



STUDENT VERSION

LINEAR, NONHOMOGENEOUS SYSTEM SOLUTION STRATEGIES

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STATEMENT

In [3] we considered a system of linear differential equations (1) of the form $\vec{X}'(t) = \mathbf{A} \cdot \vec{X}(t)$ where $\vec{X}(t)$ is the vector of dependent functions we seek to find and \mathbf{A} is the coefficient matrix which relates the dependent functions, in our case in (1), $x(t)$ and $y(t)$, and produced a solution strategy from a conjecture.

$$\begin{aligned}x'(t) &= ax(t) + by(t) \\y'(t) &= cx(t) + dy(t)\end{aligned}\tag{1}$$

with $x(0) = x_0$ and $y(0) = y_0$.

This strategy involved finding eigenvalues and eigenvectors of \mathbf{A} , namely eigenvalues of λ_1 and λ_2 , with respective eigenvectors $\vec{\mathbf{u}}_1(t)$ and $\vec{\mathbf{u}}_2(t)$, and building a solution (2) from linear combinations involving these,

$$\vec{\mathbf{X}}(t) = c_1 e^{-\lambda_1 t} \vec{\mathbf{u}}_1(t) + c_2 e^{-\lambda_2 t} \vec{\mathbf{u}}_2(t)\tag{2}$$

where c_1 and c_2 are determined by the initial conditions for the differential equations.

This system (1) is homogeneous in nature, i.e. the derivatives of our dependent functions $x'(t)$ and $y'(t)$, respectively, are offered in terms of functions of $x(t)$ and $y(t)$ only, and are linear, without any intervention from other functions. In this Technique Narrative we consider developing a solution strategy for a nonhomogeneous system of linear differential equations (3).

$$\begin{aligned}x'(t) &= ax(t) + by(t) + g_1(t) \\y'(t) &= cx(t) + dy(t) + g_2(t)\end{aligned}\tag{3}$$

with $x(0) = x_0$ and $y(0) = y_0$. We can put this in matrix form (4):

$$\vec{\mathbf{X}}'(t) = \mathbf{A} \cdot \vec{\mathbf{X}}(t) + \vec{\mathbf{G}}(t), \quad (4)$$

where $\vec{\mathbf{X}}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, while $\vec{\mathbf{G}}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$.

Some Preliminaries

We are going to stand on the shoulders of giants; those who have gone before and have laid a firm foundation on which all can stand when making claims about solutions to differential equations. In particular, with regard to possible solutions to (3) there is theory, [1, pp. 358-359] and [2, pp. 248-250] to cite two sources, which says that under broad conditions solutions to (3) exist and are unique. Moreover, if we can build a candidate which satisfies the differential equation system (3) AND the initial conditions then we have THE UNIQUE solution and need look no more. So our focus will be on building a solution and showing it satisfies both the differential equation system (3) AND the initial conditions. The approach we are going to take is called *Variation of Parameters*.

We will first find a broad form solution to the homogeneous system, $\vec{\mathbf{X}}(t)'(t) = \mathbf{A} \cdot \vec{\mathbf{X}}(t)$, calling it $\vec{\mathbf{X}}_{\mathbf{H}}(t)$. Our next step will be to produce a particular solution of the nonhomogeneous system $\vec{\mathbf{X}}'(t) = \mathbf{A} \cdot \vec{\mathbf{X}}(t) + \vec{\mathbf{G}}(t)$, calling it $\vec{\mathbf{X}}_{\mathbf{P}}(t)$. We will then add the two solutions, $\vec{\mathbf{X}}_{\mathbf{H}}(t)$ and $\vec{\mathbf{X}}_{\mathbf{P}}(t)$ and use the initial conditions to give specificity to the “broad form solution” $\vec{\mathbf{X}}_{\mathbf{H}}(t)$ to render a unique solution (5) to (3).

$$\vec{\mathbf{X}}(t) = \vec{\mathbf{X}}_{\mathbf{H}}(t) + \vec{\mathbf{X}}_{\mathbf{P}}(t) \quad (5)$$

Step by Step Analysis

We render a general analysis for producing a solution to (3) and also offer a running example problem, (6), in this case, of a system of two constant coefficient, linear, nonhomogeneous differential equations. The analysis for larger systems, say, three and four constant coefficient, linear nonhomogeneous differential equations is the same, only the computation is more intense.

$$\begin{aligned} x'(t) &= -2x(t) + 1y(t) + 2e^{-t} \\ y'(t) &= 1x(t) - 2y(t) + 3t \end{aligned} \quad (6)$$

with $x(0) = 5$ and $y(0) = 0$.

We can put this in matrix form (7):

$$\overbrace{\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}}^{\vec{\mathbf{X}}'(t)} = \overbrace{\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}}^{\mathbf{A}} \cdot \overbrace{\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}}^{\vec{\mathbf{X}}(t)} + \overbrace{\begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}}^{\vec{\mathbf{G}}(t)}. \quad (7)$$

Homogeneous Solution $\vec{\mathbf{X}}_{\mathbf{H}}(t)$

We will first build a solution to the homogeneous system, (8). The elements used to build this homogeneous solution are from the eigenvalues and eigenvectors of our system matrix \mathbf{A} .

$$\overbrace{\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}}^{\vec{\mathbf{X}}'(t)} = \overbrace{\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}}^{\mathbf{A}} \cdot \overbrace{\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}}^{\vec{\mathbf{X}}(t)}. \quad (8)$$

Assignment 1

For the system matrix

$$\mathbf{A} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \quad (9)$$

in our running problem show that the eigenvalues and corresponding eigenvectors of \mathbf{A} are $\lambda_1 = -3$ and $\lambda_2 = 1$, with respective eigenvectors $\vec{\mathbf{u}}_1(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\vec{\mathbf{u}}_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Be sure to demonstrate how you found these.

Assignment 2

Show the following:

- a) $e^{\lambda_1 t} \vec{\mathbf{u}}_1(t) = e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $e^{\lambda_2 t} \vec{\mathbf{u}}_2(t) = e^{1t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are each a solution to the homogeneous system (8) and
- b) the linear combination, which we will call our homogeneous solution, $\vec{\mathbf{X}}_{\mathbf{H}}(t) = c_1 e^{-\lambda_1 t} \vec{\mathbf{u}}_1(t) + c_2 e^{-\lambda_2 t} \vec{\mathbf{u}}_2(t) = c_1 e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a solution to the homogeneous system (8).

Particular Solution $\vec{\mathbf{X}}_{\mathbf{P}}(t)$

We now turn our attention to constructing a particular solution, $\vec{\mathbf{X}}_{\mathbf{P}}(t)$, to our nonhomogeneous system (7).

We will use the *fundamental matrix* for matrix \mathbf{A} . The fundamental matrix of \mathbf{A} consists of columns of vectors of the solution to the homogeneous portion of our system obtained from eigenvalues and eigenvectors of \mathbf{A} in Activity 2. We will use the symbol Φ or $\Phi(t)$ (the capital Greek letter phi) to denote the fundamental matrix, but some presentations [1, p. 414] use the capital Greek letter psi, Ψ .

In our problem we see that

$$\Phi(t) = \begin{pmatrix} -1e^{-3t} & 1e^t \\ 1e^{-3t} & 1e^t \end{pmatrix}. \quad (10)$$

Indeed, the solution for the homogeneous system (8) can be expressed using $\Phi(t)$:

$$\Phi(t) \cdot \vec{c} = \Phi(t) \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1e^{-3t} & 1e^t \\ 1e^{-3t} & 1e^t \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1 e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (11)$$

A reasonable conjecture for the nonhomogeneous system is

$$\vec{X}_{\mathbf{P}}(t) = \Phi(t) \cdot \vec{u} = \begin{pmatrix} -1e^{-3t} & 1e^t \\ 1e^{-3t} & 1e^t \end{pmatrix} \cdot \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \quad (12)$$

where $u_1(t)$ and $u_2(t)$ are to be determined, namely some non-constant multiples of the homogeneous solutions.

We can test the reasonableness of this conjecture by substituting $\vec{X}_{\mathbf{P}}(t)$ directly into (7). However, let us continue symbolically before getting caught up in too many numbers and functions.

To substitute $\vec{X}_{\mathbf{P}}(t)$ directly into (7) symbolically we will need to differentiate $\Phi(t)$. We now make the direct substitution of our conjecture (12) into (7), first using the product rule for differentiation:

$$\Phi'(t) \cdot \mathbf{u}(t) + \Phi(t) \cdot \mathbf{u}'(t) = \mathbf{A} \cdot \Phi(t) \cdot \mathbf{u}(t) + \mathbf{G}(t). \quad (13)$$

Notice we are dropping the vector (arrow) notation, but using the boldface notation for vectors.

Now since $\Phi(t)$ is the fundamental matrix of \mathbf{A} , consisting of column solutions to the homogeneous system, we know that $\Phi'(t) = \mathbf{A} \cdot \Phi(t)$. So we can cancel these terms, $\Phi'(t) \cdot \mathbf{u}(t) = \mathbf{A} \cdot \Phi(t) \cdot \mathbf{u}(t)$, from both sides of (13). This leaves

$$\Phi(t) \cdot \mathbf{u}'(t) = \mathbf{G}(t). \quad (14)$$

If we know the determinant of $\Phi(t)$ is not zero, i.e. $\Phi(t)$ is nonsingular (this occurs if our eigen solutions which made up $\Phi(t)$ are linearly independent - can usually safely assume this - or in the two by two cases, are not scalar multiples of each other) then we can find $\Phi^{-1}(t)$ and use it to determine our functions $\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$. Let us obtain $\mathbf{u}'(t)$ and then $\mathbf{u}(t)$ by first multiplying both sides of (14) by $\Phi^{-1}(t)$,

$$\mathbf{u}'(t) = \Phi^{-1}(t) \cdot \mathbf{G}(t). \quad (15)$$

Finally, we are in a position to determine the functions of $\mathbf{u}(t)$ by integration in (16)

$$\mathbf{u}(t) = \int \Phi^{-1}(t) \cdot \mathbf{G}(t) dt + \mathbf{k}, \quad (16)$$

where \mathbf{k} is a vector of constants of integration. Since this is indefinite integration we will pick $\mathbf{k} = \vec{\mathbf{0}}$.

We return to our nonhomogeneous system (7) and denote our success in finding a particular solution, $\mathbf{X}_P(t)$:

$$\mathbf{X}_P(t) = \Phi(t) \cdot \mathbf{u}(t) = \Phi(t) \cdot \int \Phi^{-1}(t) \cdot \mathbf{G}(t) dt. \quad (17)$$

We are now ready to build our General Solution and get back to the computations for our problem.

General Solution $\vec{\mathbf{X}}(t) = \vec{\mathbf{X}}_H(t) + \vec{\mathbf{X}}_P(t)$

From the previous two sections in which we obtained the homogeneous solution $\vec{\mathbf{X}}_H(t)$ and the nonhomogeneous (particular) solution $\vec{\mathbf{X}}_P(t)$ we summarize:

$$\vec{\mathbf{X}}(t) = \vec{\mathbf{X}}_H(t) + \vec{\mathbf{X}}_P(t) = \Phi(t) \cdot \mathbf{c} + \Phi(t) \cdot \int \Phi^{-1}(t) \cdot \mathbf{G}(t) dt, \quad (18)$$

where the constants c_1 and c_2 which make up vector \mathbf{c} can be obtained from the initial conditions of the system of differential equations.

We return to our problem now to see how this plays out. Recall that our system matrix is $\mathbf{A} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ and our fundamental matrix is $\Phi(t) = \begin{pmatrix} -1e^{-3t} & 1e^t \\ 1e^{-3t} & 1e^t \end{pmatrix}$.

Activity 3

a) Show that the inverse matrix of our fundamental matrix $\Phi(t)$ for our problem is

$$\Phi^{-1}(t) = \begin{pmatrix} -\frac{e^{3t}}{2} & \frac{e^{3t}}{2} \\ \frac{e^t}{2} & \frac{e^t}{2} \end{pmatrix} \quad (19)$$

b) Substitute the information about our problem (7) into (18) and show that we obtain,

$$\mathbf{X}(t) = \mathbf{X}_H(t) \cdot \mathbf{c} + \mathbf{X}_P(t) = c_1 \begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} + \begin{pmatrix} e^{-t}t + t + \frac{e^{-t}}{2} - \frac{4}{3} \\ e^{-t}t + 2t - \frac{e^{-t}}{2} - \frac{5}{3} \end{pmatrix}. \quad (20)$$

c) Use initial conditions $x(0) = 5$ and $y(0) = 0$ to render a complete and final solution (21) to (7).

$$\mathbf{X}(t) = \begin{pmatrix} e^{-t}t + t + \frac{11e^{-3t}}{6} + \frac{9e^{-t}}{2} - \frac{4}{3} \\ e^{-t}t + 2t - \frac{11e^{-3t}}{6} + \frac{7e^{-t}}{2} - \frac{5}{3} \end{pmatrix}. \quad (21)$$

d) Confirm that your final solution really is a solution to (7).

Activity 4

Solve completely, using the techniques developed here, the system of constant coefficient, linear, nonhomogeneous differential equations (22),

$$\begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} = \begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix} \cdot \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} + \begin{pmatrix} 2e^{-t} \\ 3t \\ 2t \end{pmatrix} \quad (22)$$

with $x(0) = 5$, $y(0) = 0$, and $z(0) = 1$.

SOME FINAL WORDS

Quite often the computations, especially for systems of three or more differential equations, become quite cumbersome when done by hand and so it is sometimes prudent to use a computer algebra system such as SAGE, Maple, or Mathematica to compute the individual pieces and steps for solutions. Indeed, it could be more appropriate to use a powerful command in that system to get a solution directly or to solve the problem numerically. In all cases it is wise to compare the solution to the situation under study, e.g., examine a plot of the solutions for their geometric significance.

REFERENCES

- [1] Boyce, W. E. and R. C. DiPrima. 2005. *Elementary Differential Equations and Boundary Value Problems, Eighth Edition*. New York: John Wiley & Sons, Inc.
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- [3] Winkel, B. 2015. 5-12-Text-S-LinearSystemConjecture. <https://www.simiode.org/resources/1237>. Accessed 25 August 2016.