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SIMIODE Systemic Initiative for Modeling Investigations and Opportunities with Differential Equations

STUDENT VERSION
PARAMETER ESTIMATION THROUGH STEADY STATE

Brian Winkel
Director SIMIODE
Cornwall NY USA

STATEMENT

We attempt to estimate parameters for a spring-mass-dashpot model differential equation (1) in which the mass is known ($m = 4$ kg) but the spring constant, k N/m, and the resistance coefficient, c N/(m/s), are unknown. We will subject the device/system to a driver force we control and use the output steady state data to estimate parameters c and k. This is a form of input output analysis in which we input a driver, analyze the output, and tell something about the device or model.

$$4 \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = f(t)\,, \quad y(0) = 0 \quad \text{and} \quad y'(0) = 0 \tag{1}$$

You will do this with the following process. Apply a driver, say $f(t) = \sin(2t)$ and collect data on the steady state output of our system. See Table 1. From these observations in the steady state time period recover c and k.

We render plots of data taken from a system with unknown c and k and known $m = 4$ that is driven by $f(t) = \sin(2t)$ with initial conditions $y'(0) = 0$ and $y(0) = 0$ in Figure 1 and Figure 2. In Figure 2 we see the oscillating motion of the steady state more clearly than in Figure 1.

In Table 1 we offer the data. In whatever software you use you should plot this data set to see that you get a reasonable plot of the steady state phenomena.

a) Consider the steady state or nonhomogeneous solution, which must be of the form, $ys(t) = A\sin(2t) + B\cos(2t)$ for our driver is $\sin(2t)$.

b) Put $ys(t)$ into (1) and collect like terms, i.e. all the $\sin(2t)$ and $\cos(2t)$ terms on the left hand side of the resulting equation.

t s	20.0	20.5	21.0	21.5	22.0	22.5	23.0	23.5	24.0	24.5	25.0
$y(t)$ m	0.006	0.038	0.036	0.001	-0.033	-0.037	-0.006	0.031	0.040	0.013	-0.026
t s	25.5	26.0	26.5	27.0	27.5	28.0	28.5	29.0	29.5	30.0	
$y(t)$ m	-0.040	-0.017	0.022	0.041	0.023	-0.016	-0.040	-0.027	0.011	0.039	

Table 1. Steady state data from a a system with unknown c and k and known $m = 4$ that is driven by $f(t) = \sin(2t)$ with initial conditions $y'(0) = 0$ and $y(0) = 0$.

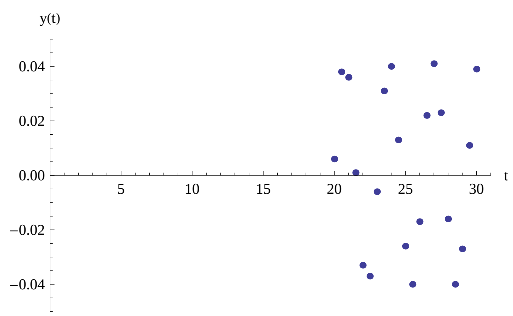

Figure 1. Plot of steady state data over entire time domain.

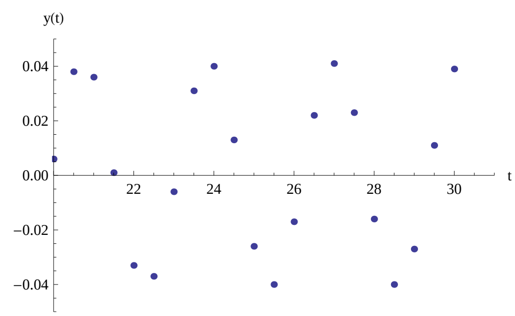

Figure 2. Plot of steady state data well into the steady state time frame.

c) Equate the coefficients from $\sin(2t)$ and $\cos(2t)$ on both sides of our equation. You should get the following:

$$\begin{aligned} 2Ac + Bk - 16B &= 0 \\ -2Bc + Ak - 16A &= 1 \end{aligned} \tag{2}$$

d) Solve (2) for A and B. Call your solutions for A and B, As and Bs respectively. These will be fractions in terms of c and k.

e) Replace A and B in $ys(t) = A\sin(2t) + B\cos(2t)$ with As and Bs into respectively and obtain a steady state solution in terms of c and k. Call this solution $yss(t)$. It will have t, c, and k, as variables.

f) Form the sum of square errors function, $SS(c, k)$, as offered in (3), between the model prediction at each data point and the observed data,

$$SS(c,k) = \sum_{i=1}^{n} (ys(t_i) - \hat{y}_i)^2 \,, \tag{3}$$

where t_i is the time of the i^{th} observation, $ys(t_i)$ is the value of our steady state solution at time t_i, and $\hat{y}_i$) is the observed dependent variable y for the i^{th} observation. Figure 3 shows a contour plot of the sum of square errors function in which you can see the region to best hunt for candidates for c and k which will produce a minimum $SS(c, k)$.

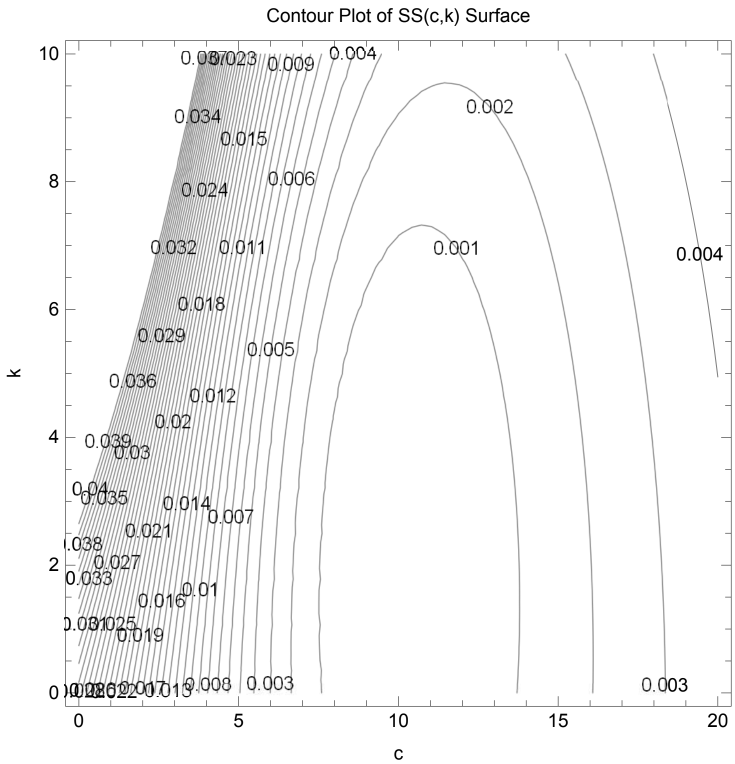

Figure 3. Contour plot of the sum of squares function $SS(c, k)$ in which showing the region to best hunt for candidates for c and k which will produce a minimum $SS(c, k)$.

g) Find the minimum value of $SS(c, k)$ using reasonable guesses for c and k for initial estimates of c and k, based on the contour plot in Figure 3.

h) Take your parameter values from (g), call them, c_{best} and k_{best} and place them in differential (1) to form differential (4):

$$4 \cdot y''(t) + c_{\text{best}} \cdot y'(t) + k_{\text{best}} \cdot y(t) = f(t)\,, \quad y(0) = 0 \quad \text{and} \quad y'(0) = 0 \tag{4}$$

i) Solve differential Equation (4) and plot your solution over the data plot you rendered above. Comments are in order.