' - 6y) &= \mathcal{L}(y'') - \mathcal{L}(y') - 6\mathcal{L}(y) \\ &= [s^2Y(s) - sy(0) - y'(0)] - [sY(s) - y(0)] - 6Y(s) \\ &= [s^2Y(s) - 2s + 1] - [sY(s) - 2] - 6Y(s) \\ &= \mathcal{L}(0) = 0. \end{aligned}$$

Solving for $Y(s)$, we get

$$Y(s) = \frac{2s-3}{(s-3)(s+2)} = \frac{3/5}{s-3} + \frac{7/5}{s+2},$$

where we have used partial fractions to get the last expression. If we know the Laplace transform of a function, we can recover the original function. We state the following theorem without proof.

Theorem 6.4. *Let f and g be two piecewise continuous exponentially bounded functions with Laplace transforms, $F(s)$ and $G(s)$, respectively. Suppose that $F(s) = G(s)$ for all $s > c$, where c is some positive number. Then $f(t) = g(t)$ for all $t \geq 0$.*

It now makes sense to define the ***inverse Laplace transform*** of a function $F(s)$, which we will denote by $\mathcal{L}^{-1}(F(s))(t)$, as the function $f(t)$ whose unique Laplace transform is $F(s)$. Furthermore, the inverse Laplace transform is linear,

$$\mathcal{L}^{-1}(\alpha F + \beta G) = \alpha\mathcal{L}^{-1}(F) + \beta\mathcal{L}^{-1}(G).$$

We can now solve our initial value problem. Since the Laplace transform of e^{at} is $1/(s-a)$, we know that

$$\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at},$$

and the solution that we seek is

$$y(t) = \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}.$$

Of course, there are much easier methods that we can use if we would like to solve this particular initial value problem, but it is nice to see that the two methods agree.

6.2.2 A Table of Laplace Transforms

To use the method of Laplace transforms effectively, we need either tables or computer software such as Sage so that we can easily find Laplace transforms and inverse Laplace transforms.

$f(t) = \mathcal{L}^{-1}(F(s))$	$F(s) = \mathcal{L}(f(t))$
1	$1/s,\ s > 0$
e^{at}	$1/(s-a),\ s > a$
$t^n,\ n \in \mathbb{N}$	$n!/s^{n+1},\ s > 0$
$t^a,\ a > -1$	$\Gamma(a+1)/s^{a+1},\ s > 0$
$\sin at$	$a/(s^2+a^2),\ s > 0$
$\cos at$	$s/(s^2+a^2),\ s > 0$
$\sinh at$	$a/(s^2-a^2),\ s > \lvert a\rvert$
$\cosh at$	$s/(s^2-a^2),\ s > \lvert a\rvert$
$e^{at}\sin bt$	$b/[(s-a)^2+b^2],\ s > a$
$e^{at}\cos bt$	$(s-a)/[(s-a)^2+b^2],\ s > a$
$t^n e^{at},\ n \in \mathbb{N}$	$n!/(s-a)^{n+1},\ s > a$
$u_c(t)$	$e^{-cs}/s,\ s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$
$f(ct)$	$(1/c)F(s/c),\ c > 0$
$\delta(t-c)$	$s^2F(s) - sf(0) - f'(0)$
$f'(t)$	$s^2F(s) - sf(0) - f'(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$

Table 6.5: Table of Laplace Transforms

6.2.3 Discontinuous Functions

In order for the Laplace transform to be useful in solving initial value problems, we need to be able to handle discontinuous functions. If $c \geq 0$, we define the function

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c. \end{cases}$$

A function of this form is called a ***step function*** or ***Heaviside function***, named for the British engineer Oliver Heaviside. The Laplace transform of u_c is given by

$$\begin{aligned} \mathcal{L}(u_c(t))(s) &= \int_0^\infty u_c(t)e^{-st}\,dt \\ &= \int_0^c u_c(t)e^{-st}\,dt + \int_c^\infty u_c(t)e^{-st}\,dt \\ &= \int_c^\infty e^{-st}\,dt \\ &= \frac{e^{-cs}}{s}. \end{aligned}$$

With this new information, we can now solve initial value problems such as

$$y' = -y + u_3(t)$$

$$y(0) = 2.$$

Notice that we can rewrite the equation $y' = -y + u_3(t)$ as

$$y' = \begin{cases} -y & t < 3 \\ -y + 1 & t \geq 3. \end{cases}$$

If we take the Laplace transform of both sides of $y' = -y + u_3(t)$, we obtain

$$sY(s) - y(0) = -Y(s) + \frac{e^{-3s}}{s}.$$

Using the fact that $y(0) = 2$ and solving for $Y(s)$, we get

$$Y(s) = \frac{2}{s+1} + \frac{e^{-3s}}{s(s+1)}.$$

Therefore,

$$y(t) = \mathcal{L}^{-1}\left(\frac{2}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{e^{-3s}}{s(s+1)}\right).$$

The inverse Laplace transform of the first term is

$$\mathcal{L}^{-1}\left(\frac{2}{s+1}\right) = 2e^{-t};$$

however, further investigation is needed to determine the inverse Laplace transform of the second term.

Given a function $f(t)$, consider the new function

$$g(t) = u_c(t)f(t-c).$$

To obtain the function g from f, we shift the graph of the f to the right by c units and let $g(t) = 0$ for $0 \leq t < c$. Let us compute the Laplace transform of g,

$$\mathcal{L}(g) = \int_0^\infty g(t)e^{-st}\,dt = \int_c^\infty f(t-c)e^{-st}\,dt.$$

If we make the substitution $u = t - c$, then

$$\mathcal{L}(g) = \int_0^\infty f(u)e^{-s(u+c)}\,dt = e^{-sc}\int_0^\infty f(u)e^{-su}\,dt = e^{-sc}\mathcal{L}(f).$$

In other words, if $\mathcal{L}(f) = F(s)$, then $\mathcal{L}(u_c(t)f(t-c)) = e^{-sc}F(s)$.

Returning to our example, we will now show how we can use this new information to compute the inverse Laplace transform of

$$\frac{e^{-3s}}{s(s+1)}.$$

First, we can use partial fractions to obtain

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s-1}.$$

Hence,

$$\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s(s+1)}\right) = \mathcal{L}^{-1}\left(\frac{e^{-3s}}{s}\right) - \mathcal{L}^{-1}\left(\frac{e^{-3s}}{s+1}\right) = u_3(t) - \mathcal{L}^{-1}\left(\frac{e^{-3s}}{s+1}\right).$$

If $g(t) = u_3(t)e^{-(t-3)}$, then the Laplace transform of $g(t)$ is

$$\mathcal{L}(g) = e^{-3s}\mathcal{L}(e^{-t}) = \frac{e^{-3s}}{s+1}.$$

Thus,

$$y(t) = 2e^{-t} + u_3(t)\left(1 - e^{-(t-3)}\right).$$

6.2.4 Forced Harmonic Oscillators

Consider the forced harmonic oscillator

$$y'' + 4y = 3\cos t$$
$$y(0) = y'(0) = 0.$$

Taking the Laplace transform of both sides of the equation $y'' + 4y = 3\cos t$, we obtain

$$\mathcal{L}(y'') + 4\mathcal{L}(y) = 3\mathcal{L}(\cos t)$$

or

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{3s}{s^2+1},$$

where $\mathcal{L}(y)(s) = Y(s)$. Substituting the initial conditions and solving for Y, we have

$$Y(s) = \frac{3s}{(s^2+4)(s^2+1)} = \frac{-s}{s^2+4} + \frac{s}{s^2+1},$$

where the last expression was obtained using partial fractions. Taking the inverse Laplace transform, we have our solution

$$y(t) = \mathcal{L}^{-1}\left(\frac{-s}{s^2+4}\right) + \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = -\cos 2t + \cos t.$$

Now let us consider a harmonic oscillator with discontinuous forcing,

$$y'' + 2y' + 5y = h(t)$$
$$y(0) = y'(0) = 0,$$

where $h(t)$ is given by

$$h(t) = \begin{cases} 5 & t < 7 \\ 0 & t \geq 7. \end{cases}$$

That is, $h(t) = 5(1 - u_7(t))$.

We may consider this to be a mass-spring system sliding on a table, where the mass is one unit, the spring constant is 5, and the damping coefficient is 2. When $t < 7$ the table is tilted so that gravity provides a force of 5 units when stretching the spring. At time $t = 7$, the table is suddenly returned to the level position.

Taking the Laplace transform of both sides of $y'' + 2y' + 5y = h(t)$, we obtain

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 5Y(s) = \mathcal{L}(h),$$

where $\mathcal{L}(y)(s) = Y(s)$. Substituting the initial conditions and evaluating the Laplace transform on the right, we have

$$(s^2 + 2s + 5)Y(s) = 5\left(\frac{1}{s} - \frac{e^{-7s}}{s}\right).$$

Solving for $Y(s)$, we have

$$Y(s) = \frac{5}{s(s^2+2s+5)} - \frac{5e^{-7s}}{s(s^2+2s+5)}$$

and

$$y = \mathcal{L}^{-1}\left(\frac{5}{s(s^2+2s+5)} - \frac{5e^{-7s}}{s(s^2+2s+5)}\right).$$

Using partial fractions, we can rewrite the first term as

$$\frac{5}{s(s^2+2s+5)} = \frac{1}{s} - \frac{s+2}{s^2+2s+5}.$$

The inverse Laplace transform of $1/s$ is 1. To find the inverse Laplace transform of the second term, we complete the square of the denominator,

$$\begin{aligned}\frac{s+2}{s^2+2s+5} &= \frac{s+2}{(s+1)^2+4} \\ &= \frac{s+1}{(s+1)^2+4} + \frac{1}{(s+1)^2+4} \\ &= \frac{s+1}{(s+1)^2+4} + \frac{1}{2}\frac{2}{(s+1)^2+4}.\end{aligned}$$

Consequently,

$$\mathcal{L}^{-1}\left(\frac{s+2}{s^2+2s+5}\right) = e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t = e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right).$$

and

$$\mathcal{L}^{-1}\left(\frac{5}{s(s^2+2s+5)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{s+2}{s^2+2s+5}\right) = 1 - e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right).$$

We can compute the inverse Laplace transform of

$$\frac{5e^{-7s}}{s(s^2+2s+5)}$$

using the Heaviside function $u_7(t)$ and the inverse Laplace transform that we just calculated to obtain

$$\mathcal{L}^{-1}\left(\frac{5e^{-7s}}{s(s^2+2s+5)}\right) = u_7(t)\left(1 - e^{-(t-7)}\left(\cos 2(t-7) + \frac{1}{2}\sin 2(t-7)\right)\right).$$

Therefore, the solution to our original initial value problem is

$$y(t) = 1 - e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right) - u_7(t)\left(1 - e^{-(t-7)}\left(\cos 2(t-7) + \frac{1}{2}\sin 2(t-7)\right)\right).$$

6.2.5 Important Lessons

- If we know the Laplace transform $F(s)$ of a function, we can recover the original function using the inverse Laplace transform of a function $\mathcal{L}^{-1}(F(s))(t)$.

- The inverse Laplace transform is linear,

$$\mathcal{L}^{-1}(\alpha F + \beta G) = \alpha \mathcal{L}^{-1}(F) + \beta \mathcal{L}^{-1}(G).$$

- If $c \geq 0$, we define the Heaviside function to be

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c. \end{cases}$$

  The Laplace transform of u_c is

$$\mathcal{L}(u_c(t))(s) = \frac{e^{-cs}}{s}.$$

- If $\mathcal{L}(f) = F(s)$, then $\mathcal{L}(u_c(t)f(t-c)) = e^{-sc}F(s)$.

6.2.6 Exercises

1. Find the solution of the initial value problem

$$2y'' + y' + 2y = 0$$
$$y(0) = y'(0) = 0$$

using the Laplace transform.

2. Find the solution of the initial value problem

$$2y'' + y' + 2y = g(t)$$
$$y(0) = y'(0) = 0$$

where $g(t)$ is defined by

$$g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1, & 5 \leq t < 20, \\ 0, & 0 \leq t < 5 \text{ and } t \geq 20. \end{cases}$$

6.3 Delta Functions and Forcing

6.3.1 Impulse Forcing

Impulse forcing is the term used to describe a very quick push or pull on a system, such as the blow of a hammer or the force of an explosion. For example, consider the equation for a damped harmonic oscillator

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 26y = g(t),$$

where $g(t)$ is a function that is very large in a very short time interval, say $|t - t_0| < \tau$ and zero otherwise. The integral

$$I(\tau) = \int_{t_0-\tau}^{t_0+\tau} g(t)\,dt$$

or since $g(t)$ is zero outside of the interval $|t - t_0| < \tau$

$$I(\tau) = \int_{-\infty}^{\infty} g(t)\,dt$$

measures the strength or ***impulse*** of the forcing function $g(t)$. In particular, assume that $t_0 = 0$ and

$$g(t) = d_\tau(t) = \begin{cases} 1/2\tau, & -\tau < t < \tau \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $I(\tau) = 1$ in this case.

Now let us realize the forcing function over shorter and shorter time intervals with τ getting closer and closer to zero, we find that $I(\tau) = 1$ in all cases. Thus,

$$\lim_{\tau \to 0} d_\tau(t) = 0$$

for $t \neq 0$; however,

$$\lim_{\tau \to 0} I(\tau) = 1.$$

We can use this information to define the ***unit impulse function***, $\delta(t)$, to be the "function" that imparts an impulse of magnitude one at $t = 0$, but is zero for all values of t other than zero. In other words, $\delta(t)$ has the properties

$$\delta(t) = 0, \qquad t \neq 0;$$
$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

Of course, we study no such function in calculus. The "function" δ is an example of what is known as a ***generalized function***. We call δ, the ***Dirac delta function***.

We can define a unit impulse at a point t_0 by considering the function $\delta(t - t_0)$. In this case,

$$\delta(t - t_0) = 0, \qquad t \neq t_0;$$
$$\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1.$$

6.3.2 The Laplace Transform of the Dirac Delta Function

Even though the Dirac delta function is not a piecewise continuous, exponentially bounded function, we can define its Laplace transform as the limit of the Laplace transform of $d_\tau(t)$ as $\tau \to 0$. More specifically, assume that $t_0 > 0$ and

$$\mathcal{L}(\delta(t - t_0)) = \lim_{\tau \to 0} \mathcal{L}(d_\tau(t - t_0)).$$

Assuming that $t_0 - \tau > 0$, the Laplace transform of $d_\tau(t - t_0)$ is

$$\begin{aligned} \mathcal{L}(d_\tau(t - t_0)) &= \int_0^\infty e^{-st} d_\tau(t - t_0)\, dt \\ &= \int_{t_0 - \tau}^{t_0 + \tau} e^{-st} d_\tau(t - t_0)\, dt \\ &= \frac{1}{2\tau} \int_{t_0 - \tau}^{t_0 + \tau} e^{-st}\, dt \\ &= \frac{1}{2s\tau} e^{-st} \Big|_{t = t_0 - \tau}^{t = t_0 + \tau} \\ &= \frac{1}{2s\tau} e^{-st_0} (e^{s\tau} - e^{-s\tau}) \end{aligned}$$

$$= \frac{\sinh s\tau}{s\tau} e^{-st_0}.$$

We can use l'Hospital's rule to evaluate $(\sinh s\tau)/s\tau$ as $\tau \to 0$,

$$\lim_{\tau\to 0} \frac{\sinh s\tau}{s\tau} = \lim_{\tau\to 0} \frac{s\cosh s\tau}{s} = 1.$$

Thus,

$$\mathcal{L}(\delta(t-t_0)) = e^{-st_0}.$$

We can extend this result to allow $t_0 = 0$, by

$$\lim_{t_0\to 0} \mathcal{L}(\delta(t-t_0)) = \lim_{t_0\to 0} e^{-st_0} = 1.$$

Let us now solve the initial value problem

$$\begin{aligned} \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 26y &= \delta_4(t) \\ y(0) &= 1 \\ y'(0) &= 0. \end{aligned}$$

We can think of this as a damped harmonic oscillator that is struck by a hammer at time $t = 4$. Let $Y(s) = \mathcal{L}(y)(s)$ and take the Laplace transform of both sides of the differential equation to obtain

$$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 26Y(s) = \mathcal{L}(\delta_4)(s)$$

or

$$s^2Y(s) - s + 2sY(s) - 2 + 26Y(s) = e^{-4s}.$$

Solving for $Y(s)$, we have

$$Y(s) = \frac{s+2}{s^2+2s+26} + \frac{e^{-4s}}{s^2+2s+26}.$$

The inverse Laplace transform of $Y(s)$ is

$$\begin{aligned} y &= \mathcal{L}\left(\frac{s+2}{s^2+2s+26}\right) + \mathcal{L}\left(\frac{e^{-4s}}{s^2+2s+26}\right) \\ &= \mathcal{L}\left(\frac{s+2}{(s+1)^2+25}\right) + \frac{1}{5}\mathcal{L}\left(\frac{5e^{-4s}}{(s+1)^2+25}\right) \\ &= e^{-t}\cos 5t + \frac{1}{5}e^{-t}\sin 5t + \frac{1}{5}u_4(t)e^{-(t-4)}\sin(5(t-4)). \end{aligned}$$

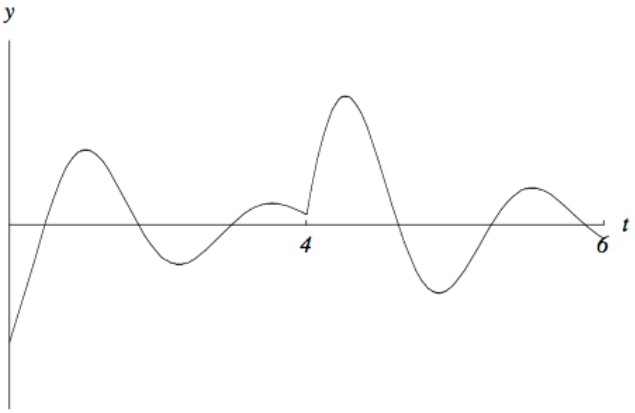

Figure 6.6: Solution to $y'' + 2y' + 26y - \delta_4(t)$

It is important to notice that we are using the Dirac delta function like an ordinary function. This requires some rigorous mathematics to justify that we can actually do this.

6.3.3 Important Lessons

- ***Impulse forcing*** is the term used to describe a very quick push or pull on a system, such as the blow of a hammer or the force of an explosion. For example, consider the equation for a damped harmonic oscillator

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t),$$

where $g(t)$ is a function that is very large in a very short time interval, say $|t - t_0| < \tau$ and zero otherwise. The integral

$$I(\tau) = \int_{t_0-\tau}^{t_0+\tau} g(t)\,dt$$

or since $g(t)$ is zero outside of the interval $|t - t_0| < \tau$

$$I(\tau) = \int_{-\infty}^{\infty} g(t)\,dt$$

measures the strength or ***impulse*** of the forcing function $g(t)$.

- We define the ***unit impulse function***, $\delta(t)$, to be the "function" that imparts an impulse of magnitude one at $t = 0$, but is zero for all values of t other than zero. In other words, $\delta(t)$ has the properties

$$\delta(t) = 0, \qquad t \neq 0;$$
$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1.$$

The "function" δ is an example of what is known as a ***generalized function***. We call δ, the ***Dirac delta function***.

- Similarly, we can define a unit impulse at a point t_0 by considering the function $\delta(t-t_0)$. In this case,

$$\delta(t-t_0)=0, \qquad t\neq 0;$$
$$\int_{-\infty}^{\infty}\delta(t-t_0)\,dt=1.$$

- The Laplace transform of the Dirac delta function is

$$\mathcal{L}(\delta(t-t_0))=e^{-st_0}.$$

  We can extend this result to allow $t_0=0$, by

$$\lim_{t_0\to 0}\mathcal{L}(\delta(t-t_0))=\lim_{t_0\to 0}e^{-st_0}=1.$$

- We can use the Dirac delta function to solve initial value problems such as

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+26y=\delta_4(t)$$
$$y(0)=1$$
$$y'(0)=0,$$

  or

$$\frac{d^2y}{dt^2}+p\frac{dy}{dt}+qy=g(t),$$

  where $g(t)$ is a function that is very large in a very short time interval.

6.3.4 Exercises

1. Find the solution of the initial value problem

$$2y''+y'+2y=\delta(t-5)$$
$$y(0)=y'(0)=0,$$

where $\delta(t)$ is the unit impulse function.

6.4 Convolution

6.4.1 ConvolutionApplying the Convolution Theorem

If f and g are two piecewise continuous exponentially bounded functions, then we define the ***convolution product*** of f and g to be

$$(f*g)(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau=\int_0^t f(\tau)g(t-\tau)\,d\tau.$$

The convolution product has many properties similar to those of ordinary multiplication.

- Commutivity: $f*g=g*f$.
- Distribution: $f*(g+h)=f*g+f*h$.
- Associativity: $f*(g*h)=(f*g)*h$.

- $0 * f = f * 0 = 0$.

All of these properties can be proven using the definition of convolution and the properties of integration.

There is, however, no multiplicative identity. In other words, $f * 1 \neq f$. For example, suppose that $f(t) = \cos t$. Then

$$(f * 1)(t) = \int_0^t \cos(t - \tau) \cdot 1 \, d\tau = \sin t.$$

Also, it may not be the case that $f * f$ is a nonnegative function.

One important property of the Laplace transform is how convolution products behave.

Theorem 6.7. *Let f and g be two piecewise continuous exponentially bounded functions, and suppose that $\mathcal{L}(f)(s) = F(s)$ and $\mathcal{L}(g)(s) = G(s)$ for $s \geq a > 0$. Then*

$$F(s)G(s) = \mathcal{L}(f * g)(s)$$

for $s > a$.

Proof. If

$$F(s) = \int_0^\infty e^{-s\xi} f(\xi) \, d\xi$$
$$G(s) = \int_0^\infty e^{-s\tau} g(\tau) \, d\tau,$$

then

$$\begin{aligned} F(s)G(s) &= \int_0^\infty e^{-s\xi} f(\xi) \, d\xi \int_0^\infty e^{-s\tau} g(\tau) \, d\tau \\ &= \int_0^\infty g(\tau) \left(\int_\tau^\infty e^{-st} f(t - \tau) \, dt \right) d\tau, \end{aligned}$$

where $\xi = t - \tau$ is the change of variable. Reversing the order of integration, we have

$$\int_0^\infty g(\tau) \left(\int_\tau^\infty e^{-st} f(t - \tau) \, dt \right) d\tau = \int_0^\infty e^{-st} \left(\int_0^t f(t - \tau) g(\tau) \, d\tau \right) dt.$$

However, this last expression is just the Laplace transform of $f * g$. □

Let us use the Convolution Theorem (Theorem 6.7) to calculate the inverse Laplace transform of

$$H(s) = \frac{a}{s^2(s^2 + a^2)} = \frac{1}{s^2} \cdot \frac{a}{s^2 + a^2}.$$

instead of using partial fractions. The inverse Laplace transform of $1/s^2$ is t, and the inverse Laplace transform of $a/(s^2 + a^2)$. is $\sin at$. Using the Convolution Theorem, the inverse Laplace transform of $H(s)$ is

$$h(t) = \int_0^t (t - \tau) \sin a\tau \, d\tau = \frac{at - \sin at}{a^2}.$$

We can also use the Convolution Theorem to solve initial value problems. Consider the IVP

$$y'' + 4y = g(t)$$

$$y(0) = 3$$
$$y'(0) = -1.$$

Taking the Laplace transform of both sides of the differential equation and applying the initial conditions, we obtain

$$s^2Y(s) - 3s + 1 + 4Y(s) = G(s),$$

where $G(s)$ is the Laplace transform of $g(t)$. Solving for $Y(s)$, we have

$$Y(s) = \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4} = 3\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}G(s).$$

The last term corresponds to the forcing term of our differential equation. Taking the inverse Laplace transform of both sides and applying the Convolution Theorem, we get

$$y = 3\cos 2t - \frac{2}{2}\sin 2t + \frac{1}{2}\int_0^t \sin 2(t-\tau)g(\tau)\,d\tau.$$

It is possible to write a solution for the initial value problem

$$ay'' + by' + cy = g(t)$$
$$y(0) = y_0$$
$$y'(0) = y_1$$

using the Convolution Theorem. Taking the Laplace transform of both sides of the differential equation and using the initial conditions, we have

$$(as^2 + bs + c)Y(s) - (as + b)y_0 - ay_1 = G(s)$$

or

$$Y(s) = \Phi(s) + \Psi(s),$$

where

$$\Phi(s) = \frac{(as+b)y_0 + ay_1}{as^2 + bs + c}$$
$$\Psi(s) = \frac{G(s)}{as^2 + bs + c}.$$

Therefore,

$$y = \phi(t) + \psi(t),$$

where $\phi(t) = \mathcal{L}^{-1}(\Phi(s))$ and $\psi(t) = \mathcal{L}^{-1}(\Psi(s))$. Observe that $\phi(t)$ is the solution to the initial value problem

$$ay'' + by' + cy = 0$$
$$y(0) = y_0$$
$$y'(0) = y_1.$$

Once we have values for a, b, and c, the function $\phi(t)$ is easy to find. To find $\psi(t)$, we first write

$$\Psi(s) = \frac{1}{as^2 + bs + c}G(s) = H(s)G(s),$$

where $H(s) = 1/(as^2 + bs + c)$. If $h(t)$ is the inverse Laplace transform of $H(s)$, then

$$\psi(t) = \mathcal{L}^{-1}(H(s)G(s)) = \int_0^t h(t-\tau)g(\tau)\,d\tau.$$

Let us consider the case where $G(s) = 1$. Consequently, $g(t) = \delta(t)$ and $\psi(s) = H(s)$. This means that $h(t)$ is a solution to the initial value problem

$$\begin{aligned} ay'' + by' + cy &= \delta(t) \\ y(0) &= y_0 \\ y'(0) &= y_1. \end{aligned}$$

For this reason, $h(t)$ is sometimes called the ***impulse response*** of the system.

6.4.2 Important Lessons

- If f and g are two piecewise continuous exponentially bounded functions, then we define the ***convolution production*** of f and g to be

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

- The convolution product has many properties similar to those of ordinary multiplication.
  - Commutivity: $f * g = g * f$
  - Distribution: $f * (g + h) = f * g + f * h$
  - Associativity: $f * (g * h) = (f * g) * h$
  - $0 * f = f * 0 = 0$

  There is, however, no multiplicative identity. In other words, $f * 1 \neq f$. Also, it may not be the case that $f * f$ is a nonnegative function.

- Let f and g be two piecewise continuous exponentially bounded functions, and suppose that $\mathcal{L}(f)(s) = F(s)$ and $\mathcal{L}(g)(s) = G(s)$ for $s \geq a > 0$. Then

$$F(s)G(s) = \mathcal{L}(f * g)(s)$$

  for $s > a$.

- It is possible to write a solution for the initial value problem

$$\begin{aligned} ay'' + by' + cy &= g(t) \\ y(0) &= y_0 \\ y'(0) &= y_1. \end{aligned}$$

  using the Convolution Theorem.

6.4.3 Exercises

1. Using convolution of functions, find the solution to the initial value problem

$$\begin{aligned} y'' + 4y' + 13y &= g(t) \\ y(0) = -5, \qquad y'(0) &= 2, \end{aligned}$$

and $g(t)$ is a piecewise continuous function.

A

Hints and Solutions to Selected Exercises

1.1.9 Exercises

1. Verify that $y(t) = 3e^{-2t}$ is a solution to the differential equation $y' = -2y$

$y' = -6e^{-2t} = -2(3e^{-2t}) = -2y$

2. Verify that $y(t) = Ce^{-t}$ is a solution to the differential equation $y' + y = 0$. Determine C if $y(0) = 3$.

We can determine that $C = 3$. Since $y' + y = -3e^{-t} + 3e^{-t} = 0$, $y(t) = 3e^{-t}$ is a solution to the initial value problem.

4. Write a differential equation that models a population P, where P changes at a rate proportional to the square root of P.

$dP/dt = k\sqrt{P}$

5. We can modify the logistic growth model to account for a threshold population. That is, if the population of a species drops below a certain level, the species is endanger of extinction. Consider the case of the black rhinoceros, once the most numerous of all rhinoceros species and now critically endangered. The black rhino, native to eastern and southern Africa, was estimated to have a population of about 100,000 around 1900. Because of hunting, habitat changes, competing species, and most of all illegal poaching, the number of black rhinos today is estimated to be below 3000. If the wild population becomes too low, the animals may not be able to find suitible mates and the black rhino will become extinct. There must be a minimum population for the species to continue. Suppose the this minimum or threshold population for the black rhino is 1000 animals and that remaining habitant in Africa will support no more that 20,000 rhinos.

(a) For what values of P is the rhino population increasing? What can be said about the value of dP/dt for these values of P?

(b) For what values of P is the rhino population decreasing? What can be said about the value of dP/dt for these values of P?

(c) For what values of P is the rhino population in equilibrium? What can be said about the value of dP/dt for these values of P?

(d) Find a differential equation that models the population of rhinos at time t.

(a) The population is increasing if $dP/dt > 0$ and $1000 < P < 20000$.

6. Given the equation $x' + px = q(t)$, where p is a constant and $q(t)$ is a continuous function defined on an interval I, show that

$$x(t) = Ce^{-at} + e^{-at}\int_{t_0}^{t} e^{as}b(s)\,ds$$

is a solution of this equation, where c is any constant and $t_0 \in I$.

Rewriting the differential equation as $x' + px - q(t) = 0$ and using the fact that

$$x'(t) = -aCe^{-at} - ae^{-at}\int_{t_0}^{t} e^{as}b(s)\,ds + b(t),$$

we see that

$$\begin{aligned} x'(t) + ax(t) - b(t) &= -aCe^{-at} - ae^{-at}\int_{t_0}^{t} e^{as}b(s)\,ds + b(t) \\ &+ \ aCe^{-at} + ae^{-at}\int_{t_0}^{t} e^{as}b(s)\,ds - b(t) \\ &= 0. \end{aligned}$$

8. Consider the following predator-prey systems of differential equations

$$\begin{aligned} \frac{dx}{dt} &= -\frac{x}{2} + \frac{xy}{2+y}, \\ \frac{dy}{dt} &= y(1-y) - \frac{xy}{2+y}. \end{aligned}$$

(a) Which equation models the prey population and which equation models the predator population?

(b) How does the prey population grow if there are no predators present?

(c) What happens if there are a lot of prey present?

Think about the limit of the interaction term as the number of prey becomes very large.

1.2.8 Exercises

1. Solve each of the following differential equations.

(a) $\dfrac{dy}{dx} = 8 - 2x$

(b) $\dfrac{dy}{dx} = \dfrac{x+1}{y}$

(c) $\dfrac{dx}{dt} = x^2t^2$, $x(0) = 1$

(d) $y' = \dfrac{t^2y - y}{y+1}$, $y(0) = -1$

(a) $y = -x^2 + 8x + C$; (c) $x = 3/(3 - t^3)$.

2. Solve each of the following equations in Exercise 1.2.8.1 using Sage.

(a)

```
x = var('x')
y = function('y')(x)
de = diff(y, x) == 8 - 2*x
desolve(de, y)
```

```
-x^2 + _C + 8*x
```

1.3.7 Exercises

2. Consider the differential equation $y' = f(y)$, where the graph of $f(y)$ is given below. Identify any equilibrium solutions and classify each equilibrium solution as a sink, source, or node.

3. What happens if we increase the harvest rate to 100 in Example 1.24.? What should be our strategy to maintain a viable population in the trout pond?

1.4.5 Exercises

1. Consider the differential equation

$$\frac{dy}{dt} = 3y - 1$$

with initial value $y(0) = 2$.

(a) Find the exact solution of the initial value problem.

(b) Use Euler's method with step size $h = 0.5$ to approximate the solution to the initial value problem on the interval $[0, 2]$ Your solution should include a table of approximate values of the dependent variable as well as the exact values of the dependent variable. Make sure that your approximations are accurate to *four* decimal places.

(c) Sketch the graph of the approximate and exact solutions.

(d) Use the error bound theorem (Theorem 1.33) to estimate the error at each approximation. Your solution should include a table of approximate values of the dependent variable the exact values of the dependent variable, the error estimates, and the actual error. Make sure that your approximations are accurate to *four* decimal places.

2. In this series of exercises, we will prove the error bound theorem for Euler's method (Theorem 1.33).

(a) Use Taylor's Theorem to show that for all $x \geq -1$ and any positive m,

$$0 \leq (1 + x)^m \leq e^{mx}.$$

(b) Use part (1) and geometric series to prove the following statement: If s and t are positive real numbers, and $\{a_i\}_{i=0}^{k}$ is a sequence satisfying

$$a_0 \geq -\frac{t}{s}$$

$$a_{i+1} \leq (1+s)a_i + t, \text{ for } i = 0, 1, 2, \ldots, k,$$

then

$$a_{i+1} \leq e^{(i+1)s}\left(\frac{t}{s} + a_0\right) - \frac{t}{s}.$$

(c) When $i = 0$, $y(t_0) = Y_0 = \alpha$ and the theorem is true. Use Euler's method and Taylor's Theorem to show that

$$|y_{i+1} - Y_{i+1}| \leq |y_i - Y_i| + h|f(t_i, y_i) - f(t_i, Y_i)| + \frac{h^2}{2}|y''(\xi_i)|$$

where $\xi_i \in (t_i, t_{i+1})$ and $y_k = y(t_k)$.

(d) Show that

$$|y_{i+1} - Y_{i+1}| \leq |y_i - Y_i|(1 + hL) + \frac{Mh^2}{2}.$$

(e) Let $a_j = |y_j - Y_j|$ for $j = 0, 1, \ldots, N$, $s = hL$, and $t = Mh^2/2$ and apply part (2) to show

$$|y_{i+1} - Y_{i+1}| \leq e^{(1+i)hL}\left(|y_0 - Y_0| + \frac{Mh^2}{2hL}\right) - \frac{Mh^2}{2hL}.$$

and derive that

$$|y_{i+1} - Y_{i+1}| \leq \frac{Mh}{2L}\left(e^{(t_{i+1}-a)L} - 1\right)$$

for each $i = 0, 1, \ldots, N-1$.

Hints for part (2):

- For fixed i show that

$$\begin{aligned} a_{i+1} &\leq (1+s)a_i + t \\ &\leq (1+s)[(1+s)a_{i-1} + t] + t \\ &\leq (1+s)\{(1+s)[(1+s)a_{i-2} + t] + t\} + t \\ &\vdots \\ &\leq (1+s)^{i+1}a_0 + [1 + (1+s) + (1+s)^2 + \cdots + (1+s)^i]t. \end{aligned}$$

- Now use a geometric sum to show that

$$a_{i+1} \leq (1+s)^{i+1}a_0 + \frac{t}{s}[(1+s)^{i+1} - 1] = (1+s)^{i+1}\left(\frac{t}{s} + a_0\right) - \frac{t}{s}.$$

- Apply part (1) to derive

$$a_{i+1} \leq e^{(i+1)s}\left(\frac{t}{s} + a_0\right) - \frac{t}{s}.$$

1.5.7 Exercises

2. Solve the initial value problem

$$y' = -\frac{2y}{x+1} + 2x$$
$$y(0) = 2.$$

$y = \dfrac{3x^4 + 8x^3 + 6x^2 + 12}{6(x+1)^2}$

3. Solve the initial value problem

$$y' = -\frac{2y}{1+x} + e^x$$
$$y(0) = 1.$$

$y = (e^x + 1)/(x+1)^2$

4. Solve the initial value problem

$$y' = -\frac{y}{x+2} + \frac{\cos x}{x+2}$$
$$y(0) = 1.$$

$y = \sin x/(x+2)$

5. Solve the initial value problem

$$y' = y\tan x + \frac{e^x}{\cos x}$$
$$y(0) = 1.$$

$y = e^x \sec x$

6. A 600-gallon tank initially contains 200 liters of water containing 10 kilograms of salt. Supposed that water containing 0.1 kilograms of salt flows into the top of the tank at a rate of 10 liters per minute. The water in the tank is kept well mixed, and 5 liters per minute are removed from the bottom of the tank. How much salt is in the tank when the tank is full?

If $x(t)$ is the amount of salt in the tank at time t, we know that $x(0) = 10$. The volume of the tank is $V = 600 + 5t$. We can model the amount of salt in the tank at time t with a differential equation,

$$\begin{aligned}\frac{dx}{dt} &= \text{rate in} - \text{rate out}\\ &= 10(0.1) - 5\frac{x}{V}\\ &= 1 - 5\frac{x}{600+5t}\\ &= 1 - \frac{x}{120+t}.\end{aligned}$$

The resulting equation

$$\frac{dx}{dt} + \frac{1}{120+t}x = 1$$

is a first order linear differential equation. An integrating factor for this equation is given by

$$\mu(t) = \exp\left(\int \frac{1}{120+t}\,dt\right) = 120 + t.$$

Multiplying both sides of the differential equation by $\mu(t)$, we have

$$\frac{d}{dt}[(120+t)x] = (120+t)\frac{dx}{dt} + x = (120+t)\left(\frac{dx}{dt} + \frac{1}{120+t}x\right) = 120 + t.$$

Integrating both sides of this equation, we obtain

$$(120+t)x = 120t + \frac{t^2}{2} + C.$$

Using the intial condition $x(0) = 10$, we can determine that $C = 1200$ or

$$x(t) = \frac{t^2 + 240t + 2400}{2t + 240}.$$

The tank is full at time $t = 400/5 = 80$, and the tank contains $x(80) = 70$ kilograms of salt when the tank is full.

7. A manager in a communications company contributes \$2400 per year into her retirement fund by making many small deposits throughout the year. The fund grows at a rate of 3.5% per year compounded continuously. After 35 years, she retires and begins and begins withdrawing from the retirement fund at a rate of \$3500 per month. If she does not make any deposits after she retires, how long will her retirement fund last? [*Hint*: Divide the problem into two smaller problems—one that deals with the situation before retirement and one that deals with the problem after retirement.]

8. Lake Baikal, located in southern Siberia, is only the seventh largest lake in the world in terms of surface area; however, it is the world's deepest lake. The lake has a depth of 1,642 meters, and the bottom lies 1,186.5 meters below sea level. Lake Baikal has a volume of 23,600 cubic kilometers and contains 20% of the world's unfrozen fresh water. In contrast, Lake Superior, the largest of the Great Lakes, has a volume of only 12,100 cubic kilometers. Although 544 rivers flow into Lake Baikal, there is only a single outlet, the Angara River. The outflow of the lake is pretty constant at 60.4 cubic kilometers per year.

Pollution is an increasing concern in Lake Baikal. One of the major polluters has been Baykalsk Pulp and Paper Mill. The mill was constructed in 1966 on the shoreline of Lake Baikal and regularly discharged waste into the lake. The plant was closed in November 2008 due to unprofitability, but production resumed in January 2010. The mill underwent a final bankruptcy in September 2013, but the longterm fate of the mill is still undecided.[1]

Suppose that we wish to understand how the pollution level changes in Lake Baikal over a period of years. Hypothetically, let us assume that the Baykalsk Pulp and Paper Mill has been responsible for most of the pollution in Lake Baikal for the last several decades. Suppose that at $t = 0$ years the mill ceases operation and there are no longer any pollutants discharged into the lake from the mill although there are still other sources of pollution. Let us assume that the lake is currently 6 times more polluted than these other sources of contaminants. We wish to know how long it will take for the pollution level to reduce to half of its current level of lake. Lake Baikal's waters are well-mixed and well-oxygenated in spite of its great depth, so we can model this situation as a simple mixing problem.

[1] Levy, Clifford J. (November 8, 2010). "Last Gasp for Factory Bequeathed by Soviets." The New York Times. Retrieved March 14, 2014 from http://www.nytimes.com/2010/11/09/world/europe/09baikal.html.

(a) The volume of water in Lake Baikal is $V = 23,600$ km^3, and

$$r_{\text{in}} = r_{\text{out}} = r = 60 \text{ km}^3/\text{year}$$

be the rates of inflow from the numerous rivers that feed the lake and outflow to the Angara River. Assume that $C = C_{\text{in}}$ is the pollutant concentration flowing into Lake Baikal and C_{out} is the concentration of the outflow into the Angara River (measured in metric tons per cubic km). If $x(t)$ is the amount of solute at time t in Lake Baikal, then

$$x_0 = x(0) = 6CV$$

is the initial amount of solute in the lake. Estimate Δx during the time interval $[t, t + \Delta t]$, where $\Delta t > 0$ is small.

(b) From your estimate of Δx in part (1), write an initial value problem that describes the amount of pollutants in the lake at time t.

(c) The equation that you found in part (2) is a first-order linear equation. Solve this equation

(d) Using part (3), predict how many years it will take to reduce the pollution in Lake Baikal to half of its current level.

9. *Variation of Parameters.* Consider the following method of solving the general linear equation of the first order,

$$y' + p(t)y = g(t). \tag{A.1}$$

(a) If $g(t)$ is identically zero, show that the solution is

$$y = A \exp\left[-\int p(t)\,dt\right],$$

where A is a constant.

(b) If $g(t)$ is not identically zero, assume that the solution is of the form

$$y = A(t) \exp\left[-\int p(t)\,dt\right], \tag{A.2}$$

where A is now a function of t. By substituting for y in the given differential equation (A.1), show that $A(t)$ must satisfy the condition

$$A'(t) = g(t) \exp\left[\int p(t)\,dt\right]. \tag{A.3}$$

(c) Find $A(t)$ from Equation (A.3). Then substitute for $A(t)$ in equation (A.2) and determine y. Verify that the solution obtain in this manner agrees with the solution given in the proof of Theorem 1.39. That is, show that this solution is equivalent to the solution

$$y = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)\,dt + C\right),$$

where

$$\mu(t) = \exp\left(\int p(t)\,dt\right).$$

This technique is know as *variation of parameters*, which we will revisit when we study second order linear differential equations.

10. *Bernoulli's equation* is an equation of the form

$$y' = a(t)y + f(t)y^n,$$

where $n \neq 0$ or 1. Bernoulli's equation is nonlinear and cannot be solved by the techniques that we have used to solve first order linear equations.[2]

(a) Using the substitution $z = y^{1-n}$, show that we can transform Bernoulli's equation into the linear equation

$$z' = (1-n)a(t)z + (1-n)f(t).$$

(b) Solve the equation $xy' + y = x^4y^3$.

11. The first-order nonlinear differential equation

$$y' = p(t) + q(t)y + r(t)y^2 \tag{A.4}$$

is known as the *Ricatti equation* and has some useful applications in control theory. If one solution $y_1(t)$ of the Ricatti equation is known, then a more general solution containing an arbitrary constant can be found by substituting $y = y_1(t) + 1/v(t)$ into equation (A.4) to find a first-order linear equation in v and t, which we can solve to find a general solution to the Ricatti equation.

(a) Show that this first-order linear equation is $v' + [q(t) + 2r(t)y_1(t)]v = -r(t)$.

(b) Find the solution to the Ricatti equation

$$y' = -\frac{1}{t^2} - \frac{1}{t}y + y^2$$

given the particular solution $y_1 = 1/t$.

(c) Find the solution to the Ricatti equation

$$y' = \cos t - y\tan t + y^2 \sec t$$

given the particular solution $y_1 = \sin t$.

(d) Find the solution to the Ricatti equation

$$y' = 2 - 3y + y^2$$

given the particular solution $y_1 \equiv 2$.

(a) If $y = y_1 + 1/v$, then $y' = y_1' - v'/v^2$. Substituting into our original equation, we obtain

$$y' = y_1' - \frac{v'}{v^2} = p + qy_1 + ry_1^2 - \frac{v'}{v^2}.$$

[2] Polking p. 63

On the other hand,

$$\begin{aligned} y' &= p + q\left(y_1 + \frac{1}{v}\right) + r\left(y_1 + \frac{1}{v}\right)^2 \\ &= p + qy_1 + \frac{q}{v} + ry_1^2 + \frac{2ry_1}{v} + \frac{r}{v^2} \\ &= y_1' + \frac{q}{v} + \frac{2ry_1}{v} + \frac{r}{v^2}. \end{aligned}$$

Therefore,

$$-\frac{v'}{v^2} = \frac{q}{v} + \frac{2ry_1}{v} + \frac{r}{v^2},$$

which is just the first-order linear equation

$$v' + [q(t) + 2r(t)y_1(t)]v = -r(t).$$

(b)

$$y = t + \frac{1}{C - t}$$

(c)

$$y(t) = \frac{1}{C\cos t - \sin t} + \sin t$$

(d)

$$y(t) = 2 + \frac{1}{Ce^t - 1}$$

12. Suppose that we have a population that not only grows logistically but also requires a minimum threshold population to survive. For example, the case of the North Pacific right whale, a species now very much on the endangered list. If the population drops too low, whales might not be able to find suitable mates and the species will eventually go extinct. In other words, the population will die out if it drops below a certain threshold. We can model this with the following equation,

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)(P - aN), \tag{A.5}$$

where P is the population of the whales at time t and N is the carrying capacity. The constants k and a are positive with $a < 1$.

(a) Find the equilibrium solutions of the this equation.

(b) Since equation (A.5) is autonomous, we can find a solution using separation of variables. Find this solution.

(c) Equation (A.5) is also a Ricatti equation (A.4). Since we know an equilibrium solution from part (1), we can use the method of the previous problem to find a general solution to (A.5). Find the general solution using the fact that we have a Ricatti equation and show that your solution agrees with the solution that you found in part (2)

1.6.4 Exercises

1. Which of the following initial value problems are guaranteed to have a unique solution by the Existence and Uniqueness Theorem (Theorem 1.40)? In each case, justify your conclusion.

(a) $y' = 4 + y^3$, $y(0) = 1$

(b) $y' = \sqrt{y}$, $y(1) = 0$

(c) $y' = \sqrt{y}$, $y(1) = 1$

(d) $x' = \dfrac{t}{x-2}$, $x(0) = 2$

(e) $x' = \dfrac{t}{x-2}$, $x(2) = 0$

(f) $y' = x \tan y$, $y(0) = 0$

(g) $y' = \dfrac{1}{t}y + 2t$, $y(0) = 1$

(a) There exists a unique solution to $y' = 4 + y^3$, $y(0) = 1$, since $f(t, y) = 4 + y^3$ and $\partial f(t, y)/\partial y = 3y^2$ are continuous at the point $(0, 1)$.

(b) The Existence and Uniqueness Theorem does not apply to $y' = \sqrt{y}$, $y(1) = 0$, since $f(t, y) = \sqrt{y}$ is not continuous at $(1, 0)$.

(c) There exists a unique solution to $y' = \sqrt{y}$, $y(1) = 1$, since $f(t, y) = \sqrt{y}$ and $\partial f(t, y)/\partial y = 1/(2\sqrt{y}\,)$ are both continuous at the point $(1, 1)$.

(d) The Existence and Uniqueness Theorem does not apply to $x' = t/(x - 2)$, $x(0) = 2$, since $f(t, x) = t/(x - 2)$ is not continuous at $(0, 2)$.

(e) There exists a unique solution to $x' = t/(x - 2)$, $x(2) = 0$, since $f(t, x) = t/(x - 2)$ and $\partial f(t, x)/\partial x = -t/(x - 2)^2$ are both continuous at the point $(2, 0)$.

(f) There exists a unique solution to $y' = x \tan y$, $y(0) = 0$, since $f(x, y) = x \tan y$ and $\partial f(x, y)/\partial y = x \sec^2 y$ are both continuous at the point $(0, 0)$.

(g) The Existence and Uniqueness Theorem does not apply to $y' = y/t + 2t$, $y(0) = 1$, since $f(t, y) = y/t + 2t$ is not continuous at $(0, 1)$.

2. Find an explicit solution to the initial value problem

$$\begin{aligned} y' &= \frac{1}{(t-1)(y+1)} \\ y(0) &= 1. \end{aligned}$$

Use your solution to determine the interval of existence.

3. Consider the initial value problem

$$\begin{aligned} y' &= 3y^{2/3} \\ y(0) &= 0. \end{aligned}$$

(a) Show that the constant function, $y(t) \equiv 0$, is a solution to the initial value problem.

(b) Show that

$$y(t) = \begin{cases} 0, & t \le t_0 \\ (t - t_0)^3, & t > t_0 \end{cases}$$

is a solution for the initial value problem, where t_0 is any real number. Hence, there exists an infinite number of solutions to the initial value problem. [*Hint*: Make sure that the derivative of $y(t)$ exists at $t = t_0$.]

(c) Explain why this example does not contradict the Existence and Uniqueness Theorem.

4. Let $\phi_n(x) = x^n$ for $0 \le x \le 1$ and show that

$$\lim_{n\to\infty} \phi_n(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1. \end{cases}$$

This is an example of a sequence of continuous functions that does not converge to a continuous function, which helps explain the need for *uniform continuity* in the proof of the Existence and Uniqueness Theorem.

5. Consider the initial value problem

$$\begin{aligned} y' &= 2ty + t \\ y(0) &= 1. \end{aligned}$$

(a) Use the fact that $y' = 2ty + t$ is a first-order linear differential equation to find a solution to the initial value problem.

(b) Let $\phi_0(t) = 1$ and use Picard iteration to find $\phi_n(t)$.

(c) Show that the sequence $\{\phi_n(t)\}$ converges to the exact solution that you found in part (a) as $n \to \infty$.

6. *The following series of exercises, prove the Existence and Uniqueness Theorem for first-order differential equations.*

Use the Fundamental Theorem of Calculus to show that the function $u = u(t)$ is a solution to the initial value problem

$$\begin{aligned} x' &= f(t, x) \\ x(t_0) &= x_0, \end{aligned}$$

if and only if u is a solution to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

7. If $\partial f/\partial x$ is continuous on the rectangle

$$R = \{(t, x) : 0 \le |t - t_0| < a, 0 \le |x - x_0| < b\},$$

prove that there exists a $K > 0$ such that

$$|f(t, x_1) - f(t, x_2)| \le K|x_1 - x_2|$$

for all (t, x_1) and (t, x_2) in R.

8. Define the sequence $\{u_n\}$ by

$$u_0(t) = x_0,$$
$$u_{n+1} = x_0 + \int_{t_0}^{t} f(s, u_n(s))\, ds, \qquad n = 1, 2, \ldots.$$

Use the result of the previous exercise to show that

$$|f(t, u_n(t)) - f(t, u_{n-1}(t))| \leq K|u_n(t) - u_{n-1}(t)|.$$

9. Show that there exists an $M > 0$ such that

$$|u_1(t) - x_0| \leq M|t - t_0|.$$

10. Show that

$$|u_2(t) - u_1(t)| \leq \frac{KM|t - t_0|^2}{2}.$$

11. Use mathematical induction to show that

$$|u_n(t) - u_{n-1}(t)| \leq \frac{K^{n-1}M|t - t_0|^n}{n!}.$$

12. Since

$$u_n(t) = u_1(t) + [u_2(t) - u_1(t)] + \cdots + [u_n(t) - u_{n-1}(t)],$$

we can view $u_n(t)$ as a partial sum for the series

$$u_0(t) + \sum_{n=1}^{\infty} [u_n(t) - u_{n-1}(t)].$$

If we can show that this series converges absolutely, then our sequence will converge to a function $u(t)$. Show that

$$\sum_{n=1}^{\infty} |u_n(t) - u_{n-1}(t)| \leq \frac{M}{K} \sum_{n=1}^{\infty} \frac{(K|t - t_0|)^n}{n!} \leq \frac{M}{K} \left(e^{K|h|} - 1 \right),$$

where h is the maximum distance between (t_0, x_0) and the boundary of the rectangle R. Since $|u_n(t) - u_{n-1}(t)| \to 0$, we know that $u_n(t)$ converges to a continuous function $u(t)$ that solves our equation.[3]

13. To show uniqueness, assume that $u(t)$ and $v(t)$ are both solutions to

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

Show that

$$|u(t) - v(t)| \leq K \int_{t_0}^{t} |u(s) - v(s)|\, ds.$$

14.

[3] We must a theorem from advanced calculus here to ensure uniform continuity. Any sequence of functions that converges uniformly, must converge to a continuous function.

(a) Define[4]

$$\phi(t) = \int_{t_0}^{t} |u(s) - v(s)|\, ds,$$

then $\phi(t_0) = 0$ and $\phi(t) \geq 0$ for $t \geq t_0$. Show that

$$\phi'(t) = |u(t) - v(t)|.$$

(b) Since

$$|u(t) - v(t)| - K \int_{t_0}^{t} |u(s) - v(s)|\, ds \leq 0,$$

we know that

$$\phi'(t) - K\phi(t) \leq 0.$$

Use this fact to show that

$$\frac{d}{dt}\left[e^{-Kt}\phi(t)\right] \leq 0.$$

Conclude that

$$\phi(t) = \int_{t_0}^{t} |u(s) - v(s)|\, ds = 0$$

for $t \geq t_0$ or for all $t \geq t_0$ and $u(t) = v(t)$.

1.7.4 Exercises

1. Describe the phase line portraits for each of the following equations and how they depend on the parameter λ. Draw the bifurcation diagram for each equation.

(a) $y' = \lambda y - \sin y$ for $\lambda > 2/\pi$.

(b) $y' = \lambda y^2 - 1$ for $\lambda \in \mathbb{R}$.

2. Outbreaks of the spruce budworm have been responsible for some major deforestations in Canada and the United States. The equation

$$x' = r\left(1 - \frac{x}{K}\right)x - c\frac{x^2}{a + x^2}$$

has been used to describe the dynamics of spruce budworm populations, where the variable x denotes the population or density of the insect [16]. One explanation that has been given for the occurrence of outbreaks is based on the multiple bifurcations that occur with this differential equation.

(a) If $a = 0.01$, $c = 1$, and $K = 1$, we have a family of differential equations parameterized by r,

$$x' = rx(1 - x) - \frac{x^2}{0.01 + x^2}.$$

Solve the equation equation

$$rx(1 - x) - \frac{x^2}{0.01 + x^2} = 0$$

and plot the result in the xr-plane for $0 \leq r < 1$.

[4] A similar argument will work for $t \leq t_0$.

(b) To find the bifurcation diagram for the spruce budworm equation, reflect the graph obtained in part (1) about the line $r = x$ line.

(c) Estimate the two bifurcation values from your graph. Explain what happens to the population as r increases. That is, when does an outbreak occur? What happens after an outbreak?

3. The differential equation

$$\frac{dy}{dt} = y - 4t + y^2 - 8yt + 16t^2 + 4.$$

is not autonomous, separable, or linear; however, we can solve this equation with a change of variable.

(a) Transform this equation into a new differential equation of the form

$$\frac{du}{dt} = f(u)$$

by letting $u = y - 4t$.

(b) Sketch the phase line for this new equation, $u' = f(u)$, and sketch several solutions.

(c) Find the solutions of the original differential equation that correspond to the equilibrium solutions of $u' = f(u)$. Graph these solutions in ty-plane. Also, sketch the graphs of the solutions that you plotted in part (b).

(d) Solve the differential new equation and use this information to solve the original differential equation.

2.1.5 Exercises

1. Consider the following predator-prey system,

$$\begin{aligned} x' &= 2x(1 - x/2) - 1.2xy \\ y' &= -y + 0.9xy, \end{aligned}$$

where $x(t)$ is the population of the prey and $y(t)$ is the population of the predator at time t.

(a) How would you modify this system to include the effect of hunting of the prey at a rate of α units of prey per unit of time?

(b) How would you modify this system to include the effect of hunting of the predators at a rate proportional to the number of predators?

(c) Suppose the predators discover a second, unlimited source of food, but they still prefer to eat the primary prey when they can catch them. How would you modify the system to include this assumption?

(d) Suppose the predators discover a second source of food that is limited in supply. How would you modify the system to include this assumption?

(e) Suppose that the predators migrate to an area if there is more than five times as many prey in that area ($x > 5y$), and that they move away if there are fewer than five times as many prey as predators. How would you modify the system to take this into account?

(f) Suppose that prey move out of an area at a rate proportional to the number of predators in the area. How would you modify the system to take this into account?

2. Consider the equation

$$\frac{d^2}{dt^2} + \frac{k}{m}y = 0$$

for the motion of a simple harmonic oscillator.

(a) Consider the function $y(t) = \cos \beta t$. Under what conditions on β is $y(t)$ a solution to the differential equation?

(b) What initial condition ($t = 0$) in the yv-plane corresponds to this solution?

(c) In terms of k and m, what is the period of this solution?

(d) Sketch the solution curve in the yv-plane that corresponds to this solution.

3. A mass weighing 4 pounds stretches a spring 4 inches.

(a) Consider the function $y(t) = \cos \beta t$. Under what conditions on β is $y(t)$ a solution to the differential equation?

(b) Formulate an initial-value problem that corresponds to the motion of this undamped mass-spring system if the mass is extended 1 foot from its rest position and released with no initial velocity.

(c) Using the result of the previous exercise, find the solution of this initial-value problem.

4. SEIR model

2.2.6 Exercises

1. Consider the competing species model

$$\frac{dx}{dt} = x(1-x) - \beta xy$$
$$\frac{dy}{dt} = y(1-y) - \beta xy,$$

where the species interact weakly, say $\beta = 1/2$.

(a) Find the x and y-nullclines for this system.

(b) Find all equilibrium points for this system.

(c) Sketch the phase plane for this system.

(d) If the inital populations are given by $x(0) = 2.4$ and $y(0) = 0.3$, what happens to the two populations as $t \to \infty$?

2. For the following two systems of equations

- Find the equilibrium points of the system.
- Sketch the phase plane and direction field for each system using technology.
- Briefly describe the behavior of typical solutions.

(a)

$$x' = \cos y$$
$$y' = -x + y$$

(b)

$$x' = y$$
$$y' = x - x^3 - y$$

3. Consider an epidemic that moves through an isolated population. We will make the following assumptions about the epidemic.

- Individuals are infected at a rate proportional to the product of the number of infected and susceptible individuals. We assume that the constant of proportionality is α.
- The length of the incubation period is negligible, and infectious individuals are immediately infectious.
- On the average, an infected individual dies after 10 days.
- Only a single individual is initially ill.
- Infected individuals do not give birth, but susceptible individuals have a birth rate of 0.0003 per individual per year. Newborns are susceptible.

If $S(t)$ is the number of susceptible and $I(t)$ is the number of infected people, then

$$\frac{dS}{dt} = -\alpha SI + 0.0003S \tag{A.6}$$
$$\frac{dI}{dt} = \alpha SI - 0.1I. \tag{A.7}$$

The constant α is a measure of the relative infectivity of the disease. Some diseases such as Ebola, a viral hemorrhagic fever, are extremely infectious with a mortaility rate of up to 90%. On the other hand, AIDS which is caused by the HIV virus, has a much lower transmission rate. The goal of this exercise is to examine the differences between the two.

(a) If $\alpha = 0.05$, draw the phase portrait. Be sure to label all nullclines and equilibrium solutions. Suppose that $S(0) = 1000$ and $I(0) = 1$. What happens to the solution curve as $t \to \infty$?

(b) If $\alpha = 0.000001$, draw the phase portrait. Be sure to label all nullclines and equilibrium solutions. Suppose that $S(0) = 30000$ and $I(0) = 1$. What happens to the solution curve as $t \to \infty$?

(c) What conclusions can you draw about the behavior of the two different epidemics?

2.3.6 Exercises

1. Consider the system

$$x' = 2x$$
$$y' = y$$

with initial conditions $x(0) = 1$ and $y(0) = 3$.[5]

(a) Show that $\mathbf{x}(t) = (e^{2t}, 3e^t)$ satisfies the initial value problem.

(b) Use Euler's method with step size $\Delta t = 0.5$ to approximate this solution, and check how close the approximation is to the real solution when $t = 2$, $t = 4$, and $t = 6$.

(c) Use Euler's method with step size $\Delta t = 0.1$ to approximate this solution, and check how close the approximation is to the real solution when $t = 2$, $t = 4$, and $t = 6$.

(d) Discuss how and why the Euler approximations differ from the real solution.

2.4.5 Exercises

1. This is an exercise.

3.1.5 Exercises

1. If

$$A = \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4 & 1 \\ -1 & -2 \end{pmatrix},$$

find each of the following.

(a) $A + B$

(b) $2A - 3B$

(c) AB

(d) BA

(e) A^{-1}

(f) B^{-1}

2. If

$$A = \begin{pmatrix} 2 & 1-i \\ 2-i & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4i & 1-i \\ 1+3i & -2-i \end{pmatrix},$$

find each of the following.

(a) $A + B$

(b) $3A - 2B$

(c) AB

[5] You will find Sage very useful for this exercise

(d) BA

3. If

$$A = \begin{pmatrix} 3 & -2 \\ 0 & -1 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} -2 \\ 3 \end{pmatrix},$$

find each of the following.

(a) $A\mathbf{x}$

(b) $A\mathbf{y}$

(c) $\mathbf{x}^T\mathbf{y}$

(d) $\mathbf{y}^T\mathbf{x}$

4. Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. Prove that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent if and only if

$$\det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \neq 0.$$

3.2.5 Exercises

5. Consider the linear system

$$\begin{aligned} x' &= x + 3y + (t - 3t^2) \\ y' &= x - y + (2 - t + t^2) \\ x(0) &= 1 \\ y(0) &= -1. \end{aligned}$$

(a) Find the general solution of the homogeneous system

$$\begin{aligned} x' &= x + 3y \\ y' &= x - y \end{aligned}$$

(b) Find a particular solution for

$$\begin{aligned} x' &= x + 3y + (t - 3t^2) \\ y' &= x - y + (1 + t - t^2) \end{aligned}$$

(c) Find the solution of

$$\begin{aligned} x' &= x + 3y + (t - 3t^2) \\ y' &= x - y + (1 + t - t^2) \\ x(0) &= 1 \\ y(0) &= -1. \end{aligned}$$

Assume that your solution must be of the form

$$\mathbf{x}_p = \begin{pmatrix} a_2t^2 + a_1t + a_0 \\ b_2t^2 + b_1t + b_0. \end{pmatrix}$$

This is called the ***method of undetermined coefficients***.

3.3.5 Exercises

1. This is an exercise.

3.4.5 Exercises

1. Solve each of the following linear systems for the given initial values. Sketch the phase plane of each system and determine the nature of the equilibrium solution(s). Do this first without the use of technology, then check your answer with *Sage*.

(a)

$$\begin{aligned} x' &= 2x + 2y \\ y' &= -4x + 6y \\ x(0) &= 2 \\ y(0) &= -3 \end{aligned}$$

(b)

$$\begin{aligned} x' &= 2x - 6y \\ y' &= 2x + y \\ x(0) &= 2 \\ y(0) &= 1 \end{aligned}$$

(c)

$$\begin{aligned} x' &= 2x + y \\ y' &= -x + 4y \\ x(0) &= 1 \\ y(0) &= -1 \end{aligned}$$

(d)

$$\begin{aligned} x' &= 4x + 2y \\ y' &= -2x - y \\ x(0) &= 1 \\ y(0) &= -1 \end{aligned}$$

2. Determine a computable condition that guarantees that, if a 2×2 matrix A has complex eigenvalues, then the solutions of $\mathbf{x}' = A\mathbf{x}$ travel around the origin in a counterclockwise direction.

3.5.4 Exercises

1.

3.6.7 Exercises

1. This is an exercise.

3.7.4 Exercises

1. Consider the one-parameter family of linear systems given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & \sqrt{2}+a/2 \\ \sqrt{2}-a/2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(a) Sketch the path traced out by this family of linear systems in the trace-determinant plane as a varies.

(b) Discuss any bifurcations that occur along this path and compute the corresponding values of a.

2. Consider the two-parameter family of linear systems

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Identify all of the regions in the ab-plane where this system possesses a saddle, a sink, a spiral sink, and so on.

3.8.5 Exercises

1. Solve the system

$$\begin{aligned} x' &= -5x + 15y - 4z \\ y' &= -x + 5y - z \\ z' &= 4x - 6y + 3z \\ x(0) &= -1 \\ y(0) &= 2 \\ z(0) &= 0 \end{aligned}$$

2. Find the general solution of the system

$$\begin{aligned} x_1' &= 6x_1 + 9x_2 + x_3 + 4x_4 \\ x_2' &= -x_1 + x_2 + x_3 \\ x_3' &= 3x_1 + 8x_2 + 4x_3 + 4x_4 \\ x_4' &= -3x_1 - 8x_2 - 7x_3 - 5x_4 \end{aligned}$$

3. Let $ax'' + bx' + cx = 0$, where $a \neq 0$ and $b^2 - 4ac = 0$.

(a) Show that $x_1(t) = e^{-bt/2a}$ is a solution to $ax'' + bx' + cx = 0$.

(b) Assume that

$$y = v(t)x_1(t) = v(t)e^{-bt/2a}$$

is a solution to $ax'' + bx' + cx = 0$ and show that $v(t) = c_1 + c_2 t$. Thus,

$$x(t) = c_1 e^{-bt/2a} + c_2 t e^{-bt/2a}$$

is a general solution for $ax'' + bx' + cx = 0$.

4. Consider the equation

$$y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0.$$

Determine all values of α, if any, for which all solutions tend toward zero as $t \to \infty$. Also, determine the values of α, if any, for which all nonzero solutions become unbounded as $t \to \infty$.

5. Solve each of the following initial value problems.

(a)

$$\begin{aligned} y'' - 2y' + 5y &= 0 \\ y(\pi/2) &= 0 \\ y'(\pi/2) &= 2 \end{aligned}$$

(b)

$$\begin{aligned} 9y'' - 12y' + 4y &= 0 \\ y(0) &= 2 \\ y'(0) &= -1 \end{aligned}$$

(c)

$$\begin{aligned} y'' + 8y' - 9y &= 0 \\ y(1) &= 1 \\ y'(1) &= 0 \end{aligned}$$

(d)

$$\begin{aligned} y'' + 2ay' + (a^2 + 1)y &= 0 \\ y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

6. *Reduction of Order.* Suppose that $x_1(t)$ is a solution (not identically zero) to the equation

$$x'' + p(t)x' + q(t)x = 0.$$

(a) Assume that $x(t) = v(t)x_1(t)$ is a solution to $x'' + p(t)x' + q(t)x = 0$ and derive the equation

$$x_1 v'' + (2x_1' + px_1)v' = 0. \tag{A.8}$$

(b) Let $u = v'$ and show that ((A.10)) is a first-order linear differential equation in u.

(c) Show that $x_1(t) = 1/t$ is a solution to

$$2t^2 x'' + 3tx' - x = 0 \tag{A.9}$$

for $t > 0$ and find a second linearly independent solution to ((A.11)).

7. *Euler Equations.* An important class of second-order linear differential equations are equations of the form

$$t^2y'' + \alpha ty' + \beta y = 0,$$

where $t > 0$ and α and β are real constants. An equation of this form is called an *Euler equation.*

(a) Show that the substitution $x = \ln t$ transforms an Euler equation into an equation of constant coefficients. *Hint*: Show that

$$\frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx}$$
$$\frac{d^2y}{dt^2} = \left(\frac{dx}{dt}\right)^2 \frac{d^2y}{dx^2} + \frac{d^2x}{dt^2}\frac{dy}{dx}.$$

(b) Solve the equation

$$t^2y'' + 4ty' + 2y = 0.$$

3.9.4 Exercises

1. Solve the system

$$\begin{aligned} x' &= -5x + 15y - 4z \\ y' &= -x + 5y - z \\ z' &= 4x - 6y + 3z \\ x(0) &= -1 \\ y(0) &= 2 \\ z(0) &= 0 \end{aligned}$$

2. Find the general solution of the system

$$\begin{aligned} x_1' &= 6x_1 + 9x_2 + x_3 + 4x_4 \\ x_2' &= -x_1 + x_2 + x_3 \\ x_3' &= 3x_1 + 8x_2 + 4x_3 + 4x_4 \\ x_4' &= -3x_1 - 8x_2 - 7x_3 - 5x_4 \end{aligned}$$

3. Let $ax'' + bx' + cx = 0$, where $a \neq 0$ and $b^2 - 4ac = 0$.

(a) Show that $x_1(t) = e^{-bt/2a}$ is a solution to $ax'' + bx' + cx = 0$.

(b) Assume that

$$y = v(t)x_1(t) = v(t)e^{-bt/2a}$$

is a solution to $ax'' + bx' + cx = 0$ and show that $v(t) = c_1 + c_2t$. Thus,

$$x(t) = c_1e^{-bt/2a} + c_2te^{-bt/2a}$$

is a general solution for $ax'' + bx' + cx = 0$.

4. Consider the equation

$$y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0.$$

Determine all values of α, if any, for which all solutions tend toward zero as $t \to \infty$. Also, determine the values of α, if any, for which all nonzero solutions become unbounded as $t \to \infty$.

5. Solve each of the following initial value problems.

(a)

$$\begin{aligned} y'' - 2y' + 5y &= 0 \\ y(\pi/2) &= 0 \\ y'(\pi/2) &= 2 \end{aligned}$$

(b)

$$\begin{aligned} 9y'' - 12y' + 4y &= 0 \\ y(0) &= 2 \\ y'(0) &= -1 \end{aligned}$$

(c)

$$\begin{aligned} y'' + 8y' - 9y &= 0 \\ y(1) &= 1 \\ y'(1) &= 0 \end{aligned}$$

(d)

$$\begin{aligned} y'' + 2ay' + (a^2 + 1)y &= 0 \\ y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

6. *Reduction of Order.* Suppose that $x_1(t)$ is a solution (not identically zero) to the equation

$$x'' + p(t)x' + q(t)x = 0.$$

(a) Assume that $x(t) = v(t)x_1(t)$ is a solution to $x'' + p(t)x' + q(t)x = 0$ and derive the equation

$$x_1 v'' + (2x_1' + px_1)v' = 0. \tag{A.10}$$

(b) Let $u = v'$ and show that ((A.10)) is a first-order linear differential equation in u.

(c) Show that $x_1(t) = 1/t$ is a solution to

$$2t^2 x'' + 3tx' - x = 0 \tag{A.11}$$

for $t > 0$ and find a second linearly independent solution to ((A.11)).

7. *Euler Equations.* An important class of second-order linear differential equations are equations of the form

$$t^2y'' + \alpha ty' + \beta y = 0,$$

where $t > 0$ and α and β are real constants. An equation of this form is called an *Euler equation.*

(a) Show that the substitution $x = \ln t$ transforms an Euler equation into an equation of constant coefficients. *Hint*: Show that

$$\frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx}$$
$$\frac{d^2y}{dt^2} = \left(\frac{dx}{dt}\right)^2 \frac{d^2y}{dx^2} + \frac{d^2x}{dt^2}\frac{dy}{dx}.$$

(b) Solve the equation

$$t^2y'' + 4ty' + 2y = 0.$$

4.2.6 Exercises

1. Find the solution of the initial-value problem

$$y'' + 6y' + 8y = \cos 3t$$
$$y(0) = 0$$
$$y'(0) = 0.$$

The solution to the homogeneous equation $y'' + 7y' + 10y = 0$ is

$$y_h = c_1e^{-4t} + c_2e^{-2t}.$$

Using the method of undetermined coefficients, we can find a particular solution

$$y_p = \frac{2}{235}(18\sin 3t - \cos 3t).$$

Thus, the general solution is

$$y = y_h + y_p = c_1e^{-4t} + c_2e^{-2t} + \frac{2}{235}(18\sin 3t - \cos 3t).$$

Applying the initial conditions $y(0) = 0$ and $y'(0) = 0$, we have

$$y = -\frac{16}{235}e^{-4t} + \frac{18}{235}e^{-2t} + \frac{2}{235}(18\sin 3t - \cos 3t).$$

2. Compute the solution of the initial value problem

$$y'' + 4y = 3\cos 2t$$
$$y(0) = y'(0) = 0.$$

The solution to the homogeneous equation, $y'' + 4y = 0$ is

$$y_h = c_1\cos 2t + c_2\sin 2t.$$

The complex version of the equation is

$$y'' + 4y = 3e^{2it}.$$

Assume that the particular solution to this equation is $y_c = ate^{2it}$. We include a factor of t, since e^{2it} is a solution to the homogeneous equation. Substituting y_c and y_c'' into the differential equation, we have

$$y_c'' + 4y_c = 4a(i-t)e^{2it} + 4ate^{2it} = 4aie^{2it} = 3e^{2it}.$$

Thus, $a = 3/(4i) = -3i/4$. Taking the real part of

$$y_c = -\frac{3}{4}it(\cos 2t + i\sin 2t),$$

we have

$$y_p = \frac{3}{4}t\sin 2t.$$

Therefore the general solution is

$$y = y_h + y_p = c_1\cos 2t + c_2\sin 2t + \frac{3}{4}t\sin 2t.$$

Applying the initial conditions $y(0) = y'(0) = 0$, we find that $c_1 = c_2 = 0$ and

$$y = \frac{3}{4}t\sin 2t.$$

3. Use the complex method to find a particular solution to

$$y'' + 7y' + 10y = -4\sin 3t.$$

Write your solution as a function of cosine. What is the amplitude, the period, and the frequency of your solution?

If we assume that the complex solution has form $y_c = Ae^{3it}$, then

$$y_c'' + 7y_c' + 10y_c = A(1+21i)e^{3it} = -4e^{3it},$$

and

$$A = \frac{-4}{1+21i} = \frac{-2+42i}{221}.$$

In polar form,

$$A = |A|e^{i\phi} = \frac{4}{\sqrt{442}}e^{i\phi},$$

where $\phi = \pi + \arctan(-21)$. Thus, $y_c = Ae^{3it} = 4e^{i(3t+\phi)}/\sqrt{442}$. Using the imaginary part of this solution, we obtain

$$y = \frac{4}{\sqrt{442}}\sin 3t + \phi = \frac{4}{\sqrt{442}}\cos(3t-\theta),$$

where $\theta = \phi - \pi/2 = \pi/2 + \arctan(-21)$. The amplitude is $4/\sqrt{442}$, the period is $2\pi/3$, and the frequency is $3/2\pi$.

4. We define two functions, $f(t)$ and $g(t)$, to be *linearly independent* on an open interval $I = (a, b)$ if there do not exist nonzero constants c_1 and c_2 such that

$$c_1 f(t) + c_2 g(t) = 0$$

for all $t \in I$. Otherwise, $f(t)$ and $f(t)$ are *linearly dependent*. Suppose that $f(t)$ and $g(t)$ are solutions to the homogeneous linear equation

$$y'' + py' + qy = 0.$$

Show that $f(t)$ and $g(t)$ are linearly dependent on an interval $I = (a, b)$ if and only if $W[f, g](t) \equiv 0$.

Suppose that that $f(t)$ and $g(t)$ are linearly dependent on an interval $I = (a, b)$. Then one function is a multiple of the other, say $f(t) = cg(t)$. Thus, $f'(t) = cg'(t)$.

$$W(f, g)(t) = \det \begin{pmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{pmatrix} = f(t)g'(t) - f'(t)g(t) = cg(t)g'(t) - cg'(t)g(t) = 0.$$

Conversely, suppose that

$$W(f, g)(t) = \det \begin{pmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{pmatrix} = 0,$$

for all t in (a, b). If $g = 0$, then $0f = g$ and the two functions are linearly dependent. Assume that $g(t_0) \neq 0$ for some t_0 in (a, b). Since g is differentiable, it must also be continuous and there is some interval (c, d) contained in (a, b) such that $t_0 \in (c, d)$ and g does not vanish on this interval. Therefore,

$$\frac{d}{dt}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2} = -\frac{W(f, g)}{g^2} = 0,$$

and f/g is constant on the interval (c, d). Thus, $f(t_0) = cg(t_0)$ and $f(t_0') = cg'(t_0)$. Since f and cg are both solutions to the differential equation $y'' + py' + qy = 0$ and have the same initial condition, $f(t) = cg(t)$ for all $t \in (a, b)$ by the existence and uniqueness theorem. Consequently, f and g are linearly dependent.

5. *Abel's Theorem.* If y_1 and y_2 are solutions of the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on an open interval $I = (a, b)$, show that

$$W[y_1, y_2](t) = c \exp\left(-\int p(t)\, dt\right),$$

for some constant c that depends on y_1 and y_2 but not on t.

(a) Use Abel's Theorem to find the Wronskian of $2t^2y'' + 3ty' - y = 0$ up to a constant multiple, where $t > 0$.

(b) Prove Abel's Theorem.

(a) We can rewrite $2t^2y'' + 3ty' - y = 0$ as

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0.$$

Since $p(t) = 1/2t$, Abel's Theorem tells us that

$$W[y_1, y_2](t) = c \exp\left(-\int \frac{3}{2t}\, dt\right) = c \exp\left(-\frac{3}{2}\ln t\right) = ct^{-3/2}.$$

(b) Since y_1 and y_2 are solutions to our differential equation, we know that

$$y_1'' + p(t)y_1' + q(t)y_1 = 0$$
$$y_2'' + p(t)y_2' + q(t)y_2 = 0.$$

Multiplying the first equation by y_2 and the second equation by y_1 and subtracting, we obtain

$$(y_1y_2'' - y_1''y_2) + p(t)(y_1y_2' - y_1'y_2) = 0. \tag{A.12}$$

If

$$W(t) = W(y_1, y_2)(t) = y_1y_2' - y_1'y_2,$$

then

$$W' = y_1y_2'' - y_1''y_2,$$

and equation (A.12) becomes

$$W'' + p(t)W = 0.$$

This equation is separable with solution

$$W(t) = c\exp\left(-\int p(t)\,dt\right),$$

6. *The Method of Variation of Parameters.* In this problem we will describe another method of finding a particular solution to a nonhomogeneous equation,

$$y'' + p(t)y' + q(t)y = f(t), \tag{A.13}$$

if we know know that the general solution to the homogeneous equation $y''+p(t)y'+q(t)y = 0$ is

$$y_h = c_1y_1 + c_2y_2.$$

(a) Assume that a particular solution of (A.13) has the form

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

where

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0.$$

Substitute y_p into the left-hand side of (A.13) to show that

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = f(t).$$

(b) Show that

$$u_1'(t) = \frac{-y_2(t)f(t)}{W[y_1, y_2](t)}$$
$$u_2'(t) = \frac{y_1(t)f(t)}{W[y_1, y_2](t)},$$

where $W[y_1, y_2](t) = y_1(t)y_2'(t) - y_2(t)y_1'(t)$ is the *Wronskian* of y_1 and y_2.

(c) If p, q, and f are continuous on an interval I, show that

$$u_1(t) = -\int_{t_0}^{t} \frac{y_2(s)f(s)}{W[y_1, y_2](s)}\, ds$$
$$u_2(t) = \int_{t_0}^{t} \frac{y_1(s)f(s)}{W[y_1, y_2](s)}\, ds$$

for any point t_0 in I. Consequently, a particular solution to (A.13) is

$$y_p = -y_1(t) \int_{t_0}^{t} \frac{y_2(s)f(s)}{W[y_1, y_2](s)}\, ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)f(s)}{W[y_1, y_2](s)}\, ds.$$

(d) Find the general solution of the differential equation

$$y'' + 4y = 3 \csc t.$$

(a) If $y_p = u_1 y_1 + u_2 y_2$, then

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2' = u_1 y_1' + u_2 y_2'$$
$$y_p'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''.$$

Substituting these expressions into equation (A.13), we have

$$\begin{aligned} y_p'' + p y_p' + q y_p &= (u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'') + p(u_1 y_1' + u_2 y_2') \\ &\quad + q(u_1 y_1 + u_2 y_2) \\ &= u_1[y_1'' + p y_1' + q y_1] + u_2[y_2'' + p y_2' + q y_2] + u_1' y_1' + u_2' y_2' \\ &= u_1' y_1' + u_2' y_2' \\ &= f(t). \end{aligned}$$

(b) If we solve the system

$$u_1'(t) y_1(t) + u_2'(t) y_2(t) = 0$$
$$u_1'(t) y_1'(t) + u_2'(t) y_2'(t) = f(t).$$

for u_1' and u_2', we obtain

$$u_1'(t) = \frac{-y_2(t)f(t)}{W[y_1, y_2](t)}$$
$$u_2'(t) = \frac{y_1(t)f(t)}{W[y_1, y_2](t)}.$$

(c) Integrate the two equations from part (2).

(d) The general solution to the homogeneous equation $y'' + 4y = 0$ is

$$y_h = c_1 \cos 2t + c_2 \sin 2t.$$

To find a particular solution, assume that the solution has the form

$$y_p = u_1(t) \cos 2t + u_2(t) \sin 2t.$$

By part (2)

$$u_1'(t) = -3\cos t$$
$$u_2'(t) = \frac{3}{2}\csc t - 3\sin t.$$

Integrating, we obtain

$$u_1(t) = -3\sin t$$
$$u_2(t) = \frac{3}{2}\ln|\csc t - \cot t| + 3\cos t.$$

Therefore,

$$\begin{aligned} y_p(t) &= u_1(t)y_1(t) + u_2(t)y_2(t) \\ &= -3\sin t\cos 2t + \left[\frac{3}{2}\ln|\csc t - \cot t| + 3\cos t\right]\sin 2t, \end{aligned}$$

and the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1\cos 2t + c_2\sin 2t - 3\sin t\cos 2t + \left[\frac{3}{2}\ln|\csc t - \cot t| + 3\cos t\right]\sin 2t. \end{aligned}$$

4.3.4 Exercises

1. Compute the solution of the initial value problem

$$y'' + 4y = 3\cos 2t$$
$$y(0) = y'(0) = 0.$$

4.4.5 Exercises

1. Given the equation

$$y'' + 10y = \sin 3t,$$

determine the following.

(a) The frequency of the beats.

(b) The frequency of the rapid oscillations.

(c) Use parts (1) and (2) to sketch a graph of a typical solution.

2.

(a) Suppose that an opera singer can break a glass by singing a particular note. Will the singer have to sing a higher note or a lower note to break an identical glass that is half full of water?

(b) Suppose that both notes are within the opera singer's range. Will it be harder or easier to break the glass when it is half full of water?

(c) Commercials for a maker of recording tape show a crystal glass being broken by the sound of a recording of an opera singer's voice. Is this a good test of the quality of the tape? Why or why not?

3. For large t, every solution of

$$y'' + py' + qy = \cos \omega t$$

oscillates with angular frequency ω and amplitude A given by

$$A(\omega, p, q) = \frac{1}{\sqrt{(q - \omega^2)^2 + p^2\omega^2}}.$$

That is, the amplitude of A is a function of the parameters ω, p, and q.

(a) Compute $\dfrac{\partial A}{\partial \omega}$.

(b) For fixed p and q, let $M(p, q)$ denote the maximum value of $A(\omega, p, q)$ as a function of ω. Compute an expression for $M(p, q)$.

(c) Set $q = 1$ and plot $M(p, q)$ as a function of p.

(d) Explain why $M(p, q)$ is proportional to $1/p$ as $p \to 0$.

4. Let us examine the amplitude and phase of the steady-state solution,

$$x_p(t) = G(\omega)A \cos(\omega t - \phi),$$

where

$$G(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}.$$

Now let $s = \omega/\omega_0$ and $D = 2c/\omega_0$. These new constants, s and D, measure the ratio of the driving frequency to the natural frequency and the effect of the damping force, respectively. Thus,

$$G = \frac{1}{\omega_0^2\sqrt{(1 - s^2)^2 + D^2s^2}}$$

or

$$\omega_0^2 G = \frac{1}{\sqrt{(1 - s^2)^2 + D^2s^2}}.$$

This expression shows us how the gain varies as $s = \omega/\omega_0$ varies. The natural frequency is fixed in

$$x'' + 2cx' + \omega_0^2 x = A \cos \omega t$$

and $D = 2c/\omega_0$ is proportional to the damping constant.

5.1.4 Exercises

1. Consider the following three systems

(a)

$$x' = 3 \sin x + y$$
$$y' = 4x + \cos y - 1$$

(b)

$$x' = -3\sin x + y$$
$$y' = 4x + \cos y - 1$$

(c)

$$x' = -3\sin x + y$$
$$y' = 4x + 3\cos y - 3$$

All three systems have an equilibrium solution at $(0, 0)$. Which two systems have phase portraits with the same "local picture" near $(0, 0)$? Justify your answer.

2. Let us consider an epidemic model for a city. We make the following additional assumptions about our model.

- The city has a constant birth rate rate of α persons per day. All new born babies are susceptible to the disease.
- Infected people either recover or die after a certain number of days. If an individual recovers, he or she is immune.

If we let $S(t)$ be the number of susceptible individuals at time t and $I(t)$ be the number of infected individuals at time t, our assumptions give rise to the following system of differential equations,

$$\frac{dS}{dt} = -\alpha SI + \beta$$
$$\frac{dI}{dt} = -\gamma I + \alpha SI.$$

The constant α is determined by the probability of an interaction between a susceptible individual and an infected individual, and γ is the rate at which infected individuals are removed from the population. If

$$\frac{dS}{dt} = -\alpha SI + \beta = 0$$
$$\frac{dI}{dt} = -\gamma I + \alpha SI = 0,$$

then both the susceptible and infected populations do not change. This will occur at

$$S_0 = \frac{\gamma}{\alpha}$$
$$I_0 = \frac{\beta}{\gamma}.$$

We are interested in the behavior of solutions near (S_0, I_0). If solutions approach this equilbrium point, then the disease will become endemic to the population.

5.3.5 Exercises

1. This is an exercise.

5.4.4 Exercises

1. Consider the system

$$\begin{aligned} x' &= y \\ y' &= -x - \frac{y}{4} + x^2. \end{aligned}$$

(a) Verify that the function

$$L(x, y) = \frac{x^2}{2} + \frac{y^2}{2} - \frac{x^3}{3}$$

is a Lyapunov function for the system.

(b) Sketch the level sets of L.

(c) What can you conclude about the phase portrait of the system from (1) and (2)?

2. Let $G(x, y) = x^3 - 3xy^2$.

(a) What is the gradient system with vector field given by the gradient of G?

(b) Sketch the graph of G and the level sets of G.

(c) Sketch the phase portrait of the gradient system in (1).

3. Suppose that the smell of a bunch of dead fish in the region $-2 \leq x \leq 2$, $-2 \leq y \leq 2$ is given by the function

$$S(x, y) = x^2 + y^2 - \frac{x^4 + y^4}{4} - 3x^2y^2 + 100.$$

(a) What is the gradient system who vector field is the gradient of S?

(b) Use Sage to graph the phase portrait of the system.

(c) How many dead fish are there and where are they?

(d) Using the results from part (2), sketch the level sets of S.

(e) Why is this model not realistic for large values of x or y?

4. Consider the system of differential equations

$$\begin{aligned} x' &= x(-x - y + 1) \\ y' &= y(-ax - y + b), \end{aligned}$$

where a and b are parameters with $a, b > 0$. Suppose that this system is only defined for $x, y \geq 0$.

(a) Use nullclines to sketch the phase portrait for this system for several values of a and b.

(b) Determine the values of a and b at which a bifurcation occurs.

5. Show that the quadratic polynomial

$$V(x, y) = ax^2 + 2bxy + cy^2$$

is positive definite with minimum at $(x, y) = (0, 0)$ if and only if $a > 0$ and $ac - b^2 > 0$.

6. Use the positive definite function $V(x, y) = x^2 + y^2$ to argue that the system

$$x' = -x - 2y^2$$
$$y' = 2xy - y^3$$

has an asymptotically stable equilibrium solution at $(x, y) = (0, 0)$.

B

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