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STUDENT VERSION MILITARY APPLICATIONS OF SPRING-MASS SYSTEMS

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STATEMENT

You have just completed BOLC (Basic Officer Leadership Course) in the branch of your choice (lucky you) and have PCS'd (Permanant Change of Station) to your first duty station, Joint Base Lewis-McChord, WA. While sitting with the other new arrivals, a Colonel from the U.S. Army Test and Evaluation Command (ATEC) at Aberdeen Proving Grounds engages the personnel clerk in a quick conversation and addresses the group.

"From the looks of things you all will be snow birding until the assignment slate opens up and I need a volunteer. You must be comfortable with some basic engineering design: mathematical modeling and differential equations. You will be making recommendations on decisions with expensive and vital ramifications."

You eagerly indicate interest. After all, you know about differential equations, but have no idea what snow birding is. The next day you report to the 51^{st} Expeditionary Signal Battalion. The Battalion Commander has assigned you to an initiatives group that has been tasked to perform testing and analysis for ATEC engineers. Your email has now been set up and you open your first email as a Mathematical Consultant, which is the following memorandum.

GENERAL MATHEMATICAL MODEL: The sprung mass of a vehicle is the mass of the vehicle minus the suspension and wheels (the unsprung mass). To simplify the analysis of a suspension system, we assume that the mass of a vehicle, and any forcing function on the suspension, are evenly distributed; in other words, the suspension system can be treated as one shock. Thus, this simplified suspension system behaves according to the differential equation

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = f(t),$$

where $x(t)$ is the displacement from its rest position in meters, m is the mass of the object in kg, β is the damping constant in $\mathrm{N} \cdot \mathrm{s/m}$, k is the spring constant in N/m, and $f(t)$ is the external force in Newtons (N).

The Rockwell Collins Satellite Transportable Terminal (STT) is a combat-proven, transportable earth terminal designed to establish secure voice, video and data communications virtually anywhere. Designed to withstand challenging conditions and demanding operations in Iraq and Afghanistan, the U.S. Army has deployed hundreds of STTs for the Joint Network Node (JNN) program.

These rugged, trailer-based earth terminals deliver up to 56 Mbps (mega bytes per second) throughput - and they can be towed anywhere a HMMWV (High Mobility Multipurpose Wheeled Vehicle) can go. Rockwell Collins has built more than 800 STT units since the JNN program's inception in 2004. [Rockwell Collins 2009] Due to recent budget cuts, more economical parts must be used in the production of STTs. Engineers have tasked you with comparing feasible options for the suspension systems of the STTs.

The suspension system for the STTs needs to be fitted with cost effective shocks that are reliable and protect the satellite while being loaded and transported. There are three shocks available:

Specifications	Spring Constant k (N/m)	Damping Coefficient β ($\mathrm{N} \cdot \mathrm{s/m}$)
Air Coasters	220	100
Cloud Riders	59	1050
Extreme Shocks	170	1110

1. Model the suspension system of a STT trailor while it is loaded with a satellite using these three different shocks. The trailer of a STT at rest will be loaded with a satellite with a mass of 1800 kg. Assuming that the loaded trailer is at equilibrium, the displacement of the shocks before the trailer is loaded is 3 cm.

(a) Determine the critically-damped value for each shock i.e. the optimal value to avoid oscillatory motion. Compare the damping coefficient of each shock to its critically-damped value.

(b) Solve the spring-mass system for each shock with the given damping value and also with the critically-damped value. For each shock, plot the solution with the given damping value and the solution with the critically-damped value together.

(c) Discuss the model of each shock with regard to the satellite being placed on the trailer. From your analyses, decide which shock is best for the STT suspension system for loading. Explain your answer.

2. ATEC engineers have specified that if the suspension system bottoms out too many times, the components of the satellite could be damaged, which could cost hundreds of thousands of dollars to repair. Therefore it is important to determine the proper shocks for the suspension system of STTs to avoid any damage to the satellite. Topographers and engineers have determined that the force on the shocks in the most severe terrain at the highest speed that a HMMWV trailering a STT will attain can be modelled by the function $f(t) = 200\sin(4t)$ N where t is in seconds. Assume that $x(0) = 0$ and $x'(0) = 0$.

   (a) For each shock, solve the associated non-homogeneous equation and plot the solutions.

   (b) Find the maximum/minimum displacement, x_{max} and x_{min} for each suspension system and present these in a table. Discuss the shock performance in the long-run.

   (c) Determine the best shock for the suspension system of the STT and explain your decision from the perspective of the satellite.

   (d) For your recommended shock, solve the associated homogeneous equation. Identify both the transient solution and the steady-state solution in your solution. Plot the solutions together to compare. Explain the effect of the transient solution on the satellite.

3. The preferred option of the budget committee for the suspension system of the STTs are the shocks: Premium Shocks (Specs: $k = 25$ N/m; $\beta = 0$ N $\cdot$ s/m). Present reasonable data persuading our budget team that undamped shocks, although cheap, will not be sufficient. Consider the same external forcing function from Part 2, but with a variable frequency (i.e. $f(t) = 200\sin(4\omega t)$ where ω is the degree of roughness of the driving conditions).

   (a) Solve the initial value problem for a function with the time, t, and frequency parameter, ω, as variables. Describe the signifigance of ω and its relation to the speed of the vehicle.

t	0	1	2	3	4	5	6	7	8
x(t)	0	0.0303	0.0581	-0.0045	-0.1188	-0.1363	0.0140	0.1995	0.2022

t	9	10	11	12	13	14	15	16
x(t)	-0.0261	-0.2696	-0.2557	0.0387	0.3272	0.2971	-0.0496	-0.3709

Table 1. Displacement Measurements of Trailer (in meters)

(b) For what positive value ω is the above solution invalid? Characterize the suspension system at this value.

(c) Create an interactive plot of the solution you found in Part 3a with respect to the frequency parameter, ω, for the first ten minutes of driving. Explain the cause of the change of frequency. Describe any damaging effects the STT might experience while being trailered.

4. In the field, the trailer of an STT was being tested with an experimental mounting of the Herstal DeFNder FN M3P high powered machine gun in place of the satellite.

While the trailer was at rest and the mounted gun was firing vertically, the vehicle began experiencing pronounced vertical motion. The soldiers operating the trailer immediately shutdown the gun and returned it to base. To help understand the phenomenon, some engineers re-fired the weapon and took measurements of the displacement of the trailer from the ground (see 1 for the reported data).

Since it appeared the structural integrity of the trailer would fail, the engineers shutdown the weapon after 60 seconds. Due to the structural restrictions of the trailer, physical experimentation is limited. The engineers have tasked you with modeling the phenomenon to see if it will stabilize or continue until the point of structural failure.

The mass of the Herstal DeFNder is 4000 kg and the engineers were able to determine the spring constant of the shocks used on the trailer to be $k = 5000$ N/m. Also, our engineers have determined that the amount of force on the trailer produced by the weapon firing is modeled by the function $f(t) = 300\cos(t)$ N.

(a) Plot the data of the phenomenon and describe the behavior of the trailer.

(b) Use data and Mathematica to estimate the damping coefficient by minimizing the sum of squared errors (SSE) between the data and the predictions of your model.

(c) A suggestion was made that the truck is experiencing resonance or beats. Is this a reasonable hypothesis?

(d) If they do not turn off the vehicle and the shocks can only extend and compress a length of 0.5 meters, will the structural integrity of the truck be compromised?

5. Before physical tests are completed by the ATEC, you have been asked to run additional simulations of the STT trailer-based system. In particular, ATEC senior engineers would like you to consider a scenario for which the trailer traverses a rugged terrain with exposed rocks, an environment the STTs will likely encounter.

A technique for modeling the rugged terrain is to express the force applied to the suspension system by each rock as an impulse. This can be done with a Dirac Delta function of the form, $w_0\delta(t-t_0)$, where w_0 is the concentrated load applied at time t_0. To model terrain with, N, *equally spaced* rocks, we can express the external force, $f(t)$, via several impulse functions, i.e.

$$f(t) = w_0 \sum_{n=1}^{N} \delta(t-(2n-1)\phi), \tag{1}$$

where 2ϕ is the time between impulses and the initial impulse occurs at $t=\phi$. Your job is to complete an analysis of the STT suspension system for a trailer traversing over rugged terrain.

(a) ATEC engineers decided that the *Extreme Shocks* with spring constant $k = 170$ N/m and damping coefficient $\beta = 1110$ N $\cdot$ s/m was the best option to prevent damage to the satellites while being loaded and transported. Model the suspension system of a STT trailer while it is loaded with a satellite and traveling over the terrain modeled by 1. For this scenario let $N = 13$, $w_0 = 150$ Newtons and $\phi = 3$ seconds. Additionally, the trailer is initially at rest at the equilibrium position (vertically) and has a mass of 1800 kg.

i. Establish the initial value problem that models the spring-mass system in the scenario described above. Clearly define your independent and dependent variables, parameters, units, and initial conditions.

ii. Solve *by hand* the IVP established in part 5a using Laplace transform methods. [Note that the *Mathematica* command `Apart` may be used for partial fraction decomposition if needed. Additionally, the Laplace transform of Equation 1 may be accomplished without writing out all the terms in the sum!]

iii. Use *Mathematica* to verify your answer for 5(a)ii [This may be done visually using graphical techniques.] Additionally, plot the solution and sketch your graph in the space provided below.

(b) Find the maximum and minimum displacement, x_{max} and x_{min} for the suspension system. Discuss the performance of the shock system as it traverses the rugged terrain over the long run.

(c) If the shock system can only extend and compress a length of 20 centimeters, will the structural integrity of the truck withstand the rough terrain? What if the time between impulses is decreased, say to $\phi = 2$ seconds? What would your guidance be in terms of traversing the STTs in rugged terrain?

REFERENCES

[1] FN Herstal. *FN Herstal deFNder FN M3P*. Virginia, South Carolina, 2016. `http://www.fnherstal.com/`.

[2] Rockwell Collins. *Satellite Transportable Terminal (STT)*. Cedar Rapids, Iowa, 2009. `http://www.rockwellcollins.com/gs`. Accessed 5 December 2016.