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SIMIODE A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

STUDENT VERSION
Models Motivating Second Order

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STATEMENT

Ordinary differential equations involve second derivatives and second derivatives appear in many contexts, chief among them are the study of forces and resulting motion. This is principly because of Newton's Second Law of Motion which says, that the sum of the forces acting on a body are equal to the product of the body's mass and acceleration.

Consider the diagram for the mass at the end of the spring in Figure 2. We will presume that the mass is settled into a static equilibrium in which the force due to gravity is countered by the restoring force of the spring to contract or expend depending upon whether the mass is below or above this static equilibrium point, respectively, and hence, pull up or push down, respectively, the mass. Suppose we let $y(t)$ be the distance the mass is from static equilibrium and $y(t) > 0$ for when the mass is below the static equilibrium, i.e. when the spring is extended, and $y(t) < 0$ for when the mass is above the static equilibrium, i.e. when the spring is compressed.

Newton's Second Law of Motion states that the sum of the forces acting on a body are equal to the product of the body's mass and acceleration.

$$m\,y''(t) = \underline{\hspace{6cm}} \tag{1}$$

We need to identify these acting forces. Remember the force due to gravity has been offset by the restoring force of the spring to contract or expand, thus putting the mass at static equilibrium. Now then, the two forces we need to consider are (1) the force to restore the spring to the static equilibrium value and (2) the force of resistance to motion due to friction and resistance.

The restoring force of the spring to restore (contract or expand) the spring can be modeled by Hooke's Law which says that "...that the force needed to extend or compress a spring by some

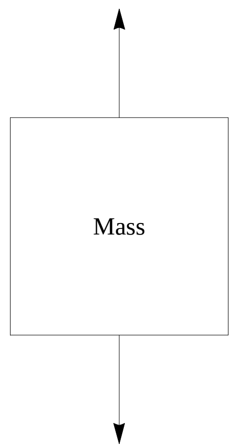

Figure 1. Base diagram for constructing a Free Body Diagram showing the forces on a mass at static equilibrium, excluding gravity and its counteracting restoring force of the spring to contract or expand depending upon whether the mass is below or above the static equilibrium point.

distance is proportional to that distance. This is a constant factor characteristic of the spring and is often referred to as its *stiffness*. The law is named after 17^{th} century British physicist Robert Hooke."[1] We offer the following historical information.

> In physics, Hooke's Law of elasticity states that the extension of an elastic spring is linearly proportional to its tension.
>
> The law holds up to a limit, called the elastic limit, or limit of elasticity, after which springs suffer plastic deformation up to the plastic limit or limit of plasticity, after which they break down.
>
> It is named after the 17^{th} century physicist Robert Hooke, who initially published it as an anagram *ceiiinosssttuv*, which he later revealed to mean *ut tensio sic vis*, or "as the extension, the force."[1]

The force of resistance to the motion of the mass is often just proportional to the velocity of the mass and is due to internal friction in the spring and external resistance due to the medium, in this case air with flat index card surface in opposition to the motion.

We will be modeling a spring mass system and seeing how well it describes data taken from an experiment. Use (1) in building a differential equation model using Newton's Second Law of Motion.

Activity 1

Draw these acting forces in Figure 2 and incorporate them into (1) according to Newton's Second Law of Motion.
where $m > 0$ is the mass in kg on the spring, $k > 0$ is the spring constant with units N/m, and $c > 0$ is the resistance coefficient in units N/(m/s). Here $y(t)$ is the displacement in meters from

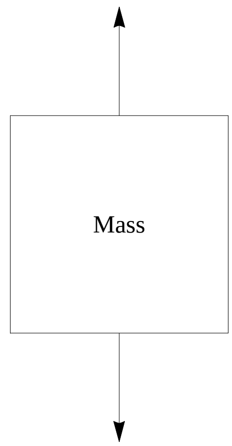

Figure 2. Base diagram for constructing a Free Body Diagram showing the forces on a mass at static equilibrium, excluding gravity and its counteracting restoring force of the spring to contract or expand depending upon whether the mass is below or above the static equilibrium point.

the static equilibrium. Of course, y_0 is the initial displacement of the mass from static equilibrium and v_0 is the initial velocity of the mass.

You should have a differential equation that looks like this:

We consider,

$$my''(t) + cy'(t) + ky(t) = 0\,, \quad y(0) = y_0\,, \quad y'(0) = v_0\,. \tag{2}$$

If you do not have an equation like (2) then you may need to redo your Free Body Diagram or rethink your application of Newton's Second Law of Motion. In any case, make sure you can convince yourself that your differential equation is correct for $y(t)$ before proceeding.

Activity 2

We collected data on 9 May 2013 at our home office in Cornwall NY USA. We set up a spring mass system depicted in Figure 3 in which a mass was suspended on a spring attached to a rod on a stand. Data was collected on the vertical motion of the mass by using a Vernier Go!Motion Detector apparatus shown on the floor below. The distance from the detector to the base of the mass was recorded in a file on a PC. The mass, m, was 0.200 kg and the spring had a spring constant of $k = 17.306$ N/m. This meant that when 17.306 N of force was applied to the spring the spring would stretch 1 meter, no matter how much it had been stretched already. The spring constant was determined experimentally by plotting force vs. displacement data for a variety of different masses and determining a linear relationship $F = 17.30 \cdot x$, where x is the displacement in m and F is the force in N necessary to obtain that displacement. As mentioned above this relationship is referred to as Hooke's Law. This is effective in a very small (say 0.3 m) range of the length of the spring beyond its resting position, for if the spring were to be stretched too far it wold lose it restorative power.

Figure 3. The apparatus for collecting data on the bouncing spring mass system with the VernierGo!Motion Detector on the floor. The spring was pulled down and released and the detector collected data on the changing distance between the mass and the collection head of the detector.

Through the interface between the motion detector and a PC, data on the position of the mass was sampled for approximately 30 seconds at 50 data points per second for total number of 1,500 observations. The data is offered in the Excel spreadsheet 3-1-S-SpringMassDataAnalysis-StudentVersion.xls which is in the supplemental files for this Modeling Scenario.

Clean up the data

Since this data offered has some noise in it initially when the spring was released, you should take a "chunk" of the data well into the data set, attempting to get the first point as a high point in the cycle as depicted in Figure 4. For when the mass is at a high (or low) point then the velocity is 0, hence $y'(0) = y_0 = 0$, as it is stopping motion in one direction and preparing to go in the other direction, at least approximately so.

Modeling the Motion of the Mass

We are going to offer several opportunities to model the motion and, in turn, determine the parameters m, k, and c, albeit we actually know m and k from measurements, namely $m = 0.200$ kg and $k = 17.306$ N/m.

We outline these approaches here and note that these will take some computing power and you should probably be using a computer algebra system. We use Mathematica here, but SAGE and Maple or MatLab will also work well.

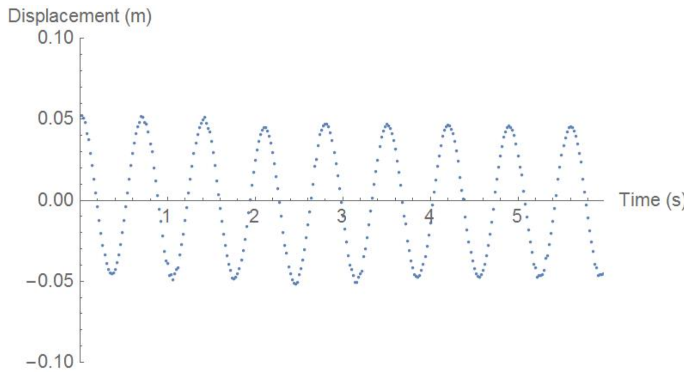

Figure 4. Plot of the data from some time, well into the motion, but at a high point for the displacement of the mass from static equilibrium.

1. Modeling based on reasonable shape of the data

Notice the plot of the full data set in Figure 5 appears to be oscillating and decaying. How might we model this function? Use an exponential decay function of the form $e^{-B\cdot t}$ $(B > 0)$ and an oscillating function which begins at its high value, i.e. a cosine function, $\cos(C \cdot t)$. We conjecture a form of a solution based on data - KEY ELEMENT - with parameters A and C, C to be determined and $A = y_0$, our initial displacement. What would that conjectured form look like? We use $-B$ in our exponential function as we believe the displacement will decay and thus B will be positive.

Somehow estimate the parameters B and C. To get a handle on B you might want to extend your data set beyond just a few ups and downs, to see (Figure 5) the amplitude of the spring's motion really decrease. How would you estimate C?

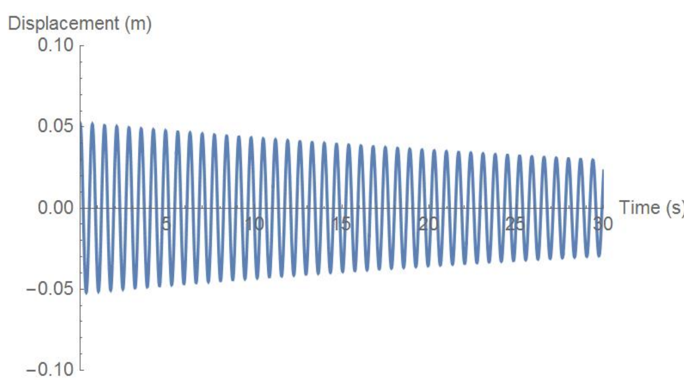

Figure 5. Plot of the data from some time, well into the motion, but at a high point for the displacement of the mass from static equilibrium and for a long duration of the motion to see the decaying amplitude.

Once you have estimates of B and C in your model then plot your model over the data and comment on your model's ability to predict the motion of the spring mass system.

2. Modeling based on the differential equation model (2) and the shape of the data

Now do a general analysis to show how m, c, and k in (2) can be estimated as they play a role in your model of the form used in the previous section, i.e. estimate all three parameters, m, c, and k, using your data and the solution to (2) when compared to your form in the previous section, namely $A \cdot e^{-B\cdot t}\cos(C \cdot t)$.

Once you have estimates of m, c, and k in (2) solve the differential equation, plot the solution over the data, and comment on your model's ability to predict the motion of the spring mass system.

Hint: Solve (2) in general and compare coefficients of like terms in the solution to your visual model that you used in section (1) above.

You should ask yourself, "What are the criteria we use to get the values for m, c, and k, for the best fit for all the data in our model?"

3. Modeling based on the differential equation model (2) and the shape of the data

Now do a general analysis to show how m, c, and k in (2) can be estimated as they play a role in this general damping and oscillating model of the form model of the form

$$\hat{y}(t) = A1 \cdot e^{-B\cdot t}\sin(C \cdot t) + A2 \cdot e^{-B\cdot t}\cos(C \cdot t)\,. \tag{3}$$

Again, since you have estimates of m, c, and k and hence $A1$, B, C, $A2$, in (2) and in (3) solve the differential equation (2), plot the solution over the data, and comment on your model's ability to predict the motion of the spring mass system.

4. Modeling based on the differential equation model (2) alone using the data

Now do a general analysis to show how m, c, and k in (2) can be estimated as they play a role in this general damping and oscillating model from the solution of (2) and the data alone.

Again, nnce you have estimates of m, c, and k in (2) solve the differential equation (2), plot the solution over the data, and comment on your model's ability to predict the motion of the spring mass system.

Conclusion

In all three cases above your model and the data, when compared, ought to look like a good fit as shown in Figure 5.

REFERENCES

[1] Wikipedia contributors. 2014. Hooke's law. Wikipedia, The Free Encyclopedia. `http://en.wikipedia.org/w/index.php?title=Hooke%27s_law&oldid=628866252`. Accessed 10 October 2014.

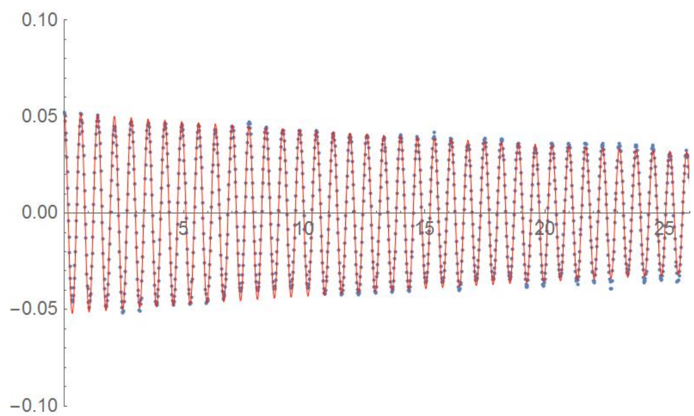

Figure 6. Plot of the data and a good model of the data from any one of the analyses performed above.