DIFFERENTIAL EQUATIONS AT WESTMINSTER COLLEGE PERSONAL ACCOUNT

Holly Zullo
Department of Mathematics
Westminster College
Salt Lake City UT 84105 USA
hzullo@westminstercollege.edu

Background

The comments here pertain to the 300-level differential equations course that Holly Zullo taught at Westminster College (Utah) in Spring 2016. No programs on campus require differential equations, although physics students are strongly encouraged to take it and mathematics majors may take it as a mathematics elective. As a purely elective course, I could choose topics as I saw fit. I started the semester with 17 students, and 13 successfully completed the course. The other four students stopped attending at various points in the semester, mainly due to personal issues (some having to do with the 8am class period). Of the 13 who completed the course, about half were math majors, with the others scattered between physics, 3-2 engineering, chemistry, and environmental science.

This was a 4-credit course that met for two 110-minute sessions each week, for a total of 29 class meetings, including the final. I used a fairly standard, print text [3]. I used Excel and Matlab, along with Dfield.jar and Pplane.jar [7]. Most students had very little experience with Excel or Matlab, although the physics majors were proficient in Mathematica and a few students knew some Java programming.

Course grades were based on the following components:

- WebWork - 15% of final grade
  - total of 24 assignments, primarily computational
- Written homework, labs, projects - 50% of final grade
  - 2 written homework assignments using problems from textbook

  - 11 labs (more on these later)
  - 1 mini project (more on this later)
  - work on labs and project in pairs
- Exams - 35% of final grade
  - midterm and final, both given in-class and weighted equally
  - involved some modeling, but largely computational

Modeling

I have a vested interest in working modeling into every possible class component, because every year I advise students in the Mathematical Contest in Modeling (MCM) [6], and my classes are about the only mathematics classes where students will see modeling. Since only two of my differential equations students were seniors, I now have 11 students who are ready to compete in the MCM next year! Competition aside, modeling is a natural fit with differential equations. So I strove to work in modeling wherever possible. I wanted students to see many models as we discussed topics in class, to explore models more deeply through labs, and to tackle a model of their choosing for a project.

Modeling Woven Through Class

Many times I would introduce a new type of differential equation to students by asking them a modeling question. This began on day 1, when after a brief discussion of what constitutes a differential equation and a solution, I posed the following question [5] to students:

> The amount of chemical in a lake is decreasing at a rate of 30% per year. If $p(t)$ is the total amount of the chemical in the lake as a function of time t (in years), which differential equation models this situation?
>
> 1. $p'(t) = -30$
> 2. $p'(t) = -0.30$
> 3. $p'(t) = p - 30$
> 4. $p'(t) = -0.3p$
> 5. $p'(t) = 0.7p$

I followed that with a question about Newton's Law of Cooling and units, and then we hit on second order differential equations with the following question:

> A branch sways back and forth with position $f(t)$. Studying its motion you find that its acceleration is proportional to its position, so that when it is 8 cm to the right, it will accelerate to the left at a rate of 2 cm/s^2. Which differential equation describes the motion of the branch?

Later in the course, I introduced systems by having students develop a simple compartment model for a medication moving from the gastrointestinal tract into the bloodstream. Students then further explored systems through a fun series of questions [5] on the Lanchester Combat model, adapted from [1]. On another day, we discussed SIR models using part of [8].

All of these models engage students and allow them to discuss the meaning behind each piece of a differential equation, thus increasing students' understanding, the rationale, the nuances, and pieces of a differential equation. Most of my students loved all of these models, although one student did complain about the combat model, as she didn't like the idea of losing her own troops or of increasing their fighting power to better kill other troops. This concern aside, using models to motivate new material was, not surprisingly, a huge success.

Modeling in Labs

Modeling was worked into the course wherever possible, but the most in-depth work occurred in the labs. Four of the labs explicitly involved modeling, while the others helped to develop analytical skills important to modeling and to understanding differential equations: qualitative analysis of graphs, understanding of numerical algorithms, analysis of long-term behavior of solutions to differential equations, and so on. Students worked in pairs (or a group of 3, if needed) on all of the labs.

Early on I used a SIMIODE Modeling Scenario on the sublimation of dry ice [10], with the Excel Solver approach for parameter estimation. For many students, this was their first exposure to the idea of trying multiple models to see which fit the best. This activity probably would have benefited from me spending a little more time on the chemistry and geometry of this situation, as I was a bit surprised by my students' general lack of familiarity with basic reaction orders. I wound up suggesting plausible values for r for the model $m'(t) = -km(t)^r$ for $m(t)$, the remaining mass of dry ice. At this stage in the course, it was challenging enough for most groups to pick a couple of values for r, solve the resulting differential equation, and then put the solution into Excel to find the best fit value of k. One group, however, did manage to use Solver to optimize over both k and r. I devoted an hour of class time to this lab, and students finished it at home.

The following week I had students do a similar lab where they used two different methods to estimate the parameters needed to fit a logistic model to population data for Yellowstone National Park's bison population. This lab was based on [2]. We again used Excel's Solver to do the minimization of the sum of squared errors in order to estimate parameters in the model. In another lab, students started with a simple exponential growth model for a hypothetical zebra mussel population, and they learned about Euler's Method by comparing their numerical solutions with various step sizes to the analytical solution, always while discussing the eventual size of the zebra mussel population. They also experimented with two different eradication plans, using Euler's Method to solve the resulting ordinary differential equations. This lab was also done with Excel, allowing students to clearly see the iterative process.

Several of the next labs had students using Matlab to solve ordinary differential equations and dfield.jar and pplane.jar [7] to provide graphical analyses, but the equations in these labs were not presented in context. While I would like to modify these labs to include real scenarios and resulting equations, they still provided students an opportunity to develop their analysis and writing skills. By week ten, students were ready for the SIMIODE Modeling Scenario on the drug, ibuprofen [11]. In this lab students develop and compare four models for how ibuprofen moves through the body. Rather than having students test the three sets of given parameters for each model, I had them do that for the first model, but then they just found their own best parameters for each later model. Through their earlier work with Excel, students had learned exactly what it means to find parameters to minimize the sum of squared errors. Wanting a bit more sophistication at this point, in this lab I showed them how to use Matlab's optimization toolbox to do this. While this is much more of a "black box" approach, students were conceptually ready for that jump, and this allowed them to use Matlab to solve the ordinary differential equations they were writing. While I had only devoted one hour of class time to most of the earlier labs, this one was more complex, and we spent our entire 2-hour class period on it, then students finished their work at home.

Modeling in Projects

Each of the labs were completed within a week's time, including usually an hour of class time and whatever other time was needed at home. I had intended to have students work on a longer, more intense project that would take them more than a week to complete, but class time ran short as did my time to develop such a project. Instead, I settled for a one-week project right before the final exam. Students worked in pairs, and the project culminated in an oral presentation to the class, with no other written submission.

Students were offered several options for the project, including any of several SIMIODE activities, further study of the SIR model we had looked at earlier, or playing with ideas of chaos through any of several explorations outlined in our textbook. Several groups chose to study chaos, two chose SIMIODE activities; one chose [9] on LSD and one chose [4], an activity centered on numerical modeling of a pendulum. By this point students were able to work quite independently on their modeling projects.

Challenges

Most students in our program have very little, if any, experience writing even short technical papers of the type I expected for these modeling activities. Despite what I thought were clear instructions, at first, many students didn't include their graphs or didn't record the actual parameter values they found. Students also struggled to discuss strengths and weaknesses of their models. In particular, when fitting a model to data, they really had to learn how to discuss where the model fit well, what characteristics of the data it captured, and what it did not capture. They were very comfortable comparing values for the sum of the squared errors (SSE) and selecting a model based solely on that numerical comparison, but their qualitative analysis skills were much slower to develop. Develop they did, though, and there were noticeable improvements by the end of the semester.

Other challenges arose simply from having enough engaging activities, both large and small, for students to work on, and also finding appropriate technology support. SIMIODE has a wide range of activities that are suitable for what I called my labs. The main technology platform for these activities, though, is Mathematica, which is too expensive for many schools. It would be helpful to see more platforms employed, including open source options and that is a goal of SIMIODE. Also, there is a need for more very short or very long activities, such as would be useful for 20-30 minutes in class or for an extended project, respectively. The MathQuest Classroom Voting questions [5] provide some help with in-class activities, but much more is needed.

Rewards

Overall, teaching differential equations with modeling was a huge success. I enjoyed it, the students enjoyed it and learned, and my physics colleague was very pleased with what his students were learning. As an applied mathematician, it is very important to me that the mathematics I teach is put into context, and there is no better way to do that than through modeling. The students were excited about the applications and about learning how to use differential equations for modeling. One student who struggled mightily through the proofs in my discrete mathematics course in the fall rediscovered his joy in this course and would frequently comment, with a huge grin on his face, "I love modeling!" Another student wrote on the end-of-semester course evaluations, "labs were the bane of my existence but i [sic] learned a lot from them!" Another highlight was a morning late in the semester when one of the physics majors said, "Peter [physics professor] really loves that you're making us do all these labs!" This was confirmed when I later ran into the same physics professor and he congratulated me on doing such a great job with the differential equations class, saying he loved what the students were learning with their labs, and he was so glad they were working with Matlab.

Conclusions and Future Work

This course played an important role in these students' development, especially since most of their other courses focus on very traditional, proof-based mathematics. As part of the modeling experience, students also learned to write short technical papers and gained basic proficiency with Excel and Matlab. The next time I teach differential equations I intend to incorporate even more modeling. There is room for me to include realistic scenarios and equations in many of the labs that currently just explore equations out of context. I can also make time for a bigger project, so that students have an opportunity to explore a model (or several models) in greater depth. As SIMIODE continues to develop, it will be even easier to collect resources to support these goals.

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