MA 211 Differential Equations

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Mathematics can do some amazing stuff!

Boeing 787 Dreamliner

Physical systems can be modeled in two ways:

- using physical models.
- using mathematical models.

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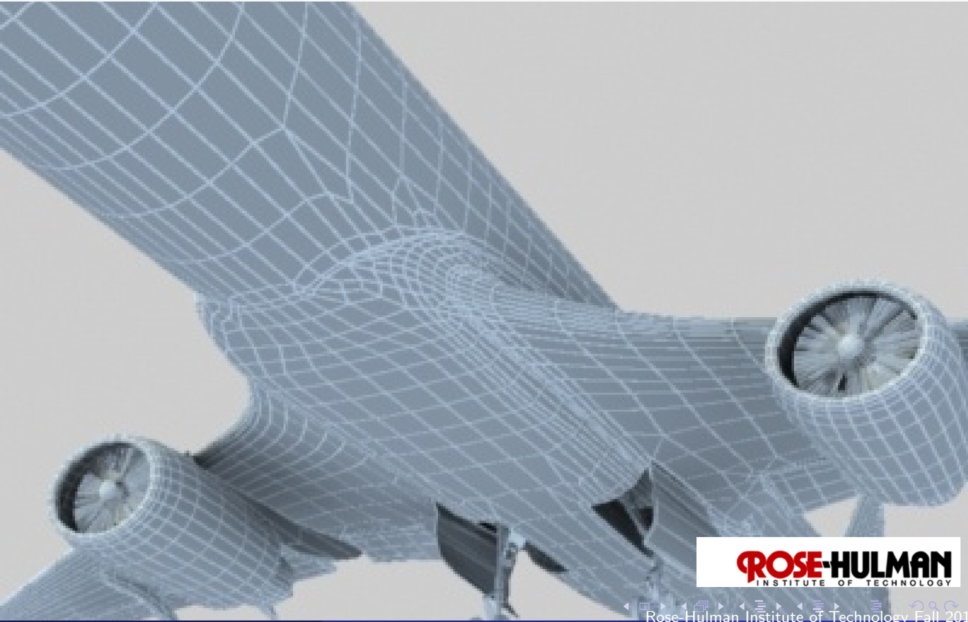
Boeing 787 Deamliner (Physical Model)

Boeing 787 Deamliner (Computer Model)



Example 1 (Physical vs Mathematical Models)

What are the advantages/disadvantages of physical vs mathematical models

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- in terms of complexity?
- in terms of accuracy?
- in terms of scaling?
- in terms of optimization?

Definition 2 (Mathematical Model)

A set of **mathematical equations** which represent a physical system is called a **mathematical model** of the physical system.

Example 3 (Dynamical Systems)

Give examples of dynamical systems.

Definition 4 (Dynamical System)

A **dynamical system** is a system that is changing.

Example 5 (Dynamical Systems)

What quantities are changing in the previous examples of dynamical systems?

- Rates of change are represented by derivatives.
- The rate at which a dynamical system is changing is specified by a *differential equation*.
- Differential equations are mathematical models of dynamical systems.

Definition 6 (Differential Equation)

An equation that contains the derivative of an unknown function is called a **differential equation** (DE for short).

Recall that if x represents a quantity, then the derivative x' represents the rate of change of the quantity.

Example 7 (Constant Rate DE)

During a forced march, a company of soldiers can march with a speed of

$$v(t) = 5 \text{ mph}.$$

How far can the company march in 3 hours?

Example 8 (Initial Value Problem)

During a forced march, a company of soldiers can march with a speed of

$$v(t) = 5e^{-0.5t} \text{ mph}.$$

How far can the company march in 3 hours?

Definition 9 (Initial Value Problem)

A differential equation with initial condition is called an **initial value problem (IVP)**.

Example 10 (Differential Equations)

Why are differential equations so useful?

Example 11 (Initial Value Problems)

Solve the following initial value problem:

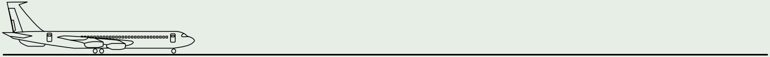
$$x' = \frac{1}{t},\ x(1) = 2$$

Then, verify that the solution is correct.

Example 12 (Motion Problem)

How much runway does an airplane need to take-off?

Assume the airplane has a mass of 150,000 kgs and has engines which can generate a total thrust of 700,000 Nts. Take-off speed is 100 m/sec. Ignore the effects of air resistance.



Mathematical models often must first be parametrized. We need to use real world data to parametrize a mathematical model.

Example 13 (Parameter Identification)

The weight of a metal alloy is directly proportional to its volume. “Model” the weight of the alloy as a function of its volume. Laboratory measurements show that 5 cm^3 of this alloy weighs 40 gms.

Example 14 (Parameter Identification)

The function $x(t) = 2e^{-3t}$ is a solution to the differential equation

$$x' + ax = 0.$$

Determine the unknown parameter a.

Example 15 (Solving DEs)

Solve $x' = 1 - x^2$.

Instead of directly solving the DE in the previous example, we can get *qualitative* information about the DE from a *slope field*.

Example 16 (Slope Field)

Plot, by hand, the slope field and several integral curves of the differential equation

$$x' = 1 - x^2.$$

Example 17 (Equilibrium)

When is a dynamical system in equilibrium?

Definition 18 (Equilibrium Solution)

A solution to a differental equation that is constant (i.e. the derivative of the solution equals zero) is called an **equilibrium solution** of the differential equation.

Example 19 (Equilibrium Solution)

Determine all *equilibrium solutions* of the differential equation

$$x' = 1 - x^2.$$

Example 20 (Integral Curves)

Use Maple to plot an **integral curve** passing through the point $x(0) = 0$ of the slope field of the differential equation

$$x' = 1 - x^2.$$

Example 21 (Solution Verification)

Verify that e^t is the solution to the initial value problem (IVP)

$$x' = x, \; x(0) = 1.$$

How was this solution obtained in the first place. Note, you can't just integrate both sides to get it. Why not?

Example 22 (Separate and Integrate)

Solve the differential equation

$$x' = x, \;\; x(0) = 1.$$

How did we get rid of the absolute value in the above solution process? Recall that (see the graph of $|x|$)

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Case 1: If $x > 0$, we have $x = e^c e^t$.
Case 2: If $x < 0$, we have $-x = e^c e^t$.
So, combining the cases we have

$$x = \pm e^c e^t.$$

Let $C = \pm e^c$ and we then have $x(t) = Ce^t$.

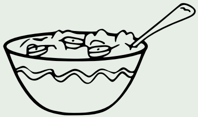
Definition 23 (Separate and Integrate Method)

1. Separate the dependent variable x and the independent variable t to different sides of the differential equation (*algebra*).
2. Integrate both sides.
3. If possible, solve for the dependent variable x.
4. Use the initial conditions to solve for the arbitrary constants of integration.

Previously, we used Newton's law of motion to solve problems. Next we apply Newton's law of cooling.

Example 24 (Newton's Law of Cooling)

A bowl of instant oatmeal comes out of the microwave at 100°C. Ten minutes later, the temperature of the oatmeal is 80°C. How long does it take for the oatmeal to cool to 50°C? Room temperature is 30°C.



Note, the oatmeal does not cool at a constant rate. Why?

Definition 25 (Newton's Law of Cooling)

The rate a body cools is proportional to the difference in temperature between the body and its surroundings.

Example 26 (Separate and Integrate)

Solve the following IVP. Leave your answer in implicit form.

$$\sqrt{t} + \sqrt{x}x' = 0, \;\; x(1) = 4$$

The last example shows that it is sometimes simplier to leave solutions in implicit form.

Not all differential equations are *separable*.

Definition 27 (Separable Differential Equation)

A differential equation which can be solved by separating variables and integrating is called **separable**.

Example 28 (Nonseparable DE)

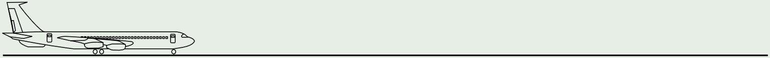
$$x' = x + \sin(t)$$

This next example is nearly identical to the next Lesson. In the example, I will use numbers. In the Lesson, you will use only letters.

Example 29 (Linear Air Resistance)

How much runway does an airplane need to take-off?

Assume the airplane has a mass of 150,000 kgs and has engines which can generate a total thrust of 700,000 Nts. Take-off speed is 100 m/sec. Maximum ground speed is 125 m/sec. Assume air resistance is directly proportional to speed.



Definition 30 (Linear Differential Equation)

A differential equation that can be put in the **standard form**

$$x' + p(t)x = f(t)$$

is called a **linear differential equation**.

Example 31 (Linear vs Nonlinear DE)

(a) $x' = t^2 x + \sin(t)$ is linear because it can be put in the standard form $x' + p(t)x = f(t)$ where $p(t) = -t^2, \; f(t) = \sin(t)$

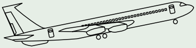
(b) $x' = tx^2 + 2$ is nonlinear.

Note that linear differential equations $x' + p(t)x = f(t)$ must have x and x' appear only to the first power. Its O.K. for $p(t)$ and $f(t)$ to be nonlinear functions.

Example 32 (Nonlinear Air Resistance)

How much runway does an airplane need to take-off?

Assume the airplane has a mass of 150,000 kgs and has engines which can generate a total thrust of 700,000 Nts. Take-off speed is 100 m/sec. Maximum ground speed is 125 m/sec. Assume air resistance is directly proportional to the *square* speed. (This assumption is a more accurate model of reality.)



Example 33 (Linear vs Nonlinear Air Resistance)

Plot velocity of linear and nonlinear models and compare. Which is which? For which model does the airplane reach maximum speed the fastest?

$$\begin{aligned}(\text{linear})\ v(t) &= 125\left(1-e^{-\frac{14}{375}t}\right)\\ (\text{nonlinear})\ v(t) &= 125\tanh(0.03733t)\end{aligned}$$

Definition 34 (Constant Coefficient Linear DEs)

Linear differential equations

$$x' + p(t)x = f(t)$$

with constant $p(t)$, e.g. $p(t) = k$, are called **constant coefficient DEs**. Non-constant coefficient DEs are called **time varying**.

Definition 35 (Forcing Function)

The function $f(t)$ in the differential equation

$$x' + p(t)x = f(t)$$

is called the **forcing function** of the differential equation.

Definition 36 (Homogeneous Differential Equation)

The differential equation

$$x' + p(t)x = 0$$

is called **homogeneous**, i.e. the forcing function $f(t) = 0$.

If both $a(t)$ and $f(t)$ are constant then the differential equation is said to be **autonomous**.

Definition 37 (Terminology)

general solution: solution which has an arbitrary constant of integration.

particular solution: solution which satisfies the initial condition.

order: highest derivative in the differential equation.

first order, linear differential equation: differential equation which can be put in the **standard form** given below. Note that the unknown function $x(t)$ and its derivative $x'(t)$ can only appear at most to the first power.

standard form: $x' + p(t)x = f(t)$

constant coefficient differential equation: when $p(t)$ is constant.

rate constant: if $p(t)$ is constant, i.e. $p(t) = k$, then $-k$ is called the rate constant of the differential equation.

forcing function: the function $f(t)$ is called the forcing function of the differential equation.

homogeneous differential equation: when the forcing function $f(t)$ is equal to zero.

first order, nonlinear differential equation: differential equation $x' = F(t, x)$ which can not be put in standard form.

time-varying differential equation: differential equation $x' = F(t, x)$ where $F(t, x)$ depends on time t.

autonomous differential equation: differential equation which does not depend on time t, i.e. a differential equation which is of the form $x' = F(x)$.

separable differential equation: differential equation that can be solved using the technique of separating variables and integrating.

Linear constant coefficient DEs are always separable. Not all linear, time varying DEs are separable. We introduce a new solution method which uses an *integrating factor* to solve time varying DEs.

Definition 38 (Integrating Factor Technique)

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2. Compute the integrating factor: $\boxed{u(t) = \exp(\int p(t)dt)}$ (You do not need the $+c$.)
3. Multiply both sides of the differential equation by the integrating factor $u(t)$.

$$\begin{aligned} \left[x' + p(t)x\right] u(t) &= f(t)u(t) \\ x' u(t) + p(t)xu(t) &= f(t)u(t) \end{aligned}$$

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$$(x(t)u(t))' = f(t)u(t)$$

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$$x(t)u(t) = \int f(t)u(t)dt$$

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6. Solve for $x(t)$.

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$$x(t)u(t) = \int f(t)u(t)dt$$

6. Solve for $x(t)$.
7. Use the initial condition to solve for the arbitrary constant of integration.

Example 39 (Integrating Factors)

Classify, then solve the following IVP:

$$x' + x - e^t = 0, \;\; x(0) = 1$$

Summary: We have learned two solution techniques.

Solution Techniques for Differential Equations

	separate & integrate	integrating factor
linear constant coefficient	yes	yes
linear time varying	maybe	yes
nonlinear	maybe	no

	separate & integrate	integrating factor
linear constant coefficient	yes	yes
linear time varying	maybe	yes
nonlinear	maybe	no

Recall that a differential equation is linear if it can be put into the standard form:

$$x' + p(t)x(t) = f(t).$$

The solution to linear differential equations have a special form.

Theorem 40 (Structure of Solutions to Linear DEs)

The general solution to linear differential equations is of the form

$$x(t) = \underbrace{x_h(t)}_{\text{general sol to homogeneous eq}} + \underbrace{x_p(t)}_{\text{any particular sol}}$$

Example 41 (Structure of Solutions)

Show that the general solution to the linear DE

$$x' = 2x - \sin(t)$$

has the special form

$$x(t) = x_h(t) + x_p(t).$$

Example 42 (Structure of Solutions)

Verify that the general solution to the linear, constant coefficient, DE

$$x' + kx = f(t)$$

is given by

$$x(t) = Ce^{-kt} + x_p(t).$$

where $x_p(t)$ is a particular solution, i.e. $x_p' + kx_p = f(t)$.

Determining $x_p(t)$ is the hard part of solving $x' + kx = f(t)$. If $f(t) = k$, i.e. if $f(t)$ is constant, then we can easily compute an equilibrium solution and use it as the particular solution.

Example 43 (Structure of Solutions)

Determine the general solution to the linear, constant coefficient, DE

$$x' = -3x + 12.$$

The general solution to autonomous, linear DEs can be written down by inspection.

Example 44 (Structure of Solutions)

Determine the general solution of the following differential equations, by inspection:

1. $x' = -x + 1$
2. $3x' - 6x = 12$
3. $x' = kx + f_0$

So far we have used Newton's law of motion, $F = mv'$, to solve motion problems and Newton's law of cooling, $x' = k(x - x_s)$, to solve heating/cooling problems. Next, we consider mixing problems.

Example 45 (Salt Tank System)

A tank contains 100 gal of salt water with initial concentration 1 lb/gal. Pure water is pumped into the tank at a rate of 10 gal/min. The solution in the tank is pumped out at 10 gal/min. Mixers keep the salt concentration uniform throughout the tank. How long does it take to flush 99% of the salt out of the tank?

Example 46 (Varying Volume)

Does increasing the pumping rate speed-up the flushing of salt?

Repeat the previous problem, except assume that pure water is pumped at 20 gal/min into the tank. Will it take a shorter or longer time to flush 99% of the salt out of the tank?

Definition 47 (Components of an Electrical Circuit)

Linear electrical circuits are built from the following components:

component	variable	units
voltage source	$E(t)$	volts (V)
resistor	R	ohms (Ω)
capacitor	C	farads (F)
inductor	L	henrys (H)

A helpful analogy to make is one involving water pipe networks.

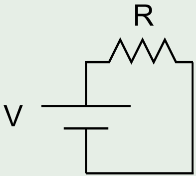
Example 48 (Pipe Network Analogy)

Analogy between electrical circuits and water pipe networks:

electrical circuit	water pipe network
voltage	water pressure
current	water flow rate
charge	water
battery	water tower
resistor	pipe flow resistance
capacitor	pressurized storage tank
inductor	water's mass (inertia)

Example 49 (R Circuit)

Consider the following battery and resistor circuit.



If $V = 10$ volts and $R = 2$ ohms, determine the current I flowing in the circuit.

Definition 50 (Ohm's Law)

Let

V_R equal voltage in volts (V).
I equal current in amperes (A).
R equal resistance in ohms (Ω).

Then, **Ohm's law** states that $V_R = IR$.

Definition 51 (Kirchoff's Laws)

Kirchoff's voltage law:

The voltage drop around a closed loop must equal zero.

Kirchoff's current law:

The current flowing into a junction must sum to zero.

Note, Ohm's law implies that as the voltage (water pressure) increases, so does the current (water flow). As the resistance increases, the current decreases.

Capacitors accumulate charge and behave like pressurized tanks that store water. As they fill with charge, they build-up voltage (pressure).

Definition 52 (Capacitance Law)

Let

V_C equal voltage drop accross capacitor in volts (V).

C equal capacitance in farads (F).

I equal current in amperes (A).

Then, the **capacitance law** states that $V_C' = \frac{1}{C} I$.

Current flowing into a small capacitor (small C) builds up the voltage E_C very quickly. The opposite is true of a large capacitor.

Definition 53 (Faraday's law)

Let

V_L equal voltage drop accross inductor in volts (V).

L equal inductance in henrys (H).

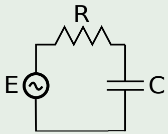
I equal current in amperes (A) .

Then, **Faraday's law** states that $E_L = LI'$.

Faraday's law is a bit like Newton's law of motion, $F = mv'$. The inductance, L, is analogous to the mass (inertia) of water flowing through pipes.

Example 54 (RC Circuit)

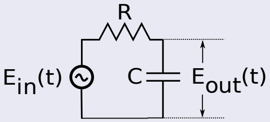
Consider the RC circuit shown below. Assume $R = 2$ ohms, $C = \frac{1}{6}$ farads and $E(t) = 12$ volts. Assume at $t = 0$ the capacitor has no charge. Determine the current $I(t)$. How long does it take for the current to drop down to 1% of its initial value?



The *low pass filter* is an important RC circuit used to filter out the noise in a noise corrupted signal.

Theorem 55 (Low Pass Filter)

The differential equation for a **low pass filter** circuit



is

$$RCE'_{\text{out}}(t) + E_{\text{out}} = E_{\text{in}}(t)$$

where $E_{\text{in}}(t)$ is the input signal to the circuit and $E_{\text{out}}(t)$ is the filtered output signal of the circuit.

Example 56 (Low Pass Filter DE)

Derive the differential equation which models the low pass filter circuit.

The name of the low pass filter comes from the fact that the low pass filter does not have a large effect on low frequence signals, letting them pass though the circuit virtually unchanges. But it stops high frequency signals from getting through the circuit. Since noise tends have a higher frequency than a signal, the low pass filter circuit filters noise from a signal.

Nonlinear differential equations do not have the special equation and solution structure that linear differential equations have.

Definition 57 (Rate Form of Differential Equations)

The **rate form** of a differential equation is

$$x' = F(t, x).$$

All differential equations, including linear differential equations, can be put in rate form.

Definition 58 (Time-Invariant Differential Equation)

Differential equations of the form

$$x' = F(x)$$

are called **time-invariant**.

We will focus on time-invariant differential equations.

Definition 59 (Equilibrium)

A system is in **equilibrium** if it is not changing.

Example 60 (Equilibrium Points)

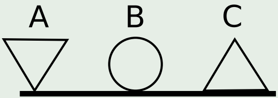
Determine all the equilibrium points of the time-invariant differential equation

$$x' = x^2 - 1.$$

Equilibrium solutions are not all the same.

Example 61 (Stability)

The three objects shown below are each in equilibrium.



object A: **unstable equilibrium**

Example 61 (Stability)

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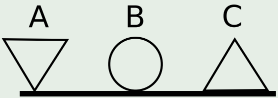

object A: **unstable equilibrium**

object B: **stable equilibrium**

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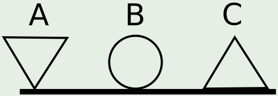


object A: **unstable equilibrium**

object B: **stable equilibrium**

object C: **asymptotically stable equilibrium**

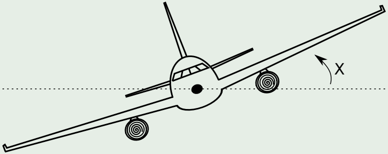
If you move an object away from an asymptotically stable equilibrium, it will return to equilibrium, so long as you don't move it too far. What happens when you move an object way from an unstable equilibrium? No matter how little you move it, it will not return to equilibrium.

Example 62 (Roll Angle)

Complete a stability analysis for the roll angle of an airplane. The differential equation which models the dynamics of the roll angle is

$$x' = x(x + \frac{\pi}{4})(x - \frac{\pi}{4})$$

where $x(t)$ represents the roll angle of the airplane.



Example 63 (Trajectory Flow)

Sketch the trajectory flow for the Roll Angle example.

Example 64 (Trajectory Flow)

Complete a stability analysis and sketch the trajectory flow for the following differential equation:

$$x' = x(x-4)$$

Check answer using Maple.

Theorem 65 (Stability Analysis)

Consider the time-invariant differential equation

$$x' = F(x).$$

If $F(x_0) = 0$ and

(a) if $F'(x_0) < 0$, then x_0 is an asymptotically stable equilibrium point.

(b) if $F'(x_0) > 0$, then x_0 is an unstable equilibrium point.

Explain why the stability analysis theorem is true.

Example 66 (Stability of Linear Systems)

Complete a stability analysis of the linear, time-invariant differential equation

$$x' + kx = f_0.$$

Example 67 (Stability of Linear Systems)

Complete a stability analysis of the following time-invariant, differential equations.

1. $x' = -5x + 10$
2. $2x' - 4x = 0$

Example 68 (Linear (Malthusian) Population Model)

Assume the growth rate of buffalo in a wild life reserve is α (buffalo/year per buffalo) and the death rate is β (buffalo/year per buffalo). Let x represent the number of buffalo and t represent time in years.

(a) Formulate a differential equation which models the population of buffalo.

(b) Complete a stability analysis.

Exponential population growth realistically can't go on forever.

Example 69 (Nonlinear (Logistic) Population Model)

Repeat the previous problem except assume that the death rate is no longer constant, but depends on the population. As population increases, food shortages and disease increase the death rate. Assume the death rate is given by $\beta + \gamma x$ (buffalo/year per buffalo).

Maple is known as a symbolic manipulation package. It can find the exact solution to all linear and many, but not all, nonlinear differential equations.

Example 70 (DE Maple can't solve)

$$x' = x^3 + t^2, \; x(0) = 1$$

In practice, rather than solving DEs exactly using a symbolic method like that used by Maple, the solution to DEs are approximated with numerical methods like Euler's method. The advantage of numerical methods is they are almost as easy to use for nonlinear DEs as linear ones.

How is Euler's method used to solve an IVP? Consider the following IVP expressed in rate form.

$$x' = F(t,x), \;\; x(0) = x_0.$$

Recall that $\Delta x = x_2 - x_1$ and $\Delta t = t_2 - t_1$.

$$x' = \text{ slope } = \frac{\text{rise}}{\text{run}} \approx \frac{\Delta x}{\Delta t}$$

Multiplying by Δt we get

$$\Delta x \approx x' \Delta t = F(t,x)\Delta t.$$

Since $\Delta x = x_2 - x_1$ we have

$$x_2 - x_1 = F(t,x)\Delta t$$

or

$$x_2 = x_1 + F(t, x)\Delta t.$$

If we let $h = \Delta t$ equal the "timestep" Δt, we have Euler's method.

Definition 71 (Euler's Method)

The solution to the initial value problem

$$x' = F(t, x), \; x(t_0) = x_0$$

can be approximated numerically by **Euler's method**,

$$x_{k+1} = x_k + h \cdot F(t_k, x_k), \; k = 0, 1, 2, \cdots$$

where h is the timestep.

Example 72 (Euler's Method)

Solve the IVP $x' = x^3 + t^2,\ x(0) = 1$ using Euler's method with a fixed time step of $h = \Delta t = 0.1$. Estimate the value of $x(0.5)$.

Example 73 (Euler's Method)

Use Euler's method to solve the IVP $x' = x + t, \; x(0) = 1$ and compute the value of $x(0.3)$. Use a time step of $h = 0.1$. Compare your approximate answer to the exact answer. How accurate is your approximation? Note: $F(t, x) = x + t$.

Use direction fields to provide a geometric explanation of Euler's method.

Example 74 (Step Size Trade-off)

What effect does the step size h have on the accuracy of Euler's method? Why is there a trade-off between accuracy and computer time?

Note, for many problems (e.g. astrophysics, molecular dynamics, large circuits, etc.) the amount of computer time need can be days, weeks even several months.

What is linearity and why is it so useful?

Definition 75 (Linear Combination of Functions)

If the function $f_3(t)$ is given by the expression

$$f_3(t) = c_1 f_1(t) + c_2 f_2(t),$$

where c_1 and c_2 are constants, then $f_3(t)$ is a **linear combination** of the functions $f_1(t)$ and $f_2(t)$.

Example 76 (Linear Combination of functions)

Let $f_3(t)$ be given by

$$f_3(t) = 5\sin(t) + 8t.$$

Then $f_3(t)$ is equal to a linear combination of the functions $\sin(t)$ and t.

Definition 77 (Linearity)

Consider the linear differential equation

$$x' + p(t)x = f(t).$$

If $x(t) = x_1(t)$ is a particular solution when the forcing function is $f(t) = f_1(t)$ and $x(t) = x_2(t)$ is a particular solution when the forcing function is $f(t) = f_2(t)$, then a particular solution for the forcing function $f_3(t)$, where $f_3(t)$ is a linear combination of forcing functions, i.e. where

$$f_3(t) = c_1 f_1(t) + c_2 f_2(t)$$

is given by

$$x_3(t) = c_1 x_1(t) + c_2 x_2(t).$$

Note that linear equations preserve linear combinations. This property of linear equations is the reason that they are so useful.

Definition 78 (Linearity Chart)

We will refer to the following chart as a **linearity chart**.

forcing function	particular solution
$f_1(t)$	$x_1(t)$
$f_2(t)$	$x_2(t)$
$f_3(t) = c_1 f_1(t) + c_2 f_2(t)$	$x_3(t) = c_1 x_1(t) + c_2 x_2(t)$

Example 79 (Linearity)

The solution to

$$x' + 2x = \sin(t) \text{ is } x_1(t) = -\frac{1}{5}\cos(t) + \frac{2}{5}\sin(t).$$

The solution to

$$x' + 2x = t \text{ is } x_2(t) = -\frac{1}{4} + \frac{1}{2}t.$$

Determine the solution to

$$x' + 2x = 5\sin(t) + 8t.$$

sectionTypes of Solutions (Responses)

Theorem 80 (Solution to Linear Const. Coef. DEs)

The solution to the linear, constant coefficient differential equation

$$x' + kx = f(t), \; x(0) = x_0$$

is given by

$$x(t) = \underbrace{x_0 e^{-kt}}_{\text{free response}} + \underbrace{\int_0^t f(\tau) e^{-k(t-\tau)}\, d\tau}_{\text{zero-state forced response (convolution integral)}} .$$

Example 81 (Solution to Linear Const. Coef. DEs)

Use an integrating factor to show that the solution to the linear, constant coefficient, differential equation

$$x' + kx = f(t),\ x(0) = x_0$$

is given by

$$x(t) = \underbrace{x_0 e^{-kt}}_{\text{free response}} + \underbrace{\int_0^t f(\tau)e^{-k(t-\tau)}\,d\tau}_{\text{zero-state forced response (convolution integral)}}.$$

Definition 82 (Free Response)

The **free response** of a system is the response of the system due only to the initial conditions of the system, i.e. the reponse with the forcing function set equal to zero.

Definition 83 (Zero-State Forced Response)

The **zero-state forced response** of a system is the response of the system due only to the forcing function of the system, i.e. the response with zero initial conditions.

The integral that gives the zero-state forced response of a system is a special integral called a *convolution integral*.

Definition 84 (Convolution)

The **convolution** of the functions $f(t)$ and $g(t)$ is represented by the symbol $\star$ and is given by

$$f(t) \star g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

Theorem 85 (Zero-State Forced Response Convolution)

The zero-state forced response of a system modeled by the differential equation

$$x' + kx = f(t)$$

is given by the convolution $f(t) \star e^{-kt}$.

Example 86 (Free and Forced Responses)

Salt water with concentration $c(t)$ is pumped into a 100 gallon tank at 10 gal/min. The well mixed solution in the tank is pumped out at 10 gal/min.

(a) Determine the free response of the system if the tank initially contains 20 lbs of salt.

(b) Determine the zero-state forced response of the system for $0 \leq t \leq 20$ minutes if $c(t) = 0.01t(20 - t)$ lbs/gal.

(c) Plot the forcing function and the forced response for $0 \leq t \leq 20$ minutes.

Definition 87 (Transient Response)

The portion of the response of a system that dies out with time is called the **transient response** of the system.

Definition 88 (Steady-State Response)

The portion of a stable response of a system that is left after the transient has died out is called the **steady-state response** of the system.

Example 89 (Transient vs Steady State)

$$x' = -2x + \sin(t), \;\; x(0) = \frac{4}{5}$$

$$x(t) = e^{-2t} + \frac{2}{5}\sin(t) - \frac{1}{5}\cos(t)$$

(Plot in Maple)

Example 90 (Transient vs Steady State)

$$x' = -2x + 14, \;\; x(0) = 2$$

$$x(t) = -5e^{-2t} + 7$$

(Plot in Maple)

What happens when you have an unstable system.

Example 91 (Unstable)

$$x' = 2x + 14,\ \ x(0) = 2$$

$$x(t) = -5e^{2t} + 7$$

No steady state solution because the transient solution does not die out.

Definition 92 (Steady-State Gain)

Assume a system is driven by a periodic forcing function. The **steady-state gain** of the system is the following ratio:

$$\text{steady-state gain} = \frac{\text{amplitude of steady-state response}}{\text{amplitude of forcing function}}.$$

Example 93 (Steady-State Gain)

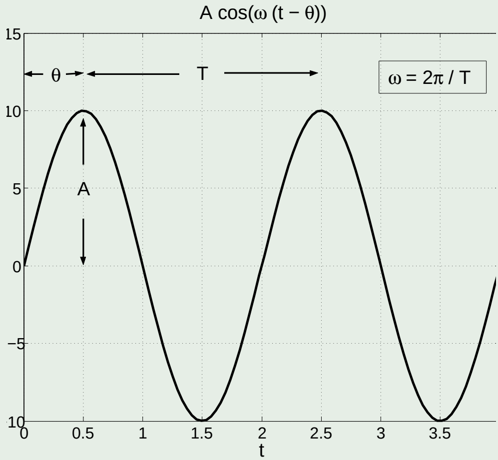
Assume a system is modeled by the linear, constant coefficient differential equation

$$x' + 2x = f(t).$$

Assume the forcing function is the periodic function $f(t) = 5\sin(t)$ and the initial condition is $x(0) = \frac{4}{5}$. Compute the steady-state gain of the system.

Example 94 (Reading Trig Graphs)

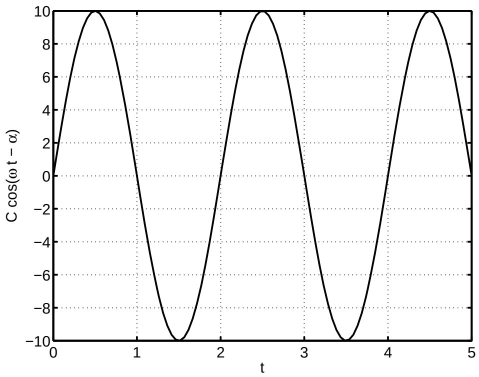
The function $A\cos(\omega(t-\theta))$ is graphed below.



The *phase angle* ϕ is the shift measured in radians, so $\phi = \omega\theta$, e.g. (rad/sec)(sec). The frequency, f, (in Hertz) is the number of cycles per second, so $f = 1/T$.

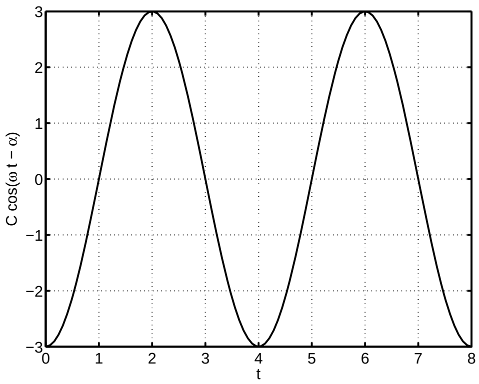
Example 95 (Reading Trig. Graphs)

Determine the amplitude, period, phase shift, radian frequency, phase angle and frequency in Hertz.



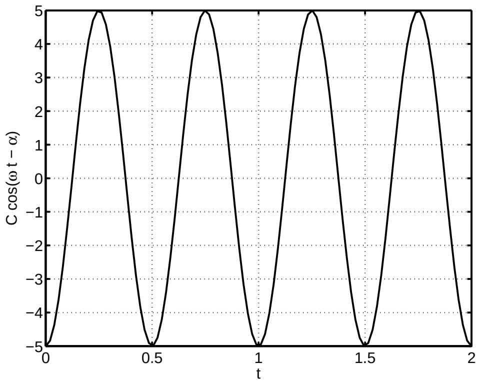
Example 96 (Reading Trig. Graphs)

Determine the amplitude, period, phase shift, radian frequency, phase angle and frequency in Hertz.



Example 97 (Reading Trig. Graphs)

Determine the amplitude, period, phase shift, radian frequency, phase angle and frequency in Hertz.



Theorem 98 (Amplitude and Phase Angle Formulas)

$$a\cos(\omega t) + b\sin(\omega t) = A\sin(\omega t + \phi_1) = A\cos(\omega t - \phi_2)$$

where the amplitude A is given by

$$A = \sqrt{a^2 + b^2}$$

and the phase angle ϕ is

$$\begin{aligned} \phi_1 &= \tan^{-1}(a, b) \\ \phi_2 &= \tan^{-1}(b, a) \end{aligned}$$

Example 99 (Computing Amplitudes and Phase Angles)

Compute the amplitude and phase angle of

$$-5\cos(3t) + 2\sin(3t)$$

Recall that the general solution to the linear differential equation

$$x' + kx = f(t)$$

is

$$x(t) = Ce^{-kt} + x_p(t)$$

where $x_p(t)$ is any particular solution. We can, for example, choose $x_p(t)$ to be the zero-state forced response, by computing the convolution $x_p(t) = f(t) \star e^{-kt}$. We can also choose $x_p(t)$ to equal the steady-state response. The *method of undetermined coefficients* is a guessing method for determining a particular solution $x_p(t)$. The method of undetermined coefficients is especially useful for computing the steady-state solution of a differential equation. The table below gives the function we should guess for $x_p(t)$ for a given forcing function $f(t)$.

forcing function $f(t)$	guess $x_p(t)$
a	A
$a+bt$	$A+Bt$
ae^{rt}	Ae^{rt}
$a\sin(\omega t)+b\cos(\omega t)$	$A\sin(\omega t)+B\cos(\omega t)$

$f(t)$	$x_p(t)$
a	A
$a+bt$	$A+Bt$
ae^{rt}	Ae^{rt}
$a\sin(\omega t)+b\cos(\omega t)$	$A\sin(\omega t)+B\cos(\omega t)$

Example 100 (Constant Forcing Function)

Use the method of undetermined coefficients to solve

$$x' + 5x = 10.$$

Example 101 (Two Tank System)

Use the method of undetermined coefficients to solve the two tank system in the earlier lesson.

The method of undetermined coefficients is especially useful for determining the steady-state solution for periodic forcing functions.

Example 102 (Periodic Forcing Function)

Use the method of undetermined coefficients to solve

$$x' + 2x = \sin(t),\ \ x(0) = 0.$$

Use the steady-state solution for the particular solution $x_p(t)$.

Definition 103 (Steady-State Gain Function)

Consider the differential equation function

$$x' + kx = R_{\text{in}} \cos(\omega t)$$

with steady-state response

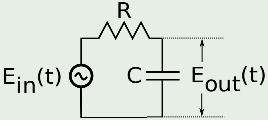
$$x(t) = R_{\text{out}}(\omega) \cos(\omega t - \phi).$$

The **steady-state gain function** $G(\omega)$ is given by

$$G(\omega) = \frac{R_{\text{out}}(\omega)}{R_{\text{in}}}.$$

Example 104 (Gain Function of Low Pass Filter)

Compute the gain function of the low pass filter with $R = 10^3\Omega$ and $C = 10^{-6}$F and use it to explain why this filter is called a *low pass* filter.



Often we are interested in reducing unwanted variations or noise in a process.

Our goal in the next example is to reduce the amplitude of the variation in the salt concentration of a salt solution used in a food processing plant. The variation can be reduced by pumping the salt solution through a mixing tank. There is a trade-off involved in the selection of the volume of the mixing tank.

Main design parameter: V — volume of mixing tank.

- What effect does a large vs small tank have on the amplitude of the steady state response?
- What effect does a large vs small tank have on the transient response?
- Does the amount by which the amplitude is reduced depend on the frequency of oscillation of salt concentration?
- What is the minimum sized tank that will reduce variations by 50%?

Example 105 (Minimum Sized Tank.)

A food processing plant has a problem. Ideally, all the jars of pickled fruit produced by the plant should have the same salt concentration. However, the concentration of the salt solution pumped into jars at 100 gal/hr is observed to vary with time. Assume the salt concentration pumped into the jars is approximated by the function

$$c_{\text{in}}(t) = 0.1 \sin(t) + 0.2 \text{ lbs/gal}$$

where t is measured in hours. What should be done to fix this problem?

Example 106 (Minimum Sized Tank—Continued)

Determine the minimum tank size, V, which will reduce the amplitude of the variations by 50%.

Solution to Tank Problem

volume: V

Solution to Tank Problem

volume: V

pumping rate: 100 gal/hr

Solution to Tank Problem

volume: V
pumping rate: 100 gal/hr
input concentration: $c_{\text{in}}(t) =$

Solution to Tank Problem

volume: V
pumping rate: 100 gal/hr
input concentration: $c_{\text{in}}(t) = 0.1\sin(t) + 0.2$ lbs/gal

Solution to Tank Problem

volume: V
pumping rate: 100 gal/hr
input concentration: $c_{\text{in}}(t) = 0.1\sin(t) + 0.2$ lbs/gal
output concentration: $c_{\text{out}}(t) =$

Solution to Tank Problem

volume: V
pumping rate: 100 gal/hr
input concentration: $c_{\text{in}}(t) = 0.1 \sin(t) + 0.2$ lbs/gal
output concentration: $c_{\text{out}}(t) = ?$

Solution to Tank Problem

volume: V
pumping rate: 100 gal/hr
input concentration: $c_{\text{in}}(t) = 0.1\sin(t) + 0.2$ lbs/gal
output concentration: $c_{\text{out}}(t) = ?$

The differential equation is:

$$x' = c_{\text{in}}(t)(100) - \frac{x(t)}{V}(100), \; x(0) = 0$$

Solution to Tank Problem

volume: V
pumping rate: 100 gal/hr
input concentration: $c_{\text{in}}(t) = 0.1\sin(t) + 0.2$ lbs/gal
output concentration: $c_{\text{out}}(t) = ?$

The differential equation is:

$$x' = c_{\text{in}}(t)(100) - \frac{x(t)}{V}(100), \;\; x(0) = 0$$

$$x' + \frac{100}{V}x = 10\sin(t) + 20, \;\; x(0) = 0$$

Solution to Tank Problem

volume: V
pumping rate: 100 gal/hr
input concentration: $c_{\text{in}}(t) = 0.1\sin(t) + 0.2$ lbs/gal
output concentration: $c_{\text{out}}(t) = ?$

The differential equation is:

$$x' = c_{\text{in}}(t)(100) - \frac{x(t)}{V}(100), \;\; x(0) = 0$$

$$x' + \frac{100}{V}x = 10\sin(t) + 20, \;\; x(0) = 0$$

But $c_{\text{out}}(t) = \dfrac{x(t)}{V}$.

Solution to Tank Problem

volume: V
pumping rate: 100 gal/hr
input concentration: $c_{\text{in}}(t) = 0.1\sin(t) + 0.2$ lbs/gal
output concentration: $c_{\text{out}}(t) = ?$

The differential equation is:

$$x' = c_{\text{in}}(t)(100) - \frac{x(t)}{V}(100), \;\; x(0) = 0$$

$$x' + \frac{100}{V}x = 10\sin(t) + 20, \;\; x(0) = 0$$

But $c_{\text{out}}(t) = \dfrac{x(t)}{V}$. So $x(t) = V\, c_{\text{out}}(t)$

Solution to Tank Problem

volume: V
pumping rate: 100 gal/hr
input concentration: $c_{\text{in}}(t) = 0.1\sin(t) + 0.2$ lbs/gal
output concentration: $c_{\text{out}}(t) = ?$

The differential equation is:

$$x' = c_{\text{in}}(t)(100) - \frac{x(t)}{V}(100), \;\; x(0) = 0$$

$$x' + \frac{100}{V}x = 10\sin(t) + 20, \;\; x(0) = 0$$

But $c_{\text{out}}(t) = \dfrac{x(t)}{V}$. So $x(t) = V\,c_{\text{out}}(t)$ and $x'(t) = V\,c'_{\text{out}}(t)$.

Solution to Tank Problem

volume: V
pumping rate: 100 gal/hr
input concentration: $c_{\text{in}}(t) = 0.1 \sin(t) + 0.2$ lbs/gal
output concentration: $c_{\text{out}}(t) = ?$

The differential equation is:

$$x' = c_{\text{in}}(t)(100) - \frac{x(t)}{V}(100), \;\; x(0) = 0$$

$$x' + \frac{100}{V}x = 10 \sin(t) + 20, \;\; x(0) = 0$$

$$\text{But } c_{\text{out}}(t) = \frac{x(t)}{V}. \text{ So } x(t) = V\, c_{\text{out}}(t) \text{ and } x'(t) = V\, c'_{\text{out}}(t).$$

Substituting for x and x' we get

Solution to Tank Problem

volume: V
pumping rate: 100 gal/hr
input concentration: $c_{\text{in}}(t) = 0.1\sin(t) + 0.2$ lbs/gal
output concentration: $c_{\text{out}}(t) = ?$

The differential equation is:

$$x' = c_{\text{in}}(t)(100) - \frac{x(t)}{V}(100), \;\; x(0) = 0$$

$$x' + \frac{100}{V}x = 10\sin(t) + 20, \;\; x(0) = 0$$

$$\text{But } c_{\text{out}}(t) = \frac{x(t)}{V}. \text{ So } x(t) = V\,c_{\text{out}}(t) \text{ and } x'(t) = V\,c'_{\text{out}}(t).$$

Substituting for x and x' we get

$$V\,c'_{\text{out}}(t) + 100c_{\text{out}} = 10\sin(t) + 20, \;\; c_{\text{out}}(0) = 0.$$

Solution to Tank Problem—Continued

The differential equation we need to solve is:

$$V\,c_{\text{out}}'(t) + 100c_{\text{out}} = 10\sin(t) + 20, \;\; c_{\text{out}}(0) = 0.$$

Solution to Tank Problem—Continued

The differential equation we need to solve is:

$$V\, c_{\text{out}}'(t) + 100 c_{\text{out}} = 10 \sin(t) + 20, \;\; c_{\text{out}}(0) = 0.$$

The solution (obtained using Maple) is very complicated, but all we care about is the amplitude of the steady state part of $c_{\text{out}}(t)$ which is:

Solution to Tank Problem—Continued

The differential equation we need to solve is:

$$V\,c'_{\text{out}}(t) + 100c_{\text{out}} = 10\sin(t) + 20, \;\; c_{\text{out}}(0) = 0.$$

The solution (obtained using Maple) is very complicated, but all we care about is the amplitude of the steady state part of $c_{\text{out}}(t)$ which is:

$$\frac{10^3}{V^2 + 10^4}\sin(t) - \frac{10V}{V^2 + 10^4}\cos(t).$$

So, the output amplitude is:

Solution to Tank Problem—Continued

The differential equation we need to solve is:

$$V\,c_{\text{out}}'(t) + 100c_{\text{out}} = 10\sin(t) + 20,\;\; c_{\text{out}}(0) = 0.$$

The solution (obtained using Maple) is very complicated, but all we care about is the amplitude of the steady state part of $c_{\text{out}}(t)$ which is:

$$\frac{10^3}{V^2 + 10^4}\sin(t) - \frac{10V}{V^2 + 10^4}\cos(t).$$

So, the output amplitude is:

$$R_{\text{out}} = \sqrt{A^2 + B^2} = \frac{10}{\sqrt{V^2 + 10^4}}.$$

Solution to Tank Problem—Continued

The differential equation we need to solve is:

$$V\, c'_{\text{out}}(t) + 100 c_{\text{out}} = 10 \sin(t) + 20, \;\; c_{\text{out}}(0) = 0.$$

The solution (obtained using Maple) is very complicated, but all we care about is the amplitude of the steady state part of $c_{\text{out}}(t)$ which is:

$$\frac{10^3}{V^2 + 10^4} \sin(t) - \frac{10V}{V^2 + 10^4} \cos(t).$$

So, the output amplitude is:

$$R_{\text{out}} = \sqrt{A^2 + B^2} = \frac{10}{\sqrt{V^2 + 10^4}}.$$

We want to choose V so that

$$\frac{R_{\text{out}}}{R_{\text{in}}} =$$

Solution to Tank Problem—Continued

The differential equation we need to solve is:

$$V\,c_{\text{out}}'(t) + 100c_{\text{out}} = 10\sin(t) + 20,\;\; c_{\text{out}}(0) = 0.$$

The solution (obtained using Maple) is very complicated, but all we care about is the amplitude of the steady state part of $c_{\text{out}}(t)$ which is:

$$\frac{10^3}{V^2+10^4}\sin(t) - \frac{10V}{V^2+10^4}\cos(t).$$

So, the output amplitude is:

$$R_{\text{out}} = \sqrt{A^2+B^2} = \frac{10}{\sqrt{V^2+10^4}}.$$

We want to choose V so that

$$\frac{R_{\text{out}}}{R_{\text{in}}} = \frac{\frac{10}{\sqrt{V^2+10^4}}}{0.1}$$

Solution to Tank Problem—Continued

The differential equation we need to solve is:

$$V\,c_{\text{out}}'(t) + 100c_{\text{out}} = 10\sin(t) + 20,\;\; c_{\text{out}}(0) = 0.$$

The solution (obtained using Maple) is very complicated, but all we care about is the amplitude of the steady state part of $c_{\text{out}}(t)$ which is:

$$\frac{10^3}{V^2 + 10^4}\sin(t) - \frac{10V}{V^2 + 10^4}\cos(t).$$

So, the output amplitude is:

$$R_{\text{out}} = \sqrt{A^2 + B^2} = \frac{10}{\sqrt{V^2 + 10^4}}.$$

We want to choose V so that

$$\frac{R_{\text{out}}}{R_{\text{in}}} = \frac{\frac{10}{\sqrt{V^2+10^4}}}{0.1} = 0.5 \text{ (i.e. } 50\%).$$

Solution to Tank Problem—Continued

The differential equation we need to solve is:

$$V\, c_{\text{out}}'(t) + 100 c_{\text{out}} = 10\sin(t) + 20, \;\; c_{\text{out}}(0) = 0.$$

The solution (obtained using Maple) is very complicated, but all we care about is the amplitude of the steady state part of $c_{\text{out}}(t)$ which is:

$$\frac{10^3}{V^2 + 10^4}\sin(t) - \frac{10V}{V^2 + 10^4}\cos(t).$$

So, the output amplitude is:

$$R_{\text{out}} = \sqrt{A^2 + B^2} = \frac{10}{\sqrt{V^2 + 10^4}}.$$

We want to choose V so that

$$\frac{R_{\text{out}}}{R_{\text{in}}} = \frac{\frac{10}{\sqrt{V^2+10^4}}}{0.1} = 0.5 \text{ (i.e. } 50\%).$$

Solving for V, we get

$$V = 173.2 \text{ gallons}.$$

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Solution to Tank Problem—Transient Response

How long does the transient response take to die out?

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For a tank of size $V = 173.2$ gallons the transient response is (using Maple):

$$-0.157e^{-0.577t}.$$

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For a tank of size $V = 173.2$ gallons the transient response is (using Maple):

$$-0.157e^{-0.577t}.$$

To reduce to 1% of its initial value we need to wait t_f hrs where t_f is the solution to the equation

$$-0.157e^{-0.577t_f} = 0.01\left(-0.157e^{0}\right).$$

Solution to Tank Problem—Transient Response

How long does the transient response take to die out?

For a tank of size $V = 173.2$ gallons the transient response is (using Maple):

$$-0.157e^{-0.577t}.$$

To reduce to 1% of its initial value we need to wait t_f hrs where t_f is the solution to the equation

$$-0.157e^{-0.577t_f} = 0.01\left(-0.157e^{0}\right).$$

The solution is

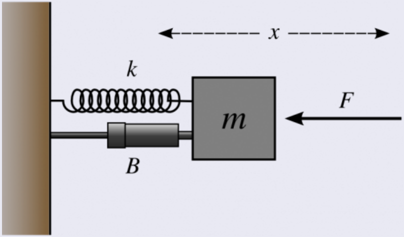
$$t_f = 7.98 \text{ hrs.}$$

Solution to Tank Problem—Tradeoffs

tank size	amplitude reduction	transient duration
100 gal	30%	4.6 hrs
173.2 gal	50%	7.98 hrs

Definition 107 (Spring-Mass-Damper System)

The **spring-mass-damper system**.



We need to use three laws from physics to derive the differential equation which models a spring-mass-damper system.

Definition 108 (Hooke's Law)

Assume

F_s — restoring force of a spring.

k — spring constant.

x — displacement of the spring.

Hooke's law state that

$$F_s = kx.$$

The restoring force of a spring is directly proportional to the displacement of the spring.

Definition 109 (Damper Law)

Assume

F_d — resistance force of a damper.
b — damping constant.
x' — velocity the damper is stretched/compressed.

The **damper law** states that

$$F_d = bx'.$$

The resistance force of a damper is directly proportional to the velocity with which the damper is stretched/compressed.

The third law we need to use is Newton's law of motion:

$$F = mx''.$$

Example 110 (Equations of the Spring-Mass-Damper System)

Show that the equations of a spring-mass-damper system with mass m, damping constant b, spring constant k and driving force $f(t)$ is

$$mx'' + bx' + kx = F(t).$$

Example 111 (Spring-Mass System)

A spring is attached to the ceiling and a 2 kg mass is attached. The mass stretches the spring 0.2 meters. The spring with mass attached is pushed up 0.1 meters and released from rest. The spring oscillates.

(a) Model the spring-mass system by a second order differential equation with two initial conditions. (The acceleration of gravity is 9.8 m/sec^2).

(b) What is the period of oscillation?

Example 112 (. . . continued)

Repeat the previous problem except attach a damper with damping constant $b = 4\sqrt{13}$.

The envelope of a damped oscillation describes what happens to the amplitude of the oscillation.

Example 113 (Time for Transient to Die Out)

Compute how long it takes for the amplitude of the transient to die down to 1 of its initial value.

Example 114 (Screen Door Closer)

A screen door closer consists of a spring with spring constant k and a damper with damping constant b. Assume the mass of the screen door is negligible. Let x represent the angular displacement of the door.

(a) Model the system by a first order DE.

(b) How long does it take for the door to close 99 of the way from its initial position?

Example 115 (Solving Second Order DEs)

Solve
$$x'' + 4x' + 3x = 6, \; x(0) = 0, \; x'(0) = 0.$$

Definition 116 (Linear Constant Coefficient DE)

A **second order** linear, constant coefficient, differential equation is a differential equation that can be put in the "standard" form

$$ax'' + bx' + cx = f(t).$$

Note: we do not divide by the leading coefficient a. This makes it easier to apply the quadratic formula to the characteristic equation of the differential equation.

The function $f(t)$ in the above definition is loosely speaking, called the forcing function—loosely, because we do not divide by the leading coefficient a.

If $f(t) = 0$, then the equation is called homogeneous.

A solution $x(t)$ is an equilibrium solution of second order differential equation if $x'(t) = x''(t) = 0$.

To obtain a unique solution to a second order differential equation, we need two initial conditions, typically $x(0)$ and $x'(0)$.

As with linear, first order differential equations, the solution to linear, second order differential equations will always have a homogeneous part and a particular part:

$$x(t) = x_h(t) + x_p(t).$$

The particular part, $x_p(t)$ can be obtained by the method of undetermined coefficients.

Example 117 (Repeated Roots)

Solve $x'' + 2x' + x = 5,\ x(0) = 0,\ x'(0) = 0.$

Definition 118 (Linearly Independent Functions)

Two functions are **linearly independent** if one function is not a linear combination of the other function.

Example 119 (Linearly Independent Functions)

Are the following functions linearly independent?

(a) $e^{-t},\ 5e^{-t}$.

(b) $e^{-t},\ e^{-5t}$.

(c) $\sin(2t),\ 2\sin(3t)$.

(d) $\sin(2t),\ \cos(2t)$.

(e) $t,\ 5t+1$.

(f) $t-1,\ 10(t-3)$.

(g) $t-1,\ 10t-10$.

Example 120 (Multiplication Rule)

Use the method of undetermined coefficients to determine the general solution to

$$x' + 2x = e^{-2t}.$$

Definition 121 (Multiplication Rule)

For repeated roots, or when the guess for the particular solution is contained in the homogeneous portion of the general solution, multiply by the independent variable to obtain a linearly independent function.

Warning: The multiplication rule only works for repeated roots and when the guess for the particular solution is in the homogeneous portion of the solution. It will not work in other situations.

Theorem 122 (General Solution for Second-Order DEs)

The general solution to the linear, second order, constant coefficient differential equation

$$ax'' + bx' + cx = f(t)$$

is a linear combination of two independent solutions plus a particular solution and is of one of the following three forms:

(i) Real Roots

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + x_p(t)$$

(ii) Repeated Roots

$$x(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t} + x_p(t)$$

(iii) Complex Roots

$$x(t) = e^{\mu t}\left[c_1 \cos(\omega t) + c_2 \sin(\omega t)\right] + x_p(t)$$

where $\mu = \mathrm{Re}(\lambda)$ and $\omega = \mathrm{Im}(\lambda)$.

Example 123 (Complex Roots)

Solve $x'' + x = 5,\ x(0) = 5,\ x'(0) = 1.$

Example 124 (Complex Roots)

Solve $\lambda^2 - 4\lambda + 29 = 0$.

i — imaginary unit

$i^2 = -1$ (by definition.)

Note, i is not a real number. We can think of i as being equal to $\sqrt{-1}$, but we must be careful.

$$i^2 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$$

The above calculation is incorrect because we are treating i as though it were a real number.

A complex number has a real and an imaginary part.

Definition 125 (Rectangular Form)

Let $z = x + iy$. Then the **real part** of z is $\mathrm{Re}(z) = x$ and the **imaginary part** of z is $\mathrm{Im}(z) = y$.

We can plot complex numbers in the complex plane. (Plot several complex numbers.)

Next, we look at the arithmetic of complex numbers.

Add/Sub: just add real part to real part and imaginary part to imaginary part. (examples)

Multiplication: use fact that $i^2 = -1$.

Example 126 (Complex Multiplication)

$(2+3i)(5+2i) = ?$

Definition 127 (Complex Conjugate)

The **complex conjugate** of $z = x + iy$ is $\overline{z} = x - iy$.

Theorem 128 (Complex Conjugate)

$\mathrm{Re}(z) = \frac{z+\overline{z}}{2}$ and $\mathrm{Im}(z) = \frac{z-\overline{z}}{2i}$.

Example 129 (Complex Conugate)

Let $z = 5 + 2i$. Use above formulas to calculate $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$.

Note: $z\bar{z}$ is always a real number. Why?

Definition 130 (Magnitude)

The **magnitude** of a complex number z is $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$.

Division: Multiply top and bottom by the complex conjugate of the denominator.

Example 131 (Division of Complex Numbers)

$$\frac{2+3i}{5+2i} = ?$$

Theorem 132 (Euler's Formula)

$$e^{\pm i\theta} = \cos(\theta) \pm i\sin(\theta)$$

Example 133 (Euler's formula)

$10e^{-i\frac{\pi}{2}} = ?$

Complex numbers can be put in polar form.

Definition 134 (Polar Form)

A complex number z is in **polar form** if $z = re^{-i\theta}$ where $r = |z|$ is the magnitude of z and θ is the angle made with respect to the positive x-axis.

Example 135 (Conver to Polar Form)

Express $z = 1 + i$ in polar form.

Example 136 (Convert to Rectangular Form)

Convert $z = 5e^{-i\pi/3}$ to rectangular form.

Add/Sub is easiest in rectangular form.
Mult/Div is easiest in polar form.

Example 137 (Division)

$z = 10e^{i\pi/2}, \; w = 5e^{i\pi/3}$. Determine z/w.

Example 138 (Complex Conjugate in Polar Form)

If $z = re^{i\theta}$ what is $\overline{z} =$?.

Theorem 139 (Complex Roots)

Assume $ax'' + bx' + cx = 0$ has complex roots $\lambda = \mu \pm i\omega$. The homogeneous solution

$$C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{(\mu + i\omega)t} + C_2 e^{(\mu - i\omega)t}$$

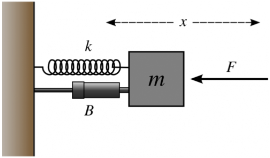
can be expressed without using any complex numbers as

$$e^{\mu t}\left(c_1 \cos(\omega t) + c_2 \sin(\omega t)\right)$$

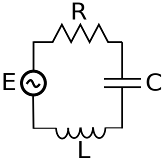
where $\mu = \mathrm{Re}(\lambda_1)$ and $\omega = \mathrm{Im}(\lambda_1)$.

The proof uses Euler's formula and that fact that $\mathrm{Re}(z) = \frac{z+\overline{z}}{2}$ and $\mathrm{Im}(z) = \frac{z-\overline{z}}{2i}$.

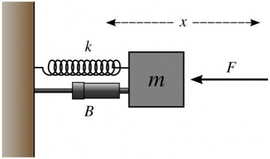
The equation for a spring-mass-damper system and the equation for an RLC circuit are both linear, second-order, constant coefficient differential equations.



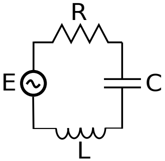
$$mx'' + bx' + kx = F(t)$$



$$Lx'' + Rx' + \frac{1}{C}x = E'(t)$$



$mx''+bx'+kx=F(t)$



$Li''+Ri'+\frac{1}{C}i=E'(t)$

We can study both systems by studying the generic equation

$$ax'' + bx' + cx = f(t).$$

Since we will look at free oscillations first, we will first study the following generic, homogeneous, second order differential equation:

$$ax'' + bx' + cx = 0$$

Note, we assume that the coefficients a, b and c are non-negative. The characteristic equation is:

$$a\lambda^2 + b\lambda + c = 0$$

The characteristic roots are:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition 140 (Free Responses for Second-Order DEs)

The linear, constant coefficient, second-order differential equation

$$ax'' + bx' + cx = 0$$

has the following types of responses:

over damped: when $b^2 - 4ac > 0$ (real roots).

Definition 140 (Free Responses for Second-Order DEs)

The linear, constant coefficient, second-order differential equation

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has the following types of responses:

over damped: when $b^2 - 4ac > 0$ (real roots).

critically damped: when $b^2 - 4ac = 0$ (repeated roots).

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The linear, constant coefficient, second-order differential equation

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over damped: when $b^2 - 4ac > 0$ (real roots).

critically damped: when $b^2 - 4ac = 0$ (repeated roots).

under damped: when $b^2 - 4ac < 0$ (complex roots).

Definition 140 (Free Responses for Second-Order DEs)

The linear, constant coefficient, second-order differential equation

$$ax'' + bx' + cx = 0$$

has the following types of responses:

over damped: when $b^2 - 4ac > 0$ (real roots).

critically damped: when $b^2 - 4ac = 0$ (repeated roots).

under damped: when $b^2 - 4ac < 0$ (complex roots).

undamped: when $b = 0$ (pure imaginary roots).

To have oscillations, the system must be under damped, i.e. the "damping constant" b must be small enough that we get complex roots so that we have sine and cosine terms in the solution.

complex roots: $b^2 - 4ac < 0$

Theorem 141 (Damped (Quasi) Frequency of Oscillation)

The **damped (quasi) frequency** of oscillation of an under damped second order system is

$$\omega = \operatorname{Im}(\lambda_1) = \frac{\sqrt{4ac - b^2}}{2a}.$$

We use the term quasi frequency, because when we have damping we really don't have a periodic function. The proof follows directly from the form of the solution for the complex root case, i.e.

$$x(t) = e^{\mu t}\left(c_1 \cos(\omega t) + c_2 \sin(\omega t)\right)$$

where $\mu = \mathrm{Re}(\lambda_1)$ and $\omega = \mathrm{Im}(\lambda_1)$.
When b is as small as it can be, i.e. $b = 0$ (zero damping) we have:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \pm i\sqrt{\frac{c}{a}}$$

The frequency of oscillation when the damping is equal to zero is called the *natural frequency*, ω_0, of the system.

Theorem 142 (Natural Frequency of Oscillation)

The **natural frequency** of oscillation of an under-damped/un-damped second order system is

$$\omega_0 = \sqrt{\frac{c}{a}}.$$

The natural frequency of oscillation for *any* spring-mass-damper system is $\omega_0 = \sqrt{k/m}$.
The natural frequency of oscillation for *any* RLC circuit is $\omega_0 = \sqrt{1/LC}$.

The actual frequency of oscillation $\omega = \mathrm{Im}(\lambda_1)$ equals the natural frequency $\omega_0 = \sqrt{\frac{c}{a}}$ only when the damping is equal to zero, i.e. only when $b = 0$.

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Example 143 (Undamped Free Oscillations)

A spring is attached to the ceiling and a 10 kg mass is attached to the spring. The mass stretches the spring 0.2 meters. The mass is then pushed up 0.3 meters and struck upwards giving it an initial velocity of 2.8 meter per second. Determine the frequency, amplitude and phase angle of the resulting oscillations.

Example 144 (Damped Free Oscillations)

Consider a horizontal spring-mass-damper system with mass $m = 1$ kg, damping constant $b = 2$ Nts-sec per meter and spring constant $k = 82$ Nts per meter. The mass is displaced 1 meter to the right and released from rest. Determine the number of oscillations before the system reaches equilibrium. Assume the system is in equilibrium when the envelope of the oscillation has died down to 1% of its initial level. Check your answer by plotting the position of the mass as a function of time.

Example 145 (Damped Free Oscillations—Shortcut)

Repeat previous problem but without using the initial conditions.

Example 146 (Undamped Forces Oscillations)

A spring-mass system has a mass $m = 1$ kg and spring constant $k = 100$ Nts/m. A time varying force $f(t) = 36\cos(8t)$ Nts is applied to the system. Assume the spring-mass system is initially at rest in its equilibrium position. At what frequency does the spring-mass system oscillate?

Theorem 147 (A Trig. Identity)

$$\cos(\omega_1 t) - \cos(\omega_0 t) = [-2\sin(\omega_b t)]\sin(\overline{\omega} t)$$

where

$$\omega_d = \frac{\omega_1 - \omega_0}{2}$$

and

$$\overline{\omega} = \frac{\omega_1 + \omega_0}{2}.$$

Note that the bracketed term $-2\sin(\omega_d t)$ is an envelope which varies at the *beat frequency*:

$$|\omega_d| = |\frac{\omega_1 - \omega_0}{2}|.$$

Example 148 (Beat Frequency)

Applying the trig identity to the previous example we have

$$\begin{aligned} x(t) &= \cos(8t) - \cos(10t) \\ &= 2\sin(t)\sin(9t) \end{aligned}$$

The beat frequency is

$$|\omega_b| = |\frac{\omega_1 - \omega_0}{2}| = |\frac{10-8}{2}| = 1.$$

We also have

$$\overline{\omega} = \frac{\omega_1 + \omega_0}{2} = \frac{10+8}{2} = 9.$$

$\overline{\omega}$ is sometimes called the *carrier frequency*. (AM radio)

Example 149 (Undamped Steady-State Gain Function)

Compute the steady-state gain function, $G(\omega)$, of the undamped spring-mass system

$$ax'' + cx = R_o \cos(\omega_1 t).$$

Solution

Recall that

$$G(\omega) = \frac{\text{amplitude of steady-state response}}{\text{amplitude of forcing function}}.$$

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$$ax'' + cx = R_o \cos(\omega_1 t)$$

where R_o is the amplitude of the forcing or driving function and ω_1 is the forcing or driving frequency for the system.

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where R_o is the amplitude of the forcing or driving function and ω_1 is the forcing or driving frequency for the system. Dividing by a, we have

$$x'' + \frac{c}{a}x \quad = \quad \frac{R_o}{a}\cos(\omega_1 t)$$

Solution

Recall that

$$G(\omega) = \frac{\text{amplitude of steady-state response}}{\text{amplitude of forcing function}}.$$

The undamped, second-order equation is:

$$ax'' + cx = R_o \cos(\omega_1 t)$$

where R_o is the amplitude of the forcing or driving function and ω_1 is the forcing or driving frequency for the system. Dividing by a, we have

$$\begin{aligned} x'' + \frac{c}{a}x &= \frac{R_o}{a}\cos(\omega_1 t) \\ &= R_{\text{in}}\cos(\omega_1 t) \end{aligned}$$

Solution

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Since $\omega_0 = \sqrt{\frac{c}{a}}$, we have that $\omega_0^2 = \frac{c}{a}$ and so

$$\begin{aligned} x'' + \frac{c}{a}x &= R_{\text{in}} \cos(\omega_1 t) \\ x'' + \omega_0^2 x &= R_{\text{in}} \cos(\omega_1 t) \end{aligned}$$

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$$x_p(t) = A\cos(\omega_1 t) + B\sin(\omega_1 t).$$

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Substituting $x_p(t)$ into the differential equation and solving for A and B gives

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The method of undetermined coefficients implies the steady state solution is of the form

$$x_p(t) = A\cos(\omega_1 t) + B\sin(\omega_1 t).$$

Substituting $x_p(t)$ into the differential equation and solving for A and B gives

$$\begin{aligned} A &= \frac{R_{\text{in}}}{\omega_0^2 - \omega_1^2} \\ B &= 0 \end{aligned}$$

Solution

Using trigonometry, we get

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$$R_{\text{out}}(\omega_1) = \sqrt{A^2 + B^2} = \left| \frac{R_{\text{in}}}{\omega_0^2 - \omega_1^2} \right|$$

Solution

Using trigonometry, we get

$$R_{\text{out}}(\omega_1) = \sqrt{A^2 + B^2} = \left| \frac{R_{\text{in}}}{\omega_0^2 - \omega_1^2} \right| \text{ and } \phi = 0.$$

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Therefore,

$$x_p(t) = \frac{R_{\text{in}}}{|\omega_0^2 - \omega_1^2|} \cos(\omega_1 t).$$

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Therefore,

$$x_p(t) = \frac{R_{\text{in}}}{|\omega_0^2 - \omega_1^2|} \cos(\omega_1 t).$$

The steady-state gain is then

$$G(\omega_1) = \frac{R_{\text{out}}(\omega_1)}{R_{\text{in}}}$$

Solution

Using trigonometry, we get

$$R_{\text{out}}(\omega_1) = \sqrt{A^2 + B^2} = \left| \frac{R_{\text{in}}}{\omega_0^2 - \omega_1^2} \right| \text{ and } \phi = 0.$$

Therefore,

$$x_p(t) = \frac{R_{\text{in}}}{|\omega_0^2 - \omega_1^2|} \cos(\omega_1 t).$$

The steady-state gain is then

$$\begin{aligned} G(\omega_1) &= \frac{R_{\text{out}}(\omega_1)}{R_{\text{in}}} \\ &= \frac{\frac{R_{\text{in}}}{|\omega_0^2 - \omega_1^2|}}{R_{\text{in}}} \end{aligned}$$

Solution

Using trigonometry, we get

$$R_{\text{out}}(\omega_1) = \sqrt{A^2 + B^2} = \left| \frac{R_{\text{in}}}{\omega_0^2 - \omega_1^2} \right| \text{ and } \phi = 0.$$

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$$\begin{aligned} G(\omega_1) &= \frac{R_{\text{out}}(\omega_1)}{R_{\text{in}}} \\ &= \frac{\frac{R_{\text{in}}}{|\omega_0^2 - \omega_1^2|}}{R_{\text{in}}} \\ &= \frac{1}{|\omega_0^2 - \omega_1^2|}. \end{aligned}$$

Solution

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The amplitude of the steady-state response is very large when the forcing function frequency, ω_1, is very close to the natural frequency, ω_0.

Solution

Thus,

$$G(\omega_1) = \frac{1}{|\omega_0^2 - \omega_1^2|}.$$

The amplitude of the steady-state response is very large when the forcing function frequency, ω_1, is very close to the natural frequency, ω_0.When $\omega_1 = \omega_0$, we have a phenomenon known as **resonance** and the method of undetermined coefficient no longer works.

Example 150 (Undamped Resonance)

Explore the forced oscillations of the undamped spring-mass system:

$$x'' + 100x = 36\cos 10t,\ \ x(0) = 0,\ \ x'(0) = 0.$$

Example 151 (Transient vs Steady State)

Consider a spring-mass-damper system with $m = 1$ kg, $b = 2$ Nt-s/m and $k = 101$ Nts/m. Assume the initial conditions are $x(0) = 2$ meters and $x'(0) = 9$ m/sec. Assume the force $f(t) = 97\cos 2t - 4\sin 2t$ Nts is applied to the mass.

(a) Determine the solution.
(b) Identify the transient and steady-state portion of the solution.
(c) What is the natural frequency of the system?
(d) What is the damped (quasi) frequency of the system?
(e) What is the frequency of the steady-state response?
(f) What is the forcing function/driving frequency?

Observe that both the driving frequency and the steady-state frequency are the same, $\omega_1 = 2$ rad/sec. This will always be the case. (A linear system can not alter the driving frequency.) The damped frequency is $\omega = 10$ rad/sec. The natural frequency is slightly higher at $\omega_0 = \sqrt{101}$ rad/sec.

Example 152 (Computing the Steady-State Gain)

Compute the gain of the spring-mass-damper system

$$x'' + 2x' + 101 = 97\cos 2t - 4\sin 2t.$$

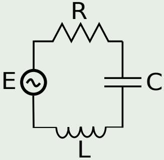
Example 153 (Computing the Steady-State Gain)

A spring-mass-damper system has mass $m = 1$ kg, damping constant $b = 5$ Nt-s/m and spring constant $k = 4$ Nts/m. The driving force $f(t) = 2\cos(2t)$ Nts is applied to the system. Assume the mass is released at rest from its equilibrium position. (a) Determine the transient and steady-state response. (b) Compute the steady-state gain.

Example 154 (Computing the Steady-State Gain)

Compute the steady-state gain of the following RLC circuit.

$$\begin{aligned} R &= 100\Omega \\ C &= 0.001\text{F} \\ L &= 0.01\text{H} \\ E(t) &= 10\cos(2t)\text{V}. \end{aligned}$$



Assume $i(0) = 0,\ i'(0) = 0.$

Finally we compute the steady-state gain function of a damped spring-mass-damper system. We will use the gain function to compute the *practical resonance frequency* of the system.

Definition 155 (Practical Resonance Frequency)

The **practical resonance frequency** of a spring-mass-damper system is the forcing function/driving frequency which maximizes the steady-state gain function.

Computing Practical Resonance

1. Compute the gain function:

$$G(\omega_1) = \frac{R_{\text{out}}(\omega_1)}{R_{\text{in}}}.$$

2. Determine the forcing function frequency, ω_1, which maximizes the gain function $G(w_1)$.

Computing Practical Resonance

1. Compute the gain function:

$$G(\omega_1) = \frac{R_{\text{out}}(\omega_1)}{R_{\text{in}}}.$$

2. Determine the forcing function frequency, ω_1, which maximizes the gain function $G(w_1)$.

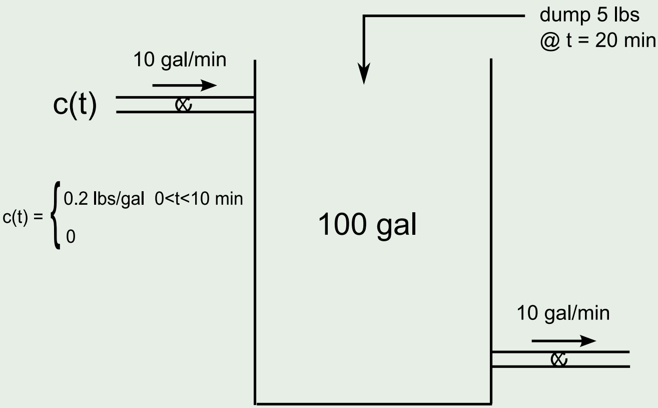
Example 156 (Computing Practical Resonance)

A spring-mass-damper system has mass $m = 1$ kg, $b = 4$ Nt-s/m, $k = 53$ Nts/m.

1. Compute the natural frequency, ω_0.
2. Compute the damped frequency, ω.
3. Compute the gain function, $G(\omega_1)$.
4. Compute the practical resonance frequency, ω_r.

Example 157 (Motivation Problem)

How much salt is in the tank shown below at $t = 30$ minutes? Assume the tank is initially filled with pure water.



It is possible to model the above tank system using the following *single* differential equation:

$$x'(t) = 2\left[u(t) - u(t-10)\right] + 5\delta(t-20) - 10\frac{x(t)}{100}, \;\; x(0) = 0$$

where

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{unit step function}$$

$$\delta(t) = \left\{ \begin{array}{ll} +\infty & t = 0 \\ 0 & \text{otherwise} \end{array} \right\} \quad \text{unit impulse function}$$

where

$$\int_{-\infty}^{\infty} \delta(t)dt = 1.$$

Laplace transforms provide a convienent way to solving the above single, differential equation. The above *differential* equation (which is in the time domain) is transformed into an *algebraic* equation in the Laplace domain.

$$\text{time domain} \quad \overset{\text{Laplace Transform}}{\Longleftrightarrow} \quad \text{Laplace domain}$$

$$x(t) \overset{\mathcal{L}}{\Longrightarrow} X(s)$$

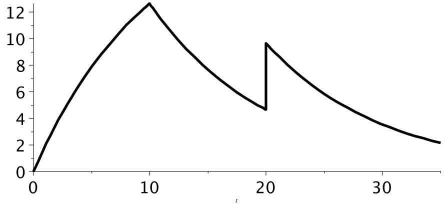
The Laplace transform of the above differential equation is:

$$sX(s) = 2\left(\frac{1}{s} - \frac{e^{-10s}}{s}\right) + 5e^{-20s} - 10\frac{X(s)}{100}.$$

Note that we use lower case letters (e.g. $x(t)$) to denote functions in the time domain and upper case letters (e.g. $X(s)$) to denote the corresponding Laplace domain functions. The above algebraic equation can easily be solved for $X(s)$.

$$X(s) = \frac{1}{s+0.1}\left[2\left(\frac{1}{s} - \frac{e^{-10s}}{s}\right) + 5e^{-20s}\right]$$

Then the inverse Laplace transform of $X(s)$ can be computed to obtain the solution $x(t)$ in the time domain. (See plot below.) Note: $x(30) = 3.55$ lbs.


12
10
8
6
4
2
0
0
10
20
30
t

Solving differential equations using Laplace transforms requires:

1. Definition of Laplace Transform:
   $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt.$
2. Linearity of Laplace Transform:
   $\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 F_1(s) + c_2 F_2(s).$
3. Laplace Transform of Derivatives:
   $\mathcal{L}\{f'(t)\} = sF(s) - f(0).$

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3. Laplace Transform of Derivatives:
   $\mathcal{L}\{f'(t)\} = sF(s) - f(0).$

Definition 158 (Laplace Transforms)

The **Laplace transform** of $f(t)$ is defined to be

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

The Laplace transform tables gives Laplace transforms of commonly used functions.

The above definition of the Laplace transform involves an *improper integral*.

Example 159 (Improper Integral)

Compute $\int_1^\infty \frac{1}{t^2}\,dt$.

Example 160 (Computing the Laplace Transform)

Compute the Laplace transform of $f(t) = 1$

Example 161 (Computing the Laplace Transform)

Compute the Laplace transform of $f(t) = e^{-t}$

Example 162 (Computing the Laplace Transform)

Compute the Laplace transform of $f(t) = t$.

Normally, we do not compute inverse Laplace transforms by hand. We use either the tables or Maple to compute inverse Laplace transforms.

Example 163 (Determining Inverse Laplace Transform)

Assume $F(s) = \frac{3}{s^2+9}$. Determine $f(t) = \mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$.

Example 164 (Inverse Laplace Transform)

Assume $F(s) = \frac{1}{s+5}$. Determine $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$.

Example 165 (Maple Example)

Use Maple to compute the Laplace transform of

$$e^{at}\cos(bt).$$

Example 166 (Linear Behavior)

Consider a linear spring. Hooke's law states that $f = kx$. For example, assume $k = 5$ Nts/m. Then

x (meters)	2	4	1	$4+1=5$	$(1/4)(4)+7(1)=8$
f (Newtons)	10	20	5	$20+5=25$	$(1/4)(20)+7(5)=40$

An operation is linear if the operation preserves *linear combinations*. Examples of linear operators are differentiation, integration and the Laplace transform. What are some examples of nonlinear operators?

Example 167 (Nonlinear Function)

$$\sin(c_1x_1 + c_2x_2) \neq c_1 \sin(x_1) + c_2 \sin(x_2)$$

Definition 168 (Linear Operator)

$\mathcal{F}$ is a **linear operator** if

$$\mathcal{F}\{c_1x_1 + c_2x_2\} = c_1\mathcal{F}\{x_1\} + c_2\mathcal{F}\{x_2\}.$$

Note that linear functions/operators behave as though they have the distributive property, so linear functions/operators behave as though they multiply their arguments.

$$\mathcal{F}\{c_1x_1 + c_2x_2\} = c_1\mathcal{F}x_1 + c_2\mathcal{F}x_2.$$

Definition 169 (Linearity Chart)

x	x_1	x_2	$x_3 = c_1 x_1 + c_2 x_2$
$\mathcal{F}$	$\mathcal{F}(x_1)$	$\mathcal{F}(x_2)$	$\mathcal{F}(x_3) = c_1\mathcal{F}(x_1) + c_2\mathcal{F}(x_2)$

Theorem 170 (Linearity of Laplace Transform)

The Laplace transform is a linear operator. Specifically,

$$\mathcal{L}\left\{c_1 f_1(t) + c_2 f_2(t)\right\} = c_1 \mathcal{L}\left\{f_1(t)\right\} + c_2 \mathcal{L}\left\{f_2(t)\right\}.$$

Example 171 (Linearity of Laplace Transform)

Compute $\mathcal{L}\left\{2e^{-t} + 3t\right\}$.

Example 172 (Linearity of Inverse Laplace Transform)

Compute $\mathcal{L}^{-1}\left\{\frac{2s+10}{s^2+4}\right\}$.

Laplace transforms convert differentiation with respect to t into multiplication by s. This property makes Laplace transforms well suited for linear solving differential equations. We can use Laplace transforms to convert a differential equation into an algebraic one.

Theorem 173 (Laplace Transform of First Derivative)

$$\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0)$$

The proof is postponed.

Example 174 (Laplace of Derivative)

$$
\begin{aligned}
\mathcal{L}\left\{\frac{d}{dt}\sin t\right\} &= s\mathcal{L}\{\sin t\} - \sin 0 \\
&= s\frac{1}{s^2+1} - 0 \\
&= \frac{s}{s^2+1}
\end{aligned}
$$

Check: $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$.

Example 175 (Laplace of a Derivative)

$$\begin{aligned}
\mathcal{L}\left\{\frac{d}{dt}e^t\right\} &= s\mathcal{L}\left\{e^t\right\} - e^0 \\
&= s\frac{1}{s-1} - 1 \\
&= \frac{1}{s-1}
\end{aligned}$$

Check: $\mathcal{L}\left\{e^t\right\} = \frac{1}{s-1}$.

The Laplace transform converts differential equations into algebraic equations. In order to solve DEs, we will need the following three properties.

Solving Differential Equations using Laplace Transforms

1. Definition: $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$.
2. Linearity: $\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 F_1(s) + c_2 F_2(s)$.
3. 1st Derivative: $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$.

Solving Differential Equations using Laplace Transforms

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3. 1st Derivative: $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$.

Example 176 (Solving First Order DEs)

Use Laplace transforms to solve $x' + 2x = e^{-t},\ x(0) = 1.$

If differentiation can be achieved in the Laplace domain by multiplying by s, how can integration be achieved?

Theorem 177 (Laplace Transform of Integrals)

$$\mathcal{L}\left\{\int_0^t f(\tau),\ d\tau\right\} = \frac{F(s)}{s}$$

Proof postponed.

Example 178 (Laplace of Integrals)

$$\begin{aligned} \mathcal{L}\left\{1\right\} &= \frac{1}{s} \\ \mathcal{L}\left\{\int_0^t 1,\ d\tau\right\} &= \mathcal{L}\left\{t\right\} = \frac{1}{s^2} \end{aligned}$$

In order to solve second order differential equations, we need the following additional property.

Theorem 179 (Laplace Transform of Second Derivatives)

$$\mathcal{L}\left\{f''(t)\right\} = s^2F(s) - sf(0) - f'(0)$$

Proof is postponed.

Definition 180 (Unit Step Function)

The **unit step function** is define to be

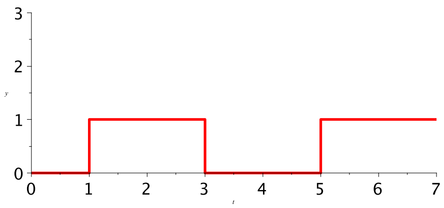
$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

(sketch)

Sketch $u(t-3)$. Note that the graph is shifted 3 units to the right.

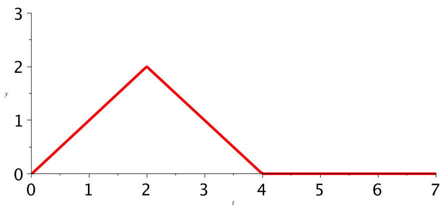
Example 181 (Switching Function)

The switching signal $s(t)$ graphed below is used to turn a light on and off. Represent $s(t)$ using unit step functions.



Example 182 (Switching Function)

Use unit step functions to represent the function $f(t)$ graphed below.



Definition 183 (Unit Pulse Function)

The **unit pulse function** is defined to be

$$p(t, w) = \begin{cases} 1 & 0 \leq t \leq w \\ 0 & \text{otherwise} \end{cases}$$

Note that w is the width of the pulse. (sketch)

The unit pulse function can be expressed in terms of the unit step function

$$p(t, w) = u(t) - u(t - w).$$

Example 184 (Unit Pulse Function)

Sketch (a) $p(t, 2)$ (b) $p(t-5, 1)$.

The pulse function can be used to cut out pieces of other functions.

Example 185 (Cut Out Piece of the Sin Function)

Use a pulse function to cut out the piece of the function $f(x) = \sin(x)$ between $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

What does an "impulse purchase" mean?

Systems are sometimes subject to "impulsive" effects.

Example 186 (Impulsive Effects)

Impulsive effects:

– lightning strikes

Characteristics of impulsive effects:

Example 186 (Impulsive Effects)

Impulsive effects:

- lightning strikes
- explosions

Characteristics of impulsive effects:

Example 186 (Impulsive Effects)

Impulsive effects:

- lightning strikes
- explosions
- car crashes

Characteristics of impulsive effects:

Example 186 (Impulsive Effects)

Impulsive effects:

- lightning strikes
- explosions
- car crashes
- salt dumps

Characteristics of impulsive effects:

Example 186 (Impulsive Effects)

Impulsive effects:

- lightning strikes
- explosions
- car crashes
- salt dumps

Characteristics of impulsive effects:

- large magnitude

Example 186 (Impulsive Effects)

Impulsive effects:

- lightning strikes
- explosions
- car crashes
- salt dumps

Characteristics of impulsive effects:

- large magnitude
- short duration

Definition 187 (Impulse)

The **impulse** of a function $f(t)$ over the interval $t \leq \tau \leq t+h$ is given by the integral

$$I = \int_t^{t+h} f(\tau)d\tau.$$

Note that the impulse is just equal to the area under the graph of $f(t)$.
Typically $f(t)$ has a large magnitude, but small duration h.

Example 188 (Impulse)

Determine the impulse of the pulse function

$$f(t) = p(t-5, 10) = u(t-5) - u(t-15)$$

for $-\infty \leq t \leq \infty$.

Note that the impulse just equals the total amount of salt dumped into the tank, i.e. 10 lbs.

Impulsive effects have large magnitude and short duration. Lets repeat the previous lesson, except dump all of the salt in shorter and shorter intervals of time. In the limit of zero duration, we get an *impulse function*.

Example 189 (Impulse Function Limit)

Discuss the following table.

duration (min)	rate (lbs/min)	forcing function $f(t)$	impulse (lbs)
10	1	$p(t-5, 10)$	10
5	2	$2p(t-5, 5)$	10
1	10	$10p(t-5, 1)$	10
0.01	1000	$1000p(t-5, 0.01)$	10
0	$+\infty$	$10\delta(t-5)$	10

(sketch)

We can think of the impulse function $\delta(t)$ as the limit of a sequence of pulse functions $\frac{1}{\varepsilon}p(t,\varepsilon)$ with decreasing duration, ε, increasing magnitude, $1/\varepsilon$, but with fixed area (impulse) equal to $\varepsilon(1/\varepsilon) = 1$. (sketch)

$$\delta(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} p(t,\varepsilon) = \begin{cases} +\infty & t = 0 \\ 0 & \text{otherwise} \end{cases}$$

Note, the impulse of $\delta(t)$ is given by

$$I = \int_{-\infty}^{\infty} \delta(\tau)\, d\tau = \int_{0^-}^{0^+} \delta(\tau)\, d\tau = 1$$

Definition 190 (Unit Impulse Function)

The **unit impulse function** is defined to equal

$$\delta(t) = \begin{cases} +\infty & t = 0 \\ 0 & \text{otherwise} \end{cases}$$

where, by definition, the impulse of the unit impulse function equals one, i.e.

$$I = \int_{-\infty}^{\infty} \delta(\tau)d\tau = \int_{0-}^{0+} \delta(\tau)d\tau = 1.$$

The unit impulse function $\delta(t)$ is a pulse "function" with infinite magnitude, zero duration, but a *finite area (impulse) equal to 1*.

Example 191 (Graph of Impulse Function)

Graph the impulse function $f(t) = 6\delta(t-1)$.

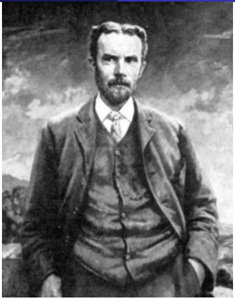


Figure: At the beginning of the 20th century, Oliver Heaviside, who was an engineer, developed the basic idea of Laplace transforms. But his idea lacked mathematical rigor.

Just like the pulse function, the impulse function can be used to cut-out (sample) a portion of a function. The pulse function cuts out intervals of a function. The impulse function can only cut-out point.

Example 192 (Sampling Property of the Impulse Function)

Evaluate $\int_{-\infty}^{\infty} t^2 \delta(t-5)dt$.

In general, we have the following result which states that $\delta(t - t_1)$ samples the value $f(t_1)$ of the function $f(t)$.

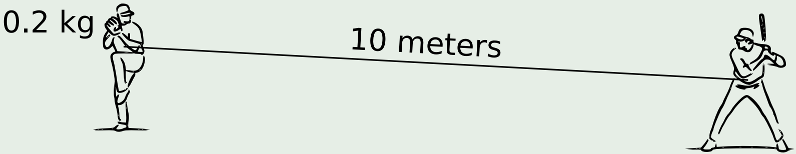
Theorem 193 (Sampling Property of Impulse Function)

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_1)dt = f(t_1).$$

Why is the theorem true?

Example 194 (Impulse Function)

A pitcher standing 10 meters from a batter pitches the ball at the batter at a speed of 100 m/sec. The ball is batted away at 250 m/sec. What is the force of the ball on the bat? Assume the mass of the ball is 0.2 kg.



Hint: Use an impulse function.

Example 195 (Salt Tank Pulse Function Revisited)

Assume all 10 lbs in the salt tank problem is dumped in at $t = 5$ minutes. Compare the pulse response to the impulse response.

Example 196 (Factoring Out Constants)

Compute the inverse Laplace transform of

$$\frac{1}{2s+10}.$$

Example 197 (Factoring Out Constants)

Compute the inverse Laplace transform of

$$\frac{10}{s^2 + 25}.$$

Example 198 (Completing the Square)

Compute the inverse Laplace transform of

$$\frac{16}{s^2 - 6s + 25}.$$

Example 199 (Completing the Square)

Compute the inverse Laplace transform of

$$\frac{2s+2}{s^2+2s+5}.$$

Example 200 (Completing the Square)

Compute the inverse Laplace transform of

$$\frac{2s+5}{s^2+2s+5}.$$

Example 201 (Partial Fraction Expansion)

Compute the inverse Laplace transform of

$$\frac{1}{s^2 - 5s + 6}.$$

When should you factor? When should you complete the square? If you can factor the denominator, don't try to complete the square.

Example 202 (Partial Fraction Expansion)

Compute the inverse Laplace transform of

$$\frac{12s}{s^2 - 4s - 5}.$$

Example 203 (Partial Fraction Expansion—Quadratic Terms)

Compute the inverse Laplace transform of

$$\frac{2s^3 - 4s - 8}{(s^2 - s)(s^2 + 4)}.$$

Example 204 (Grouping Terms Approach)

Compute the inverse Laplace transform of

$$\frac{1}{s(s^2+1)}.$$

Example 205 (Delay Operator)

$$\sin(t) \xrightarrow{\mathcal{L}} \frac{1}{s^2+1} \xrightarrow{\text{delay operator}} \frac{e^{-2s}}{s^2+1} \xrightarrow{\mathcal{L}^{-1}} \sin(t-2)u(t-2)$$

Example 206 (Delay Operator: e^{-as})

What does multiplying by e^{-as} in the Laplace domain do in the time domain? Plot the following functions in the time domain for $a = 2$ on the same axes.

$x(t)$	$X(s)$	$e^{-as}X(s)$	$\mathcal{L}^{-1}\left\{e^{-as}X(s)\right\}$
$\sin(t)$	$\frac{1}{s^2+1}$	$\frac{e^{-as}}{s^2+1}$	$\sin(t-a)u(t-a)$
e^{-t}	$\frac{1}{s+1}$	$\frac{e^{-as}}{s+1}$	$e^{-(t-a)}u(t-a)$

Theorem 207 (t-Shift Theorem)

If $F(s) \xrightarrow{\mathcal{L}^{-1}} f(t)$, then $e^{-as}F(s) \xrightarrow{\mathcal{L}^{-1}} f(t-a)u(t-a)$.
(Note: t-shift preserves the sign.)

Example 208 (t-Shift)

$\mathcal{L}\left\{u(t-a)\right\} = ?$

Example 209 (t-Shift)

$\mathcal{L}\left\{10p(t-2,4)\right\} = ?$

Example 210 (t-Shift)

Which of the following functions represents a time shift of the function $f(t) = \sin(t)$?
(a) $\sin(t)u(t-1)$ (b) $\sin(t-1)u(t-1)$. (Plot each function.)

Theorem 211 (s-Shift Theorem)

If $f(t) \xrightarrow{\mathcal{L}} F(s)$, then $e^{at}f(t) \xrightarrow{\mathcal{L}} F(s-a)$.
(Note: s-shift reverses the sign.)

Example 212 (s-shift)

$\mathcal{L}\left\{e^{at}\cos(bt)\right\} = ?$

Example 213 (s-shift)

$\mathcal{L}\left\{te^{-t}\right\} = ?$

Example 214 (s-shift)

$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+5}\right\} = ?$

Solution: Completing the square we have

$$\frac{s+1}{s^2+2s+5} = \frac{s+1}{(s+1)^2+2^2}.$$

Now $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} = \cos(2t)$. So

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} = e^{-t}\cos(2t).$$

Proof: (t-Shift Thm)

$$\begin{aligned}
e^{-as}F(s) &= e^{-as}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau \\
&= \int_0^\infty f(\tau)e^{-s(\tau+a)}\,d\tau \\
&= \int_a^\infty f(t-a)e^{-st}\,dt \\
&= \int_0^\infty f(t-a)u(t-a)e^{-st}\,dt \\
&= \mathcal{L}\left\{f(t-a)u(t-a)\right\}.
\end{aligned}$$

Proof: (s-Shift Theorem) Since $\mathcal{L}\left\{f(t)\right\} = \int_0^\infty f(t)e^{-st}dt$

$$\begin{aligned}
\mathcal{L}\left\{e^{at}f(t)\right\} &= \int_0^\infty e^{at}f(t)e^{-st}\,dt \\
&= \int_0^\infty f(t)e^{-(s-a)t}\,dt \\
&= F(s-a).
\end{aligned}$$

Therefore, $e^{at}f(t) = \mathcal{L}^{-1}\left\{F(s-a)\right\}$.

Recall the definition for the convolution of two functions.

Definition 215 (Convolution)

The **convolution** of $f(t)$ and $g(t)$ is defined to be

$$f(t) \star g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Multiplying Laplace transforms is a basic operation. *Multiplication* in the Laplace domain corresponds to *convolution* in the time domain.

time domain	Laplace domain
$x(t) = f(t) \star g(t)$	$X(s) = F(s)G(s)$

time domain	Laplace domain
$x(t) = f(t) \star g(t)$	$X(s) = F(s)G(s)$

Next we give an example which illustrates that convolution in the time domain corresponds to multiplication in the Laplace domain.

Example 216 (Convolution)

Show that $\mathcal{L}\left\{e^t \star e^{-t}\right\} = \mathcal{L}\left\{e^t\right\}\mathcal{L}\left\{e^{-t}\right\}$.

The order of convolution does not matter.

Theorem 217 (Order of Convolution)

$$f(t) \star g(t) = g(t) \star f(t).$$

Example 218 (Convolution)

Compute the convolution $e^{-t} \star u(t)$ by hand

1. using the definition of convolution.
2. using the Laplace transform tables.

Example 219 (Convolution)

Compute $e^t \star \delta(t-2)$ by hand:

1. using the Laplace transform tables.
2. using the definition of convolution.

Solution: Using the Laplace tables we have:

$$\mathcal{L}\left\{e^t\right\}\mathcal{L}\left\{\delta(t-2)\right\} = \frac{1}{s-1}e^{-2s} = \frac{e^{-2s}}{s-1}$$

and

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-1}\right\} = e^{t-2}u(t-2).$$

Using the definition of convolution we have

$$e^t \star \delta(t-2) = \int_0^t e^{\tau}\delta(t-\tau-2)d\tau \text{ tricky to do}$$

Its easier to switch the order:

Transfer functions are used to represent a system in the Laplace domain. They convert a *forcing* or *input function* into a *response* or *output function*.

Definition 220 (Transfer Function)

Assume all initial conditions are equal to zero. The **transfer function**, $H(s)$, of a system is defined to be the ratio

$$H(s) = \frac{X(s)}{F(s)} = \frac{\mathcal{L}\left\{\text{response (output)}\right\}}{\mathcal{L}\left\{\text{forcing function (input)}\right\}}.$$

Transfer functions generalize the steady-state gain function defined earlier. Recall the following definition.

Definition 221 (Steady-State Gain Function)

Consider the differential equation function

$$x' + kx = R_{\text{in}} \cos(\omega t)$$

with steady-state response

$$x(t) = R_{\text{out}}(\omega) \cos(\omega t - \phi).$$

The **steady-state gain function** $G(\omega)$ is given by

$$G(\omega) = \frac{R_{\text{out}}(\omega)}{R_{\text{in}}}.$$

In fact we have the following interesting connection between transfer functions and the steady state gain function.

Theorem 222 (Transfer Functions and Steady-State Gain Functions)

If $H(s)$ is the transfer function of a system, then

$$H(i\omega) = G(\omega)e^{-i\phi(\omega)}$$

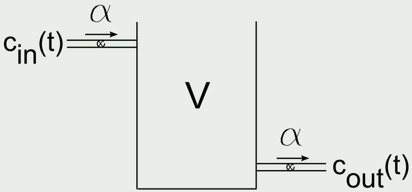
where $G(\omega)$ is the steady-state gain function and $\phi(\omega)$ is the steady-state phase angle function.

The above formula provides a fast way for computing the steady-state gain function $G(\omega)$ using complex numbers. First compute the transfer function. Then plug in $i\omega$ for s and compute the magnitude, i.e.

$$G(\omega) = \sqrt{H(i\omega)\overline{H(i\omega)}}.$$

Example 223 (Transfer Function — Salt Tank System)

Salt solution at a concentration of $c_{\text{in}}(t)$ is pumped at a rate α into a mixing tank with volume V. The well mixed solution in the tank is pumped out at the same rate α. (See diagram below.) Determine the transfer function for input $c_{\text{in}}(t)$ and output $c_{\text{out}}(t)$.



Example 224 (Minimum Sized Tank.)

A food processing plant has a problem. Ideally, all the jars of pickled fruit produced by the plant should have the same salt concentration. However, the concentration of the salt solution pumped into jars at 100 gal/hr is observed to vary with time. Assume the salt concentration pumped into the jars is approximated by the function

$$c_{\text{in}}(t) = 0.1\sin(t) + 0.2 \text{ lbs/gal}$$

where t is measured in hours. What should be done to fix this problem?

Determine the minimum tank size, V, which will reduce the amplitude of the variations by 50%.

Solution I

Solution: Since the pumping rate must be $\alpha = 100$ gal/hr, we have the transfer function is

$$H(s) = \frac{100}{s + 100/V}.$$

The gain function is

$$G(\omega) = \sqrt{H(i\omega)\overline{H(i\omega)}} = \frac{100}{\sqrt{\omega^2 V^2 + 1000}}$$

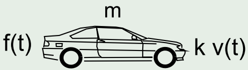
which is obtained using Maple's `conjugate` and `simplify, Assume Real` commands. Solving

$$G(1) = \frac{100}{\sqrt{V^2 + 1000}} = 0.5$$

using Maple yields $V = 173.2$ gallons.

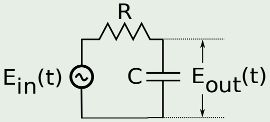
Example 225 (Transfer Function — Car Speed)

Consider the car shown below. Let $f(t)$ equal the forward force generated by the car's engine. Assume the air resistance drag force impeding the car's forward motion is directly proportional to the cars speed, i.e. assume the drag force is equal to $kv(t)$. The mass of the car is m. Let $f(t)$ be the forcing function and $v(t)$ be the response function. Determine the transfer function for this system.



Example 226 (Transfer Function — Low Pass Filter)

A low pass filter is used to filter out the noise in a signal. (See diagram below.) The unfiltered signal is the input signal $v_{\text{in}}(t)$ and the filtered signal is the output singal $v_{\text{out}}(t)$.



The relationship between $v_{\text{in}}(t)$ and $v_{\text{out}}(t)$ is given by the integral equation

$$v_{\text{out}}(t) = \frac{1}{C}\int_0^t \frac{v_{\text{in}}(t) - v_{\text{out}}(t)}{R}\,dt.$$

Determine the transfer function for this system.

Definition 227 (Impulse Response)

The **impulse response**, $h(t)$, of a system is the inverse Laplace transform of the system's transfer function, i.e.

$$h(t) = \mathcal{L}^{-1}\left\{H(s)\right\}$$

The above definition makes sense because if $x(t)$ is the response to the impulse forcing function $f(t) = \delta(t)$, then the transfer function is

$$H(s) = \frac{X(s)}{F(s)} = \frac{X(s)}{1} = X(s)$$

and

$$h(t) = \mathcal{L}^{-1}\left\{(H(s)\right\} = \mathcal{L}^{-1}\left\{X(s\right\} = x(t).$$

Example 228 (Impulse Response)

The response of a spring-mass-damper system to a unit impulse forcing function (hammer blow) is

$$h(t) = \frac{2}{\sqrt{39}} e^{-\frac{1}{4}t} \sin(\frac{\sqrt{39}}{4} t).$$

Determine the mass, m, damping constant μ and spring constant k. (Assume the forcing function is the force applied to the mass and the response is the position of the mass.)

Theorem 229 (General Solution—Laplace Domain)

The **general solution** of a linear differential equation can be expressed in the Laplace domain as follows:

$$X(s) = \underbrace{H(s)F(s)}_{\text{forced reponse}} + \underbrace{H(s)\;(\text{ I.C. })}_{\text{free response}}$$

where $H(s)$ is the transfer function, $F(s)$ is the Laplace transform of the forcing function and $(I.C.)$ refers to the initial conditions.

Example 230 (General Solution Salt Tank)

Pure water is pumped at a rate r into a tank with volume V. The well mixed solution in the tank is drained at a rate r. The rate at which salt is poured into the tank is $f(t)$. The tank initially contains x_0 pounds of salt. Write down the general solution in the Laplace domain. Let $f(t)$ be the forcing function and $x(t)$, the amount of salt in the tank, be the response.

Example 231 (Free vs Forced Response)

Consider the previous problem with volume $V = 100$ gal, pumping rate $r = 20$ gal/min, and forcing function $f(t) = u(t-5) - u(t-15)$. Assume the tank initially contains $x_0 = 10$ lbs of salt. Determine the following: (a) zero-state forced response (b) free response (c) total response.

1. Definition:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

2. Linearity:

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 F_1(s) + c_2 F_2(s)$$

3. First Derivative:

$$\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0)$$

4. Second Derivative:

$$\mathcal{L}\left\{f''(t)\right\} = s^2 F(s) - sf(0) - f'(0)$$

5. Integral:

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

6. Convolution:

$$\mathcal{L}\{f(t) \star g(t)\} = F(s)G(s)$$

7. t-Shift:

$$\mathcal{L}\{f(t-\tau)u(t-\tau)\} = e^{-\tau s}F(s)$$

8. s-Shift:

$$\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)$$