“What’s on the test?” is not a silly question. Its answer defines what students will learn. The problems we pose to our students are the ultimate measure of what we hope to teach.

If we aim in the introductory ordinary differential equations course for more than mechanics—for problem solving, modeling, analysis, interpretation, and the exercise of critical judgment—we must pose problems that demand these skills. Indeed, we need a spectrum of problems ranging from mimic-the-example exercises to open-ended projects so that students are supported and challenged as their skills and confidence develop.

This article describes some design principles for constructing ranges of such exercises. The next four sections present a variety of strategies; they are illustrated with examples. The last section states the design principles upon which those strategies are based.

The use of interesting problems benefits everyone. Instructors can break away from a dull routine. (What if solving constant coefficient linear equations were like radioactive plutonium? What if exposure beyond a modest maximal lifetime dose were fatal?!) Students rise to the challenge of problems that require thinking. And interesting problems let mathematics present a better face, showing itself much more than a dull concoction of recipes and rules.

**Bad Mathematics ⇒ Good Problems**

The precision at the heart of our discipline serves us poorly when we try to ease away from sterile, mechanical problems. In an introductory course, well-posed questions are often boring; they invite mechanical responses from students, who simply search for an isomorph among the examples in the chapter under study. Assigning too many well-posed problems teaches trivial pattern recognition, not analysis or problem solving or any other skill our discipline demands.

Good problems contain too much or too little or contradictory information. Good problems may not have unique solutions, and they do not define the desired solution. They often require that students reformulate a better posed problem from that which is first put to them. For example, a carefully posed problem is

Find the unique solution of the initial value problem \( y' = y, \ y(0) = 1 \).
Some more interesting variants that omit or add information or contradict themselves are

Find the unique solution of \( y' = y, \ y(0) = 1, \ y(1) = 0. \)
Find the unique solution of \( y' = y. \)
Find the general solution of \( y' = y, \ y(0) = 1. \)
Find a nontrivial solution of \( y' = y, \ y(0) = 0. \)

We should always ask the well-posed question first to build confidence. We should not stop there, though. Students will better grasp concepts if they see them in different—even wrong—settings. And they will develop critical judgment and problem solving skills if they are forced to formulate and answer a better question than the one they were asked. (There is considerable anecdotal evidence that this process of finding the right problem among a myriad of questions is important in the practice of mathematics in industry [3].)

From a student's perspective, problems that involve parameters instead of numbers are almost as ambiguous as if they were ill posed. As their maturity increases, students realize that the vague ambiguity of a poorly posed problem is much different from the clearly limited range of options introduced by the variability of a parameter. Developing a sense of that distinction is another argument for posing problems that include parameters. For example,

Find the unique solution of \( y' = y, \ y(0) = a \)
is no harder than if the initial condition were \( y(0) = 1. \) But the parameter is a small additional challenge. The parameter can be explicit or implicit:

Do solutions of \( y' = y, \ y(0) = a, \) increase or decrease?
Do solutions of \( y' = y \) increase or decrease?

Of course, the first question would be more straightforward if it were worded, "Does the solution of . . . ."

Students should have to make decisions about what values of the parameter need to be considered and about the effect of parameter values on both processes and outcomes. Compare these three problems:

\[
\begin{align*}
\text{Solve } y' &= y - 3e^{2t}. \\
\text{Solve } y' &= y - ae^{2t}. \\
\text{Solve } y' &= y - 3e^{at}.
\end{align*}
\]

The first is straightforward, the second asks only that the parameter \( a \) be dragged along, with perhaps a brief look at \( a = 0, \) and the last demands separate consideration of \( a = 1. \)

**Ask about, Not for**

We too often ask for programmed responses. We seldom ask for judgments. Asking about processes, methods, and results, instead of asking specifically for a given process, method, or result gives students the freedom to break out of the Pavlovian mode. A typical exercise that asks for the application of a specific process is

Find the inverse of the Laplace transform \( \frac{4}{1 - s}. \)
A problem that asks about the process is

The Laplace transform of a function is \( \frac{4}{1 - s} \). Does the function grow, decay, or oscillate?

Students could choose to invert the transform, or they could simply observe that the form of the denominator forces an exponential in time.

Of course, that last exercise is not perfectly posed, for the adjectives grow, decay, and oscillate hardly exhaust the set of possible behaviors. To add to the ambiguity, give the transform of a function that does not fall cleanly into one of these three classes:

The Laplace transform of a function is \( \frac{4}{1 + (s - 2)^2} \). Does the function grow, decay, or oscillate?

Are grow, decay, and oscillate mutually exclusive choices?

In the same spirit, an instructor can ask about an outcome rather than for a specific outcome. Instead of

Find the formula for the solution of \( y' = y, \ y(0) = 1 \)

ask

Does the solution of \( y' = y, \ y(0) = 1 \), increase or decrease?

Leave to the student the choice of the route to an answer. The outcome might be a solution formula, or it might be some alternative analysis of solution behavior. The fog of ambiguity can be made successively denser by asking

Do the solutions of \( y' = y \) increase or decrease?

Do the solutions of \( y'' = -y \) increase or decrease?

Do the solutions of \( y'' = y \) increase or decrease?

Ask about methods rather than for their specific application. Don’t place in front of a list of equations an instruction like

Solve the following equations using the methods of this chapter.

Instead, try

For each of the following equations, list every method you know that can be used to solve it. Justify each claim you make. Use one of those methods to solve each equation.

Include in the list some equations to which no method applies and others that hark back to chapters past; e.g., nonlinear equations when the methods are linear or first-order equations in a chapter on second-order methods. Both of the preceding problems exercise the solution processes, but the second requires judgments about which methods are appropriate and why. The art of making such judgments is innately more important than temporary memorized mastery of the steps in a solution process.

Explore the Vocabulary

We take the vocabulary of mathematics for granted and, more fundamentally, we accept from experience the need for a well-defined vocabulary. Students frequently refuse to believe in either the words or their necessity. Problems that explore and
reinforce the meaning of mathematical vocabulary develop mathematical literacy, a fundamental requirement of successful analysis, problem solving, and communication.

Like a new language, the words must be heard and spoken. We frequently ask

Which of the following equations are linear (or homogeneous or ...)?

Mastery of the vocabulary demands the converse, though:

Give three examples of linear (or homogeneous or ...) equations.
Give an example of a linear homogeneous equation.
Give an example of a linear equation that has the trivial solution.
Give an example of a nonlinear homogeneous equation.
Give an example of an equation that is nonhomogeneous and has the solution y = 0.

Ask for connections among ideas upon which the vocabulary is based:

What is the amplitude of the trivial solution?
What is the amplitude of the solution of a linear homogeneous second-order equation subject to zero initial conditions?
Find the amplitude and the maximum displacement of the mass in the spring–mass system governed by \( x'' + x = 0 \), \( x(0) = 1 \), \( x'(0) = 0 \).
Find the period of the mass in the preceding exercise. Would doubling the initial displacement change the period? Would it change the amplitude?

The Laplace transform of a function is \( \frac{4}{1 + (s - 2)^2} \). Is it oscillatory, periodic or neither?

The Rule of Three Plus One

The rule of three. The modern rule of three suggests looking at mathematical concepts from each of the numerical, algebraic, and graphical perspectives. The purely algebraic perspective is the most common in the introductory course:

Find the solution of the initial value problem \( P' = 0.015P - 0.209 \), \( P(0) = 10 \).

The geometric perspective is too rich to ignore:

Sketch graphs of solutions of \( P' = 0.015P - 0.209 \) for various positive initial values. Determine the regions where the solution curves are increasing, decreasing, concave up, and concave down.
Does \( P' = 0.015P - 0.209 \) have any constant solutions? If so, what initial condition(s) lead to such solutions?
Sketch the direction field of \( P' = 0.015P - 0.209 \) in the first quadrant. Identify ranges of initial conditions that lead to qualitatively similar solution behavior.
Which of the curves shown in the figure (referring to a family of curves, some increasing and some decreasing, of varying concavity, including a horizontal line) might be solutions of \( P' = 0.015P - 0.209 \)? Justify accepting or rejecting each curve.
Repeat the previous exercise using tables of solution values rather than graphs.

Similarly, the numerical view is too important to postpone until late in the course. And its exercise need not demand extensive computation:

The previous exercises suggest that \( P' = 0.015P - 0.209 \), \( P(0) = 10 \), will have a decreasing solution. Use one step of Euler’s method to estimate the time required for the solution to reach \( P = 0 \).
Use Euler’s method with \( \Delta t = 1 \) to construct a table of values of the solution of
\[ P' = 0.015P - 0.209, \quad P(0) = 10 \] for \( t = 0, 1, \ldots, 10 \). Include in the table the values of
\( P'(0), P'(1), \ldots, P'(10) \). Is the behavior you see in that table consistent with a direction
field diagram for this equation? How could you use the entries in that table to help you
draw an accurate direction field diagram? Are the entries in the table consistent with the
exact solution?

Use Euler’s method with \( \Delta t = 1 \) to estimate \( P(10) \). Compare that value with the exact
solution. Using available software, repeat the computation with \( \Delta t = 0.5 \) and \( \Delta t = 0.25 \).
How does the error in the estimated value of \( P(10) \) vary with \( \Delta t \)? What return in
improved accuracy does Euler’s method give for the increased work caused by halving
the step size?

The graphical and numerical perspectives are also abundant sources of “real”
mathematics—that material we would love to teach but for the stupor it induces in
our students. For example, a graphical examination of solution curves can motivate
a uniqueness theorem. After students have sketched solutions of the logistic
equation \( P' = aP - sP^2 \), ask whether the (stable) steady state \( P = a/s \) is reached in
finite or infinite time. The uniqueness theorem shows that the approach is indeed
asymptotic because two solution curves cannot cross.

Analysis of the error in initial value solvers like Euler’s method requires the
mean value theorem and Taylor polynomials. (Taylor polynomials also motivate
higher order Runge-Kutta methods.) When students have explored the numerical
relation between step size and error, they are prepared to approach error analysis
as a way of understanding the potential payback in improved accuracy from the
added computational effort of reducing step size. Central ideas of analysis are
brought into play because they help answer questions important to the students,
not just because the professor likes the concepts.

Good exercises and projects can be formulated in both the graphical and
numerical settings by letting students first discover for themselves the basic
phenomena—the possibility of an asymptote among logistic equation solution
graphs or the proportionality between numerical error and step size in Euler’s
method. Then build on that motivation to introduce the mathematical ideas that
will permit students to answer the questions their experiences have raised.

Three plus one: Modeling. Supplementing the rule of three with the physical
perspective leads to most of the principal ingredients of mathematical modeling:
deriving the model, analyzing the resulting mathematical problem, and interpreting
the results in light of the original physical problem [8]. While some might resist
including all three steps in an introductory differential equations course, the last
step, interpreting analytical results in a physical context, is certainly central to any
such course.

These kinds of problems can ask directly about the behavior predicted by a
specific model, or they can ask about generic behavior:

Beginning in 1847, many residents of Ireland left the island to escape the terrible potato
famine. The population of Ireland at that time could be modeled by \( P' = 0.015P - 0.209, \)
\( P(1847) = 8 \). (\( P \) is population in millions; time is measured in years.) Will the population
grow or decline?

Suppose \( y' = y + b, \quad b > 0 \), is a model of some physical process. Does the part of the
process modeled by the term \( b \) act to raise or lower \( y \)? What if \( b < 0 \)?
Questions like these are harder than they look. To ease the door open in the context of the population question, for example, one could also ask

Is the equation \( P' = 0.015P - 0.209 \) a reasonable model of growth coupled with emigration? Begin your answer by determining the units of each term. What does each term represent? What are the units of the coefficients 0.015 and 0.209 and what do they represent? Are these values reasonable? How do they compare with those for the U.S.? For China? Are the values given here consistent with claims like, “Birth rates of human populations seldom exceed 4%”? What is the connection between birth rate and the number of surviving children born to each woman?

In many ways, the modeling approach to teaching differential equations is the culmination of developing a graduated spectrum of challenging exercises. Besides offering students obvious links to their studies in other disciplines, modeling encompasses all of the principles demonstrated here for developing better exercises.

Physical problems are naturally ambiguous, for it is seldom obvious which questions are reasonable and which are not. Furthermore, the mathematics is often too well hidden at the start to offer useful clues. Using mathematics to understand a physical process always demands asking about; such problems never ask for the use of a particular method. Physical problems demand mastery of the mathematical vocabulary in order to marshal the proper tools for analysis. These problems also demand mastery of a minimal vocabulary in the target discipline, demonstrating that technical vocabulary is not a demand peculiar to mathematics.

Finally, a good understanding of a mathematical model usually demands full use of all of the perspectives of the rule of three: algebraic, graphical, and numerical.

Finding models. Exercises that begin, “A model of such and such is the equation…” quickly become unsatisfactory because the model has come magically onto the scene, deus ex machina. Nonetheless, an instructor seeking such examples can hunt for equations among undergraduate texts in science and engineering or talk with colleagues in those disciplines. Many population models require minimal background, and the mathematics and the physical behavior are rich. Useful references include those in the review [5] (at a higher level) as well as [6] and [9] (simpler but more limited).

An excellent introduction to the ideas of mathematical modeling in a variety of physical and biological settings is Lin and Segel’s classic text [8]. Other texts are discussed in the review [2]. The COMAP library is a source of models and exercises of many sorts, e.g. [6]. Several introductory differential equations texts treat modeling, including [1], [4], [7].

Design Principles

Principle I. Be ambiguous.

• Use poorly posed problems. Provide too much or too little data in otherwise standard problems.
• Formulate questions with many solutions or no solution.
• Replace numbers with parameters.
• Admit multiple solution strategies.
Principle II. Ask about, not for.
- Ask about the results of a process or method, not for its application.
- Ask students to match problems and methods with outcomes.

Principle III. Explore vocabulary.
- Ask students to provide examples of concepts, properties, and terms.
- Inquire about the interdependence of concepts, properties, and terms.

Principle IV. Shift context and perspective.
- Employ the rule of three: pose questions from the graphical and numerical perspectives, not just the algebraic.
- Use mathematical modeling to set questions in a physical context.

These principles are neither foolproof nor exhaustive. But they lead to exercises that make teaching more rewarding and learning more complete.

References

If It’s Tuesday, It Must Be Calculus
The [Army Specialized Training Program] course I had to give…met 2 hours each day for 5 days a week. The syllabus called for covering algebra, trigonometry, analytic geometry, and differential and integral calculus in one semester. The….students swore that if someone dropped his pencil and leaned over to pick it up, he missed analytic geometry.