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STUDENT VERSION
AN INTERACTIVE ILLUSTRATION OF THE VAN DER POL OSCILLATOR

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STATEMENT

The purpose of this paper is to present an interactive spreadsheet application that allows the reader to investigate the Van der Pol oscillator. Through a number of suggested activities, the reader will gain a better understanding of the qualitative behavior of solutions as the parameters of the governing differential equation are varied.

A Brief Description of the Van der Pol Oscillator

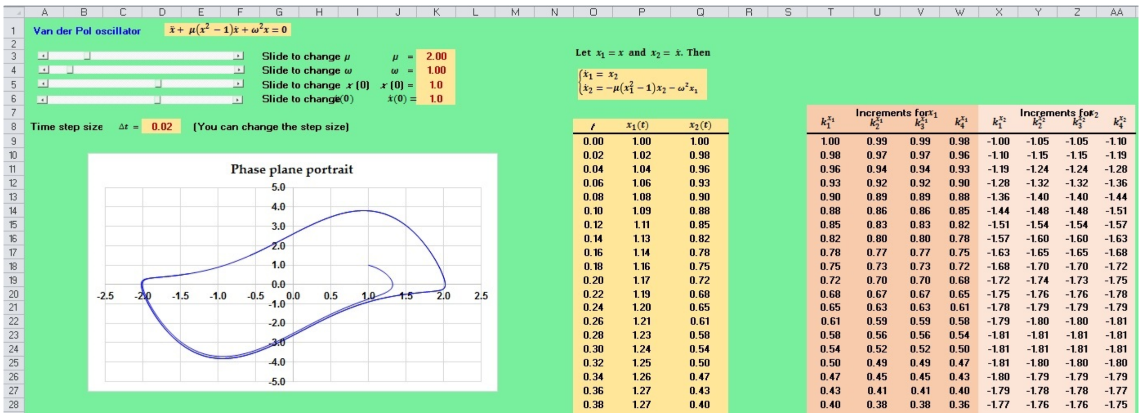
The Van der Pol oscillator is described by the nonlinear, second-order ordinary differential equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + \omega^2 x = 0, \tag{1}$$

where μ and ω are positive parameters, and x (a function of time t) is the quantity of interest (e.g., position or voltage). The quantities $\dot{x}$ and $\ddot{x}$ denote the first- and second-order derivative with respect to time t, respectively.

This celebrated equation (1) is attributed to the Dutch electrical engineer Balthasar Van der Pol, who pioneered experimental investigations in nonlinear dynamics in the early days of radio and telecommunications. He proposed his equation in order to describe the nonlinear oscillations produced by self-sustained oscillating triode circuits [5]. His equation has been extensively studied and finds numerous applications in engineering, physics, biology, sociology, and economics.

The Van der Pol equation (1) introduces a nonlinear damping term, namely $\mu(x^2 - 1)\dot{x}$. Unlike the familiar linearly damped harmonic oscillator, the presence of the nonlinear damping term can



t	$x_1(t)$	$x_2(t)$	$k_1^{x_1}$	$k_2^{x_1}$	$k_3^{x_1}$	$k_4^{x_1}$	$k_1^{x_2}$	$k_2^{x_2}$	$k_3^{x_2}$	$k_4^{x_2}$
0.00	1.00	1.00	1.00	0.99	0.99	0.98	-1.00	-1.05	-1.05	-1.10
0.02	1.02	0.98	0.98	0.97	0.97	0.96	-1.10	-1.15	-1.15	-1.19
0.04	1.04	0.96	0.96	0.94	0.94	0.93	-1.19	-1.24	-1.24	-1.28
0.06	1.06	0.93	0.93	0.92	0.92	0.90	-1.28	-1.32	-1.32	-1.36
0.08	1.08	0.90	0.90	0.89	0.89	0.88	-1.36	-1.40	-1.40	-1.44
0.10	1.09	0.88	0.88	0.86	0.86	0.85	-1.44	-1.48	-1.48	-1.51
0.12	1.11	0.85	0.85	0.83	0.83	0.82	-1.51	-1.54	-1.54	-1.57
0.14	1.13	0.82	0.82	0.80	0.80	0.78	-1.57	-1.60	-1.60	-1.63
0.16	1.14	0.78	0.78	0.77	0.77	0.75	-1.63	-1.65	-1.65	-1.68
0.18	1.16	0.75	0.75	0.73	0.73	0.72	-1.68	-1.70	-1.70	-1.72
0.20	1.17	0.72	0.72	0.70	0.70	0.68	-1.72	-1.74	-1.73	-1.75
0.22	1.19	0.68	0.68	0.67	0.67	0.65	-1.75	-1.76	-1.76	-1.78
0.24	1.20	0.65	0.65	0.63	0.63	0.61	-1.78	-1.79	-1.79	-1.79
0.26	1.21	0.61	0.61	0.59	0.59	0.58	-1.79	-1.80	-1.80	-1.81
0.28	1.23	0.58	0.58	0.56	0.56	0.54	-1.81	-1.81	-1.81	-1.81
0.30	1.24	0.54	0.54	0.52	0.52	0.50	-1.81	-1.81	-1.81	-1.81
0.32	1.25	0.50	0.50	0.49	0.49	0.47	-1.81	-1.80	-1.80	-1.80
0.34	1.26	0.47	0.47	0.45	0.45	0.43	-1.80	-1.79	-1.79	-1.79
0.36	1.27	0.43	0.43	0.41	0.41	0.40	-1.79	-1.78	-1.78	-1.77
0.38	1.27	0.40	0.40	0.38	0.38	0.36	-1.77	-1.76	-1.76	-1.75

Figure 1. Main user interface of the Microsoft Excel spreadsheet for investigating the behavior of the Van der Pol oscillator. The screen capture displays the limit cycle for $\mu = 2$ (only the first 28 rows of the spreadsheet are shown here).

introduce negative damping depending on the magnitude of x. More specifically, if $|x| > 1$, there is *positive* damping and energy is dissipated from the system resulting in decaying motion (as in the linearly damped harmonic oscillator). On the other hand, if $|x| < 1$, there is *negative* damping and energy is pumped into the system resulting in amplification of motion. Therefore, it is reasonable to expect that the steady-state motion consists of sustained periodic oscillations (e.g., the system may start at a small value of x, then be driven to a large value of x by the amplification, then damped back to a small x by the decay, and repeat this process *ad infinitum*). These periodic oscillations are referred to as *limit cycles*. The interested reader can learn more about nonlinear dynamics from these excellent sources [2], [4].

The Interactive Spreadsheet Model for Investigating the Van der Pol Oscillator

The accompanying electronic spreadsheet model has been created using Microsoft Excel. The user interface is shown in Figure 1. The top left section of the spreadsheet shows a number of active controls for changing the parameters and initial conditions for the Van der Pol equation (1). To change the damping μ, the frequency ω, the initial position $x(0)$, or the initial velocity $\dot{x}(0)$, the user simply slides the thumb on the corresponding scroll bar; the values selected by the user will be automatically adjusted and displayed in the cell range K3:K6. As the user varies the parameters and/or the initial conditions, the phase plane portrait of the limit cycle is also automatically adjusted to reflect the changes introduced by the user.

To obtain the limit cycle, the Van der Pol equation (1) is solved numerically within the spreadsheet. The numerical solution requires rewriting (1) as a set of first-order differential equations.

Defining $x_1 = x$ and $x_2 = \dot{x}$, we obtain the following system:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\mu(x_1^2 - 1)x_2 - \omega^2 x_1. \end{aligned} \tag{2}$$

The set of differential equations (2) is solved in the spreadsheet using the Runge-Kutta method. The tabulated solutions for the state variables $x_1(t)$ and $x_2(t)$, as well as the Runge-Kutta increments, are shown in the spreadsheet. If necessary, the user can manually alter the time step size, Δt, stored in cell D8 to obtain more refined solutions; making Δt too small may require the user to copy the pre-programmed formulas occupying the cell range O10:AA10 (or any complete set of formulas below row 10 spanning columns O through AA) in order to generate sufficient points to plot the limit cycle.

Activity 1. Harmonic Oscillator

Assume the initial conditions $x_1(0) = x(0) = x_0$ and $x_2(0) = \dot{x}(0) = y_0$ (not both zero), and set $\mu = 0$ in (1), which will then become the *harmonic oscillator*

$$\ddot{x} + \omega^2 x = 0. \tag{3}$$

Show that the trajectory (solution to (3)) is an ellipse described by

$$\frac{x_1^2}{x_0^2 + \dfrac{y_0^2}{\omega^2}} + \frac{x_2^2}{\omega^2 x_0^2 + y_0^2} = 1. \tag{4}$$

Confirm this with the spreadsheet application by setting the parameters $\mu = 0$, $\omega \neq 0$, and the initial conditions to any allowable values (not both zero).

For what value of ω does the trajectory become a circle? For this value of ω, what is the radius of the circle?

Activity 2. Rigid Body

Assume the initial conditions $x_1(0) = x(0) = x_0$ and $x_2(0) = \dot{x}(0) = y_0$ (not both zero), and set both $\mu = 0$ and $\omega = 0$ in (1), which will result in the *rigid body* described by

$$\ddot{x} = 0. \tag{5}$$

Observe the phase plane portrait displayed by the spreadsheet application. Provide an analytical justification to your observations.

Activity 3. Empty Phase Portrait

Set both initial conditions to zero $\big(x_1(0) = x(0) = 0$ and $x_2(0) = \dot{x}(0) = 0\big)$ on the spreadsheet application; any values of μ and ω allowed by the spreadsheet can be used. In this case, the phase portrait will appear empty. Give an explanation.

Activity 4. Behavior of Van der Pol Oscillator for Small μ

Describe the trajectory and the shape of the limit cycle for the Van der Pol oscillator when μ is small. To do this using the spreadsheet model, set μ to a small value, say $\mu = 0.2$ or less; set the other parameters ω, $x(0)$, and $\dot{x}(0)$ to any nonzero values permitted by the spreadsheet (remember not to set both initial conditions to zero). If desired, use the Scattered Chart feature of the spreadsheet to obtain the time history plots for $x_1(t)$ and $x_2(t)$ from their tabulated values.

Activity 5. Behavior of Van der Pol Oscillator for Large μ

Describe the shape of the limit cycle for the Van der Pol oscillator when μ is large. To see this on the spreadsheet model, set μ to a large value, say $\mu = 5$ or greater; set the other parameters ω, $x(0)$, and $\dot{x}(0)$ to any nonzero values permitted by the spreadsheet (remember not to set both initial conditions to zero). If desired, use the Scattered Chart feature of the spreadsheet to obtain the time history plots for $x_1(t)$ and $x_2(t)$ from their tabulated values.

Activity 6. Existence of Limit Cycle

In this section, the reader will be able to show the existence of a stable limit cycle for the Van der Pol equation (1). To do this, consider the *Liénard equation*

$$\ddot{x} + u(x)\dot{x} + v(x) = 0, \tag{6}$$

which can be written as the system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -u(x)x_2 - v(x). \end{aligned} \tag{7}$$

Then apply the *Levinson-Smith theorem* [3] given below to show the existence of a limit cycle for (1).

Levinson-Smith Theorem.[1] Given equation (6), suppose the following conditions are satisfied:

(a) $u(x)$ is even and continuous.

(b) $v(x)$ is odd, $v(x) > 0$ if $x > 0$, and $v(x)$ is continuous for all x.

(c) $V(x) \to \infty$ as $x \to \infty$, where

$$V(x) = \int_0^x v(t)\,dt.$$

(d) For some $k > 0$, the following holds:

- $U(x) < 0$, for $0 < x < k$,
- $U(x) > 0$ and increasing, for $x > k$,

[1] The Levinson-Smith theorem is reproduced from [3] and is provided here for the convenience of the reader in showing the existence of a limit cycle for the Van der Pol equation (1).

- $U(x) \to \infty$ as $x \to \infty$,

where
$$U(x) = \int_0^x u(t)\,dt.$$

Then, the system given in (7), and hence (6), has

i. a unique critical point at the origin;

ii. a unique non-zero closed trajectory C, which is a stable limit cycle around the origin;

iii. all other non-zero trajectories spiraling toward C as $t \to \infty$.

Activity 7. Model of Aortic Blood Flow

One application of the Van der Pol oscillator is the modeling of biological systems in which periodic (or nearly periodic) phenomena occur. The work presented in [1] uses precisely (1) to model blood flow in the human aorta. To run a simulation on the accompanying spreadsheet, set $\mu = 1$, $\omega = 2.37$ ($\omega^2 = 5.6$), $x(0) = 1$, and $\dot{x}(0) = -3$. The reader can use the tabulated values to obtain a pattern of blood flow (x_1 vs. t) using the Scattered Chart feature of Excel. The pattern will not match that of [1] because the solution to the Van der Pol equation requires some post-processing in order to produce a more meaningful physiological signal. Nevertheless, this example serves to illustrate the usefulness of the Van der Pol oscillator in real-life applications.

CONCLUDING REMARKS

This paper presented the Van der Pol oscillator and how its qualitative behavior can be investigated using an electronic spreadsheet model. The spreadsheet features a number of dynamic controls that permit the user to alter the parameters of the Van der Pol equation and interactively explore the effect of such changes on the solution and the shape of the limit cycle as the damping and other parameters change. The accompanying spreadsheet can be used to investigate other problems arising from physics, engineering, and biology, among others, in which periodic (or nearly periodic) phenomena occur and the Van der Pol oscillator can be profitably used to gain further insight or to obtain approximate solutions.

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